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Valuations and hyperplanes of dual polar spaces

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Abstract

Valuations were introduced in De Bruyn and Vandecasteele (Valuations of near polygons, preprint, 2004) as a very important tool for classifying near polygons. In the present paper we study valuations of dual polar spaces. We will introduce the class of the SDPS-valuations and characterize these valuations. We will show that a valuation of a finite thick dual polar space is the extension of an SDPS-valuation if and only if no induced hex valuation is ovoidal or semi-classical. Each SDPS-valuation will also give rise to a geometric hyperplane of the dual polar space.
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1. Introduction

A near polygon [14] is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}),\ \mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point p and every line L there exists a unique point on L nearest to p. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph $\Gamma_{\mathcal{S}}$ of $\mathcal{S}$. If d denotes the diameter of $\Gamma_{\mathcal{S}}$, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line.

If X_1 and X_2 are two nonempty set of points of a near polygon, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x_1\}$, then we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. For every $i \in \mathbb{N}$ and for every

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nonempty set X of points, $\Gamma_i(X)$ denotes the set of all points y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near $2d$-gon, $d \geqslant 2$, is called a *generalized* $2d$*-gon* [16] if $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for every $i \in \{1, \ldots, d-1\}$ and every two points x and y at distance i from each other. The generalized quadrangles [9] are precisely the near quadrangles. A generalized $2d$-gon is called *degenerate* if it does not contain an ordinary $2d$-gon as subgeometry. A degenerate generalized quadrangle consists of a number of lines through a point.

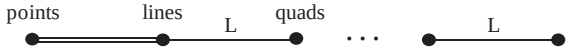
A subspace X of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called *geodetically closed* if every point on a shortest path between two points of X also belongs to X. Suppose X is geodetically closed. Let $\mathcal{L}_X$ denote the set of lines of $\mathcal{S}$ which are completely contained in X and put $\mathrm{I}_X := \mathrm{I} \cap (X \times \mathcal{L}_X)$. Then $(X, \mathcal{L}_X, \mathrm{I}_X)$ is a (geodetically closed) sub near polygon of $\mathcal{S}$. A nondegenerate geodetically closed sub near quadrangle of a near polygon is called a *quad*. Sufficient conditions for the existence of quads were given in Proposition 2.5 of [14].

If $X_1, \ldots, X_k$ are nonempty sets of points, then $\mathcal{C}(X_1, \ldots, X_k)$ denotes the smallest geodetically closed sub near polygon through $X_1 \cup \cdots \cup X_k$, i.e. the intersection of all geodetically closed sub near polygons through $X_1 \cup \cdots \cup X_k$. If one of these sets is a singleton $\{x\}$, then we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \ldots)$.

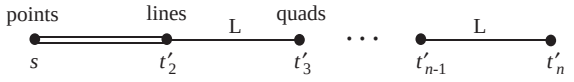
A geodetically closed sub near polygon F of a dense near polygon $\mathcal{S}$ is called *classical in* $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. We call π_F the *projection* on F.

A near polygon is said to have *order* (s, t) if every line is incident with $s+1$ points and if every point is incident with precisely $t+1$ lines. A near $2n$-gon, $n \geqslant 2$, is called *regular* if it has an order (s, t) and if there exist constants t_i, $i \in \{0, \ldots, n\}$, such that for every two points x and y at distance i from each other, there are precisely $t_i + 1$ lines through y containing a (necessarily unique) point at distance $i-1$ from x. Obviously, $t_0 = -1$, $t_1 = 0$ and $t_n = t$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. If x and y are two points of a dense near polygon at distance δ from each other, then by Theorem 4 of [2], x and y are contained in a unique geodetically closed sub near 2δ-gon (which necessarily coincides with $\mathcal{C}(x, y)$). These sub near polygons are called *hexes* if δ is equal to 3. For δ equal to 0, 1, respectively 2, we find the points, lines, respectively quads, of $\mathcal{S}$. With every dense near $2n$-gon $\mathcal{S}$, there is associated a rank n geometry Δ. The elements of type $i \in \{1, \ldots, n\}$ are the geodetically closed sub near $2(i-1)$-gons of $\mathcal{S}$. The geometry Δ has the following diagram:



If $\mathcal{S}$ is a regular near $2n$-gon with parameters s, t, t_i $(0 \leqslant i \leqslant n)$ such that $s \geqslant 2$ and $t_2 \geqslant 1$, then we can parametrize the diagram as follows:

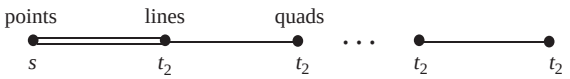


Here $t_i' := \frac{t_i - t_{i-1}}{t_{i-1} - t_{i-2}}$ for every $i \in \{2, \ldots, n\}$. Note that $t_2' = t_2$.

An important class of near polygons is the one of the dual polar spaces [3]. Let P be a polar space of rank $n \geqslant 2$. We define P^{D} as the incidence structure with points, respectively lines, the $(n-1)$-dimensional, respectively $(n-2)$-dimensional, singular subspaces of P. P^{D} is a near $2n$-gon. If α is an $(n-i)$-dimensional singular subspace of P, then the $(n-1)$-dimensional singular subspaces through α determine a geodetically closed sub near $2(i-1)$-gon of P^{D}. Conversely, every geodetically closed sub near $2(i-1)$-gon of P^{D} is obtained in this way. Every geodetically closed sub near polygon of P^{D} is classical in P^{D}. If P is a finite thick polar space of rank $n \geqslant 3$, then P^{D} is a regular near $2n$-gon with $t_i = t_2(t_{i-1}+1)$ for every $i \in \{1, \ldots, n\}$. By Tits' classification of polar spaces [15], P^{D} is then isomorphic to one of the following examples:

Dual polar space	Quads	(s, t_2)
$W^{\mathrm{D}}(2n-1, q)$ (symplectic)	$Q(4, q)$	(q, q)
$Q^{\mathrm{D}}(2n, q)$ (orthogonal)	$W(q)$	(q, q)
$[Q^{-}(2n+2, q)]^{\mathrm{D}}$ (orthogonal)	$H(3, q^2)$	(q^2, q)
$H^{\mathrm{D}}(2n-1, q^2)$ (hermitean)	$Q(5, q)$	(q, q^2)
$H^{\mathrm{D}}(2n, q^2)$ (hermitean)	$H^{\mathrm{D}}(4, q^2)$	(q^3, q^2)

Notice that $Q^{\mathrm{D}}(2n, q)$ is isomorphic to $W^{\mathrm{D}}(2n-1, q)$ if and only if q is even. The diagram of each of the above-mentioned dual polar spaces looks as follows:



We will end this introduction with an important characterization of dual polar spaces due to Cameron.

Definition. We call a near polygon $\mathcal{S}$ *classical* if it satisfies the following properties:

- every two points at distance 2 are contained in a unique quad;
- every quad is classical in $\mathcal{S}$.

Theorem 1 (*Cameron [3]*). *The classical near $2n$-gons, $n \geqslant 2$, are precisely the dual polar spaces of rank n.*

2. (Geometric) hyperplanes and valuations

Definition. A proper subspace H of a near polygon is called a *(geometric) hyperplane* if every line has at least one point in common with H.

Examples. (1) If x is a point of a near $2n$-gon $\mathcal{S}$ $(n \geqslant 1)$ then the set of points at distance at most $n-1$ from x is a hyperplane of $\mathcal{S}$.

(2) A set of points in a near polygon is called an *ovoid* if it meets every line in a unique point. Obviously, every ovoid is a hyperplane.

(3) In Section 5, we will define a class of hyperplanes of dual polar spaces arising from so-called SDPS-sets.

The above-mentioned classes (2) and (3) can give rise to other hyperplanes by the following proposition.

