Journal of Combinatorial Theory, Series A 147 (2017) 75–118

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Nonsymmetric Askey–Wilson polynomials and Q-polynomial distance-regular graphs

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A R T I C L E   I N F O

Article history:
Received 18 September 2015
Available online 7 December 2016

Dedicated to Professor Paul Terwilliger on his 60th birthday

Keywords:
Askey–Wilson polynomial
Nonsymmetric Askey–Wilson polynomial
q-Racah polynomial
Nonsymmetric q-Racah polynomial
DAHA of rank one
Distance-regular graph
Q-polynomial

A B S T R A C T

In his famous theorem (1982), Douglas Leonard characterized the q-Racah polynomials and their relatives in the Askey scheme from the duality property of Q-polynomial distance-regular graphs. In this paper we consider a nonsymmetric (or Laurent) version of the q-Racah polynomials in the above situation. Let Γ denote a Q-polynomial distance-regular graph that contains a Delsarte clique C. Assume that Γ has q-Racah type. Fix a vertex $x \in C$. We partition the vertex set of Γ according to the path-length distance to both x and C. The linear span of the characteristic vectors corresponding to the cells in this partition has an irreducible module structure for the universal double affine Hecke algebra $\hat{H}_q$ of type $(C_1^\vee, C_1)$. From this module, we naturally obtain a finite sequence of orthogonal Laurent polynomials. We prove the orthogonality relations for these polynomials, using the $\hat{H}_q$-module and the theory of Leonard systems. Changing $\hat{H}_q$ by $\hat{H}_{q^{-1}}$ we show how our Laurent polynomials are related to the nonsymmetric Askey–Wilson polynomials, and therefore how our Laurent polynomials can be viewed as nonsymmetric q-Racah polynomials.

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http://dx.doi.org/10.1016/j.jcta.2016.11.006

1. Introduction

The nonsymmetric Askey–Wilson polynomials were first treated by Sahi [23]. They are expressed as certain Laurent polynomials, and are obtained in the double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$ as eigenfunctions of the Cherednik–Dunkl operator on the basic representation for the algebra. The nonsymmetric Askey–Wilson polynomials along with the DAHA of rank one were studied in algebraic aspects by Noumi and Stokman [22], Macdonald [21, Section 6.6] and Koornwinder [17,18]. In the present paper we study the nonsymmetric Askey–Wilson polynomials in a combinatorial aspect, using a Q-polynomial distance-regular graph that contains a Delsarte clique.

The Q-polynomial property for distance-regular graphs was introduced by Delsarte [7]. Since then, this property has been receiving substantial attention from many mathematicians; see e.g. [3,4,6,10,20,27]. In [20], Leonard characterized the q-Racah polynomials and their relatives in the Askey scheme using the duality property of Q-polynomial distance-regular graphs (see also [3, Section III.5]). Terwilliger defined the subconstituent algebra (or Terwilliger algebra) as a method of the study of Q-polynomial distance-regular graphs [27–29]. This algebra has been a significant tool in the study of Q-polynomial distance-regular graphs and its connections to Lie theory, quantum algebras, and coding theory have also been revealed; see e.g. [9,11–14,24,26].

In [19] the author showed a relationship between Q-polynomial distance-regular graphs and the universal DAHA $\hat{H}_q$ of type $(C_1^\vee, C_1)$ using the Terwilliger algebra. We briefly summarize this result. Let Γ denote a Q-polynomial distance-regular graph that contains a Delsarte clique C. Assume that Γ has q-Racah type. Fix a vertex $x \in C$. Partitioning the vertex set of Γ according to the path-length distance to both x and C gives a two-dimensional equitable partition, which takes a staircase shape consisting of nodes and edges. Let $\mathbf{W}$ denote the $\mathbb{C}$-vector space spanned by the characteristic vectors corresponding to the nodes of the staircase shape of the partition. Then $\mathbf{W}$ has an irreducible module structure for the algebra $\hat{H}_q$ [19, Sections 11,12].

From the above staircase picture of $\mathbf{W}$, the q-Racah polynomials [1] arise naturally as follows. Roughly speaking, horizontal edges correspond to a sequence of q-Racah polynomials and vertical edges correspond to another sequence of q-Racah polynomials. In the present paper, using the irreducible $\hat{H}_q$-module $\mathbf{W}$, we define a finite sequence of certain Laurent polynomials that correspond to nodes of the staircase picture. We denote these polynomials by ε_i^σ, where $\sigma \in \{+, -\}$. The ε_i^σ are considered as nonsymmetric q-Racah polynomials, a discrete version of the nonsymmetric Askey–Wilson polynomials. And then we treat the orthogonality relations for ε_i^σ, using the $\hat{H}_q$-module $\mathbf{W}$ and the theory of Leonard systems [32]. This orthogonality is new and it can be viewed as a discrete version of the orthogonality for the nonsymmetric Askey–Wilson polynomials, which was worked by Koornwinder and Bouzeffour [18, Section 5].

The paper is organized as follows. In Section 2, we review some basic definitions, concepts and notation regarding nonsymmetric Askey–Wilson polynomials and DAHAs of type $(C_1^\vee, C_1)$. In Sections 3 and 4, we review some backgrounds concerning Q-polynomial distance-regular graphs, the Terwilliger algebra, Leonard systems, and parameter arrays. Our vector space $\mathbf{W}$ appears along with a comprehensible picture in Section 3. In Section 5, we study the module for the Terwilliger algebra T on $\mathbf{W}$ and the associated q-Racah polynomials. The T-module $\mathbf{W}$ decomposes into the direct sum of two irreducible T-modules, and the Leonard system corresponding to each T-module gives rise to a sequence of the q-Racah polynomials. We express these polynomials and the related formulae in terms of certain scalars a, b, c, d. In Section 6, we recall the algebra $\hat{H}_q$ and its properties. And we display the $\hat{H}_q$-module $\mathbf{W}$ in terms of the scalars a, b, c, d. For this module, we describe the action of $\mathbf{X} := t_3 t_0 \in \hat{H}_q$.

In Section 7, we define the Laurent polynomial g that plays a role to connect the above two irreducible T-submodules of $\mathbf{W}$. Using g and the $\hat{H}_q$-module $\mathbf{W}$, we define a finite sequence of Laurent polynomials $\varepsilon_i^\sigma (\sigma \in \{+, -\})$. Moreover, for the element $\mathbf{Y} := t_0 t_1 \in \hat{H}_q$ we describe the action of $\varepsilon_i^\sigma[\mathbf{Y}]$ on the $\hat{H}_q$-module $\mathbf{W}$. In Section 8, we compute the eigenvalues/eigenvectors of $\mathbf{Y}$ on $\mathbf{W}$. Using the results, in Section 9 we define a bilinear form on the vector space L spanned by $\{\varepsilon_i^\sigma\}_{i=0}^{D-1}$. With respect to this bilinear form we prove the orthogonality relations for the Laurent polynomials ε_i^σ. In Section 10, we consider the algebra $\hat{H}_{q^{-1}}$ by changing q by q^{-1}. We discuss how the algebra $\hat{H}_{q^{-1}}$ is related to the (ordinary) DAHA $\tilde{\mathfrak{H}}$ of type $(C_1^\vee, C_1)$. We, further, redescribe the Laurent polynomials ε_i^σ and the associated formulae in terms of q^{-1}-version. In Section 11, we make a normalization for ε_i^σ of q^{-1}-version and discuss how these polynomials are related to the nonsymmetric Askey–Wilson polynomials. The paper ends with a brief summary and direction for future work in Section 12. An Appendix provides some explicit data involving the $\hat{H}_q$-action on $\mathbf{W}$.

Notation 1.1. Throughout this paper we assume $q \in \mathbb{C}^*$ is not a root of unity. For $a \in \mathbb{C}$,

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1}$$

where $n = 0, 1, 2, \ldots$. For $a_1, a_2, \ldots, a_r \in \mathbb{C}$,

$$(a_1, a_2, \ldots, a_r; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

Let $\mathbb{C}[z, z^{-1}]$ denote the space of the Laurent polynomials with a variable z. We write an element of $\mathbb{C}[z, z^{-1}]$ by $f[z]$. We say $f[z]$ is *symmetric* if $f[z] = f[z^{-1}]$, otherwise *nonsymmetric*. Note that a symmetric Laurent polynomial $f[z]$ can be viewed as an ordinary polynomial $f(x)$ in $x = z + z^{-1}$.

2. Nonsymmetric Askey–Wilson polynomials

In this section we review some backgrounds concerning the Askey–Wilson polynomials, DAHAs of type $(C_1^\vee, C_1)$, and the nonsymmetric Askey–Wilson polynomials. For more background, see [2,17,21,22]. We acknowledge that notation and presentations of the nonsymmetric Askey–Wilson polynomials and the DAHA of type $(C_1^\vee, C_1)$ are taken from Koornwinder's papers [17,18]. Throughout this section, let $a, b, c, d \in \mathbb{C}^*$ be such that

$$ab, ac, ad, bc, bd, cd, abcd \notin \{q^{-m} \mid m = 0, 1, 2, \ldots\}.$$

We now recall the Askey–Wilson polynomials [2]. For $n = 0, 1, 2, \ldots$ define a polynomial

$$\begin{aligned} p_n(x) = p_n[z; a, b, c, d \mid q] &:= \sum_{i=0}^{\infty} \frac{(q^{-n}, abcdq^{n-1}, az, az^{-1}; q)_i}{(ab, ac, ad, q; q)_i} q^i \\ &= {}_4\phi_3 \left(\begin{matrix} q^{-n}, \ abcdq^{n-1}, \ az, \ az^{-1} \\ ab, \ ac, \ ad \end{matrix} \,\middle|\, q, \ q \right), \end{aligned} \tag{2}$$

where $x = z + z^{-1}$. The last equality follows from the definition of basic hypergeometric series [8, p. 4]. Observe that $(q^{-n}; q)_i = 0$ if $i > n$. We call p_n the n-th *Askey–Wilson polynomial*. Consider the *monic* Askey–Wilson polynomials

$$P_n = P_n[z; a, b, c, d \mid q] := \frac{(ab, ac, ad; q)_n}{a^n (abcdq^{n-1}; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \ abcdq^{n-1}, \ az, \ az^{-1} \\ ab, \ ac, \ ad \end{matrix} \,\middle|\, q, \ q \right).$$

Note that P_n is symmetric. For $n = 1, 2, \ldots$, define a Laurent polynomial [18, Section 4]

$$Q_n := a^{-1} b^{-1} z^{-1} (1 - az)(1 - bz) P_{n-1}[z; qa, qb, c, d \mid q]. \tag{3}$$

Definition 2.1. [18, §4 (4.2)–(4.3)] The *nonsymmetric Askey–Wilson polynomials* are defined by

$$\begin{aligned} E_{-n} &:= P_n - Q_n && (n = 1, 2, \ldots), \\ E_n &:= P_n - \frac{ab(1 - q^n)(1 - cdq^{n-1})}{(1 - abq^n)(1 - abcdq^{n-1})} Q_n && (n = 0, 1, 2, \ldots), \end{aligned}$$

where $(1 - q^n) Q_n := 0$ for $n = 0$.

