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Random Walk in an Alcove of an Affine Weyl Group, and Non-colliding Random Walks on an Interval

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We use a reflection argument, introduced by Gessel and Zeilberger, to count the number of k-step walks between two points which stay within a chamber of a Weyl group. We apply this technique to walks in the alcoves of the classical affine Weyl groups. In all cases, we get determinant formulas for the number of k-step walks. One important example is the region $m > x_1 > x_2 > \cdots > x_n > 0$, which is a rescaled alcove of the affine Weyl group $\tilde{C}_n$. If each coordinate is considered to be an independent particle, this models n non-colliding random walks on the interval $(0, m)$. Another case models n non-colliding random walks on a circle.

Key Words: random walk; affine Weyl group; lattice path enumeration; reflection principle; Weyl chamber; alcove.

1. INTRODUCTION

The *ballot problem*, a classical problem in random walks, asks how many ways there are to walk from the origin to a point $(\lambda_1, \ldots, \lambda_n)$, taking k unit-length steps in the positive coordinate directions while staying in the region $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. The solution is known and leads to the hook-length formula for Young tableaux; a combinatorial proof, using a reflection argument, is given in [17, 19].

The same reflection argument has also been applied to the case of n independent diffusions, or n discrete processes which cannot pass each other without first colliding. Using this method, Karlin and McGregor [13, 12] give a determinant formula for the probability or measure for the n particles, starting at known positions, not to have collided up to time t and to be in given positions. Hobson and Werner [10] generalize this argument to n independent Brownian motions in an interval or on a circle.

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Gessel and Zeilberger [6], and independently Biane [1], consider a more general question, for which some of the same techniques apply. For certain "reflectable" walk-types, we can count the number of k-step walks between two points of a lattice, staying within a Weyl chamber. The argument generalizes naturally to "reflectable" diffusions [7]; we can compute the density function for the diffusion started at a point η to stay within the Weyl chamber up to time t and be at a point λ.

Grabiner and Magyar [8] classify all reflectable random walks for finite Weyl groups, and compute determinant formulas for many important cases, including walks with steps $\pm e_i$ or with steps $\pm\frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_n$ in all of the classical Weyl groups.

We prove analogous results for the alcoves of affine Weyl groups. In contrast to the chambers of classical Weyl groups, these are bounded regions, such as $m > x_1 > \cdots > x_n > 0$. The reflectable random walks for the affine Weyl groups are the same as for the corresponding classical Weyl groups. We use these reflection arguments to find determinant formulas for the number of walks of length k which stay within the alcoves of the classical affine Weyl groups. We then simplify the determinant of infinite sums to get a determinant of finite sums of sines and cosines or of exponentials of cosines, depending on the random walk.

Many results are known in the $\tilde{A}_{n-1}$ cases, in which the region in $\mathbb{R}^2$ is $x_1 > x_2 > \cdots > x_n > x_1 - m$. The $\tilde{A}_1$ case (or equivalently $\tilde{B}_1$, region $m > x > 0$) is a single random walk on an interval. This is the classical gambler's ruin problem; gamblers with initial stakes of η and $N-\eta$ chips bet one chip at a time until one is broke. The probability that the gambler who started with η will first go broke after k bets is

$$\frac{1}{N}\sum_{r=1}^{N-1} \cos^{k-1}(\pi r/N)\sin(\pi r/N)\sin(\pi\eta r/N). \tag{1}$$

This formula goes back to Lagrange [3, p. 353]. Similar calculations show that the probability that the gambler who started with η will have λ left after k bets, with neither player going broke, is

$$\frac{2}{N}\sum_{r=1}^{N-1} \cos^{k}(\pi r/N)\sin(\pi\lambda r/N)\sin(\pi\eta r/N). \tag{2}$$

The reflection principle was applied to this problem by Grossman [9]. A q-analogue of this formula was computed by Krattenthaler and Mohanty [15].

The n-dimensional case, with steps only in the positive coordinate directions, was solved by Filaseta [4], and a q-analogue was proved by a reflection argument by Krattenthaler [14]. This case can be viewed as a variation of the n-candidate ballot problem: how many ways are there to arrange the ballots so that the candidates stay in order, with the difference between the first-place and last-place candidates also limited? (In the ballot problem, equal coordinates are allowed, but we can translate the start and end by $(n-1, n-2, ..., 0)$ to make the inequalities strict.)

Our most important case is the model of n non-colliding particles in discrete random walks in the interval $(0, m)$. Equivalently, we can consider a single random walk in n dimensions in the region $m > x_1 > \cdots > x_n > 0$, with each coordinate of the n-dimensional walk corresponding to one of the n particles, and permitted steps in the positive and negative coordinate directions. We then simplify the determinant of infinite sums, computing the exponential generating function in the number of steps. This gives the number of walks for one particle to go from η to λ while staying in this region, or for n particles to go from η_i to λ_i while staying in the interval $(0, m)$ and not colliding. The exponential generating function is

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| \frac{1}{2m} \sum_{r=0}^{2m-1} 2 \sin(\pi r(\lambda_j)/m) \sin(\pi r(\eta_i)/m) \exp(2x \cos(\pi r/m)) \right|. \tag{3}$$

Using a reflection argument of Gessel and Krattenthaler [5] which uses the methods of [10, 13], we can also give a formula for the model of n independent discrete random walks on the circle. As a one-particle model, this is a variation of the affine Weyl group $\tilde{A}_{n-1}$ counting walks to multiple destinations in $\mathbb{R}^n$ which become equivalent when projected the circle.

2. REFLECTABLE RANDOM WALKS

A walk-type is defined by a lattice L, a set S of allowable steps between lattice points, and a region C to which the walks are confined. Without affecting the walk problems, we may restrict L and C to the linear span of S, so that L, S, and C have the same linear span.

We will assume C is a *Weyl chamber* of a finite Weyl group W, or an *alcove* of an affine Weyl group. The following definitions and results are given in [11].

For a finite Weyl group, we have $L, S, C \subset \mathbb{R}^n$; C is defined by a system of simple roots $\Delta \subset \mathbb{R}^n$ as

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid (\alpha, \mathbf{x}) \geqslant 0 \text{ for all } \alpha \in \Delta\}; \tag{4}$$

the orthogonal reflections r_α: $\mathbf{x} \mapsto \mathbf{x} - \frac{2(\alpha, \mathbf{x})}{(\alpha, \alpha)} \alpha$ preserve L and S for all α in Δ; and the r_α generate a finite group W of linear transformations, the *Weyl group*.