Proposition 1. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2n$-gon and let F be a geodetically closed sub near $2m$-gon of $\mathcal{S}$ which is classical in $\mathcal{S}$. If H is a hyperplane of F, then the extension $\overline{H} := \{x \in \mathcal{P} \mid \mathrm{d}(x, H) \leqslant n - m\}$ of H is a hyperplane of $\mathcal{S}$.*

Proof. Let L denote an arbitrary line of $\mathcal{S}$ and put $\delta := \mathrm{d}(L, F)$. It is known (and easy to prove, see e.g. Lemma 3 of [5]) that one of the following cases occurs:

(a) Every point of L has distance δ from F. In this case $\pi_F(L) := \{\pi_F(x) \mid x \in L\}$ is a line of F.
(b) There exists a unique point x in L at distance δ from F. In this case $\pi_F(L) = \{\pi_F(x)\}$.

Suppose first that case (a) occurs. If $\pi_F(L)$ is contained in H, then L is contained in $\overline{H}$. If $|\pi_F(L) \cap H| = 1$ and $\delta < n - m$, then L is contained in $\overline{H}$. If $|\pi_F(L) \cap H| = 1$ and $\delta = n - m$, then $|L \cap \overline{H}| = 1$. Suppose next that case (b) occurs. If $\pi_F(x) \in H$, then L is contained in $\overline{H}$. If $\pi_F(x) \notin H$ and $\delta < n - m - 1$, then L is contained in $\overline{H}$. If $\pi_F(x) \notin H$ and $\delta = n - m - 1$, then $|L \cap \overline{H}| = 1$. This proves that $\overline{H}$ is a hyperplane of $\mathcal{S}$. □

Definition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon. A function f from $\mathcal{P}$ to $\mathbb{N}$ is called a *valuation* if it satisfies the following properties ($f(x)$ is called the *value* of x):

(V_1) there exists at least one point with value 0;
(V_2) every line L of $\mathcal{S}$ contains a unique point x_L with smallest value and $f(x) = f(x_L) + 1$ for every point x of L different from x_L;
(V_3) every point x of $\mathcal{S}$ is contained in a geodetically closed sub near polygon F_x which satisfies the following properties:
- $f(y) \leqslant f(x)$ for every point y of F_x,
- every point z of $\mathcal{S}$ which is collinear with a point y of F_x and which satisfies $f(z) = f(y) - 1$ also belongs to F_x.

It can be shown, see Proposition 7 of [5], that the geodetically closed sub near polygon F_x in property (V_3) is unique. If f is a valuation, then we denote by O_f the set of points with value 0. Valuations were introduced in [5] and form a very important tool for classifying near polygons. In [7] the classification of the valuations of the 11 dense near hexagons [1] of order $(2, t)$ was used to obtain a complete classification of all dense near octagons of order $(2, t)$. Several classes of valuations are known, see [5]. We list the ones which we will need in the present paper.

[1] See [1] for the classification of these hexagons.

Examples. (a) Let x denote an arbitrary point of a dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. For every point y of $\mathcal{S}$, we define $f(y) := \mathrm{d}(x, y)$. Then f is a valuation of $\mathcal{S}$. We call any such valuation a *classical valuation*.

(b) Let O be an ovoid of $\mathcal{S}$. If y is a point of $\mathcal{S}$, then we define $f(y) = 0$ if $y \in O$ and $f(y) = 1$ otherwise. Then f is a valuation of $\mathcal{S}$. We call any such valuation an *ovoidal valuation*.

(c) Let x be a point of a dense near $2n$-gon $\mathcal{S}$ ($n \geqslant 2$) and let O be a set of points at distance n from x with the property that every line at distance $n-1$ from x has a unique point in common with O. If y is a point of $\mathcal{S}$, then we define $f(y) = \mathrm{d}(x, y)$ if $\mathrm{d}(x, y) \leqslant n-1$, $f(y) = n-2$ if $y \in O$ and $f(y) = n-1$ otherwise. Then f is a valuation of $\mathcal{S}$. We call any such valuation a *semi-classical valuation*.

(d) Let $\mathcal{S}$ be a dense near polygon, let F be a geodetically closed sub near polygon of $\mathcal{S}$ which is classical in $\mathcal{S}$ and let f' be a valuation of F. For every point y of $\mathcal{S}$, we define $f(y) = \mathrm{d}(y, \pi_F(y)) + f'(\pi_F(y))$. Then f is a valuation of $\mathcal{S}$. We call f an *extension* of f'. If $F = \mathcal{S}$, then we say that f is the *trivial extension* of f'.

Lemma 1 (*Corollary 2 and Proposition 17 of [5]*). (a) *Every valuation of a dense generalized quadrangle is either classical or ovoidal.*

(b) *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geqslant 2$, let x be a point of $\mathcal{S}$ and let f be a valuation of $\mathcal{S}$ such that $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from x. Then f is classical or semi-classical.*

Valuations can induce others as indicated by the following lemma.

Lemma 2 (*Proposition 8 of [5]*). *Let f be a valuation of a dense near polygon $\mathcal{S}$ and let F be a geodetically closed sub near polygon of $\mathcal{S}$. Put $m := \min\{f(x) \mid x \in F\}$ and $f'(x) := f(x) - m$ for every point x of F. Then f' is a valuation of F.*

If f is a valuation of a dense near polygon and if Q is a quad containing two points of O_f, then the valuation induced in Q cannot be classical and hence must be ovoidal. So, $Q \cap O_f$ is an ovoid of Q in that case.

The following proposition shows that there is a connection between valuations and hyperplanes.

Proposition 2. *Let f be valuation of a dense near polygon $\mathcal{S}$ and let M denote the maximal value attained by f. Then the set of points with value at most $M-1$ is a hyperplane H_f of $\mathcal{S}$. If Q is a quad of $\mathcal{S}$ not contained in H_f, then H_f intersects Q is either an ovoid or the perp of a point.*

Proof. By property (V_2) it immediately follows that H_f is a hyperplane. Let Q denote a quad which is not entirely contained in H_f. Let f' denote the valuation of Q induced by f. By Lemma 1(a), f' is either classical or ovoidal. The proposition now immediately follows. □

Hyperplanes H of dual polar spaces in which each quad $Q \not\subseteq H$ intersects H in an ovoid or the perp of a point were already studied in [10,12] for the rank 3 case. Very recently,

Pralle and Shpectorov [11] also considered the rank 4 case. They call a hyperplane of a rank 3 dual polar space *ovoidal* if it is the extension (see Proposition 1) of an ovoid in a quad.

Suppose P^{D} is a (possibly infinite) thick dual polar space of rank 4 with quads admitting ovoids. Let H be a hyperplane of P^{D} with the property that each hex Σ of P^{D} intersects H in an ovoidal hyperplane of Σ. Let $\delta(\Sigma)$ be the unique quad of Σ for which $H \cap \Sigma$ is the extension of a (uniquely determined) ovoid $O(\Sigma)$ of $\delta(\Sigma)$.

Theorem 2 (*Theorem 1 of [11]*). *The geometry* $\Gamma = (\mathcal{L}, \mathcal{D}, \mathrm{I})$ (*natural incidence*) *is a generalized quadrangle. If* P^{D} *is finite and if quads have order* (s, t_2), *then* Γ *has order* (t_2^2, st_2).

Theorem 3 (*Theorem 2 of [11]*). *If* Δ *is finite, then one of the following holds*:

- $\Delta \cong W^{\mathrm{D}}(7, q)$ *and* $\Gamma \cong W(q)$;
- $\Delta \cong [Q^-(9, q)]^{\mathrm{D}}$ *and* $\Gamma \cong H(4, q^2)$.

3. Motivation

In the previous section, we already motivated the study of valuations: these are important objects for classification purposes. There is an extra motivation in the case of dual polar spaces. With every valuation (of a dual polar space) there is associated a hyperplane and recently there has been quiet some interest in these objects (see e.g. [4,8,10–12]). Valuations of general dense near hexagons were studied in Section 4 of [5]. In the case of dual polar spaces of rank 3, we can say more.

Proposition 3. *Every valuation of a classical near hexagon* $\mathcal{S}$ *is either classical, ovoidal, semi-classical or the extension of an ovoidal valuation in a quad of* $\mathcal{S}$.