The DAHA of type $(C_1^\vee, C_1)$, denoted by $\tilde{\mathfrak{H}}$ [17, Section 3], is defined by the generators Z, Z^{-1}, T_0, T_1 and relations

$$\begin{aligned} (T_1 + ab)(T_1 + 1) &= 0, & (T_0 + q^{-1}cd)(T_0 + 1) &= 0, \\ (T_1 Z + a)(T_1 Z + b) &= 0, & (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) &= 0. \end{aligned}$$

The algebra $\tilde{\mathfrak{H}}$ has a faithful representation on $\mathbb{C}[z, z^{-1}]$, which is called the *basic representation* [17, Section 3]. On the basic representation, by [17, Theorem 4.1] each of $E_{\pm n}$

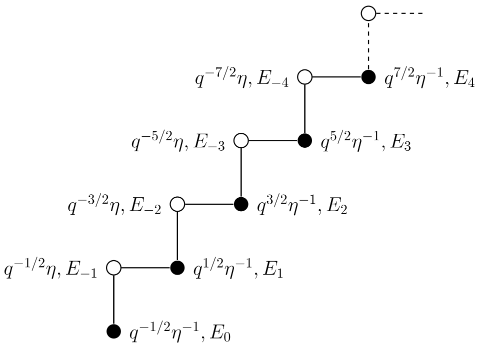

Fig. 1. The eigenspaces of $q^{1/2}\eta Y$.

is the eigenfunction for $Y = T_1T_0$;

$$YE_{-n} = q^{-n}E_{-n} \qquad (n = 1, 2, \ldots), \tag{4}$$

$$YE_n = q^{n-1}abcdE_n \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

Fix square roots $a^{1/2}, b^{1/2}, c^{1/2}, d^{1/2}$ and $q^{1/2}$. Consider $q^{1/2}\eta Y$, where $\eta = a^{-1/2}b^{-1/2}c^{-1/2}d^{-1/2}$. From (4) and (5), it follows

$$q^{1/2}\eta YE_{-n} = q^{-n+\frac{1}{2}}\eta E_{-n} \qquad (n = 1, 2, \ldots), \tag{6}$$

$$q^{1/2}\eta YE_n = q^{n-\frac{1}{2}}\eta^{-1}E_n \qquad (n = 0, 1, 2, \ldots). \tag{7}$$

By (6) and (7), we give a staircase diagram that describes the structure of eigenspaces of $q^{1/2}\eta Y$.

We remark that each white node represents the eigenspace of $q^{1/2}\eta Y$ corresponding the eigenvalue $q^{-n+\frac{1}{2}}\eta$ and the eigenvector E_{-n} for $n = 1, 2, \ldots$, and each black node represents the eigenspace of $q^{1/2}\eta Y$ corresponding the eigenvalue $q^{n-\frac{1}{2}}\eta^{-1}$ and the eigenvector E_n for $n = 0, 1, 2, \ldots$. Observe that the product of eigenvalues of each vertical edge is equal to q^{-1} and the product of eigenvalues of each horizontal edge is equal to 1.

We discuss the orthogonality relations for the Askey–Wilson polynomials. By [5, Theorems I.4.4 and II.3.2] (or [18, (3.6)–(3.8)]), there exists a positive Borel measure $\mu = \mu_{a,b,c,d;q}$ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ such that

$$\langle P_m, P_n\rangle_{a,b,c,d;q} := \int_{\mathbb{R}} P_m(x)P_n(x)d\mu(x) = h_n\delta_{m,n}, \tag{8}$$

where

$$h_n = h_n^{a,b,c,d;q} = \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(abcdq^{n-1}; q)_n}.$$

In [18] Koorwinder and Bouzeffour introduced a presentation of nonsymmetric Laurent polynomials as two-dimensional vector-valued polynomials. By [18, p. 7], we can identify a Laurent polynomial f with 2-vector-valued symmetric Laurent polynomial $(f_1, f_2)^t$, where t denotes transpose. In particular, from [18, (4.10) and (4.11)]

$$E_{-n} = \begin{pmatrix} P_n[z; a, b, c, d \mid q] \\ -a^{-1}b^{-1}P_{n-1}[z; aq, bq, c, d \mid q] \end{pmatrix} \qquad (n = 1, 2, \ldots), \tag{9}$$

$$E_n = \begin{pmatrix} P_n[z; a, b, c, d \mid q] \\ -\dfrac{(1-q^n)(1-cdq^{n-1})}{(1-abq^n)(1-abcdq^{n-1})} P_{n-1}[z; aq, bq, c, d \mid q] \end{pmatrix} \qquad (n = 0, 1, 2, \ldots), \tag{10}$$

where $(1-q^n)P_{n-1} := 0$ for $n = 0$. In [18, Section 5], the authors introduced a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[z, z^{-1}]$:

$$\langle g, h \rangle = \langle (g_1, g_2)^t, (h_1, h_2)^t \rangle = \langle g_1, h_1 \rangle_{a,b,c,d;q} + C \langle g_2, h_2 \rangle_{aq,bq,c,d;q}, \tag{11}$$

where $\langle \cdot, \cdot \rangle_{a,b,c,d;q}$ is from (8) and

$$C = -ab \frac{(1-ab)(1-abq)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-abcdq)}.$$

Note that the nonsymmetric Askey–Wilson polynomials $E_n (n \in \mathbb{Z})$ are orthogonal with respect to the bilinear form (11). This bilinear form is positive definite with some conditions for the scalars a, b, c, d; see [18, Proposition 5.1] for details.

Lemma 2.2. *With respect to the bilinear form (11),*

(i) *for* $n = 1, 2, \ldots,$

$$\langle E_{-n}, E_{-n} \rangle = \frac{(ab-1)(1-abcdq^{2n-1})}{ab(1-q^n)(1-cdq^{n-1})} \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(abcdq^{n-1}; q)_n},$$

(ii) *for* $n = 0, 1, 2, \ldots,$

$$\langle E_n, E_n \rangle = \frac{(1-ab)(1-abcdq^{2n-1})}{(1-abq^n)(1-abcdq^{n-1})} \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(abcdq^{n-1}; q)_n}.$$

Proof. (i) From (9), we set $f_1 = P_n[z; a, b, c, d \mid q]$ and $f_2 = -a^{-1}b^{-1}P_{n-1}[z; aq, bq, c, d \mid q]$. Then by (11)

$$\langle E_{-n}, E_{-n} \rangle = \langle (f_1, f_2)^t, (f_1, f_2)^t \rangle = \langle f_1, f_1 \rangle_{a,b,c,d;q} + C \langle f_2, f_2 \rangle_{aq,bq,c,d;q}.$$

Compute the right-hand side of the above equation by using (8). The result follows. (ii) Similar to (i). □

3. Q-polynomial distance-regular graphs

We recall some basic concepts and notation concerning Q-polynomial distance-regular graphs. For more information we refer the reader to [3,4,27]. Let X denote a nonempty finite set. Define $\mathrm{Mat}_X(\mathbb{C})$ to be the $\mathbb{C}$-algebra consisting of the square matrices indexed by X with entries in $\mathbb{C}$. Let V denote the $\mathbb{C}$-vector space consisting of column vectors indexed by X with entries in $\mathbb{C}$. View V as a left $\mathrm{Mat}_X(\mathbb{C})$-module. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle_V$ such that $\langle u, v\rangle_V = u^t \bar{v}$, where t denotes transpose and $^-$ denotes complex conjugate. We abbreviate $\|u\|^2 = \langle u, u\rangle_V$ for all $u \in V$. For $y \in X$ let $\hat{y}$ denote the vector in V with a 1 in the y-coordinate and 0 in all other coordinates. For $Y \subseteq X$ define $\hat{Y} = \sum_{y\in Y} \hat{y}$, called the *characteristic vector* of Y.

Let Γ denote a simple connected graph with vertex set X and diameter $D \geq 3$, where $D := \max\{\partial(x, y) \mid x, y \in X\}$ and where ∂ is the shortest path-length distance function. For $x \in X$, define

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\} \qquad (0 \leq i \leq D). \tag{12}$$

We say that Γ is *distance-regular* whenever for $0 \leq i \leq D$ and vertices $x, y \in X$ with $\partial(x, y) = i$ the numbers

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|, \qquad a_i = |\Gamma_i(x) \cap \Gamma_1(y)|, \qquad b_i = |\Gamma_{i+1} \cap \Gamma_1(y)|, \tag{13}$$

are independent of x and y. Define the matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ by $(A_i)_{xy} = 1$ if $\partial(x, y) = i$ and 0 otherwise. We call A_i the i-th *distance matrix* of Γ. In particular, $A = A_1$ is called the *adjacency matrix* of Γ. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, called the *adjacency algebra*. By definition, every element in M forms a polynomial in A. The graph Γ satisfies the *P-polynomial property*, that is, for $0 \leq i \leq D$ there exists a polynomial $f_i \in \mathbb{C}[x]$ such that $\deg(f_i) = i$ and $f_i(A) = A_i$.

We recall the notion of Q-polynomial property. By [4, p. 127], the elements $\{A_i\}_{i=0}^D$ form a basis for M. Since A is real symmetric and generates M, A has $D + 1$ mutually distinct real eigenvalues, denoted by $\theta_0, \theta_1, \ldots, \theta_D$. Let $E_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the orthogonal projection onto the eigenspace of $\theta_i (0 \leq i \leq D)$. We call E_i the *i-th primitive idempotent* of Γ. Note that $\{E_i\}_{i=0}^D$ form a basis for M. We say that Γ is *Q-polynomial* with respect to the ordering $E_0, E_1, \ldots E_D$ whenever there exists $f_i^* \in \mathbb{C}[x]$ such that $\deg(f_i^*) = i$ and $f_i^*(E_1) = E_i$, where the multiplication of M is under the entrywise product [3, p. 193]. Throughout the paper we assume that Γ is a Q-polynomial distance-regular graph.

By a *clique* we mean a nonempty subset C of X such that any two distinct vertices in C are adjacent. It is known that $|C| \leq 1 - k/\theta_{\min}$ [4, Proposition 4.4.6], where $\theta_{\min}$ is the minimum eigenvalue of A. We say that C is *Delsarte* when $|C| = 1 - k/\theta_{\min}$. Assume that Γ contains a Delsarte clique C. For $0 \leq i \leq D - 1$, we define

$$C_i := \{y \in X \mid \partial(y, C) = i\}, \tag{14}$$

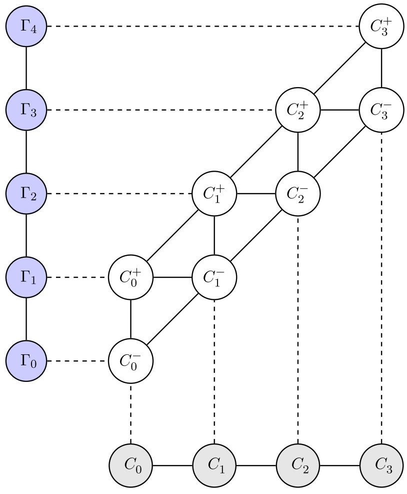

Fig. 2. The set $\{C_i^{\pm}\}_{i=0}^{3}$ of X when $d = 4$.

where $\partial(y, C) = \min\{\partial(y, z) \mid z \in C\}$. For the rest of the paper we fix a vertex $x \in C$. Recall $\Gamma_i = \Gamma_i(x)$ $(0 \le i \le D)$ and C_i $(0 \le i \le D-1)$ from (12) and (14). For $0 \le i \le D-1$ define

$$C_i^- = \Gamma_i \cap C_i, \qquad C_i^+ = \Gamma_{i+1} \cap C_i \qquad (0 \le i \le D-1), \tag{15}$$

see Fig. 2. Note that each of $C_i^{\pm}$ $(0 \le i \le D-1)$ is nonempty, and by construction the $\{C_i^{\pm}\}_{i=0}^{D-1}$ is an *equitable* partition of X in the sense of [10, p. 75]; see [19, Proposition 5.6]. Define $\mathbf{W}$ to be the subspace of V spanned by $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$. By the previous comments one readily sees that $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ is an orthogonal basis for $\mathbf{W}$.