The full root system Φ consists of the images of all roots under W; these roots come in pairs, and we can take the system Φ^+ of *positive roots* to be the set of all roots which are positive linear combinations of the simple roots; this includes one root from each pair. The hyperplanes orthogonal to the positive roots are all reflections in W, and they partition the space into $|W|$ disjoint Weyl chambers.

For any root α, the corresponding *coroot* is $2\alpha/(\alpha, \alpha)$. The corresponding affine Weyl group $\tilde{W}$ is the semidirect product of W and the translation group of the coroot lattice; that is, it is generated by reflections

$$r_{\alpha, k}: \mathbf{x} \mapsto \mathbf{x} - \frac{2(\alpha, \mathbf{x}) - k}{(\alpha, \alpha)} \alpha$$

for all roots α and all integers k. Again, the group $\tilde{W}$ contains the reflections $r_{\alpha, k}$ not only for simple α but for all α, and these hyperplanes of reflection partition space into *alcoves*. The alcoves are the regions bounded by $|\Phi^+|$ simultaneous inequalities $k_\alpha < (\lambda, \alpha) < k_\alpha + 1$, as α runs over all roots in Φ^+, for k_α in $\mathbb{Z}$. The *principal alcove A* is bounded by the inequalities $0 < (\lambda, \alpha) < 1$ for all positive roots α; this can be shown to be non-empty.

EXAMPLE. Let W be the symmetric group S_n permuting the n coordinates in $\mathbb{R}^n$; this is generated by reflections in the simple roots $\Delta = \{e_i - e_{i+1}, 1 \leqslant i \leqslant n-1\}$ (where e_i is the ith coordinate vector), which gives Weyl chamber $x_1 > x_2 > \cdots > x_n$. The positive roots are $e_i - e_j$ for $i < j$; their hyperplanes $x_i = x_j$ give $n!$ Weyl chambers, one for each permutation of the coordinates. The corresponding affine Weyl group contains a reflection in the hyperplane $x_i - x_j = k$ for any integer k; the principal alcove is thus $x_1 > x_2 > \cdots > x_n > x_1 - 1$. (While this alcove is unbounded, W really acts only on the subspace in which $\sum x_i = 0$, and the alcove is bounded in that subspace.)

We would like to classify those Weyl chambers and alcoves which allow us to reflect a walk from the point at which it hits a wall. The definition of a reflectable walk from [8] generalizes easily to the affine case.

DEFINITION. A walk-type (L, S, C) is *reflectable* with respect to the finite or affine Weyl group W if the steps S are symmetric under the finite Weyl group, and the following equivalent conditions hold:

(1) Any step $s \in S$ from any lattice point in the interior of C will not exit C.

(2) For each simple root α_i, there is a real number k_i such that $(\alpha_i, s) = \pm k_i$ or 0 for all steps $s \in S$; (α_i, λ) is an integer multiple of k_i for all $\lambda \in L$; and if C is an alcove of an affine Weyl group, then $1/k_i$ must be an integer.

To see that these conditions are equivalent, note that if α_j is any root which is symmetric to α_i, then $k_j = k_i$. Thus the second condition guarantees than L contains only points whose dot-product with α_j is an integer multiple of k_j, and a single step can change the dot product 0 or $\pm k_j$. Therefore, the walk cannot go from one side of the wall $(\alpha_j, \lambda) = 0$ to the other without stopping on a wall. Likewise, since each alcove wall is the set of points with $(\alpha_j, \lambda) = m$ for some m, and m is a multiple of k_j, the walk cannot go from one side to the other in a single step.

EXAMPLE. In the example above, in which W is the symmetric group, the steps $\pm e_i/t$ on $L = \mathbb{Z}^n$ give a reflectable random walk, with each $k_i = 1/t$. This is reflectable for the affine alcove as well provided that t is an integer.

By a similar argument, all of the reflectable random walks on the finite Weyl groups give reflectable walks on the corresponding affine Weyl groups. These steps are enumerated in [8]; they turn out to be precisely the Weyl group images of the *minuscule weights* [2], those weights with dot-product 0 or ± 1 with every root. The reflectable walks include the weights which are minuscule for only one of B_n and C_n. In the Bourbaki numbering [2], the allowed step sets are the Weyl group images of the following $\check{\omega}_i$, the duals of the fundamental roots:

A_n:	$\check{\omega}_1, \ldots, \check{\omega}_n$, all compatible,
B_n, C_n:	$\check{\omega}_1, \check{\omega}_n$, not compatible,
D_n:	$\check{\omega}_1, \check{\omega}_{n-1}, \check{\omega}_n$, all compatible,
E_6 :	$\check{\omega}_1, \check{\omega}_6$, compatible,
E_7:	$\check{\omega}_7$.
E_8, F_4, G_2:	none.

We can also get a reflectable walk by taking any union of compatible step sets (step sets which give the same k_i), and we can add the zero step.

For the minuscule weights, which include all of the above weights for the Weyl groups A_n, D_n, E_6 and E_7, and also include the weight $\check{\omega}_1$ of B_n, and $\check{\omega}_n$ of C_n, the allowed steps have dot product 0 or ± 1 with every root; thus multiplying the steps by $1/t$ for any positive integer t gives a reflectable random walk for the affine Weyl group.

For the weights $\check{\omega}_n$ of B_n and $\check{\omega}_1$ of C_n, the dot product with the short roots is 0 or ± 1, and with the long roots is 0 or ± 2. Thus, for these cases, we must multiply the steps by $1/2t$ for a positive integer t to satisfy the reflectability condition.

In the natural cases we study later, it will be clear from the definitions that the first reflectability condition is satisfied.

3. THE REFLECTION ARGUMENT OF GESSEL AND ZEILBERGER

In a reflectable random walk problem, we want to compute $b_{\eta\lambda, k}$, the number of walks from η to λ of length k which stay in the interior of a Weyl chamber or alcove. For example, the ballot problem can be converted to this form by starting at the point $(n-1, n-2, \ldots, 0)$ instead of the origin, and requiring the coordinates to remain strictly ordered.

Let $c_{\gamma, k}$ denote the number of random walks of length k, with steps in S, from the origin to γ, but *unconstrained* by a chamber. The fundamental result of Gessel and Zeilberger [6] (also proved independently by Biane [1] for finite Weyl groups), is

THEOREM 3.1. *If the walk from η to λ is reflectable, then*

$$b_{\eta\lambda, k} = \sum_{w \in W} \operatorname{sgn}(w)\, c_{w(\lambda)-\eta, k}. \tag{5}$$

If W is an affine group, this is an infinite sum, but only finitely many terms are nonzero for any fixed k.