Proof. By Proposition 14 of [5], $f(x) = \mathrm{d}(x, O_f)$ for every point x at distance at most 2 from O_f. If $|O_f| = 1$, then f must be classical or semi-classical by Lemma 1(b). Suppose therefore that $|O_f| \geqslant 2$. Suppose also that f is not ovoidal, then every two different points of O_f lie at distance 2 from each other by Proposition 24 of [5]. Let o_1 and o_2 be two different points of O_f and let Q denote the unique quad through o_1 and o_2. Then $Q \cap O_f$ is an ovoid of Q. For every point x outside Q, there exists a point in $O_f \cap Q$ at distance 3 from x. Hence $O_f = O_f \cap Q$. Now, every point of $\mathcal{S}$ lies at distance at most 2 from O_f. So, $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), O_f)$ and f is the extension of an ovoidal valuation of Q. □

In Table 1, we list what is known about the existence of valuations in finite thick dual polar spaces of rank 3. Regarding this table, we can make the following remarks:

(1) The quads of $W^{\mathrm{D}}(5, q)$ and $[Q^-(7, q)]^{\mathrm{D}}$ have ovoids, while the quads of $Q^{\mathrm{D}}(6, q)$, q odd, and $H^{\mathrm{D}}(5, q^2)$ have no ovoids. We refer to [9] for more details.
(2) The dual polar space $W^{\mathrm{D}}(5, q)$ has no ovoid as has been shown by Shult for q even (see e.g. [8, 2.8]) and by Cooperstein and Pasini [4] for q odd.

Table 1
Existence of valuations in thick dual polar spaces of rank 3

	Class.	Extended	Semi-class.	Ovoidal
$W^{\mathrm{D}}(5,q)$	Yes	Yes (1)	? (3)	No (2)
$Q^{\mathrm{D}}(6,q)$, q odd	Yes	No (1)	No (1)	No (1)
$[Q^-(7,q)]^{\mathrm{D}}$	Yes	Yes (1)	?	?
$H^{\mathrm{D}}(5,q^2)$	Yes	No (1)	No (1)	No (1)
$H^{\mathrm{D}}(6,q^2)$	Yes	? (4)	? (4)	? (4)

(3) In general, it is not known whether $W^{\mathrm{D}}(5,q)$ can have semi-classical valuations. For certain values of q (like $q=4$ or q prime), it is known that $W^{\mathrm{D}}(5,q)$ cannot have semi-classical valuations (see e.g. Corollary 6.3 of [6]).
(4) The dual polar space $H^{\mathrm{D}}(4,4)$ has no extended, semi-classical and ovoidal valuations, since $H(4,4)$ has no spread (computer result of Brouwer). For $q>4$, it is not known whether extended, semi-classical or ovoidal valuations exist.

As we see in the table, no semi-classical or ovoidal valuation is known for each of the listed dual polar spaces of rank 3. This observation stimulated us to study valuations of finite thick dual polar spaces with the property that no induced hex valuation is ovoidal or semi-classical. In this way we found new valuations which we called SDPS-valuations. An SDPS-valuation of a dual polar space P^{D} of rank n arises from a set of points of P^{D} on which a certain dual polar space of rank $\frac{n}{2}$ can be defined (so n must be even). The terminology SDPS refers to the fact that there is a Sub Dual Polar Space associated with every SDPS-set or SDPS-valuation. We will define SDPS-sets in the following section.

4. SDPS-sets in dual polar spaces

Let $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be one of the following classical near $4n$-gons:

(a) a point ($n=0$);
(b) a dense generalized quadrangle ($n=1$);
(c) $W^{\mathrm{D}}(4n-1,q)$ with $n\geqslant 2$;
(d) $[Q^-(4n+1,q)]^{\mathrm{D}}$ with $n\geqslant 2$.

A subset X of $\mathcal{P}$ is called an *SDPS-set* of $\mathcal{A}$ if it satisfies the following properties:

(1) No two points of X are collinear in $\mathcal{A}$.
(2) If $x,y\in X$ such that $\mathrm{d}(x,y)=2$, then $X\cap\mathcal{C}(x,y)$ is an ovoid of the quad $\mathcal{C}(x,y)$.
(3) The point-line incidence structure $\tilde{\mathcal{A}}$ whose points are the elements of X and whose lines are quads of $\mathcal{A}$ containing at least two points of X (natural incidence) is isomorphic to one of the following near polygons:
  (a) a point if $\mathcal{A}$ is a point;
  (b) a line if $\mathcal{A}$ is a generalized quadrangle;

(c) $W^{\mathrm{D}}(2n-1,q^2)$ if $\mathcal{A}\cong W^{\mathrm{D}}(4n-1,q)$ with $n\geqslant 2$;
(d) $H^{\mathrm{D}}(2n,q^2)$ if $\mathcal{A}\cong[Q^-(4n+1,q)]^{\mathrm{D}}$ with $n\geqslant 2$.

(4) For all $x, y\in X$, $\mathrm{d}(x,y)=2\cdot\delta(x,y)$, where $\delta(\cdot,\cdot)$ denotes the distance in $\tilde{\mathcal{A}}$.

An SDPS-set of the near 0-gon consists of the unique point of the near 0-gon. An SDPS-set of a dense generalized quadrangle is just an ovoid of that generalized quadrangle. SDPS-sets of type (c) and (d) do also exist (Section 9 of [13], Section 1.3 of [11]). Consider the finite field $\mathbb{F}_{q^2}$ with q^2 elements and let $\mathbb{F}_q$ denote the unique subfield of order q of $\mathbb{F}_{q^2}$. Let η denote an arbitrary element of $\mathbb{F}_{q^2}\setminus\mathbb{F}_q$. Then $\mathbb{F}_{q^2}=\{x_1+x_2\eta\mid x_1,x_2\in\mathbb{F}_q\}$; define $\tau:\mathbb{F}_{q^2}\to\mathbb{F}_q, x_1+x_2\eta\mapsto x_1$.

- Consider the following bijection ϕ between the vector spaces $\mathbb{F}_q^{4n}$ and $\mathbb{F}_{q^2}^{2n}$:

$$\phi(x_1,x_2,\ldots,x_{4n})=(x_1+\eta x_2,\ldots,x_{4n-1}+\eta x_{4n}).$$

  If $\langle\cdot,\cdot\rangle$ is a nondegenerate symplectic form of $\mathbb{F}_{q^2}^{2n}$, then $\tau(\langle\phi(\cdot),\phi(\cdot)\rangle)$ is a nondegenerate symplectic form in $\mathbb{F}_q^{4n}$. If α is a totally isotropic n-dimensional subspace of $\mathbb{F}_{q^2}^{2n}$, then $\phi^{-1}(\alpha)$ is a $2n$-dimensional totally isotropic subspace of $\mathbb{F}_q^{4n}$. In this way we obtain an "embedding" of $W^{\mathrm{D}}(2n-1,q^2)$ in $W^{\mathrm{D}}(4n-1,q)$, giving rise to an SDPS-set of type (c).
- Consider the following bijection ϕ between the vector spaces $\mathbb{F}_q^{4n+2}$ and $\mathbb{F}_{q^2}^{2n+1}$:

$$\phi(x_1,x_2,\ldots,x_{4n+2})=(x_1+\eta x_2,\ldots,x_{4n+1}+\eta x_{4n+2}).$$

  Let $\langle\cdot,\cdot\rangle$ be a nondegenerate hermitean form of $\mathbb{F}_{q^2}^{2n+1}$. For every $x\in\mathbb{F}_{q^2}^{2n+1}$, we define $h(x):=\langle x,x\rangle$ and for every $x\in\mathbb{F}_q^{4n+2}$, we define $q(x):=\langle\phi(x),\phi(x)\rangle$. The equation $h(x)=0$, respectively $q(x)=0$, defines a nonsingular hermitean variety $H(2n,q^2)$ in $\mathrm{PG}(2n,q^2)$, respectively a nonsingular elliptic quadric $Q^-(4n+1,q)$ in $\mathrm{PG}(4n+1,q)$. With every generator of $H(2n,q^2)$, there corresponds a generator of $Q^-(4n+1,q)$. In this way we obtain an "embedding" of $H^{\mathrm{D}}(2n,q^2)$ in $[Q^-(4n+1,q)]^{\mathrm{D}}$, giving rise to an SDPS-set of type (d).

5. SDPS-valuations

In this section, we will prove the following results.