For $0 \le i \le D$ define the diagonal matrix $E_i^* = E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ by $(E_i^*)_{yy} = 1$ if $\partial(x, y) = i$ and 0 otherwise. We call E_i^* the *i-th dual primitive idempotent* of Γ with respect to x. Observe that $I = \sum_{i=0}^{D} E_i^*$ and $E_i^* E_j^* = \delta_{i,j} E_i^*$ for $0 \le i, j \le D$. So the set $\{E_i^*\}_{i=0}^{D}$ forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual adjacency algebra* of Γ with respect to x. Define the diagonal matrix $A_i^* = A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ by $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$, called the *i-th dual distance matrix* of Γ with respect to x. By [27, p. 379] $\{A_i^*\}_{i=0}^{D}$ is a basis for M^*. We abbreviate $A^* = A_1^*$, called the *dual adjacency matrix* of Γ with respect to x. By [27, Lemma 3.11] A^* generates M^*. By these comments A^* has $D+1$ mutually distinct real eigenvalues, denoted by $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ and called θ_i^* the i-th *dual eigenvalue* of A^*.

Terwilliger algebra $T = T(x)$ with respect to x is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^* [27]. By T-module, we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. We define $\widetilde{A}^* = \widetilde{A}^*(C) = |C|^{-1} \sum_{y \in C} A^*(y) \in \mathrm{Mat}_X(\mathbb{C})$, called the *dual adjacency matrix* of Γ with respect to C. The Terwilliger algebra $\widetilde{T} = \widetilde{T}(C)$ with respect to C is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, $\widetilde{A}^*$ [25]. In [19, Definition 5.20] we defined the generalized Terwilliger algebra $\mathbf{T} = \mathbf{T}(x, C)$. The algebra $\mathbf{T}$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by T, $\widetilde{T}$. Observe that A, A^* and $\widetilde{A}^*$ generate $\mathbf{T}$. Note that $\mathbf{W}$ has a module structure for both T and $\widetilde{T}$, and so is a $\mathbf{T}$-module [19, Proposition 5.25]. The T-submodule (resp. $\widetilde{T}$-submodule) of $\mathbf{W}$ generated by $\hat{x}$ (resp. $\hat{C}$) will be called the *primary T-module* (resp. primary $\widetilde{T}$-module), denoted by $M\hat{x}$ (resp. $M\hat{C}$). The $\{A_i\hat{x}\}_{i=0}^{D}$ (resp. $\{\hat{C}_i\}_{i=0}^{D-1}$) is a basis for $M\hat{x}$ (resp. $M\hat{C}$). In Section 5, we will discuss the T-module $\mathbf{W}$ in more detail.

4. Leonard systems and parameter arrays

Let d denote a positive integer. Let $M_{\mathsf{d}+1}(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all $(\mathsf{d}+1)\times(\mathsf{d}+1)$ matrices that have entries in $\mathbb{C}$. Let $\mathcal{A}$ denote a $\mathbb{C}$-algebra isomorphic to $M_{\mathsf{d}+1}(\mathbb{C})$. Let V denote an irreducible left $\mathcal{A}$-module. Remark that V is unique up to isomorphism of $\mathcal{A}$-modules and V has dimension $\mathsf{d}+1$. For $\mathsf{A}\in\mathcal{A}$, A is called *multiplicity-free* whenever A has $\mathsf{d}+1$ mutually distinct eigenvalues. Assume A is multiplicity-free. Let $\{\theta_i\}_{i=0}^{\mathsf{d}}$ denote an ordering of distinct eigenvalues of A. For $0\le i\le \mathsf{d}$ let V_i denote the eigenspace of A associated with θ_i. Define $\mathsf{E}_i\in\mathcal{A}$ by $(\mathsf{E}_i-\mathrm{I})\mathsf{V}_i=0$ and $\mathsf{E}_i\mathsf{V}_j=0$ for $j\neq i$ $(0\le j\le \mathsf{d})$, where I is the identity of $\mathcal{A}$. We call E_i the *primitive idempotent* of A associated with θ_i. Observe that (i) $\mathsf{A}\mathsf{E}_i=\theta_i\mathsf{E}_i$, (ii) $\mathsf{E}_i\mathsf{E}_j=\delta_{i,j}\mathsf{E}_i$, (iii) $\sum_{i=0}^{\mathsf{d}}\mathsf{E}_i=\mathrm{I}$. We now define a Leonard system in $\mathcal{A}$.

Definition 4.1. [30, Definition 1.4] By a *Leonard system* on V, we mean a sequence

$$\Phi=(\mathsf{A};\mathsf{A}^*;\{\mathsf{E}_i\}_{i=0}^{\mathsf{d}};\{\mathsf{E}_i^*\}_{i=0}^{\mathsf{d}})$$

that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathcal{A}$.
(ii) $\{\mathsf{E}_i\}_{i=0}^{\mathsf{d}}$ is an ordering of the primitive idempotents of A.
(iii) $\{\mathsf{E}_i^*\}_{i=0}^{\mathsf{d}}$ is an ordering of the primitive idempotents of A^*.
(iv) For $0\le i,j\le \mathsf{d}$,

$$\mathsf{E}_i\mathsf{A}^*\mathsf{E}_j=\begin{cases}0 & \text{if}\quad |i-j|>1,\\ \neq 0 & \text{if}\quad |i-j|=1.\end{cases}$$

(v) For $0\le i,j\le \mathsf{d}$,

$$\mathsf{E}_i^*\mathsf{A}\mathsf{E}_j^*=\begin{cases}0 & \text{if}\quad |i-j|>1,\\ \neq 0 & \text{if}\quad |i-j|=1.\end{cases}$$

We call d the *diameter* of Φ, and say Φ is *over* $\mathbb{C}$.

Example 4.2. Recall from Section 3 that Γ is a Q-polynomial distance-regular graph and T is the Terwilliger algebra of Γ with respect to x. Referring to Section 3, consider a sequence of elements of T

$$(A; A^*; \{E_i\}_{i=0}^{D}; \{E_i^*\}_{i=0}^{D}), \tag{16}$$

where A (resp. A^*) is the adjacency matrix (resp. dual adjacency matrix) of Γ and E_i (resp. E_i^*) is the i-th primitive idempotent (resp. dual primitive idempotent) of Γ. Then the sequence (16) is a Leonard system on $M\hat{x}$.

Let Φ be a Leonard system in Definition 4.1. Each of the following is a Leonard system on V:

$$\Phi^* := (\mathsf{A}^*; \mathsf{A}; \{\mathsf{E}_i^*\}_{i=0}^{\mathsf{d}}; \{\mathsf{E}_i\}_{i=0}^{\mathsf{d}}),$$

$$\Phi^{\downarrow} := (\mathsf{A}; \mathsf{A}^*; \{\mathsf{E}_i\}_{i=0}^{\mathsf{d}}; \{\mathsf{E}_{\mathsf{d}-i}^*\}_{i=0}^{\mathsf{d}}), \qquad \Phi^{\Downarrow} := (\mathsf{A}; \mathsf{A}^*; \{\mathsf{E}_{\mathsf{d}-i}\}_{i=0}^{\mathsf{d}}; \{\mathsf{E}_i^*\}_{i=0}^{\mathsf{d}}).$$

For $0 \le i \le \mathsf{d}$, let θ_i^* denote the eigenvalue of A^* associated with E_i^*. By [30, Theorem 3.2] there exist nonzero scalars $\{\varphi_i\}_{i=0}^{\mathsf{d}}$ and a $\mathbb{C}$-algebra homomorphism $\natural : \mathcal{A} \to M_{\mathsf{d}+1}(\mathbb{C})$ such that

$$\mathsf{A}^{\natural} = \begin{bmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_{\mathsf{d}} \end{bmatrix}, \qquad \mathsf{A}^{*\natural} = \begin{bmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_{\mathsf{d}} \\ \mathbf{0} & & & & \theta_{\mathsf{d}}^* \end{bmatrix}.$$

We call the sequence $\{\varphi_i\}_{i=1}^{\mathsf{d}}$ the *first split sequence* of Φ. We let $\{\phi_i\}_{i=0}^{\mathsf{d}}$ denote the first split sequence of $\Phi^{\Downarrow}$ and call this the *second split sequence* of Φ. By the *parameter array* of Φ we mean the sequence

$$p(\Phi) := (\{\theta_i\}_{i=0}^{\mathsf{d}}, \{\theta_i^*\}_{i=0}^{\mathsf{d}}, \{\varphi_i\}_{i=1}^{\mathsf{d}}, \{\phi_i\}_{i=1}^{\mathsf{d}}).$$

Let Ψ denote a Leonard system in a $\mathbb{C}$-algebra $\mathcal{B}$. We say that Ψ is *isomorphic* to Φ whenever there is a $\mathbb{C}$-algebra isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\Psi = \Phi^{\alpha} := (\mathsf{A}^{\alpha}; \mathsf{A}^{*\alpha}; \{\mathsf{E}_i^{\alpha}\}_{i=0}^{\mathsf{d}}; \{\mathsf{E}_i^{*\alpha}\}_{i=0}^{\mathsf{d}})$. In [30, Theorem 1.9] Terwilliger classified Leonard systems by using parameter arrays, and characterized the set of parameter arrays of Leonard systems with diameter d. Moreover, he displayed all the parameter arrays over $\mathbb{C}$ in [33]. We recall the q-Racah family of parameter arrays, that is the most general family.

Example 4.3. [33, Example 5.3] (q-Racah type) For $0 \le i \le \mathsf{d}$ define

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \tag{17}$$

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}, \tag{18}$$

and for $1 \le i \le \mathsf{d}$ define

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-\mathsf{d}-1})(1-r_1 q^i)(1-r_2 q^i), \tag{19}$$

$$\phi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-\mathsf{d}-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^*, \tag{20}$$

where θ_0 and θ_0^* are scalars in $\mathbb{C}$, and where h, h^*, s, s^*, r_1, r_2 are nonzero scalars in $\mathbb{C}$ such that $r_1 r_2 = ss^* q^{\mathsf{d}+1}$. To avoid degenerate situations assume that

(i) none of $q^i, r_1 q^i, r_2 q^i, s^* q^i/r_1, s^* q^i/r_2$ is equal to 1 for $1 \le i \le \mathsf{d}$,
(ii) neither of $sq^i, s^* q^i$ is equal to 1 for $2 \le i \le 2\mathsf{d}$.