Proof. Every walk from η to any $w(\lambda)$ which does touch at least one wall has some first step j at which it touches a wall. Let the wall be a hyperplane perpendicular to α_i, choosing the largest i if there are several choices [16]; the reflection in that wall is a reflection $r_{\alpha_i, k}$. Reflect all steps of the walk after step j across that hyperplane; the resulting walk is a walk from η to $r_{\alpha_i, k} w(\lambda)$ which touches the same wall at step j. This clearly gives a pairing of walks, and since $r_{\alpha_i, k}$ has sign -1, these two walks cancel out in (5). The only walks which do not cancel in these pairs are the walks which stay within the Weyl chamber or alcove, and since $w(\lambda)$ is inside the Weyl chamber or alcove only if w is the identity, this is the desired number of walks. ∎

The specific case of the theorem in which W is the symmetric group S_n was proved by Karlin and McGregor [13]. We can view the S_n process as n separate walks which are not allowed to collide, rather than one walk restricted to the region $x_1 > \cdots > x_n$, and interchange two particles when they collide. If $n_{ij,k}$ is the number of walks of length k from η_i to λ_j, then the formula (5) becomes

$$b_{\eta\lambda,k} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, n_{i,\sigma(i),k} = \det_{n\times n} |n_{ij,k}|. \tag{6}$$

4. UNCONSTRAINED WALKS

Two natural choices of step sets are the positive and negative coordinate directions $\pm e_i$, or the 2^n diagonals $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$. Both cases give reflectable walks for all the classical Weyl chambers. To get a non-trivial walk for the alcoves of the affine Weyl groups, we have to re-scale either the steps or the alcoves. It is more natural to state the problem if we leave the steps alone and re-scale the alcoves by a factor of m or $2m$, so that we study the steps $\pm e_i$ in the alcoves of the Weyl group mW or $2mW$.

For the diagonals, $c_{\gamma,k}$ is easy to compute, and for the coordinate directions, the exponential generating function

$$h_\gamma(x) = \sum_{k=0}^{\infty} c_{\gamma,k} x^k/k!$$

is easy to compute.

If the steps S are the diagonals $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$, then the walk is essentially n independent walks in the coordinate directions. Each step involves a step of $\pm\frac{1}{2}$ in each coordinate direction, and thus the walk will go forwards a distance of γ_i if there are γ_i more forward steps than backward steps. That is,

$$c_{\gamma,k} = \prod_{i=1}^{n} \binom{k}{(k/2)+\gamma_i}. \tag{7}$$

If the steps S are $\pm e_i$, the positive and negative coordinate directions, let $\chi(\mathbf{u}) = \sum_{i=1}^{n} u_i + u_i^{-1}$, the generating function for the steps in the formal monomials $u^{(x_1, \dots, x_n)} = u_1^{x_1} \cdots u_n^{x_n}$. Then we have

$$c_{\gamma, k} = \chi(\mathbf{u})^k|_{\mathbf{u}^\gamma},$$

where $|_{\mathbf{u}^\gamma}$ denotes the coefficient of $\mathbf{u}^\gamma$ in the polynomial. This gives

$$h_\gamma(x) = \sum_{k=0}^{\infty} c_{\gamma, k} x^k / k! = \exp(x\chi(\mathbf{u}))|_{\mathbf{u}^\gamma}$$

for any walk, and for this walk, we have

$$h_\gamma(x) = \sum_{k=0}^{\infty} \prod_{i=1}^{n} \exp(x(u_i + u_i^{-1}))|_{\mathbf{u}^\gamma}.$$

This infinite sum can be written as a product of hyperbolic Bessel functions, using the generating-function definition of the Bessel functions [18]. We have

$$\begin{aligned} \exp(x(u+u^{-1})) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=-k}^{k} \binom{k}{j} u^{k-2j} \\ &= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^k}{k!} \binom{k}{(k+m)/2} \\ &= \sum_{m=-\infty}^{\infty} u^m \sum_{t=0}^{\infty} \frac{x^{2t+m}}{t!(t+m)!} \\ &= \sum_{m=-\infty}^{\infty} u^m I_m(2x). \end{aligned}$$

Thus, in this case, the exponential generating function for the unconstrained walks is

$$h_\gamma(x) = \prod_{i=1}^{n} I_{\gamma_i}(2x). \tag{8}$$

5. NON-COLLIDING RANDOM WALKS ON AN INTERVAL, AND FORMULAS FOR $\tilde{C}_n$

The basic techniques are similar for all the classical groups; the case of $\tilde{C}_n$ is the simplest as well as the most important case.

The walk with steps $\pm e_i$ in the chamber $m > x_1 > x_2 > \cdots > x_n > 0$ is equivalent to a walk of n independent particles in the interval $(0, m)$, with the process terminating if two particles collide or if one particle hits an end of the interval. It is reflectable if m is an integer. The Weyl chamber for $\tilde{C}_n$ is $\frac{1}{2} > x_1 > x_2 > \cdots > x_n > 0$ because one of the roots is $2e_1$. We thus rescale the group to $2m\tilde{C}_n$, to get the chamber $m > x_1 > x_2 > \cdots > x_n > 0$.

We will also consider the walk for the diagonal steps $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$. This is reflectable in the same chamber if m is an integer or half-integer.

Since the coroots of $\tilde{C}_n$ are $e_i \pm e_j$ and $\pm e_i$, the affine Weyl group $2m\tilde{C}_n$ includes all permutations with any number of sign changes, and translations of any coordinates by multiples of $2m$.

We will first consider the walk for the diagonal steps $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$. We write an element of the Weyl group as the product of σ in the symmetric group, reflections $\varepsilon_i = \pm 1$ in the coordinate directions, and translations by $2mt_i$ in the coordinate directions. Thus Theorem 3.1 gives

$$b_{\eta\lambda,k} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{t_i \in \mathbb{Z}} \sum_{\varepsilon_i = \pm 1} \prod \varepsilon_i c_{(\varepsilon_1 \lambda_{\sigma(1)} + 2mt_1, \ldots, \varepsilon_n \lambda_{\sigma(n)} + 2mt_n) - \eta, k}. \tag{9}$$

Using the formula (7) gives

$$b_{\eta\lambda,k} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{t_i \in \mathbb{Z}} \sum_{\varepsilon_i = \pm 1} \prod_{i=1}^{n} \varepsilon_i \binom{k}{(k/2) + \varepsilon_i \lambda_{\sigma(i)} - \eta_i + 2mt_i}. \tag{10}$$

Splitting each factor into terms with $\varepsilon_i = 1$ and $\varepsilon_i = -1$ gives

$$b_{\eta\lambda,k} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{t_i \in \mathbb{Z}} \prod_{i=1}^{n} \left[\binom{k}{(k/2) + \lambda_{\sigma(i)} - \eta_i + 2mt_i} - \binom{k}{(k/2) - \lambda_{\sigma(i)} - \eta_i + 2mt_i} \right]. \tag{11}$$