Theorem 4 (*Section 5.1*). *Let $\mathcal{A}$ be isomorphic to either the point, a dense generalized quadrangle, $W^{\mathrm{D}}(4n-1,q)$ or $[Q^-(4n+1,q)]^{\mathrm{D}}$ $(n\geqslant 2)$, and let X be an SDPS-set of $\mathcal{A}$. For every point x of $\mathcal{A}$, define $f(x)=\mathrm{d}(x,X)$. Then f is a valuation of $\mathcal{A}$.*

Definition. A valuation which can be obtained from an SDPS-set as described in Theorem 4 is called an *SDPS-valuation*.

Theorem 5 (*Section 5.2*). *Let $\mathcal{A}$ be a finite thick dual polar space and let f be a valuation of $\mathcal{A}$ which is the* (*possibly trivial*) *extension of an SDPS-valuation in a geodetically closed sub near polygon $\mathcal{B}$ of $\mathcal{A}$. Then the following holds*:

(i) *$f(x) = \mathrm{d}(x, O_f)$ for every point x of $\mathcal{A}$;*
(ii) *if H is a hex of $\mathcal{A}$, then the valuation induced in H is either classical or the extension of an ovoidal valuation in a quad of H.*

5.1. Proof of Theorem 4

We will prove the Theorem 4 in several lemmas. We will use the same notations as in Section 4. We call a quad of $\mathcal{A}$ *special* if it contains at least two points of X. If Q is a special quad, then $Q \cap X$ is an ovoid of Q.

Lemma 3. *Suppose $n \geqslant 2$. If (s, t_2), respectively $(\tilde{s}, \tilde{t}_2)$, denotes the order of the quads of $\mathcal{A}$, respectively $\tilde{\mathcal{A}}$, then $(\tilde{s}, \tilde{t}_2) = (st_2, t_2^2)$.*

Proof. If $\mathcal{A} \cong W^{\mathrm{D}}(4n-1, q)$, then $(s, t_2) = (q, q)$ and $(\tilde{s}, \tilde{t}_2) = (q^2, q^2)$. If $\mathcal{A} \cong [Q^-(4n+1, q)^{\mathrm{D}}$, then $(s, t_2) = (q^2, q)$ and $(\tilde{s}, \tilde{t}_2) = (q^3, q^2)$. So, $(\tilde{s}, \tilde{t}_2) = (st_2, t_2^2)$ in any case. □

Lemma 4. *No two special quads intersect in a line.*

Proof. Suppose the contrary. Let Q_1 and Q_2 denote two special quads which intersect in a line L. Let x denote the unique point of X on L and let x_1, x_2 denote two other points of L. Let y_i, $i \in \{1, 2\}$, denote a point of $(Q_i \cap X) \setminus \{x\}$ collinear with x_i. Then $\mathrm{d}(y_1, y_2) = \mathrm{d}(y_1, x_1) + \mathrm{d}(x_1, y_2) = 1 + 2 = 3$, contradicting property (4) in the definition of SDPS-set. □

Lemma 5. *If $x \in X$, then every line of $\mathcal{A}$ through x is contained in a unique special quad.*

Proof. Obviously, the lemma holds if $n \leqslant 1$. Suppose therefore that $n \geqslant 2$. Put $t = t_2 + t_2^2 + \cdots + t_2^{2n-1}$ and $\tilde{t} = \tilde{t}_2 + \tilde{t}_2^2 + \cdots + \tilde{t}_2^{n-1}$. Then $\mathcal{A}$ has order (s, t) and $\tilde{\mathcal{A}}$ has order $(\tilde{s}, \tilde{t})$. By Lemma 4, the number of lines through x which are contained in a special quad is equal to $(t_2+1)(\tilde{t}+1) = (t_2+1)(1+\tilde{t}_2+\tilde{t}_2^2+\cdots+\tilde{t}_2^{n-1}) = (t_2+1)(1+t_2^2+\cdots+t_2^{2n-2}) = 1 + t_2 + \cdots + t_2^{2n-1} = t + 1$. This proves the lemma. □

Lemma 6. *Let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathcal{A}$. Then $\mathcal{F} \cap X$ is a geodetically closed subspace of $\tilde{\mathcal{A}}$.*

Proof. We suppose that $\mathcal{F} \cap X$ is nonempty. Let x_1 and x_2 denote two points of $\mathcal{F} \cap X$ such that $\delta(x_1, x_2) = 1$. Then $\mathrm{d}(x_1, x_2) = 2$. Since $\mathcal{F}$ is geodetically closed, $\mathcal{C}(x_1, x_2) \subseteq \mathcal{F}$ and hence $\mathcal{C}(x_1, x_2) \cap X \subseteq \mathcal{F} \cap X$. This proves that $\mathcal{F} \cap X$ is a subspace of $\tilde{\mathcal{A}}$. Now, let a, b, c denote points of X such that $a, b \in \mathcal{F} \cap X$, $\delta(a, c) = \delta(a, b) - 1$ and $\delta(c, b) = 1$. Then $\mathrm{d}(a, c) = \mathrm{d}(a, b) - 2$ and $\mathrm{d}(c, b) = 2$. Since $\mathcal{F}$ is geodetically closed and $\mathrm{d}(a, c) + \mathrm{d}(c, b) = \mathrm{d}(a, b)$, $c \in \mathcal{F} \cap X$. Hence $\mathcal{F} \cap X$ is also geodetically closed (in $\tilde{\mathcal{A}}$). □

Lemma 7. *Let $\tilde{\mathcal{F}}$ be a geodetically closed sub near polygon of $\tilde{\mathcal{A}}$. Then there exists a unique geodetically closed sub near polygon $\mathcal{F}$ of $\mathcal{A}$ such that* $\mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$ *and $\tilde{\mathcal{F}} = \mathcal{F} \cap X$. Moreover, $\tilde{\mathcal{F}}$ is an SDPS-set of $\mathcal{F}$.*

Proof. Let x and y be two points of $\tilde{\mathcal{F}}$ such that $\delta(x, y) = \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$. If $\mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{A}$ such that $\mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$ and $\tilde{\mathcal{F}} = \mathcal{F} \cap X$, then $\mathcal{F}$ necessarily equals $\mathcal{C}(x, y)$ since $x, y \in \mathcal{F}$ and $\mathrm{d}(x, y) = 2 \cdot \delta(x, y) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}}) = \mathrm{diam}_{\mathcal{A}}(\mathcal{F})$. We will now show that $\mathcal{F} := \mathcal{C}(x, y)$ satisfies all required properties. By Lemma 6, $\mathcal{F} \cap X$ is a geodetically closed sub near polygon of $\tilde{\mathcal{A}}$ containing x and y and hence also $\tilde{\mathcal{F}}$. On the other hand, we have $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) = \frac{1}{2}\mathrm{diam}_{\mathcal{A}}(\mathcal{F} \cap X) \leqslant \frac{1}{2}\mathrm{diam}(\mathcal{F}) = \frac{1}{2}\mathrm{d}(x, y) = \delta(x, y) = \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$. This proves that $\mathcal{F} \cap X = \tilde{\mathcal{F}}$.

One now easily sees that $(\mathcal{F}, \tilde{\mathcal{F}})$ is an admissible pair of classical near polygons (i.e. $(\mathcal{F}, \tilde{\mathcal{F}})$ is of one of the types (a), (b), (c) or (d) as described in the definition of SDPS-set). So, $\tilde{\mathcal{F}}$ is an SDPS-set of $\mathcal{F}$. □

Definition. A geodetically closed sub near polygon $\mathcal{F}$ of $\mathcal{A}$ is called *special* if $\mathcal{F} \cap X$ is an SDPS-set of $\mathcal{F}$. The special sub near polygons of $\mathcal{A}$ are those sub near polygons of the form $\mathcal{C}(x_1, x_2)$ where $x_1, x_2 \in X$.

From Lemmas 6 and 7, we have the following corollary.