Then the sequence $(\{\theta_i\}_{i=0}^{\mathsf{d}}, \{\theta_i^*\}_{i=0}^{\mathsf{d}}, \{\varphi_i\}_{i=1}^{\mathsf{d}}, \{\phi_i\}_{i=1}^{\mathsf{d}})$ is a parameter array over $\mathbb{C}$. This parameter array is said to have *q-Racah type*.

We say that Φ *has q-Racah type* whenever its parameter array has q-Racah type.

Let u be a nonzero vector in $\mathsf{E}_0\mathsf{V}$. By [32, Lemma 10.2], the sequence $\{\mathsf{E}_i^* u\}_{i=0}^{\mathsf{d}}$ is a basis for V, called a Φ-*standard basis* for V. The following is a characterization of the Φ-standard basis.

Lemma 4.4. [32, Lemma 10.4] *Let $\{v_i\}_{i=0}^{\mathsf{d}}$ denote a sequence of vectors in V, not all 0. Then this sequence is a Φ-standard basis for V if and only if both* (i) $v_i \in \mathsf{E}_i^*\mathsf{V}$ *for* $0 \le i \le \mathsf{d}$; (ii) $\sum_{i=0}^{\mathsf{d}} v_i \in \mathsf{E}_0\mathsf{V}$.

Consider the Leonard system Φ from Definition 4.1 and its corresponding parameter array $p(\Phi)$. The matrix representing A^* relative to a Φ-standard basis is

$$\operatorname{diag}(\theta_0^*, \theta_1^*, \theta_2^*, \ldots, \theta_{\mathsf{d}}^*).$$

Moreover, the matrix representing A relative to a Φ-standard basis is the tridiagonal matrix

$$\begin{bmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_{\mathsf{d}-1} \\ \mathbf{0} & & & c_{\mathsf{d}} & a_{\mathsf{d}} \end{bmatrix}, \tag{21}$$

where $\{a_i\}_{i=0}^{\mathsf{d}}$, $\{b_i\}_{i=0}^{\mathsf{d}-1}$, $\{c_i\}_{i=1}^{\mathsf{d}}$ are some scalars in $\mathbb{C}$. We call a_i, b_i, c_i the *intersection numbers of* Φ. Note that the matrix (21) has constant row sum θ_0 [32, Lemma 10.5].

Example 4.5. Let Φ be the Leonard system in Example 4.2. Then $\{A_i\hat{x}\}_{i=0}^{D}$ form a Φ-standard basis for $M\hat{x}$. Recall the scalars a_i, b_i, c_i from (13). These are the intersection numbers of Φ [28, Theorem 4.1(vi)].

Note 4.6. Recall from Section 3 that Γ is a Q-polynomial distance-regular graph. Let $\Phi = \Phi(\Gamma)$ denote the Leonard system (16) associated with Γ. We say that Γ *has q-Racah type* when Φ has q-Racah type. For the rest of the paper, assume that Γ has q-Racah type. Because $p(\Phi)$ has q-Racah type, it satisfies (17)–(20) for some scalars h, h^*, s, s^*, r_1, r_2. We fix this notation for the rest of the paper. Referring to this notation, whenever we encounter square roots, these are interpreted as follows. We fix square roots $s^{1/2}, s^{*1/2}, r_1^{1/2}, r_2^{1/2}$ such that $r_1^{1/2} r_2^{1/2} = s^{1/2} s^{*1/2} q^{(D+1)/2}$.

Let Φ denote a Leonard system in Definition 4.1. Let $p(\Phi) = p(\Phi; q)$ denote the parameter array of Φ that has q-Racah type in Example 4.3. In the following proposition we describe the parameter array that has q^{-1}-Racah type.

Proposition 4.7. (q^{-1}-Racah type) *For* $0 \le i \le \mathsf{d}$ *define*

$$\theta_i' = \theta_0' + h'(1 - q^{-i})(1 - s' q^{-i-1}) q^i, \tag{22}$$

$$\theta_i^{*\prime} = \theta_0^{*\prime} + h^{*\prime}(1 - q^{-i})(1 - s^{*\prime} q^{-i-1}) q^i, \tag{23}$$

and for $1 \le i \le \mathsf{d}$ *define*

$$\varphi_i{}' = h' h^{*\prime} q^{-1+2i}(1 - q^{-i})(1 - q^{-i+\mathsf{d}+1})(1 - r_1' q^{-i})(1 - r_2' q^{-i}), \tag{24}$$

$$\phi_i{}' = h' h^{*\prime} q^{-1+2i}(1 - q^{-i})(1 - q^{-i+\mathsf{d}+1})(r_1' - s^{*\prime} q^{-i})(r_2' - s^{*\prime} q^{-i})/s^{*\prime}, \tag{25}$$

where

$$\theta_0' = \theta_0, \qquad \theta_0^{*\prime} = \theta_0^*, \tag{26}$$

$$h' = hsq, \qquad s' = s^{-1}, \qquad r_1' = r_1^{-1}, \tag{27}$$

$$h^{*\prime} = h^* s^* q, \qquad s^{*\prime} = s^{*-1}, \qquad r_2' = r_2^{-1}. \tag{28}$$

Then the sequence $p(\Phi; q^{-1}) := (\{\theta_i'\}_{i=0}^{\mathsf{d}}, \{\theta_i^{*\prime}\}_{i=0}^{\mathsf{d}}, \{\varphi_i'\}_{i=1}^{\mathsf{d}}, \{\phi_i'\}_{i=1}^{\mathsf{d}})$ *is equal to* $p(\Phi; q)$. *Therefore* $p(\Phi; q^{-1})$ *is the parameter array that has* q^{-1}*-Racah type.*

Proof. Using (26)–(28) one checks that $\theta_i' = \theta_i$, $\theta_i^{*\prime} = \theta_i^*$ for $0 \le i \le \mathsf{d}$ and $\varphi_i' = \varphi_i$, $\phi_i' = \phi_i$ for $1 \le i \le \mathsf{d}$. It follows that $p(\Phi; q^{-1})$ is the parameter array of Φ. By definition of q-Racah type in Example 4.3, $p(\Phi; q^{-1})$ has q^{-1}-Racah type. $\square$

5. T-module W

We recall the T-module $\mathbf{W}$ from the last paragraph in Section 3. Note that $\mathbf{W}$ is decomposed into the direct sum of two irreducible T-modules $M\hat{x}$ and $M\hat{x}^{\perp}$ [19, Section 5]. We first discuss $M\hat{x}$ and its associated polynomials. Recall from Example 4.2 that

$\Phi := (A, A^*, \{E_i\}_{i=0}^D, \{E_i^*\}_{i=0}^D)$ is a Leonard system on $M\hat{x}$. Also recall from Example 4.5 that $\{A_i\hat{x}\}_{i=0}^D$ is the Φ-standard basis for $M\hat{x}$ and the intersection numbers a_i, b_i, c_i of Φ. Abbreviate $v_i = A_i\hat{x}$ for $0 \leq i \leq D$. Observe that $v_0 = \hat{C}_0^- = \hat{x}$, $v_i = \hat{C}_{i-1}^+ + \hat{C}_i^-$ $(1 \leq i \leq D-1)$, and $v_D = \hat{C}_{D-1}^+$. We now define a sequence of polynomials $f_0, f_1, \ldots, f_D$ by $f_0 := 1$ and

$$xf_i = b_{i-1}f_{i-1} + a_if_i + c_{i+1}f_{i+1} \qquad (0 \leq i \leq D-1),$$

where $f_{-1} = 0$. Then by [32, Theorem 13.4] we have

$$f_i(A)v_0 = v_i \qquad (0 \leq i \leq D). \tag{29}$$

For $0 \leq i \leq D$, define the scalars k_i by

$$k_i = b_0b_1 \cdots b_{i-1}/c_1c_2 \cdots c_i. \tag{30}$$

With the scalars k_i and the polynomials f_i we define the polynomial F_i by

$$F_i = f_i/k_i \qquad (0 \leq i \leq D). \tag{31}$$

One routinely checks that

$$xF_i = b_iF_{i+1} + a_iF_i + c_iF_{i-1} \qquad (0 \leq i \leq D-1),$$

where $F_{-1} = 0$. By [32, Theorem 23.2], it follows that for $0 \leq i \leq D$

$$F_i(x) = \sum_{j=0}^{i} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{j-1}^*)}{\varphi_1\varphi_2\cdots\varphi_j}(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{j-1}). \tag{32}$$

Definition 5.1. With reference to the parameters s, s^*, r_1, r_2, D associated with $p(\Phi) = p(\Phi; q)$, define the scalars a, b, c, d by

$$a = \left(\frac{r_1r_2}{s^*q^D}\right)^{1/2}, \quad b = \left(\frac{s^*}{r_1r_2q^D}\right)^{1/2}, \quad c = \left(\frac{r_2s^*q^{D+2}}{r_1}\right)^{1/2}, \quad d = \left(\frac{r_1s^*q^{D+2}}{r_2}\right)^{1/2}. \tag{33}$$

We say that the scalars a, b, c, d *are associated with* $p(\Phi)$.

Referring to Definition 5.1, we have the following equations which are useful for our calculation.

$$\begin{aligned} &ab = q^{-D}, && ac = r_2q, && ad = r_1q, && \\ &bc = s^*q/r_1, && bd = s^*q/r_2, && cd = s^*q^{D+2}, && abcd = s^*q^2. \end{aligned} \tag{34}$$

Lemma 5.2. *Let the scalars a, b, c, d be as in Definition 5.1. Then the following hold:*

(i) *none of $abq^i, acq^i, adq^i, bcq^i, bdq^i$ is equal to 1 for $0 \leq i \leq D-1$,*
(ii) *$cdq^i \neq 1$ for $-D \leq i \leq D-2$,*
(iii) *$abcdq^i \neq 1$ for $0 \leq i \leq 2D-2$.*

Proof. Use assumption (i), (ii) in Example 4.3 and (34). □

Consider a finite sequence of the polynomials $\{p_i(y+y^{-1})\}_{i=0}^D$ which are defined by the scalars a, b, c, d associated with $p(\Phi)$:

$$p_i(y+y^{-1}) = p_i(y+y^{-1}; a, b, c, d \mid q) = {}_4\phi_3\left(\begin{matrix} q^{-i},\ abcdq^{i-1},\ ay,\ ay^{-1} \\ ab,\ ac,\ ad \end{matrix} \,\middle|\, q,\ q\right), \tag{35}$$

where y is indeterminate; cf. (2). Applying (33) and the equation $r_1 r_2 = ss^* q^{D+1}$ to (35) gives

$$p_i(y+y^{-1}) = {}_4\phi_3\left(\begin{matrix} q^{-i},\ s^*q^{i+1},\ (sq)^{1/2}y,\ (sq)^{1/2}y^{-1} \\ q^{-D},\ r_1 q,\ r_2 q \end{matrix} \,\middle|\, q,\ q\right). \tag{36}$$

Lemma 5.3. *Recall the polynomial sequences $\{F_i\}_{i=0}^D$ from (32) and $\{p_i(y+y^{-1})\}_{i=0}^D$ from (36). Let x be of the form*

$$h(sq)^{1/2}(y+y^{-1}) + (\theta_0 - h - hsq), \tag{37}$$

where h, s, θ_0 are associated with $p(\Phi)$. Then

$$F_i(x) = p_i(y+y^{-1}), \qquad i = 0, 1, 2, \ldots D. \tag{38}$$

Proof. First we compute the left-hand side in (38). Put (37) for x in (32) and evaluate the result to obtain

$$\sum_{j=0}^{i} \frac{(q^{-i};q)_j (s^*q^{i+1};q)_j (s^{1/2}q^{1/2}y;q)_j (s^{1/2}q^{1/2}y^{-1};q)_j}{(r_1 q;q)_j (r_2 q;q)_j (q^{-D};q)_j (q;q)_n} q^j.$$

This is equal to the right-hand side of (36) by the definition of basic hypergeometric series. Therefore the result follows. □

Remark 5.4. With the above discussion, pick an integer j $(0 \leq i \leq D)$. Evaluating (36) at $y = s^{1/2}q^{1/2+j}$ (or $y = s^{-1/2}q^{-1/2-j}$) we get

$${}_4\phi_3\left(\begin{matrix} q^{-i},\ s^*q^{i+1},\ q^{-j},\ sq^{j+1} \\ q^{-D},\ r_1 q,\ r_2 q \end{matrix} \,\middle|\, q,\ q\right).$$

By this and definition of the q-Racah polynomials [1], $\{p_i(y+y^{-1})\}_{i=0}^{D-1}$ are the q-Racah polynomials.