We interchange the inner sum and the product to get

$$b_{\eta\lambda,k} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{t_i \in \mathbb{Z}} \left[\binom{k}{(k/2) + \lambda_{\sigma(i)} - \eta_i + 2mt_i} - \binom{k}{(k/2) - \lambda_{\sigma(i)} - \eta_i + 2mt_i} \right]. \tag{12}$$

And the signed sum over permutations is the definition of a determinant, which gives

$$b_{\eta\lambda,k} = \det_{n \times n} \left| \sum_{t_i \in \mathbb{Z}} \binom{k}{(k/2) + \lambda_j - \eta_i + 2mt_i} - \binom{k}{(k/2) - \lambda_j - \eta_i + 2mt_i} \right|. \tag{13}$$

The infinite periodic sums of binomial coefficients can be written as a finite sum of powers of roots of unity, and thus of cosines. Let $\omega = e^{2\pi i/4m}$; then for fixed r and s, we have

$$\omega^{-rs}(\omega^r + \omega^{-r})^k = \sum_j \omega^{r(2j-s)} \binom{k}{(k/2)+j}. \tag{14}$$

Thus, if we take the sum over all r from 0 to $4m-1$, all terms for which $2j \not\equiv s \pmod{4m}$ (that is, $j \not\equiv \frac{s}{2} \pmod{2m}$) will cancel out because of the roots of unity; that is,

$$\frac{1}{4m} \sum_{r=0}^{4m-1} \omega^{-rs}(\omega^r + \omega^{-r})^k = \sum_{j \equiv \frac{s}{2} \pmod{2m}} \binom{k}{(k/2)+j}. \tag{15}$$

Since the original sum was real, we can eliminate the roots of unity in this sum by taking the real part of ω^{-rs} and writing everything in terms of cosines. This gives

$$\frac{1}{4m} \sum_{r=0}^{4m-1} \cos(2\pi rs/4m)(2\cos(2\pi r/4m))^k = \sum_{j \equiv \frac{s}{2} \pmod{2m}} \binom{k}{(k/2)+j}. \tag{16}$$

Substituting this formula into (13) gives a determinant formula for this case.

$$b_{\eta\lambda,k} = \det_{n\times n} \left| \frac{2^k}{4m} \sum_{r=0}^{4m-1} (\cos(2\pi r 2(\lambda_j - \eta_i)/4m) - \cos(2\pi r 2(-\lambda_j - \eta_i)/4m)) \right.$$
$$\left. \cdot \cos^k(2\pi r/4m) \right|. \tag{17}$$

We can convert the difference of cosines to a product of sines by the identity $\cos(\alpha - \beta) - \cos(-\alpha - \beta) = 2\sin\alpha \sin\beta$; doing this and simplifying factors of 2 where possible gives the simplified formula

$$b_{\eta\lambda,k} = \det_{n\times n} \left| \frac{2^{k-1}}{m} \sum_{r=0}^{4m-1} (\sin(\pi r\lambda_j/m)\sin(\pi r\eta_i/m)) \cos^k(\pi r/2m) \right|. \tag{18}$$

For $n = 1$, this is equivalent to the gambler's ruin formula (1) and the similar formula (2). The sum of the stakes N is our $2m$ (and m can be a half-integer), since our steps are $\pm\frac{1}{2}$ rather than ± 1. The gambler's ruin formula counts the probability a walk will end at 0 on the kth step, which is half the probability it will be at 1 after the $(k-1)$st step, so it corresponds to walks from η to $\lambda=\frac{1}{2}$ of length $k-1$. In addition, we are counting walks rather than determining the probabilities for k-step walks, so we have an extra factor of 2^{k-1}.

The final difference is that our sum goes from $r=0$ to $4m-1$ rather than $2m-1$. The $r=0$ term is zero. Adding $2m$ to r changes the sign of $\cos^k(\pi r/2m)$ if k is odd, and changes the signs of $\sin(\pi r\lambda/2m)$ (and likewise $\sin(\pi r\eta/2m)$) if λ (respectively, η) is a half-integer. Thus, if k is odd and $\lambda-\eta$ is an integer, or k is even and $\lambda-\eta$ is a half-integer, the terms from $r=2m$ to $4m-1$ cancel out the terms from $r=0$ to $2m-1$, so the total sum is zero, which is correct because parity makes such a walk impossible. Otherwise, the terms are duplicates, and thus we can multiply by 2 and take only the terms from $r=0$ to $2m-1$. The formulas (1) and (2) as stated are only valid for possible walks, and we must add the extra terms so that we get zero for impossible walks.

The procedure for the steps $\pm e_i$ is similar. The same argument that gave us the determinant (13) of periodic sums of binomial coefficients for the case of diagonals shows that the exponential generating function is a product of periodic sums of Bessel functions. We get

$$g_{\eta\lambda}(x)=\sum_{\sigma\in S_n}\sum_{t_i\in\mathbb{Z}}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\left(I_{\lambda_{\sigma(i)}-\eta_i+2mt_i}(2x)-I_{-\lambda_{\sigma(i)}-\eta_i+2mt_i}(2x)\right). \tag{19}$$

We simplify these periodic sums of Bessel functions by again using the generating-function definition for the hyperbolic Bessel functions

$$\exp(x(z+z^{-1}))=\sum_{j\in\mathbb{Z}} z^j I_j(2x).$$

If we let $z=\omega^r$, where $\omega=e^{2\pi i/2m}$ here rather than $e^{2\pi i/4m}$ as for the diagonals, we have

$$\omega^{-rs}\exp(x(\omega^r+\omega^{-r}))=\sum_{j\in\mathbb{Z}}\omega^{r(j-s)}I_j(2x). \tag{20}$$

As before, we now take $1/2m$ times the sum over all values of r to eliminate the terms for which $j\not\equiv s \pmod{2m}$, and then take the real part of ω^{-rs} to write everything in terms of cosines. This gives

$$\frac{1}{2m}\sum_{r=0}^{2m-1}\cos(2\pi rs/2m)\exp(2x\cos(2\pi r/2m))=\sum_{j\equiv s\,(\mathrm{mod}\,2m)} I_j(2x). \tag{21}$$

Substituting this into our determinant gives the exponential generating function for this random walk.