Corollary 1. *If $\mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{A}$, then $\mathcal{F} \cap X$ is either empty or an SDPS-set in a geodetically closed sub near polygon of $\mathcal{F}$.*

Lemma 8. *Let $n \geqslant 1$. If $\mathcal{F}$ is a geodetically closed sub near $(4n-2)$-gon of $\mathcal{A}$, then $\mathcal{F} \cap X$ is a geodetically closed sub near $(2n-2)$-gon of $\tilde{\mathcal{A}}$.*

Proof. We first show that $\mathcal{F} \cap X$ is nonempty. Let x be an arbitrary point of X. If $x \in \mathcal{F}$, then we are done. If $x \notin \mathcal{F}$, let Q denote the unique special quad through the line $x\pi_{\mathcal{F}}(x)$. Then $\mathcal{F} \cap Q$ is a line. Since this line contains a point of X, also $\mathcal{F}$ contains a point of X.

Now, let x^* denote a point of $\mathcal{F} \cap X$ and let y denote a point of X at maximal distance from x^*. Then $\mathrm{d}(x^*, y) = 2n$ and $y \notin \mathcal{F}$. Let Q denote the unique special quad through the line $y\pi_{\mathcal{F}}(y)$. The quad Q intersects $\mathcal{F}$ in a line which contains a point y'' of X. Now, $\mathrm{d}(x^*, y'') \geqslant 2n-2$ and hence $\delta(x^*, y'') \geqslant n-1$. This proves that $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) \geqslant n-1$. On the other hand, we have $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) = \frac{1}{2}\mathrm{diam}_{\mathcal{A}}(\mathcal{F} \cap X) \leqslant \frac{1}{2}\mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = \frac{2n-1}{2}$. Hence, $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) = n-1$. In Lemma 6, we have already shown that $\mathcal{F} \cap X$ is geodetically closed. □

Lemma 9. *Let $n \geqslant 1$. Let x be a point of $\mathcal{A}$ and let $\mathcal{F}$ denote a geodetically closed sub near $(4n-2)$-gon through x. Then* $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F} \cap X)$.

Proof. Let y be a point of $X \setminus \mathcal{F}$, let Q denote the unique special quad through the line $y\pi_{\mathcal{F}}(y)$ and let y' denote the unique point of X on the line $Q \cap \mathcal{F}$. Then $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_{\mathcal{F}}(y)) + 1 \geqslant \mathrm{d}(x, y')$. As a consequence, $\mathrm{d}(x, X \setminus \mathcal{F}) \geqslant \mathrm{d}(x, \mathcal{F} \cap X)$ and $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F} \cap X)$. □

Lemma 10. *Let x be a point of $\mathcal{A}$ and let $\mathcal{F}$ denote a special sub near polygon through x. Then* $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F} \cap X)$.

Proof. The lemma holds if $\mathcal{F} = \mathcal{A}$. So, suppose that $\mathcal{F} \neq \mathcal{A}$ and that the lemma holds for any special sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 2$. Let $\mathcal{F}''$ denote a special sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 2$ through $\mathcal{F}$ and let $\mathcal{F}'$ denote a geodetically closed sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 1$ such that $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{F}''$. By our assumption, $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F}'' \cap X)$. By Lemma 9, $\mathrm{d}(x, \mathcal{F}'' \cap X) = \mathrm{d}(x, \mathcal{F}'' \cap X \cap \mathcal{F}')$ and by Lemma 8, $\mathcal{F}'' \cap X \cap \mathcal{F}' = \mathcal{F} \cap X$. The lemma now follows. □

Lemma 11. *For every pointx of $\mathcal{A}$,* $\mathrm{d}(x, X) \leqslant n$. *Moreover, there exists a point x^* of $\mathcal{A}$ such that* $\mathrm{d}(x^*, X) = n$.

Proof. We will prove this by induction on n. Obviously, the lemma holds if n is equal to 0 or 1. Suppose therefore that $n \geqslant 2$. Let $\mathcal{F}$ denote a geodetically closed sub near $(4n-2)$-gon through x and let $\mathcal{F}'$ denote the unique geodetically closed sub near $(4n-4)$-gon of $\mathcal{A}$ containing all points of $\mathcal{F} \cap X$. Then $\mathrm{d}(x, \pi_{\mathcal{F}'}(x)) \leqslant 1$. Since $\mathcal{F} \cap X$ is an SDPS-set in $\mathcal{F}'$, we have $\mathrm{d}(\pi_{\mathcal{F}'}(x), \mathcal{F} \cap X) \leqslant n-1$ and hence $\mathrm{d}(x, X) \leqslant n$. By the induction hypothesis, we know that there exists a point $y \in \mathcal{F}'$ such that $\mathrm{d}(y, X \cap \mathcal{F}) = n-1$. If x^* denotes a point of $\mathcal{F} \setminus \mathcal{F}'$ collinear with y, then $\mathrm{d}(x^*, X) = \mathrm{d}(x^*, X \cap \mathcal{F}) = 1 + \mathrm{d}(y, X \cap \mathcal{F}) = n$. This proves the lemma. □

Lemma 12. *Let x be a point of $\mathcal{A}$. Then there exist two points $x_1, x_2 \in X$ such that* $\mathrm{d}(x, x_1) = \mathrm{d}(x, x_2) = \mathrm{d}(x, X)$ *and* $\mathrm{d}(x_1, x_2) = 2 \cdot d(x, X)$. *As a consequence, every point is contained in a special geodetically closed sub near* $4 \cdot d(x, X)$*-gon.*

Proof. We will prove this by induction on the distance $\mathrm{d}(x, X)$. Obviously, the property holds if $\mathrm{d}(x, X) = 0$. Suppose therefore that $\mathrm{d}(x, X) \geqslant 1$ and that the property holds for any point at distance at most $\mathrm{d}(x, X) - 1$ from X. Take a point x' collinear with x at distance $\mathrm{d}(x, X) - 1$ from X and let x_1' and x_2' denote two points of X such that $\mathrm{d}(x', x_1') = \mathrm{d}(x', x_2') = \mathrm{d}(x, X) - 1$ and $\mathrm{d}(x_1', x_2') = 2 \cdot \mathrm{d}(x, X) - 2$. Put $\mathcal{F}' := \mathcal{C}(x_1', x_2')$. Then $x' \in \mathcal{F}'$ since x' is on a shortest path between x_1' and x_2'. If $x \in \mathcal{F}'$, then by Lemma 11, x would have distance at most $\mathrm{d}(x, X) - 1$ to $X \cap \mathcal{F}'$, a contradiction. So, x is not contained in $\mathcal{F}'$ and $\mathrm{d}(x, x_1') = \mathrm{d}(x, x_2') = 1 + \mathrm{d}(x', x_1') = 1 + \mathrm{d}(x', x_2') = \mathrm{d}(x, X)$. Let $\mathcal{F}$ denote the unique special geodetically closed sub near $4 \cdot \mathrm{d}(x, X)$-gon through $\mathcal{F}'$ containing the line xx'. (The special geodetically closed sub near $4 \cdot \mathrm{d}(x, X)$-gons through $\mathcal{F}'$ partition the set of lines through x_2' not contained in $\mathcal{F}'$ and hence also the set of lines through x' not contained in $\mathcal{F}'$.) Let L_2 denote a line of $\mathcal{F}$ through x_2' not contained in $\mathcal{F}'$ and containing a point y_2 at distance $\mathrm{d}(x, X) - 1$ from x. Let Q_2 denote the unique special quad through L_2. Then $Q_2 \subseteq \mathcal{F}$. Now, put $x_1 := x_1'$ and let x_2 denote a point of $(X \cap Q_2) \setminus \{x_2'\}$ collinear with y_2. Then $\mathrm{d}(x, x_2) \leqslant \mathrm{d}(x, y_2) + \mathrm{d}(y_2, x_2) = \mathrm{d}(x, X)$ and hence $\mathrm{d}(x, x_2) = \mathrm{d}(x, X)$. Since the quad Q_2 intersects $\mathcal{F}'$ only in the point x_2', we have $\mathrm{d}(x_1, x_2) = \mathrm{d}(x_1, x_2') + \mathrm{d}(x_2', x_2) = 2 \cdot \mathrm{d}(x, X)$. We have already noticed that $\mathrm{d}(x, x_1) = \mathrm{d}(x, x_1') = \mathrm{d}(x, X)$. This proves the lemma. □