We now consider $M\hat{x}^\perp$, the orthogonal complement of $M\hat{x}$ in $\mathbf{W}$. From [19, Section 6] the sequence $\Phi_i^\perp := (A, A^*, \{E_i^\perp\}_{i=0}^{D-2}, \{E_i^{*\perp}\}_{i=0}^{D-2})$ acts as a Leonard system on $M\hat{x}^\perp$, where $E_i^\perp = E_{i+1}$ and $E_i^{*\perp} = E_{i+1}^*$ for $0 \le i \le D-2$. Define the vectors $\{v_i^\perp\}_{i=0}^{D-2}$ by

$$v_i^\perp = \xi_{i+1}\hat{C}_i^+ + \xi_{i+1}\epsilon_{i+1}\hat{C}_{i+1}^-, \tag{39}$$

where for $1 \le i \le D-1$

$$\xi_i = q^{1-i}(1-q^{i-D})(1-s^*q^{i+1}), \qquad \epsilon_i = \frac{(1-q^i)(1-s^*q^{D+i+1})}{q^D(1-q^{i-D})(1-s^*q^{i+1})}. \tag{40}$$

Then the sequence $\{v_i^\perp\}_{i=0}^{D-2}$ is a $\Phi^\perp$-standard basis for $M\hat{x}^\perp$ [19, Lemma 6.6]. Let $a_i^\perp$, $b_i^\perp$, $c_i^\perp$ denote the intersection numbers of $\Phi^\perp$ [19, (83)]. We define a sequence of polynomials $f_0^\perp, f_1^\perp, \ldots, f_{D-2}^\perp$ by $f_0^\perp := 1$ and

$$xf_i^\perp = b_{i-1}^\perp f_{i-1}^\perp + a_i^\perp f_i^\perp + c_{i+1}^\perp f_{i+1}^\perp \qquad (0 \le i \le D-3),$$

where $f_{-1} = 0$. By [32, Theorem 13.4] we have

$$f_i^\perp(A)v_0^\perp = v_i^\perp \qquad (0 \le i \le D-2). \tag{41}$$

For $0 \le i \le D-2$ define the scalars $k_i^\perp$ by

$$k_i^\perp = b_0^\perp b_1^\perp \cdots b_{i-1}^\perp / c_1^\perp c_2^\perp \cdots c_i^\perp. \tag{42}$$

With the scalars $k_i^\perp$ and the polynomials $f_i^\perp$ we define the polynomial $F_i^\perp$ by

$$F_i^\perp = f_i^\perp / k_i^\perp \qquad (0 \le i \le D-2). \tag{43}$$

One routinely checks that

$$xF_i^\perp = b_i^\perp F_{i+1}^\perp + a_i^\perp F_i^\perp + c_i^\perp F_{i-1}^\perp \qquad (0 \le i \le D-3),$$

where $F_{-1}^\perp = 0$. Consider the parameter array $p(\Phi^\perp) = (\{\theta_i^\perp\}_{i=0}^{D-2}, \{\theta_i^{*\perp}\}_{i=0}^{D-2}, \{\varphi_i^\perp\}_{i=0}^{D-2}, \{\phi_i^\perp\}_{i=0}^{D-2})$. Applying [32, Theorem 23.2] to $\Phi^\perp$ we find that for $0 \le i \le D-2$

$$F_i^\perp(x) = \sum_{j=0}^{i} \frac{(\theta_i^{*\perp} - \theta_0^{*\perp})(\theta_i^{*\perp} - \theta_1^{*\perp}) \cdots (\theta_i^{*\perp} - \theta_{j-1}^{*\perp})}{\varphi_1^\perp \varphi_2^\perp \cdots \varphi_j^\perp}(x-\theta_0^\perp)(x-\theta_1^\perp)\cdots(x-\theta_{j-1}^\perp). \tag{44}$$

Lemma 5.5. [19, Theorem 6.10, Corollary 6.11] *The parameter array $p(\Phi^\perp)$ has q-Racah type. With reference to the scalars h, h^*, s, s^*, r_1, r_2 associated with $p(\Phi)$ and the scalars $h^\perp, h^{*\perp}, s^\perp, s^{*\perp}, r_1^\perp, r_2^\perp$ associated with $p(\Phi^\perp)$, their relations are as follows.*

$$h^\perp = hq^{-1}, \qquad s^\perp = sq^2, \qquad r_1^\perp = r_1 q, \tag{45}$$

$$h^{*\perp} = h^* q^{-1}, \qquad s^{*\perp} = s^* q^2, \qquad r_2^\perp = r_2 q. \tag{46}$$

Consider the scalars $a^\perp, b^\perp, c^\perp, d^\perp$ associated with $p(\Phi^\perp)$ in a view of Definition 5.1. Then by Lemma 5.5,

$$\begin{aligned} a^\perp &= \left(\tfrac{r_1 r_2}{s^* q^{D-2}}\right)^{1/2}, \quad b^\perp = \left(\tfrac{s^*}{r_1 r_2 q^{D-2}}\right)^{1/2}, \\ c^\perp &= \left(\tfrac{r_2 s^* q^{D+2}}{r_1}\right)^{1/2}, \quad d^\perp = \left(\tfrac{r_1 s^* q^{D+2}}{r_2}\right)^{1/2}. \end{aligned} \tag{47}$$

Using Definition 5.1 one can readily check that

$$a^\perp = aq, \qquad b^\perp = bq, \qquad c^\perp = c, \qquad d^\perp = d.$$

We define a finite sequence of the polynomials $\{p_i^\perp(y+y)\}_{i=0}^{D-2}$ which are defined by the scalars $a^\perp, b^\perp, c^\perp, d^\perp$ associated with $p(\Phi^\perp)$:

$$p_i^\perp(y+y^{-1}) := p_i(y+y^{-1}; a^\perp, b^\perp, c^\perp, d^\perp \mid q) = p_i(y+y^{-1}; aq, bq, c, d \mid q). \tag{48}$$

Using (47) we find

$$p_i^\perp(y+y^{-1}) = {}_4\phi_3\left(\begin{matrix} q^{-i}, \ s^* q^{i+3}, \ (sq)^{1/2} qy, \ (sq)^{1/2} qy^{-1} \\ q^{-D+2}, \ r_1 q^2, \ r_2 q^2 \end{matrix} \,\middle|\, q, \ q\right). \tag{49}$$

Lemma 5.6. *Recall the polynomial sequences $\{F_i^\perp\}_{i=0}^{D-2}$ from (44) and $\{p_i^\perp(y+y^{-1})\}_{i=0}^{D-2}$ from (49). Let x be of the form*

$$h(sq)^{1/2}(y+y^{-1}) + (\theta_0 - h - hsq),$$

where h, s, θ_0 are associated with $p(\Phi)$. Then

$$F_i^\perp(x) = p_i^\perp(y+y^{-1}), \qquad i = 0, 1, 2, \dots D-2.$$

Proof. Similar to Lemma 5.3. $\square$

Recall the Φ-standard basis $\{v_i\}_{i=0}^D$ for $M\hat{x}$ and the $\Phi^\perp$-standard basis $\{v_i^\perp\}_{i=0}^{D-2}$ for $M\hat{x}^\perp$. By [19, Lemma 8.3], for $1 \le i \le D-1$

$$\hat{C}_{i-1}^+ = \frac{\epsilon_i}{\epsilon_i - 1} v_i + \frac{1}{\xi_i(1-\epsilon_i)} v_{i-1}^\perp, \qquad \hat{C}_i^- = \frac{1}{1-\epsilon_i} v_i + \frac{1}{\xi_i(\epsilon_i - 1)} v_{i-1}^\perp. \tag{50}$$

Lemma 5.7. *For* $1 \leq i \leq D-1$,

$$\hat{C}^+_{i-1} = \frac{\epsilon_i}{\epsilon_i - 1} f_i(A)v_0 + \frac{1}{\xi_i(1-\epsilon_i)} f^\perp_{i-1}(A)v^\perp_0, \tag{51}$$

$$\hat{C}^-_i = \frac{1}{1-\epsilon_i} f_i(A)v_0 + \frac{1}{\xi_i(\epsilon_i - 1)} f^\perp_{i-1}(A)v^\perp_0. \tag{52}$$

And

$$\hat{C}^-_0 = f_0(A)v_0, \qquad \hat{C}^+_{D-1} = f_D(A)v_0.$$

Proof. To get (51), (52) use (29), (41) together with (50). Note that $\hat{C}^-_0 = v_0$ and $\hat{C}^+_{D-1} = v_D$. The result follows. □

We finish this section with a few comments. The following lemmas will be useful in the sequel.

Lemma 5.8. *Recall the Hermitian inner product* $\langle \cdot, \cdot \rangle_V$ *from the first paragraph in Section* 3. *For* $0 \leq i \leq D-1$,

$$\|\hat{C}^-_i\|^2 = \left(\frac{s^* q^D}{r_1 r_2}\right)^i \frac{(q^{1-D}, s^*q^2, r_1q, r_2q; q)_i}{(q, s^*q^{D+2}, s^*q/r_1, s^*q/r_2; q)_i},$$

$$\|\hat{C}^+_i\|^2 = -\frac{s^*(1-r_1q)(1-r_2q)}{(r_1 - s^*q)(r_2 - s^*q)} \left(\frac{s^* q^D}{r_1 r_2}\right)^i \frac{(q^{1-D}, s^*q^2, r_1q^2, r_2q^2; q)_i}{(q, s^*q^{D+2}, s^*q^2/r_1, s^*q^2/r_2; q)_i}.$$

Proof. Evaluate [19, Lemma 5.7] using [19, (17)–(20)] and [19, Corollary 4.9]. The results routinely follow. □

Note that $\|\hat{C}^-_i\|^2$, $\|\hat{C}^+_i\|^2$ are all positive integral, since they are equal to the cardinality of C^-_i, C^+_i, respectively.

Lemma 5.9. [31, Section 18]

(i) *Recall the scalar* k_i $(0 \leq i \leq D)$ *from* (30). *Then*

$$k_i = \frac{(r_1q;q)_i (r_2q;q)_i (q^{-D};q)_i (s^*q;q)_i (1 - s^*q^{2i+1})}{s^i q^i (q;q)_i (s^*q/r_1;q)_i (s^*q/r_2;q)_i (s^*q^{D+2};q)_i (1-s^*q)}.$$

To get k^*_i, *replace* (s, s^*) *with* (s^*, s).