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| \frac{1}{2m} \sum_{r=0}^{2m-1} (\cos(2\pi r(\lambda_j - \eta_i)/2m) - \cos(2\pi r(-\lambda_j - \eta_i)/2m)) \right.$$
$$\left. \cdot \exp(2x \cos(2\pi r/2m)) \right|. \tag{22}$$

As before, we simplify factors of 2 and the difference of cosines to get the final formula

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| \frac{1}{m} \sum_{r=0}^{2m-1} \sin(\pi r\lambda_j/m) \sin(\pi r\eta_i/m) \right.$$
$$\left. \cdot \exp(2x \cos(\pi r/m)) \right|. \tag{23}$$

Since this is a model of n non-colliding particles, this formula could also be derived from the argument of Karlin and McGregor [12, 13]. The possibility of a walk restricted to an interval is not mentioned there, but the argument is still valid. The formula above could thus be derived from the sum of Bessel functions for the one-dimensional walks using a particle-interchange argument analogous to the one we use below in Theorem 6.1.

6. FORMULAS FOR $\tilde{A}_{n-1}$, AND NON-COLLIDING RANDOM WALKS ON THE CIRCLE

The Weyl group $\tilde{A}_{n-1}$ gives Weyl chamber $x_1 > x_2 > \cdots > x_n > x_1 - 1$, so we rescale by a factor of m to get $x_1 > x_2 > \cdots > x_n > x_1 - m$. The Weyl group acts only on the hyperplane $\sum x_i = 0$. It is easier to study walks in $\mathbb{R}^n$ with steps in the coordinate or diagonal directions as before; we will also project these walks onto the hyperplane on which $\tilde{A}_{n-1}$ acts to study walks on this hyperplane.

This Weyl group gives another important reflectable case, the n steps e_i, in addition to the two reflectable cases we had for $\tilde{C}_n$. The step set e_i is the primary case that has been studied previously [4, 14]. This step set does not give a reflectable walk for $\tilde{B}_n$, $\tilde{C}_n$, or $\tilde{D}_n$ because the step set is not symmetric under the Weyl group.

In addition, we will study the case of n independent random walks on the circle (the interval $[0, m]$ with endpoints identified). We will use the same notation for this problem as for a single n-dimensional walk, as it is analogous to a single n-dimensional walk in $m\tilde{A}_{n-1}$. The desired number of walks $b'_{\eta\lambda,k}$ is the number of ways for particle i to go from η_i to λ_i in k steps such that no two particles collide. The walk is reflectable if it is symmetric under an interchange of particles at any time that two particles are in the same place.

For the steps e_i, the problem is the same for the Weyl group $m\tilde{A}_{n-1}$ acting on $\mathbb{R}^n$ or the hyperplane, and for n non-colliding particles on the circle (the interval $[0, m]$ with endpoints identified) with one moving forward at a time. The points η and $(\eta_1+c, ..., \eta_n+c)$ in $\mathbb{R}^n$ project to the same point on the hyperplane $\sum x_i=0$, but if there are walks from λ to η of k steps, then $\sum(\eta_i-\lambda_i)=k$, and there cannot be walks to $(\eta_1+c, ..., \eta_n+c)$ for $c\neq 0$. (Note that this does not hold for the other step sets because there are backward steps.) Thus there is a bijection between walks in $\mathbb{R}^n$ from λ to η and walks in the hyperplane from the projection of λ to the projection of η. The walks in the hyperplane have steps which are $(n-1)/n$ in one coordinate direction and $-1/n$ in the other coordinate directions; that is, they are the Weyl group images of the fundamental weight $\check{\omega}_1$.

To see that the walks restricted to the alcove on $\mathbb{R}^n$ and of n non-colliding particles on the circle are equivalent, let each coordinate of a walk in $\mathbb{R}^n$ represent an individual particle. The first and last particles will collide if $x_1-m=x_n$, and two adjacent particles will collide if $x_i=x_{i+1}$; these walls are the same as the walls of the Weyl chamber. We want to count the number of k-step walks in the alcove in $\mathbb{R}^n$ in which one particle goes from λ to η. This is the same as the number of sets of non-colliding walks of n particles in which particle i goes from λ_i to η_i, provided that the coordinates of η are translated by multiples of m if necessary so that $\sum(\eta_i-\lambda_i)=k$, and so that they are in decreasing order in an interval of length m. (Again, this argument is not valid for the other step sets, because we do not have the condition $\sum(\eta_i-\lambda_i)=k$.)

For example, a walk of 59 steps from (2, 1, 0) to (4, 3, 5) on a circle of length $m=10$ corresponds to a walk in the alcove in $\mathbb{R}^n$ ending at (24, 23, 15); a walk of 49 steps is impossible without permutation of particles, as the walk in the alcove in $\mathbb{R}^n$ cannot end at (14, 23, 15), and a walk which ends at (23, 15, 14) corresponds to a walk on the circle in which the first particle goes from 2 to 3, not from 2 to 4.

The number of unconstrained walks from η to $\eta+\gamma$ with steps e_i is the multinomial coefficient $k!/\gamma_1!\cdots\gamma_n!$. The Weyl group $m\tilde{A}_{n-1}$ has coroots e_i-e_j, so the Weyl group contains all permutations, with translations of all coordinates by multiples of m such that the sum of translations is zero. Thus, if we look at the constrained walks on $\mathbb{R}^n$ which end at λ, then Theorem 3.1 gives us the sum for any step set

$$b_{\eta\lambda, k} = \sum_{\sigma \in S_n} \sum_{\sum t_i = 0} \operatorname{sgn}(\sigma)\, c_{(mt_1, \ldots, mt_n) + \sigma(\lambda) - \eta, k}, \tag{24}$$

which in our case is

$$b_{\eta\lambda, k} = \sum_{\sigma \in S_n} \sum_{\sum t_i = 0} \operatorname{sgn}(\sigma) \frac{k!}{\prod_{i=1}^{n} (mt_i + \lambda_{\sigma(i)} - \eta_i)!}. \tag{25}$$

This is the formula computed by Filaseta and Krattenthaler [4, 14].

We cannot convert this sum to a single determinant in this case; the condition $\sum t_i = 0$ means that we do not have a periodic infinite sum. We can convert the sum to a sum of determinants by interchanging the order of summation and taking out the constant factor $k!$. This gives

$$b_{\eta\lambda, k} = k! \sum_{\sum t_i = 0} \det_{n \times n} |1/(mt_i + \lambda_j - \eta_i)!|. \tag{26}$$

For the other step sets, the problems of n particles on the circle, of one particle in $\mathbb{R}^n$, and of one particle in the hyperplane $\sum x_i = 0$ on which the Weyl group acts, are not equivalent. We can use our methods to get single determinant formulas for walks of n particles in the circle, or of one particle on the hyperplane.

It is most natural to start with the case of n particles on the circle, as we will use the results of this case in our formulas for the walk on the hyperplane. We will use a reflection argument from [13] closely related to Theorem 3.1, and a technique used in [10] for the analogous problem for Brownian motion. This theorem is essentially a case of [5, Theorem 2].