Lemma 13. *Every line L of $\mathcal{A}$ contains a unique point at smallest distance from X.*

Proof. The proof happens by induction on n. Obviously, the lemma holds if n is equal to 0 or 1. Suppose therefore that $n \geqslant 2$. Let $\mathcal{F}$ denote a geodetically closed sub near $(4n-2)$-gon through L and let $\mathcal{F}'$ denote the unique special sub near $(4n-4)$-gon contained in $\mathcal{F}$. For every point x of L, $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F}' \cap X)$ by Lemma 9. If L is contained in $\mathcal{F}'$, then the lemma holds for the line L by the induction hypothesis. If L is disjoint from $\mathcal{F}'$, then the lemma holds for the line $\pi_{\mathcal{F}'}(L)$ and hence also for the line L since $\mathrm{d}(x, \mathcal{F}' \cap X) = 1 + \mathrm{d}(\pi_{\mathcal{F}'}(x), \mathcal{F}' \cap X)$ for every point x of L. If L intersects $\mathcal{F}'$ in a unique point, then this point is the unique point of L at smallest distance from X. □

Lemma 14. *Let x be a point of $\mathcal{A}$ and let F denote a special sub near $4 \cdot \mathrm{d}(x, X)$-gon through x. Then a line L through x contains a point at distance $\mathrm{d}(x, X) - 1$ from X if and only if L is contained in F. As a consequence, x is contained in a unique special sub near $4 \cdot \mathrm{d}(x, X)$-gon F_x.*

Proof. Suppose that L is contained in F. By Lemma 13, L contains a unique point nearest to X and by Lemma 11 every point of F has distance at most $\mathrm{d}(x, X)$ to X. Hence L contains a unique point at distance $\mathrm{d}(x, X) - 1$ from x. Conversely, suppose that L is not contained in F and contains a point y at distance $\mathrm{d}(x, X) - 1$ from X. Let F' denote the unique special $(4 \cdot \mathrm{d}(x, X) + 4)$-gon through $\mathcal{C}(y, F)$. Then $\mathrm{d}(y, X) = \mathrm{d}(y, F' \cap X)$ by Lemma 10. Let y' denote a point of $X \cap F'$ at distance $\mathrm{d}(x, X) - 1$ from y. Then $y' \notin F$ and hence also $y' \notin \mathcal{C}(y, F)$. Let y'' denote the unique point of $\mathcal{C}(y, F)$ collinear with y' and let Q denote the unique special quad through the line $y'y''$. Then the line $Q \cap \mathcal{C}(y, F)$ contains a point z of X which necessarily belongs to F. We have $\mathrm{d}(y, z) \geqslant \mathrm{d}(x, X) + 1$ and $\mathrm{d}(y, z) \leqslant \mathrm{d}(y, y') + \mathrm{d}(y', z) = \mathrm{d}(x, X) - 1 + 2 = \mathrm{d}(x, X) + 1$. Hence y' is contained in a shortest path between y and z. But this is impossible since $y' \notin \mathcal{C}(y, F)$. □

Lemma 15. *For every point x of $\mathcal{A}$ and every point $y \in F_x$, $F_y \subseteq F_x$.*

Proof. In the near polygon F_x, there exists a unique special sub near $4 \cdot \mathrm{d}(y, F_x \cap X)$-gon F_y' through y. By Lemma 10, $\mathrm{d}(y, F_x \cap X) = \mathrm{d}(y, X)$. Hence F_y' must coincide with F_y. This proves the lemma. □

The following lemma completes the proof of Theorem 4.

Lemma 16. *The map $f : \mathcal{P} \to \mathbb{N}; x \mapsto \mathrm{d}(x, X)$ is a valuation of $\mathcal{A}$.*

Proof. Obviously, there exists a point with value 0. By Lemma 13, every line contains a unique point with smallest value. We will now show that f also satisfies property (V_3). For every point y of F_x, we have $\mathrm{d}(y, X) \leqslant \mathrm{d}(x, X)$ by Lemma 11. If y is a point of F_x and if z is a point of $\mathcal{A}$ collinear with y for which $\mathrm{d}(z, X) = \mathrm{d}(y, X) - 1$, then $z \in F_y$ by Lemma 14. By Lemma 15, it now follows that $z \in F_x$. Hence f is a valuation of $\mathcal{A}$. □

5.2. Proof of Theorem 5

For every point x of $\mathcal{A}$, $f(x) = \mathrm{d}(x, \pi_{\mathcal{B}}(x)) + \mathrm{d}(\pi_{\mathcal{B}}(x), O_f) = \mathrm{d}(x, O_f)$. This proves (i). We will prove property (ii) by induction on the diameter n of $\mathcal{A}$. Obviously, the property holds if $n \leqslant 3$. So, suppose $n \geqslant 4$ and that the property holds for any thick dual polar space of diameter less than n (= induction hypothesis). We distinguish the following possibilities:

- $\mathcal{B} \neq \mathcal{A}$: Let $\mathcal{A}'$ denote a geodetically closed sub near $(2n-2)$-gon of $\mathcal{A}$ through $\mathcal{B}$. For every point x of $\mathcal{A}$, we have $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_{\mathcal{A}'}(x)) + \mathrm{d}(\pi_{\mathcal{A}'}(x), O_f)$ $(*)$. If H is contained in $\mathcal{A}'$, then the valuation induced in H is either classical or the extension of an ovoidal valuation in a quad of H since property (ii) holds for $\mathcal{A}'$. If H meets $\mathcal{A}'$ in a quad Q, then by $(*)$, the valuation induced in H is the extension of the valuation induced in Q. So, the property also holds in this case. If H is disjoint from $\mathcal{A}'$, then by the induction hypothesis applied to the dual polar space $\mathcal{A}'$, the property holds for the hex $\pi_{\mathcal{A}'}(H)$. Hence, the property also holds for H by $(*)$.
- $\mathcal{B} = \mathcal{A}$: Let $\mathcal{A}'$ denote a geodetically closed sub near $2(n-1)$-gon through H and let $\mathcal{A}''$ denote the unique special geodetically closed sub near $(2n-4)$-gon of $\mathcal{A}'$. The valuation f' induced by f in $\mathcal{A}'$ is the extension of an SDPS-valuation in $\mathcal{A}''$. By the induction hypothesis applied to $\mathcal{A}'$ it follows that the valuation f'' induced by f' in H is classical or the extension of an SDPS-valuation in a quad of H. The theorem then holds since f'' is also the valuation induced by f in H.

6. Characterization of SDPS-valuations

In this section, we will prove the also the converse of Theorem 5 holds.

Theorem 6. *Let $\mathcal{A}$ be a finite thick dual polar space and let f be a valuation of $\mathcal{A}$ with the property that every induced hex valuation is either classical or the extension of a valuation in a quad. Then f is the* (*possibly trivial*) *extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{A}$.*

Let n be the diameter of $\mathcal{A}$. We will prove the theorem by induction on n. Obviously, the lemma holds if $n = 3$. So, suppose that $n \geqslant 4$ and that the lemma holds for any finite thick dual polar space of diameter less than n (= induction hypothesis). Call a quad of $\mathcal{A}$ special if it contains two points of O_f, or equivalently, if it intersects O_f in an ovoid.

Lemma 17. *No two special quads intersect in a line.*

Proof. Suppose that Q_1 and Q_2 are two special quads intersecting in a line. Then $\mathcal{C}(Q_1, Q_2)$ is a hex H. By Proposition 3 and our assumptions, the valuation induced in H is either classical or extended. It is easily seen that none of these two possibilities can occur. □

Lemma 18. *For all $x_1, x_2 \in O_f$, $\mathrm{d}(x_1, x_2)$ is even.*

Proof. We distinguish three possibilities.