(ii) *Recall the scalar* $k^\perp_i$ $(0 \leq i \leq D-2)$ *from* (42). *Then*

$$k^\perp_i = \frac{(r_1q^2;q)_i (r_2q^2;q)_i (q^{-D+2};q)_i (s^*q^3;q)_i (1 - s^*q^{2i+3})}{s^i q^{3i} (q;q)_i (s^*q^2/r_1;q)_i (s^*q^2/r_2;q)_i (s^*q^{D+2};q)_i (1-s^*q^3)}.$$

To get $k^{*\perp}_i$, *replace* (s, s^*) *with* (s^*, s).

(iii) *Let* ν *be the scalar such that* $tr(E_0E_0^*) = \nu^{-1}$. *Then*

$$\nu = \frac{(sq^2;q)_D(s^*q^2;q)_D}{r_1^D q^D (sq/r_1;q)_D(s^*q/r_1)_D}.$$

(iv) *Let* $\nu^\perp$ *be the scalar such that* $tr(E_0^\perp E_0^{*\perp}) = \nu^{\perp -1}$. *Then*

$$\nu^\perp = \frac{(sq^4;q)_{D-2}(s^*q^4;q)_{D-2}}{r_1^{D-2} q^{2D-4}(sq^2/r_1;q)_{D-2}(s^*q^2/r_1)_{D-2}}.$$

Proof. (i), (iii): From [31, p. 37].
(ii): In part (i) replace D by $D-2$ and use (45), (46) to get the result.
(iv): In part (iii) replace D by $D-2$ and use (45), (46) to get the result. □

6. The universal DAHA of type $(C_1^\vee, C_1)$

In this section we discuss the universal DAHA of type $(C_1^\vee, C_1)$ and its properties. For notational convenience define an index set $\mathbb{I} = \{0, 1, 2, 3\}$.

Definition 6.1. [34, Definition 3.1] The *universal DAHA* of type $(C_1^\vee, C_1)$ is the $\mathbb{C}$-algebra $\hat{H}_q$ defined by generators $\{t_n^{\pm 1}\}_{n\in\mathbb{I}}$ and relations

(i) $t_nt_n^{-1} = t_n^{-1}t_n = 1$ $(n \in \mathbb{I})$; (ii) $t_n + t_n^{-1}$ is central $(n \in \mathbb{I})$; (iii) $t_0t_1t_2t_3 = q^{-1/2}$.

For notational convenience define the following elements in $\hat{H}_q$:

$$\mathbf{Y} = t_0t_1, \qquad \mathbf{X} = t_3t_0, \qquad \widetilde{\mathbf{X}} = t_1t_2 = q^{-1/2}(t_3t_0)^{-1},$$
$$\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}, \qquad \mathbf{B} = \mathbf{X} + \mathbf{X}^{-1}, \qquad \widetilde{\mathbf{B}} = \widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}^{-1}.$$

Lemma 6.2. *There exists a unique antiautomorphism* $\dagger$ *of* $\hat{H}_q$ *that sends*

$$t_0 \mapsto t_1, \qquad t_1 \mapsto t_0, \qquad t_2 \mapsto t_3, \qquad t_3 \mapsto t_2.$$

Moreover $\dagger^2 = 1$.

Proof. Use Definition 6.1. □

By Lemma 6.2 we have the following:

$$\mathbf{Y}^\dagger = \mathbf{Y}, \qquad \mathbf{X}^\dagger = \widetilde{\mathbf{X}}, \qquad \widetilde{\mathbf{X}}^\dagger = \mathbf{X}. \tag{53}$$

In [19, Section 11] the author showed that the space $\mathbf{W}$ is an $\hat{H}_q$-module as well as a $\mathbf{T}$-module, and displayed its module structure in detail. In the present paper, for the purpose of our study we will twist $\mathbf{W}$ via a certain $\mathbb{C}$-algebra automorphism of $\hat{H}_q$. Recall the $\hat{H}_q$-module $\mathbf{W}$ from [19, Section 11]. Consider a $\mathbb{C}$-algebra automorphism $\rho : \hat{H}_q \to \hat{H}_q$ that sends

$$t_0 \mapsto t_1, \qquad t_1 \mapsto t_0, \qquad t_2 \mapsto t_0^{-1} t_3 t_0, \qquad t_3 \mapsto t_1 t_2 t_1^{-1}.$$

Observe that $\rho^2 = 1$. There exists an $\hat{H}_q$-module structure on $\mathbf{W}$, called $\mathbf{W}$ *twisted via* ρ, that behaves as follows; for all $h \in \hat{H}_q$, $w \in \mathbf{W}$, the vector $h.w$ computed in $\mathbf{W}$ twisted via ρ coincides with the vector $h^\rho.w$ computed in the original $\hat{H}_q$-module $\mathbf{W}$. We display the $\hat{H}_q$-module structure $\mathbf{W}$ twisted via ρ in Appendix 13.1 in detail. For the rest of the paper, we regard an $\hat{H}_q$-module $\mathbf{W}$ as the $\hat{H}_q$-module $\mathbf{W}$ twisted via ρ. The $\hat{H}_q$-module structure on $\mathbf{W}$ is determined by the scalars q, s, s^*, r_1, r_2, D associated with $p(\Phi)$. We denote this module by $\mathbf{W}(s, s^*, r_1, r_2, D; q)$.

Define the scalars $\{\kappa_n\}_{n\in\mathbb{I}}$ by

$$\kappa_0 = \left(\frac{1}{q^D}\right)^{1/2}, \qquad \kappa_1 = \left(\frac{r_1 r_2}{s^*}\right)^{1/2}, \qquad \kappa_2 = \left(\frac{r_2}{r_1}\right)^{1/2}, \qquad \kappa_3 = \left(s^* q^{D+1}\right)^{1/2}. \tag{54}$$

Lemma 6.3. *For each $n \in \mathbb{I}$, the scalar κ_n is not equal to ± 1.*

Proof. Use assumption (i), (ii) in Example 4.3. □

Lemma 6.4. *For each $n \in \mathbb{I}$, t_n is diagonalizable on $\mathbf{W}(s, s^*, r_1, r_2, D; q)$.*

Proof. By [19, Lemma 11.9] for each $n \in \mathbb{I}$ it follows $(t_n + t_n^{-1}).w = (\kappa_n + \kappa_n^{-1}).w$ for all $w \in \mathbf{W}$. So the minimal polynomial of t_n is $(x - \kappa_n)(x - \kappa_n^{-1})$, and this has distinct roots by Lemma 6.3. The result follows. □

On $\mathbf{W}(s, s^*, r_1, r_2, D; q)$ the action of $\mathbf{X}$ on $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ as follows [19, Lemma 11.8]:

$$\mathbf{X}.\hat{C}_i^- = q^i (s^* q)^{1/2} \hat{C}_i^-, \qquad \mathbf{X}.\hat{C}_i^+ = q^{-i-1}(s^* q)^{-1/2} \hat{C}_i^+ \qquad (0 \le i \le D-1). \tag{55}$$

The action of $\mathbf{Y}$ on $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ is given as a linear combination of four terms; see Appendix 13.2. Recall the generators A, A^*, $\widetilde{A}^*$ of the algebra $\mathbf{T}$ and the elements $\mathbf{A}$, $\mathbf{B}$, $\widetilde{\mathbf{B}}$ of $\hat{H}_q$. The following theorem explains how the $\mathbf{T}$-action on $\mathbf{W}$ is related to the $\hat{H}_q$-action.

Theorem 6.5. [19, Theorem 12.1] *On* $\mathbf{W}$,

(i) *A acts as $h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)$;*
(ii) *A^* acts as $h^*(s^*q)^{1/2}\mathbf{B} + (\theta_0^* - h^* - h^* s^* q)$;*
(iii) *$\widetilde{A}^*$ acts as $\widetilde{h}^*(\widetilde{s}^*q)^{1/2}\widetilde{\mathbf{B}} + (\widetilde{\theta}_0^* - \widetilde{h}^* - \widetilde{h}^*\widetilde{s}^*q)$;*

(iv) $(t_0 - \kappa_0^{-1})(\kappa_0 - \kappa_0^{-1})^{-1}$ *acts as the projection of* $\mathbf{W}$ *onto* $M\hat{x}$;
(v) $(t_1 - \kappa_1^{-1})(\kappa_1 - \kappa_1^{-1})^{-1}$ *acts as the projection of* $\mathbf{W}$ *onto* $M\hat{C}$.

We now consider the scalars a, b, c, d from Definition 5.1. Using the relations (33) we can describe the $\hat{H}_q$-module $\mathbf{W}(s, s^*, r_1, r_2, D; q)$ in terms of a, b, c, d and q as follows.

Lemma 6.6. *Let the scalars* a, b, c, d *be as in Definition 5.1. Consider the block diagonal matrices* $\mathbf{T}_n (n \in \mathbb{I})$:

$$\mathbf{T}_0 = \text{blockdiag}\Big[\tau_0(0), \tau_0(1), \dots, \tau_0(D-1), [(ab)^{1/2}]\Big],$$
$$\mathbf{T}_1 = \text{blockdiag}\Big[\tau_1(0), \tau_1(1), \dots, \tau_1(D-1)\Big],$$
$$\mathbf{T}_2 = \text{blockdiag}\Big[\tau_2(0), \tau_2(1), \dots, \tau_2(D-1)\Big],$$
$$\mathbf{T}_3 = \text{blockdiag}\Big[\tau_3(0), \tau_3(1), \dots, \tau_3(D-1), [(cdq^{-1})^{-1/2}]\Big],$$

where $\tau_0(0) = \big[(ab)^{1/2}\big]$ *and for* $1 \le i \le D-1$,

$$\tau_0(i) = (ab)^{-1/2} \begin{bmatrix} \frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} + ab & -\frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} \\ \frac{ab(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} & -\frac{ab(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} + ab \end{bmatrix},$$

and for $0 \le i \le D-1$,

$$\tau_1(i) = (ab^{-1})^{1/2} \begin{bmatrix} \frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} + \frac{b}{a} & -\frac{b}{a}\frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}} \\ \frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} & \frac{b}{a}\left(1 - \frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}}\right) \end{bmatrix},$$

and for $0 \le i \le D-1$,

$$\tau_2(i) = (cd)^{-1/2} \begin{bmatrix} \frac{1}{aq^i}\left(1 - \frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}}\right) & \frac{bcdq^i(1-adq^i)(1-acq^i)}{1-abcdq^{2i}} \\ -\frac{1}{bq^i}\frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} & acdq^i\left(\frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} + \frac{b}{a}\right) \end{bmatrix},$$

and $\tau_3(0) = \big[(cdq^{-1})^{1/2}\big]$ *and for* $1 \le i \le D-1$,

$$\tau_3(i) = (cdq^{-1})^{-1/2} \begin{bmatrix} \frac{1}{q^i}\left(1 - \frac{(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\right) & \frac{1}{abq^i}\frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} \\ -\frac{abcdq^{i-1}(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} & cdq^{i-1}\left(\frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} + ab\right) \end{bmatrix}.$$

Then there exists an $\hat{H}_q$*-module structure on* $\mathbf{W}$ *such that for* $n \in \mathbb{I}$ *the matrix* $\mathbf{T}_n$ *represents the generator* t_n *relative to* $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$.