THEOREM 6.1. *Let $c'_{\gamma, k}$ be the number of unconstrained walks in $\mathbb{R}^n$ (not on the circle) of length k from η to $\eta + \gamma$. We may assume the coordinates of η are in decreasing order; let λ_s be the smallest coordinate of λ. If the walk from η to λ on the circle of size m is reflectable, then the number of constrained walks of length k on the circle is*

$$b'_{\eta\lambda, k} = \sum_{\sigma \in S_n} \sum_{\sum t_i \equiv s \pmod{n}} \operatorname{sgn}(\sigma)\, c'_{(mt_1, \ldots, mt_n) + \sigma(\lambda) - \eta, k}. \tag{27}$$

Proof. The walks counted by $c'_{\gamma,k}$ on $\mathbb{R}^n$ project to walks of n particles on the circle by taking coordinates modulo m. In a walk which projects to a good walk, if particle s goes from η_s to $\lambda_s + mt_s$, then particle i for $i < s$, which starts at $\eta_i > \eta_s$, must end between $\lambda_s + mt_s$ and $\lambda_s + (m+1)\,t_s$ in order not to collide with particle s when the walk is projected to the circle. Since $\lambda_i > \lambda_s$, we must have $t_i = t_s$. Similarly, if $i > s$, we have $t_i = t_s - 1$. Thus the sum of all the t_i, corresponding to the total number of revolutions, is congruent to s modulo n.

Now, consider a bad walk counted by (27). Consider the first time at which two particles collide, and the first pair of particles i and j which collide at that time. Pair it with the corresponding walk obtained by switching particles i and j after the collision. The new σ will differ from the old σ by a transposition; the values of t_i and t_j may change, but the sum $t_i + t_j$ will not because the total forward distance covered by the two particles does not change. Thus the paired walk is still counted in (27), and these walks cancel out.

The only walks which do not cancel out are those which have no collisions, and the only walks which have no collisions and are counted are those in which the particles end in the correct positions. ▮

We can convert the sum from this theorem to a finite sum in the same way as before. For the steps $\pm e_i$, which corresponds to one particle at a time moving independently, we have the exponential generating function

$$g'_{\eta\lambda}(x) = \sum_{\sigma \in S_n} \sum_{\sum t_i \equiv s \,(\mathrm{mod}\, n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} I_{\lambda_{\sigma(i)} - \eta_i + mt_i}(2x). \tag{28}$$

Let $\zeta = e^{2\pi i/mn}$; then we can eliminate the condition $\sum t_i \equiv s \,(\mathrm{mod}\, n)$ by using the fact that

$$\frac{1}{n} \sum_{u=0}^{n-1} \zeta^{um(-s+\sum t_i)}$$

is 1 if $\sum t_i \equiv s \,(\mathrm{mod}\, n)$ and 0 otherwise. Thus we have

$$g'_{\eta\lambda}(x) = \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-ums} \sum_{\sigma \in S_n} \sum_{t_i \in \mathbb{Z}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \zeta^{umt_i} I_{\lambda_{\sigma(i)} - \eta_i + mt_i}(2x). \tag{29}$$

For each value of u, we get a determinant as before, but this time we have period m rather than $2m$. This gives

$$g'_{\eta\lambda}(x) = \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-ums} \det_{n \times n} \left| \sum_{t_i \in \mathbb{Z}} \zeta^{umt_i} I_{\lambda_j - \eta_i + mt_i}(2x) \right|. \tag{30}$$

Except for the factor ζ^{umt_i}, these determinants are the same type as as (19). We can again use $\exp(x(z+1/z)) = \sum_{j \in \mathbb{Z}} z^j I_j(2x)$, and take a sum over different values of z; this time, we will use $z = \zeta^{u+nr}$, and take the sum of only m terms; that is,

$$\frac{1}{m} \sum_{r=0}^{m-1} \zeta^{-(u+nr)\,v} \exp(x(\zeta^{u+nr} + \zeta^{-(u+nr)})) = \sum_{j \equiv v \pmod m} \zeta^{ju} I_j(2x). \tag{31}$$

We cannot eliminate all of the complex terms by taking the real part, but we can get a formula which still contains the complex roots of unity,

$$g'_{\eta\lambda}(x) = \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-ums} \det_{n \times n} \left| \frac{1}{m} \sum_{r=0}^{m-1} \zeta^{-(u+nr)(\lambda_j - \eta_i)} \exp(2x \cos(2\pi(u+nr)/(mn))) \right|. \tag{32}$$

An analogous method works for the diagonal walk. In this walk on the circle, the particles all start at integer positions, or all start at half-integer positions, and at each step, all particles move simultaneously forwards or backwards by $\frac{1}{2}$. We define s as before and $\zeta = e^{2\pi i/2mn}$ since each periodic sum becomes $2m$ terms rather than m, and follow a similar process to get

$$b'_{\eta\lambda,\,k} = \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-2ums} \det_{n \times n} \left| \frac{2^{k-1}}{m} \sum_{r=0}^{2m-1} \zeta^{-(u+nr)(\lambda_j - \eta_i)} \cos^k(\pi(u+nr)/(mn)) \right|. \tag{33}$$

For the case of walks in the $m\tilde{A}_{n-1}$ Weyl chamber on $\mathbb{R}^n$, restricted to end at λ, we have the formula (24). We have the same difficulty in getting a single determinant for all step sets, but we can use the same argument as for (26) to get a sum of determinants. For the steps $\pm e_i$, we get the exponential generating function

$$g_{\eta\lambda}(x) = \sum_{\sum t_i = 0} \det_{n \times n} |I_{mt_i + \lambda_j - \eta_i}(2x)|, \tag{34}$$

and for the diagonal steps $\pm\frac{1}{2} e_1 \cdots \pm\frac{1}{2} e_n$, we get

$$b_{\eta\lambda,\,k} = \sum_{\sum t_i = 0} \det_{n \times n} \left| \binom{k}{(k/2) + mt_i + \lambda_j - \eta_i} \right|. \tag{35}$$

In contrast, we can get a sum of determinants for the walk restricted to the hyperplane $\sum x_i = 0$ restricted to the alcove, by using the formulas for walks on the circle. We look at the walks on $\mathbb{R}^n$, and then project them back to the hyperplane; a walk which ends at $(\lambda_1+c, ..., \lambda_n+c)$ on $\mathbb{R}^n$ projects to a walk which ends at λ on the hyperplane. In particular, the destinations $(\lambda_1+mt, ..., \lambda_n+mt)$ for all integers t must all be considered, and these are exactly the destinations we had in the problem on the circle. Summing the formula (24) over all such destinations gives exactly the same sum as in (27), allowing us to use our previous formulas. We will get one term for each $(\lambda_1+c+mt, ..., \lambda_n+c+mt)$ for $0 \leqslant c < m$ for which the walk can reach such destinations. Thus c may be any integer with $0 \leqslant c < m$ if the steps in $\mathbb{R}^n$ are $\pm e_i$. It must also be an integer if the steps are the diagonals, provided that we choose our λ so that $\eta_i - \lambda_i - k/2$ is an integer; that is, so that λ itself has the correct coordinates to make a walk to λ possible in $\mathbb{R}^n$ by parity; if η were unreachable by parity, then c would have to be a half-integer.