(1) $\mathrm{d}(x_1, x_2) < n$. If f' denotes the valuation induced in $\mathcal{C}(x_1, x_2)$, then by the induction hypothesis, f' is an SDPS-valuation in $\mathcal{C}(x_1, x_2)$. Since $x_1, x_2 \in O_{f'}$, $\mathrm{d}(x_1, x_2)$ is even.
(2) $\mathrm{d}(x_1, x_2) = n$ and n is even.
(3) $\mathrm{d}(x_1, x_2) = n$ and n is odd. Let F denote a geodetically closed sub near $(2n-2)$-gon through x_1 and let x_2' denote the unique point of F collinear with x_2. Then $f(x_2') = 1$. By the induction hypothesis, the valuation f' induced in F is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of F. Since $f(x_2') = 1$, x_2' is collinear with a point x_2'' of $O_f \cap F$, see Theorem 5. Since $\mathrm{d}(x_1, x_2'')$ is even and $\mathrm{d}(x_1, x_2'') \geqslant n-2$, $\mathrm{d}(x_1, x_2'') = n-1$. So, f' is an SDPS-valuation of $F = \mathcal{C}(x_1, x_2'')$. Now, the line $x_2'x_2''$ is contained in at least two special quads ($\mathcal{C}(x_2, x_2'')$ + the unique special quad through $x_2'x_2''$ contained in F, see Lemma 5), contradicting Lemma 17. □

Lemma 19. *If $x_1, x_2 \in O_f$ with $\mathrm{d}(x_1, x_2)$ as big as possible, then $O_f \subseteq \mathcal{C}(x_1, x_2)$.*

Proof. Obviously, this holds if $\mathrm{d}(x_1, x_2) = n$. Suppose therefore that $\mathrm{d}(x_1, x_2) < n$ and let F denote a geodetically closed sub near $(2n-2)$-gon through $\mathcal{C}(x_1, x_2)$. By the induction hypothesis applied to F, it follows that every point of $O_f \cap F$ is contained in $\mathcal{C}(x_1, x_2)$. Suppose now that there exists a point $y \in O_f$ not contained in F and let y' denote the unique point of F collinear with y. Since $f(y') = 1$, there exists a point $x_3 \in O_f \cap F$ collinear with y'. Then $x_3 \in \mathcal{C}(x_1, x_2)$. Since the valuation induced in $\mathcal{C}(x_1, x_2)$ is an SDPS-valuation, there exists a point $x_4 \in O_f \cap \mathcal{C}(x_1, x_2)$ at distance $\mathrm{d}(x_1, x_2)$ from x_3. We now distinguish two possibilities.

(1) The quad $\mathcal{C}(y, x_3)$ intersects $\mathcal{C}(x_1, x_2)$ in a line L. Then L would be contained in two special quads ($\mathcal{C}(y, x_3)$ + the unique special quad through L contained in $\mathcal{C}(x_1, x_2)$), contradicting Lemma 17.
(2) The quad $\mathcal{C}(y, x_3)$ intersects $\mathcal{C}(x_1, x_2)$ in the point x_3. Then $\mathrm{d}(y, x_4) = 2 + \mathrm{d}(x_1, x_2)$, contradicting the maximality of $\mathrm{d}(x_1, x_2)$. □

Lemma 20. *The valuation f is not semi-classical.*

Proof. Suppose the contrary. Let x denote the unique point of $\mathcal{A}$ with value 0 and let H denote a hex containing a point at maximal distance n from x. Then it is easily seen that the valuation induced in H is semi-classical, contradicting our assumptions. □

Lemma 21. *If $|O_f| = 1$, then f is a classical valuation.*

Proof. Let x denote the unique point of O_f. If y is a point at distance at most $n-1$ from x, then by the induction hypothesis, the valuation induced in $\mathcal{C}(x, y)$ is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{C}(x, y)$. Since $|O_f| = 1$, this induced valuation must be classical. Hence, $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from x. The lemma now follows from Lemmas 1 and 20. □

Lemma 22. *If $|O_f| \geqslant 2$, then every point x of O_f is contained in a special quad.*

Proof. Suppose the contrary. Then, by the induction hypothesis, the valuation induced in every geodetically closed sub near $(2n-2)$-gon through x must be classical. Hence, $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from x. By Lemmas 1 and 20, it now follows that f is classical, contradicting $|O_f| \geqslant 2$. □

Lemma 23. *For every point x of $\mathcal{A}$, $f(x) = \mathrm{d}(x, O_f)$.*

Proof. By Lemma 21, this holds if $|O_f| = 1$. Suppose therefore that $|O_f| \geqslant 2$. Let x^* denote a point of O_f nearest to x and let Q denote a special quad through x^*. Then either $\pi_Q(x) = x^*$ or $\pi_Q(x) \sim x^*$. Since $\mathrm{d}(x, \pi_Q(x)) \leqslant n-2$, we have $\mathrm{d}(x, x^*) \leqslant n-1$. By the induction hypothesis, the valuation f' induced in $\mathcal{C}(x, x^*)$ is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{C}(x, x^*)$. So, $f(x) = f'(x) = \mathrm{d}(x, O_f \cap \mathcal{C}(x, x^*)) = \mathrm{d}(x, x^*) = \mathrm{d}(x, O_f)$. □

Lemma 24. *If the maximal distance between two points of O_f is smaller than n, than f is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{A}$.*

Proof. Let $x_1, x_2 \in O_f$ with $\mathrm{d}(x_1, x_2)$ as big as possible. Since $\mathrm{d}(x_1, x_2) < n$, the valuation induced in $F := \mathcal{C}(x_1, x_2)$ is an SDPS-valuation f' with $O_{f'} = O_f$ (see Lemma 19). For every point x of $\mathcal{A}$, we have $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), O_{f'})$, proving the lemma. □

In the sequel, we will suppose that the maximal distance between two points of O_f is equal to n. This implies that n is even, see Lemma 18. Let $\tilde{S}$ denote the following partial linear space:

- the points of $\tilde{\mathcal{A}}$ are the elements of O_f,
- the lines of $\tilde{\mathcal{A}}$ are the special quads,
- incidence is containment.

If x_1 and x_2 are two points of O_f, then (as before) we denote by $\mathrm{d}(x_1, x_2)$ the distance between x_1 and x_2 in the geometry $\mathcal{A}$ and by $\delta(x_1, x_2)$ the distance between x_1 and x_2 in the geometry $\tilde{\mathcal{A}}$.

Lemma 25. *For all points $x_1, x_2 \in O_f$, $\mathrm{d}(x_1, x_2) = 2 \cdot \delta(x_1, x_2)$. As a consequence, the diameter of $\tilde{\mathcal{A}}$ is half the diameter of $\mathcal{A}$.*

Proof. Every path of $\tilde{\mathcal{A}}$ between x_1 and x_2 can be turned into a path of $\mathcal{A}$ with double length. This proves that $\mathrm{d}(x_1, x_2) \leqslant 2 \cdot \delta(x_1, x_2)$ for all points $x_1, x_2 \in O_f$.

We will prove the lemma by induction on the distance $\mathrm{d}(x_1, x_2)$ which is always even by Lemma 18. Obviously, the lemma holds if $\mathrm{d}(x_1, x_2)$ is 0 or 2. Suppose therefore that $\mathrm{d}(x_1, x_2) = 2k \geqslant 4$. Then we already know that $\delta(x_1, x_2) \geqslant k$. We will now show that there exists a special quad Q through x_2 containing a point x_3 at distance $2k-2$ from x_1. If

$2k < n$, this follows from the fact that the valuation induced in $\mathcal{C}(x_1, x_2)$ is an SDPS-valuation. If $2k = n$, this follows from Lemma 22. If $x_3 \notin O_f$, then there exists a point in $Q \cap O_f$ at distance $2k-1$ from x_1, contradicting Lemma 18. Hence $x_3 \in O_f$. Since $\mathrm{d}(x_1, x_3) = 2k-2$, $\delta(x_1, x_3) = k-1$ and $\delta(x_1, x_2) \leqslant k$. Together with $\delta(x_1, x_2) \geqslant k$, this implies that $\mathrm{d}(x_1, x_2) = 2 \cdot \delta(x_1, x_2)$. □

Lemma 26. *$\tilde{\mathcal{A}}$ is a near polygon* (*and hence a near n-gon*).