Proof. In Definition 13.1, replace s, s^*, r_1, r_2, D by a, b, c, d using relations (33). $\square$

For the rest of the paper we let $\mathbf{W}(a, b, c, d; q)$ denote the $\hat{H}_q$-module in Lemma 6.6.

Lemma 6.7. *Referring to the $\hat{H}_q$-module $\mathbf{W}(a, b, c, d; q)$ in Lemma 6.6, the action of $\mathbf{X}$ on $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ is as follows.*

$$\begin{cases} \mathbf{X}.\hat{C}_i^- = q^{i-\frac{1}{2}}(abcd)^{1/2}\hat{C}_i^- & \text{for} \quad i = 0, 1, \ldots, D-1, \\ \mathbf{X}.\hat{C}_{i-1}^+ = q^{-i+\frac{1}{2}}(abcd)^{-1/2}\hat{C}_{i-1}^+ & \text{for} \quad i = 1, 2, \ldots, D. \end{cases}$$

Proof. By Definition 5.1, $(s^*q)^{1/2} = q^{-1/2}(abcd)^{1/2}$. By this and (55) the result follows. $\square$

7. Nonsymmetric Laurent polynomials $\varepsilon_i^{\pm}$

In this section we define certain nonsymmetric Laurent polynomials $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ and discuss their properties. Let $\mathbb{C}[y, y^{-1}]$ denote the space of Laurent polynomials with a variable y. Recall the Φ-standard basis $\{v_i\}_{i=0}^{D}$ for $M\hat{x}$ and the $\Phi^{\perp}$-standard basis $\{v_i^{\perp}\}_{i=0}^{D-2}$ for $M\hat{x}^{\perp}$. Define the Laurent polynomial

$$g[y] := \left(\frac{s^*}{r_1 r_2 q^D}\right)^{1/2} \frac{(1-s^*q^2)(1-s^*q^3)}{(1-s^*q/r_1)(1-s^*q/r_2)} \left(y - \left(\frac{r_1 r_2}{s^* q^D}\right)^{1/2} - \left(\frac{s^*}{r_1 r_2 q^D}\right)^{1/2} + \frac{1}{q^D} y^{-1}\right). \tag{56}$$

Lemma 7.1. *Recall $\mathbf{Y} = t_0 t_1$. On the $\hat{H}_q$-module $\mathbf{W}(s, s^*, r_1, r_2, D; q)$,*

$$g[\mathbf{Y}]v_0 = v_0^{\perp}.$$

Proof. Abbreviate $u = v_0^{\perp} - g[\mathbf{Y}]v_0$. We show that $u = 0$. By (39), $v_0^{\perp} = \xi_1 \hat{C}_0^+ + \xi_1 \epsilon_1 \hat{C}_1^-$. Next evaluate $g[\mathbf{Y}]v_0$ using Lemma 13.2(a), Lemma 13.3(a) and $v_0 = \hat{C}_0^-$. Using these comments, we evaluate u to get zero. The result follows. $\square$

We now define nonsymmetric Laurent polynomials $\varepsilon_i^-, \varepsilon_i^+$ $(0 \leq i \leq D-1)$. Recall the equation (51), that is,

$$\hat{C}_{i-1}^+ = \frac{\epsilon_i}{\epsilon_i - 1} f_i(A)v_0 + \frac{1}{\xi_i(1-\epsilon_i)} f_{i-1}^{\perp}(A)v_0^{\perp}.$$

Applying (31) and (43) to the right-hand side of the above equation gives

$$\frac{\epsilon_i}{\epsilon_i - 1} k_i F_i(A)v_0 + \frac{1}{\xi_i(1-\epsilon_i)} k_{i-1}^{\perp} F_{i-1}^{\perp}(A)v_0^{\perp}. \tag{57}$$

Applying Theorem 6.5(i) to (57) we find

$$
\begin{aligned}
\hat{C}^{+}_{i-1} &= \frac{\epsilon_i}{\epsilon_i - 1} k_i F_i \Big(h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)\Big) v_0 \\
&\quad + \frac{1}{\xi_i(1-\epsilon_i)} k^{\perp}_{i-1} F^{\perp}_{i-1} \Big(h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)\Big) v^{\perp}_0 \\
\text{(by Lemma 5.3 and Lemma 5.6)} &= \frac{\epsilon_i}{\epsilon_i - 1} k_i p_i(\mathbf{A}) v_0 + \frac{1}{\xi_i(1-\epsilon_i)} k^{\perp}_{i-1} p^{\perp}_{i-1}(\mathbf{A}) v^{\perp}_0 \\
\text{(by Lemma 7.1)} &= \left(\frac{\epsilon_i}{\epsilon_i - 1} k_i p_i(\mathbf{A}) + \frac{1}{\xi_i(1-\epsilon_i)} k^{\perp}_{i-1} p^{\perp}_{i-1}(\mathbf{A}) g[\mathbf{Y}] \right) v_0,
\end{aligned}
\tag{58}
$$

where $\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}$. Similarly we find

$$
\hat{C}^{-}_{i} = \left(\frac{1}{1-\epsilon_i} k_i p_i(\mathbf{A}) + \frac{1}{\xi_i(\epsilon_i - 1)} k^{\perp}_{i-1} p^{\perp}_{i-1}(\mathbf{A}) g[\mathbf{Y}] \right) v_0. \tag{59}
$$

Motivated by (58) and (59) we make a definition as follows.

Definition 7.2. For $1 \leq i \leq D-1$, define

$$
\begin{aligned}
\varepsilon^{+}_{i-1}[y] &:= \frac{\epsilon_i}{\epsilon_i - 1} k_i p_i(y + y^{-1}) + \frac{1}{\xi_i(1-\epsilon_i)} k^{\perp}_{i-1} p^{\perp}_{i-1}(y + y^{-1}) g[y], \\
\varepsilon^{-}_{i}[y] &:= \frac{1}{1-\epsilon_i} k_i p_i(y + y^{-1}) + \frac{1}{\xi_i(\epsilon_i - 1)} k^{\perp}_{i-1} p^{\perp}_{i-1}(y + y^{-1}) g[y].
\end{aligned}
$$

Moreover, we define

$$
\varepsilon^{-}_0 := 1, \qquad \varepsilon^{+}_{D-1} := k_D p_D.
$$

Remark 7.3. With reference to Definition 7.2, the Laurent polynomials $\{\varepsilon^{\pm}_i\}_{i=0}^{D-1}$ are considered as *nonsymmetric q-Racah polynomials* in a view of Remark 5.4. The explicit forms are as follows. For $1 \leq i \leq D-1$

$$
\begin{aligned}
&\varepsilon^{+}_{i-1} := \\
&\frac{(1-q^i)(1-s^*q^{D+i+1})(q^{-D}, r_1 q, r_2 q, s^* q; q)_i}{s^i q^i (1-q^D)(1-s^*q)(q, s^*q/r_1, s^*q/r_2, s^*q^{D+2}; q)_i} \, {}_4\phi_3 \left(\left. \begin{matrix} q^{-i}, s^*q^{i+1}, (sq)^{1/2}y, (sq)^{1/2}y^{-1} \\ q^{-D}, r_1 q, r_2 q \end{matrix} \, \right| \, q, q \right) \\
&+ \frac{s^{1-i} q^{2-2i} (q^{-D+2}, r_1 q^2, r_2 q^2, s^* q^3; q)_{i-1} g[y]}{(1-q^{-D})(1-s^*q^3)(q, s^*q^2/r_1, s^*q^2/r_2, s^*q^{D+2}; q)_{i-1}} \, {}_4\phi_3 \left(\left. \begin{matrix} q^{-i+1}, s^*q^{i+2}, (sq)^{1/2}qy, (sq)^{1/2}qy^{-1} \\ q^{-D+2}, r_1 q^2, r_2 q^2 \end{matrix} \, \right| \, q, q \right), \\
&\varepsilon^{-}_{i} := \\
&\frac{(1-q^{i-D})(1-s^*q^{i+1})(q^{-D}, r_1 q, r_2 q, s^* q; q)_i}{s^i q^i (1-q^{-D})(1-s^*q)(q, s^*q/r_1, s^*q/r_2, s^*q^{D+2}; q)_i} \, {}_4\phi_3 \left(\left. \begin{matrix} q^{-i}, s^*q^{i+1}, (sq)^{1/2}y, (sq)^{1/2}y^{-1} \\ q^{-D}, r_1 q, r_2 q \end{matrix} \, \right| \, q, q \right) \\
&- \frac{s^{1-i} q^{2-2i} (q^{-D+2}, r_1 q^2, r_2 q^2, s^* q^3; q)_{i-1} g[y]}{(1-q^{-D})(1-s^*q^3)(q, s^*q^2/r_1, s^*q^2/r_2, s^*q^{D+2}; q)_{i-1}} \, {}_4\phi_3 \left(\left. \begin{matrix} q^{-i+1}, s^*q^{i+2}, (sq)^{1/2}qy, (sq)^{1/2}qy^{-1} \\ q^{-D+2}, r_1 q^2, r_2 q^2 \end{matrix} \, \right| \, q, q \right),
\end{aligned}
$$

where $g[y]$ is from (56).

Proposition 7.4. *Referring to the $\hat{H}_q$-module $\mathbf{W}(s, s^*, r_1, r_2, D; q)$, for $0 \le i \le D-1$*

$$\varepsilon_i^-[\mathbf{Y}].v_0 = \hat{C}_i^-, \qquad \varepsilon_i^+[\mathbf{Y}].v_0 = \hat{C}_i^+.$$

Proof. By (58) and (59) along with Definition 7.2, the result follows. □

By Proposition 7.4 we can see that the Laurent polynomials $\varepsilon_i^+, \varepsilon_i^-$ play a role of a map that sends v_0 to each $\hat{C}_i^+, \hat{C}_i^-$. Recall the scalars a, b, c, d from Definition 5.1. We will express the nonsymmetric Laurent polynomials $\{\varepsilon_i^\pm\}_{i=0}^{D-1}$ in terms of the scalars a, b, c, d. To this end, we need some preliminary lemmas.