For the walk on the hyperplane equivalent to the walk with steps $\pm e_i$, the $2n$ allowed steps are the projections of $\pm e_i$ on this hyperplane, which are $(n-1)/n$ in one coordinate direction and $-1/n$ in all others, or $-(n-1)/n$ in one coordinate direction and $1/n$ in all others. That is, they are the Weyl group images of the weights $\check{\omega}_1$ and $\check{\omega}_{n-1}$. We define s in (32) as before (it is not changed when we translate all coordinates of η by c), and take the sum of the m terms to get

$$g_{\eta\lambda}(x) = \sum_{c=0}^{m-1} \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-ums} \cdot \det_{n \times n} \left| \frac{1}{m} \sum_{r=0}^{m-1} \zeta^{-(u+nr)(\lambda_j-\eta_i-c)} \exp(2x\cos(2\pi(u+nr)/(mn))) \right|. \tag{36}$$

The walk on $m\tilde{A}_{n-1}$ corresponding to the diagonals is more natural. If a particular diagonal step has p coordinates $\frac{1}{2}$ and $n-p$ coordinates $-1/2$, then it projects to the vector in the hyperplane with p coordinates $(n-p)/n$ and $n-p$ coordinates $-p/n$. These include all the fundamental weights $\check{\omega}_1, ..., \check{\omega}_{n-1}$ of the Weyl group A_{n-1}, including their images under permutations; we also get the zero step with multiplicity 2 from the two diagonals with all coordinates equal:

$$b_{\eta\lambda,k} = \sum_{c=0}^{m-1} \frac{1}{n} \sum_{u=0}^{n-1} \zeta^{-2ums} \cdot \det_{n \times n} \left| \frac{2^{k-1}}{m} \sum_{r=0}^{2m-1} \zeta^{-(u+nr)(\lambda_j-\eta_i-c)} \cos^k(\pi(u+nr)/(mn)) \right|. \tag{37}$$

7. FORMULAS FOR $\tilde{B}_n$ AND $\tilde{D}_n$

The affine Weyl group $2m\tilde{B}_n$ gives the chamber $x_1 > x_2 > \cdots > x_n > 0$, $x_1 + x_2 < 2m$. The affine Weyl group $2m\tilde{D}_n$ gives the chamber $x_1 > x_2 > \cdots > x_n$, $x_1 + x_2 < 2m$, $x_{n-1} > -x_n$. Neither one gives a natural model for n independent particles in one dimension.

The Weyl group $2m\tilde{B}_n$ contains all permutations with any number of sign changes, and with translations of coordinates by multiples of $2m$ such that the total translation is a multiple of $4m$ (since $\tilde{B}_n$ does not have $2e_i$ as a root and thus does not have e_i as a coroot). The Weyl group $2m\tilde{D}_n$ contains all permutations with an even number of sign changes, and with translations of coordinates by multiples of $2m$ such that the total translation is a multiple of $4m$.

We will find the formulas for these cases by using the fact that $2m\tilde{B}_n$ and $2m\tilde{D}_n$ are subgroups of $2m\tilde{C}_n$, of index 2 and 4. We can thus modify the formula (9) by including a term which is 1 if the potential endpoint of the walk is in the Weyl group image of the appropriate subgroup, and 0 otherwise; this is analogous to the technique used to get D_n formulas from B_n formulas in [7, 8].

If an element of $2m\tilde{C}_n$ contains translations by $2mt_i$, then it is in $2m\tilde{B}_n$ if $\sum t_i$ is even, and thus $(1+(-1)^{\sum t_i})/2$ is a factor which is 1 if the element is in $2m\tilde{B}_n$ and 0 otherwise. Likewise, if it contains reflections ε_i, it is only in $2m\tilde{D}_n$ if it is in $2m\tilde{B}_n$ and $\prod \varepsilon_i = 1$, so the appropriate factor is $(1+\prod \varepsilon_i)/2$.

Thus, for $2m\tilde{B}_n$, we have

$$g_{\eta\lambda}(x) = \sum_{\sigma \in S_n} \sum_{\varepsilon_i = \pm 1} \sum_{t_i \in \mathbb{Z}} \frac{1+(-1)^{\sum t_i}}{2} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \varepsilon_i I_{\varepsilon_i \lambda_{\sigma(i)} - \eta_i + 2mt_i}(2x). \tag{38}$$

We take the $1/2$ and $(-1)^{\sum t_i}/2$ terms separately. The $1/2$ term is the term for $2m\tilde{C}_n$. The other term also gives a determinant,

$$\det_{n\times n} \left| \sum_{t_i \in \mathbb{Z}} (-1)^{t_i} \left(I_{\lambda_j - \eta_i + 2mt_i}(2x) - I_{-\lambda_j - \eta_i + 2mt_i}(2x)\right)\right|. \tag{39}$$

These sums are periodic, but with period $4m$ rather than $2m$. We can express the terms with odd and even t_i as separate periodic sums as before,

$$\begin{aligned} \frac{1}{4m} \sum_{u=0}^{4m-1} (\cos(2\pi us/4m) - \cos(2\pi u(2m+s)/4m)) \exp(2x\cos(2\pi u/4m)) \\ = \sum_{j \equiv s \,(\mathrm{mod}\, 4m)} I_j(2x) - I_{2m+j}(2x). \end{aligned} \tag{40}$$