Proof. Let x denote a point of O_f and let Q denote a special quad. If $\pi_Q(x) \notin O_f$, then there exists a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_Q(x)) + 1$ from x and a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_Q(x)) + 2$ from x, contradicting Lemma 18. So, $\pi_Q(x) \in O_f$ and $\pi_Q(x)$ is the unique point of $O_f \cap Q$ nearest to x. The statement now follows from Lemma 25. □

Lemma 27. *Let (s, t_2) denote the order of the quads of $\mathcal{A}$. Then every line of $\tilde{\mathcal{A}}$ is incident with $\tilde{s}+1 := st_2+1$ points and every two points at distance 2 in $\tilde{\mathcal{A}}$ have $\tilde{t}_2 + 1 := t_2^2 + 1$ common neighbours. So, $\tilde{\mathcal{A}}$ is a dense near polygon.*

Proof. Obviously, every line of $\tilde{\mathcal{A}}$ is incident with $st_2 + 1$ points. Now, choose points $x_1, x_2 \in O_f$ at distance 4 from each other. Let H be a hex through x_1 contained in $\mathcal{C}(x_1, x_2)$ and let x_2' denote the unique point of H collinear with x_2. Since $f(x_1) = 0$ and $f(x_2') = 1$, the valuation induced in H is the extension of an ovoidal valuation of a quad Q of H. Hence, H contains a unique special quad. Now, we calculate the number $\tilde{t}_2 + 1$ of common neighbours of x_1 and x_2 (in $\tilde{\mathcal{A}}$). Any such common neighbour is contained in $\mathcal{C}(x_1, x_2)$. The number $\tilde{t}_2 + 1$ is equal to the number of special quads of $\mathcal{C}(x_1, x_2)$ through x_1. Since every hex of $\mathcal{C}(x_1, x_2)$ through x_1 contains a unique such quad, we have $\tilde{t}_2 + 1 = \frac{t_2^3 + t_2^2 + t_2 + 1}{t_2 + 1} = t_2^2 + 1$ or $\tilde{t}_2 = t_2^2$. □

If x_1 and x_2 are two points of O_f at distance 4 from each other, then $O_f \cap \mathcal{C}(x_1, x_2)$ is a geodetically closed subspace and hence a quad of $\tilde{\mathcal{A}}$. Conversely, every quad is obtained in this way.

Lemma 28. *The near polygon $\tilde{\mathcal{A}}$ is classical.*

Proof. The lemma holds trivially if $\tilde{\mathcal{A}}$ is a generalized quadrangle. So, suppose $n > 4$. Let x denote a point of $\tilde{\mathcal{A}}$ and let $\tilde{Q}$ denote a quad of $\tilde{\mathcal{A}}$. Let F denote the geodetically closed sub near octagon of $\mathcal{A}$ containing all points of $\tilde{Q}$. Then, by the induction hypothesis, the valuation induced in F is an SDPS-valuation. Let H denote an arbitrary hex of F through $\pi_F(x)$. Then $H \cap O_f$ is an ovoid in a quad Q. Let x' denote the unique point of Q nearest to $\pi_F(x)$. Then $\mathrm{d}(\pi_F(x), O_f \cap F) = \mathrm{d}(\pi_F(x), H \cap O_f) = \mathrm{d}(\pi_F(x), x') + \mathrm{d}(x', H \cap O_f) \leqslant 2$. So, we have the following possibilities.

(a) $\pi_F(x) \in O_f \cap F$. Then $\pi_F(x)$ is indeed the unique point of $\tilde{Q}$ nearest to x.
(b) $\mathrm{d}(\pi_F(x), O_f \cap F) = 1$. Then $\pi_F(x)$ is contained in a special quad Q of F, see Lemma 5. Then there exists a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_F(x)) + 1$ from x and a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_F(x)) + 2$ from x, contradicting Lemma 18.

(c) $\mathrm{d}(\pi_F(x), O_f \cap F) = 2$. Let x' denote a neighbour of $\pi_F(x)$ collinear with a point of $O_f \cap F$, then x' is contained in a special quad Q of F. Then there exists a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_F(x)) + 2$ from x and a point in $O_f \cap Q$ at distance $\mathrm{d}(x, \pi_F(x)) + 3$ from x, contradicting Lemma 18. □

The following lemma completes the proof of Theorem 6.

Lemma 29. *One of the following cases occurs*:

- $\mathcal{A} \cong W^{\mathrm{D}}(2n-1, q)$ *and* $\tilde{\mathcal{A}} \cong W^{\mathrm{D}}(n-1, q^2)$;
- $\mathcal{A} \cong [Q^-(2n+1, q)]^{\mathrm{D}}$ *and* $\tilde{\mathcal{A}} \cong H^{\mathrm{D}}(n, q^2)$.

Proof. The quads of $\mathcal{A}$ admit ovoids. So, $\mathcal{A}$ is isomorphic to one of the following dual polar spaces: $W^{\mathrm{D}}(2n-1, q)$, $[Q^-(2n+1, q)]^{\mathrm{D}}$ and (possibly) $H^{\mathrm{D}}(2n, q^2)$.

- Suppose $\mathcal{A} \cong W^{\mathrm{D}}(2n-1, q)$. The quads of $\tilde{\mathcal{A}}$ have order (q^2, q^2). So, $\tilde{\mathcal{A}} \cong W^{\mathrm{D}}(n-1, q^2)$ or $\tilde{\mathcal{A}} \cong Q^{\mathrm{D}}(n, q^2)$ if $n \geqslant 6$. If $n = 4$, then $\tilde{\mathcal{A}}$ is a generalized quadrangle of order (q^2, q^2).
- Suppose $\mathcal{A} \cong [Q^-(2n+1, q)]^{\mathrm{D}}$. The quads of $\tilde{\mathcal{A}}$ have order (q^3, q^2). So, $\tilde{\mathcal{A}} \cong H^{\mathrm{D}}(n, q^2)$ if $n \geqslant 6$. If $n = 4$, then $\tilde{\mathcal{A}}$ is a generalized quadrangle of order (q^3, q^2).
- Suppose $\mathcal{A} \cong H^{\mathrm{D}}(2n, q^2)$. Since the quads of $\tilde{\mathcal{A}}$ have order (q^5, q^4), $\mathcal{A} \cong H^{\mathrm{D}}(8, q)$ and $\tilde{\mathcal{A}}$ is a generalized quadrangle of order (q^5, q^4).

Now, consider a geodetically closed sub near octagon F of $\mathcal{A}$ containing two points of O_f at distance 4 from each other and put $Y := F \cap O_f$.

We will show that every hex of F intersects Y in an ovoid of a quad. Let y denote an arbitrary point of Y. Since no two special quads through y intersect in a line, $(1+\tilde{t}_2)(1+t_2) \leqslant t_F + 1$. Since $t_F + 1 = t_2^3 + t_2^2 + t_2 + 1 = (t_2+1)(\tilde{t}_2+1)$, every line of F through y is contained in a unique special quad. The number of hexes of F through y containing a special quad through y is equal to $(1+\tilde{t}_2)(1+t_2)$. This is the total number of hexes of F through y. So, every hex of F through y intersects Y in an ovoid of a quad. Now, let G denote a hex of F not containing y, let y' denote the unique point of G collinear with y and let Q denote the unique special quad through the line yy'. The quad Q intersects G in a line which contains a point of Y. As before, we then know that $G \cap Y$ is an ovoid in a quad.

Now, put $H := \cup_{y \in Y} y^{\perp} \cap F$. Let L be a line of F and let G be a hex of F through L. Since $G \cap Y$ is an ovoid in a quad, L is either contained in H or intersects H in precisely 1 point (every point of $G \setminus H$ has distance 2 from Y). This proves that H is a hyperplane of F intersecting each hex in an ovoidal hyperplane. By Theorem 3, (i) $\mathcal{A} \cong H^{\mathrm{D}}(8, q^2)$ is impossible, (ii) $\tilde{\mathcal{A}} \cong W^{\mathrm{D}}(n-1, q^2)$ if $\mathcal{A} \cong W^{\mathrm{D}}(2n-1, q)$ and (iii) $\tilde{\mathcal{A}} \cong H^{\mathrm{D}}(n, q^2)$ if $\mathcal{A} \cong [Q^-(2n+1, q)]^{\mathrm{D}}$. This proves the lemma. □

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