For even u, $\cos(2\pi us/4m)=\cos(2\pi u(2m+s)/4m)$, so these terms are all zero. For odd u, $\cos(2\pi us/4m)=-\cos(2\pi u(2m+s)/4m)$, so the two cosines combine to one term. Thus we can let $u=2r+1$ and write the sum as

$$\frac{1}{2m}\sum_{r=0}^{2m-1}\cos(2\pi(2r+1)\,s/4m)\exp(2x\cos(2\pi(2r+1)/4m))$$

$$=\sum_{j\equiv s\,(\mathrm{mod}\,4m)} I_j(2x)-I_{2m+j}(2x). \tag{41}$$

We put both of these terms together in the determinant, to get

$$\begin{aligned} g_{\eta\lambda}(x)=\frac{1}{2}\Bigg[&\det_{n\times n}\left|\frac{1}{2m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi r(\lambda_j-\eta_i)}{2m}\right)-\cos\left(\frac{2\pi r(-\lambda_j-\eta_i)}{2m}\right)\right]\right.\\ &\left.\cdot\exp(2x\cos(2\pi r/2m))\right|\\ &+\det_{n\times n}\left|\frac{1}{2m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi(2r+1)(\lambda_j-\eta_i)}{4m}\right)\right.\right.\\ &\left.\left.-\cos\left(\frac{2\pi(2r+1)(-\lambda_j-\eta_i)}{4m}\right)\right]\exp(2x\cos(2\pi(2r+1)/4m))\right|\Bigg]. \end{aligned} \tag{42}$$

Again, we simplify this to

$$\begin{aligned} g_{\eta\lambda}(x)=\frac{1}{2}\Bigg[&\det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{2m-1}\sin(\pi r\lambda_j/m)\sin(\pi r\eta_i/m)\exp(2x\cos(\pi r/m))\right|\\ &+\det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{2m-1}\sin(\pi(2r+1)\,\lambda_j/2m)\sin(\pi(2r+1)\,\eta_i/2m)\right.\\ &\left.\cdot\exp(2x\cos(\pi(2r+1)/2m))\right|\Bigg]. \end{aligned} \tag{43}$$

Likewise, for $2m\tilde{D}_n$, we have as our initial formula

$$g_{\eta\lambda}(x)=\sum_{\sigma\in S_n}\sum_{\varepsilon_i=\pm 1}\sum_{t_i\in\mathbb{Z}}\frac{1+(-1)^{\Sigma t_i}}{2}\,\frac{1+\prod\varepsilon_i}{2}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}I_{\varepsilon_i\lambda_{\sigma(i)}-\eta_i+mt_i}(2x). \tag{44}$$

We split this into four terms, taking each combination of the $\frac{1}{2}$ or the other term. When we use $\frac{1}{2}$ rather than $\prod\varepsilon_i/2$, and continue as in (9) or (38), we have a plus sign rather than a minus sign before the term of $I_{-\lambda_{\sigma(i)}-\eta_i+mt_i}(2x)$. Everything else carries out just as before; we have a plus sign rather than a minus sign between the two terms in this determinant. This gives us the determinant formula for $2m\tilde{D}_n$,

$$\begin{aligned} g_{\eta\lambda}(x) \\ =\frac{1}{4}\Bigg[&\det_{n\times n}\Bigg|\frac{1}{2m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi r(\lambda_j-\eta_i)}{2m}\right)-\cos\left(\frac{2\pi r(-\lambda_j-\eta_i)}{2m}\right)\right] \\ &\cdot\exp(2x\cos(2\pi r/2m))\Bigg| \\ &+\det_{n\times n}\Bigg|\frac{1}{2m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi(2r+1)(\lambda_j-\eta_i)}{4m}\right)-\cos\left(\frac{2\pi(2r+1)(-\lambda_j-\eta_i)}{4m}\right)\right] \\ &\cdot\exp(2x\cos(2\pi(2r+1)/4m))\Bigg| \\ &+\det_{n\times n}\Bigg|\frac{1}{2m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi r(\lambda_j-\eta_i)}{2m}\right)+\cos\left(\frac{2\pi r(-\lambda_j-\eta_i)}{2m}\right)\right] \\ &\cdot\exp(2x\cos(2\pi r/2m))\Bigg| \\ &+\det_{n\times n}\Bigg|\frac{1}{m}\sum_{r=0}^{2m-1}\left[\cos\left(\frac{2\pi(2r+1)(\lambda_j-\eta_i)}{4m}\right)+\cos\left(\frac{2\pi(2r+1)(-\lambda_j-\eta_i)}{4m}\right)\right] \\ &\cdot\exp(2x\cos(2\pi(2r+1)/4m))\Bigg|\Bigg]. \end{aligned} \tag{45}$$

We simplify as before, with the sum of cosines simplifying by $\cos(\alpha-\beta)+\cos(-\alpha-\beta)=2\cos\alpha\cos\beta$, to get our final formula for $2m\tilde{D}_n$,

$$\begin{aligned} g_{\eta\lambda}(x) \\ =\frac{1}{4}\Bigg[&\det_{n\times n}\Bigg|\frac{1}{m}\sum_{r=0}^{2m-1}\sin(\pi r\lambda_j/m)\sin(\pi r\eta_i/m) \\ &\cdot\exp(2x\cos(\pi r/m))\Bigg| \\ &+\det_{n\times n}\Bigg|\frac{1}{m}\sum_{r=0}^{2m-1}\sin(\pi(2r+1)\,\lambda_j/2m)\sin(\pi(2r+1)\,\eta_i/2m) \\ &\cdot\exp(2x\cos(\pi(2r+1)/2m))\Bigg| \end{aligned}$$

$$+\det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{2m-1}\cos(\pi r\lambda_j/m)\cos(\pi r\eta_i/m)\right.$$

$$\left.\cdot\exp(2x\cos(\pi r/m))\right|$$

$$+\det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{2m-1}\cos(\pi(2r+1)\,\lambda_j/2m)\cos(\pi(2r+1)\,\eta_i/2m)\right.$$

$$\left.\left.\cdot\exp(2x\cos(\pi(2r+1)/2m))\right|\right]. \tag{46}$$

In both cases, we have analogous formulas for the diagonal walk, which again allows m to be a half-integer.

8. OPEN QUESTIONS

The number of reflectable random walks in a classical Weyl chamber is of interest in representation theory. Let the step set be the set of weights of a representation of the corresponding Lie group. Let the starting point η be ρ, half the sum of the positive roots, and the end point λ be $\rho+\mu$. If the walk is reflectable, then the number of walks from ρ to $\rho+\mu$ is the multiplicity of the representation with highest weight μ in the kth tensor power of the representation whose weights are the step set [8]. Does the number of walks in an alcove of an affine Weyl chamber have a similar meaning in representation theory, either in the representations of other Lie groups or Lie algebras, or of the classical Lie groups over some other field?

We have a formula for the probability that n particles in an interval or on a circle will not collide in k steps. Can we get a single general formula, or an asymptotic, for the total probability that there will be no collision?

There are formulas and interpretations of the q-analogue of the case with Weyl group $\tilde{A}_{n-1}$ and steps $+e_i$ [14]. Can q-analogues of the other cases be defined and are they of interest?

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