Lemma 7.5. *Recall the Laurent polynomial $g[y]$ from (56). Then*

$$g[y] = \frac{b(1-abcd)(1-abcdq)}{(1-bc)(1-bd)} y(1-ay^{-1})(1-by^{-1}).$$

Proof. Apply the relations (33) to (56). The result follows. □

Lemma 7.6. *The following hold.*

(i) *For $1 \le i \le D-1$,*

$$\frac{\epsilon_i}{\epsilon_i - 1} = \frac{ab(1-q^i)(1-cdq^{i-1})}{(ab-1)(1-abcdq^{2i-1})}, \qquad \frac{1}{1-\epsilon_i} = \frac{(1-abq^i)(1-abcdq^{i-1})}{(1-ab)(1-abcdq^{2i-1})},$$
$$\frac{1}{\xi_i(\epsilon_i - 1)} = \frac{q^{i-1}}{(ab-1)(1-abcdq^{2i-1})}, \qquad \frac{1}{\xi_i(1-\epsilon_i)} = \frac{q^{i-1}}{(1-ab)(1-abcdq^{2i-1})}.$$

(ii) *For $0 \le i \le D-1$,*

$$k_i = \frac{(ab, ac, ad; q)_i (abcd; q)_{2i}}{a^{2i}(q, bc, bd, cd, abcdq^{i-1}; q)_i}.$$

For $i = D$,

$$k_D = \frac{(ab, ac, ad; q)_D (abcd; q)_{2D-1}}{a^{2D}(q, bc, bd, abcdq^{D-1}; q)_D (cd; q)_{D-1}}.$$

(iii) *For $0 \le i \le D-2$,*

$$k_i^\perp = \frac{(abq^2, acq, adq; q)_i (abcdq^2; q)_{2i}}{(aq)^{2i}(q, bcq, bdq, cd, abcdq^{i+1}; q)_i}.$$

Proof. (i): Use (33) and (40).
(ii), (iii): Use (33) and Lemma 5.9 (i), (ii). □

Recall the polynomial sequences $\{p_i\}_{i=0}^{D}$ and $\{p_i^\perp\}_{i=0}^{D-2}$ from (35) and (48), respectively. Denote their monic polynomials by

$$P_i = \frac{(ab,ac,ad;q)_i}{a^i(abcdq^{i-1};q)_i}p_i, \qquad P_i^\perp = \frac{(abq^2,acq,adq;q)_i}{(aq)^i(abcdq^{i+1};q)_i}p_i^\perp. \tag{60}$$

One checks that $P_i^\perp = P_i[y;aq,bq,c,d \mid q]$.

Lemma 7.7.

(i) *For $0 \le i \le D-1$,*

$$k_i p_i = \frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}P_i.$$

For $i = D$,

$$k_D p_D = \frac{(abcd;q)_{2D-1}}{a^D(q,bc,bd;q)_D(cd;q)_{D-1}}P_D.$$

(ii) *For $0 \le i \le D-2$,*

$$k_i^\perp p_i^\perp = \frac{(abcdq^2;q)_{2i}}{a^i q^i(q,bcq,bdq,cd;q)_i}P_i^\perp.$$

Proof. Use Lemma 7.6 (ii), (iii) and (60). □

Proposition 7.8. *Let $\{\varepsilon_i^\pm\}_{i=0}^{D-1}$ be as in Definition 7.2. Referring to the scalars a,b,c,d associated with $p(\Phi)$, the $\{\varepsilon_i^\pm\}_{i=0}^{D-1}$ are equal to the following: For $1 \le i \le D-1$,*

$$\begin{aligned}
\varepsilon_{i-1}^+ &= \frac{ab(1-q^i)(1-cdq^{i-1})}{(ab-1)(1-abcdq^{2i-1})}\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\left(P_i - y(1-ay^{-1})(1-by^{-1})P_{i-1}^\perp\right),\\
\varepsilon_i^- &= \frac{(1-abq^i)(1-abcdq^{i-1})}{(1-ab)(1-abcdq^{2i-1})}\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\\
&\quad\times\left(P_i - \frac{ab(1-q^i)(1-cdq^{i-1})}{(1-abq^i)(1-abcdq^{i-1})}y(1-ay^{-1})(1-by^{-1})P_{i-1}^\perp\right).
\end{aligned}$$

Moreover,

$$\varepsilon_0^- = 1, \qquad \varepsilon_{D-1}^+ = \frac{(abcd;q)_{2D-1}}{a^D(q,bc,bd;q)_D(cd;q)_{D-1}}P_D.$$

Proof. Apply Lemma 7.6 and Lemma 7.7 to Definition 7.2. The result follows. □

We give a comment. For $1 \le i \le D-1$, the ε_{i-1}^{+} has of the form

$$\frac{ab(1-q^i)(1-cdq^{i-1})}{(ab-1)(1-abcdq^{2i-1})}\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\Big((\text{constant})y^{i-1}+\cdots+(1-ab)y^{-i}\Big),$$

and the ε_i^- has of the form

$$\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\left(y^i+\cdots+\frac{1+ab-abq^i-abcdq^{i-1}}{1-abcdq^{2i-1}}y^{-i}\right).$$

Therefore the set $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ is linearly independent in $\mathbb{C}[y,y^{-1}]$.

Remark 7.9. Let L denote the subspace of $\mathbb{C}[y,y^{-1}]$ spanned by $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$. By the above comment, $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ are a basis for L. Recall the $\hat{H}_q$-module $\mathbf{W}(a,b,c,d;q)$ in Lemma 6.6. The space L is isomorphic to $\mathbf{W}$ via a $\mathbb{C}$-vector space isomorphism that sends ε_i^σ to $\hat{C}_i^\sigma$ for $\sigma \in \{+,-\}$ and $i=0,1,\dots,D-1$. By these comments we can endow a module structure for $\hat{H}_q$ with L. On this module L, the matrix representing t_n relative to a basis $\varepsilon_0^-,\varepsilon_0^+,\varepsilon_0^-,\varepsilon_0^+,\dots,\varepsilon_{D-1}^-,\varepsilon_{D-1}^+$ coincides with the matrix $\mathbf{T}_n$ in Lemma 6.6. We denote this module by $L(a,b,c,d;q)$.

8. The operator $\mathbf{Y}$

In Section 6 we mentioned the eigenvalues/eigenvectors of $\mathbf{X}$ on $\mathbf{W}(s,s^*,r_1,r_2,D;q)$. In this section we will find eigenvectors of $\mathbf{Y}$ along with the corresponding eigenvalues on $\mathbf{W}(s,s^*,r_1,r_2,D;q)$. First we find the eigenvalues of $\mathbf{A}=\mathbf{Y}+\mathbf{Y}^{-1}$. Throughout this section we work on the $\hat{H}_q$-module $\mathbf{W}=\mathbf{W}(s,s^*,r_1,r_2,D;q)$.

Lemma 8.1. *The eigenvalues of* $\mathbf{A}$ *are* $q^i(sq)^{1/2}+q^{-i}(sq)^{-1/2}$ *for* $0 \le i \le D$.

Proof. By Theorem 6.5(i), we find the equation $\mathbf{A}=h^{-1}(sq)^{-1/2}(A-(\theta_0-h-hsq)I)$ on $\mathbf{W}$. Recall that A has the eigenvalues $\{\theta_i\}_{i=0}^D$ and each θ_i has the form (17). By these comments, the result follows. $\square$

For notational convenience we denote $\ell_i = q^i(sq)^{1/2}+q^{-i}(sq)^{-1/2}$ for $0 \le i \le D$. Let W_{ℓ_i} denote the eigenspace of $\mathbf{A}$ for ℓ_i. Then $W_{\ell_i}=E_i\mathbf{W}$ for $0 \le i \le D$ since A and $\mathbf{A}$ share a common eigenvector by Theorem 6.5(i). Observe that $\mathbf{W}=\sum_{i=0}^D W_{\ell_i}$, the orthogonal direct sum. Moreover, by construction of the T-module $\mathbf{W}$ the following lemma holds.

Lemma 8.2. *For* $1 \le i \le D-1$, *we have*

$$W_{\ell_i}=\operatorname{span}\{E_iv_0, E_{i-1}^{\perp}v_0^{\perp}\}.$$

Moreover, $W_{\ell_0}=\operatorname{span}\{E_0v_0\}$ *and* $W_{\ell_D}=\operatorname{span}\{E_Dv_0\}$.

By Lemma 6.4 each of t_0 and t_1 is diagonalizable. By [15, Lemma 3.8] t_0 and t_1 commute with $\mathbf{A} = t_0t_1 + (t_0t_1)^{-1}$, so each of t_0 and t_1 shares the eigenspaces of $\mathbf{A}$. It follows that $W_{\ell_i}(0 \le i \le D)$ is invariant under $t_n(n = 0, 1)$. By these comments and Lemma 8.2 the matrix representing $t_n(n = 0, 1)$ relative to the ordered basis

$$\mathcal{B} := \{E_0v_0, E_1v_0, E_0^\perp v_0^\perp, E_2v_0, E_1^\perp v_0^\perp, \dots, E_{D-1}v_0, E_{D-2}^\perp v_0^\perp, E_Dv_0\}$$

for $\mathbf{W}$ takes the form

$$\begin{pmatrix} * & & & & & & & & \\ & * & * & & & & & & \\ & * & * & & & & & & \\ & & & * & * & & & & \\ & & & * & * & & & & \\ & & & & & \ddots & & & \\ & & & & & & * & * & \\ & & & & & & * & * & \\ & & & & & & & & * \end{pmatrix}. \tag{61}$$

We explicitly find the matrix representing $t_n(n = 0, 1)$ relative to $\mathcal{B}$, in order to find the eigenvectors of $\mathbf{Y}$. First we find the eigenvectors of t_0 in terms of vectors in $\mathcal{B}$. For notational convenience we define $\hat{C}_{-1}^- = 0$, $\hat{C}_{-1}^+ = 0$ and $\hat{C}_D^- = 0$, $\hat{C}_D^+ = 0$.

Lemma 8.3. *For* $1 \le i \le D - 1$,

$$t_0.(\hat{C}_{i-1}^+ + \hat{C}_i^-) = \kappa_0(\hat{C}_{i-1}^+ + \hat{C}_i^-), \tag{62}$$

$$t_0.(\epsilon_i^{-1}\hat{C}_{i-1}^+ + \hat{C}_i^-) = \kappa_0^{-1}(\epsilon_i^{-1}\hat{C}_{i-1}^+ + \hat{C}_i^-), \tag{63}$$

where ϵ_i *is from* (40) *and* κ_0 *is from* (54).

Proof. To show (62) it suffices to show that $(t_0-\kappa_0).(\hat{C}_{i-1}^+ + \hat{C}_i^-) = 0$. By Theorem 6.5(iv) the $(t_0 - \kappa_0^{-1})(\kappa_0 - \kappa_0^{-1})^{-1}$ acts as the projection onto $M\hat{x}$. By this and since $\hat{C}_{i-1}^+ + \hat{C}_i^- = v_i \in M\hat{x}$, we find (62). Next we show (63). Similar to (62), it suffices to show $(t_0 - \kappa_0^{-1}).(\epsilon^{-1}\hat{C}_{i-1}^+ + \hat{C}_i^-) = 0$. Since $(t_0 - \kappa_0)(\kappa_0^{-1} - \kappa_0)^{-1} = 1 - (t_0 - \kappa_0^{-1})(\kappa_0 - \kappa_0^{-1})^{-1}$ acts as the projection onto $M\hat{x}^\perp$ and $\epsilon_i^{-1}\hat{C}_{i-1}^+ + \hat{C}_i^- = \epsilon_i^{-1}\xi_i^{-1}v_{i-1}^\perp \in M\hat{x}^\perp$ for $1 \le i \le D - 1$, the desired result follows. □

Consider the eigenvector $\hat{C}_0^+ + \hat{C}_1^-$ of t_0 for κ_0. Observe that $\hat{C}_0^+ + \hat{C}_1^- = v_1 = Av_0 = \sum_{r=0}^{D} \theta_r E_r v_0$, where the last equation is from $A = \sum_{r=0}^{D} \theta_r E_r$. Therefore the coordinate vector of $\hat{C}_0^+ + \hat{C}_1^-$ relative to $\mathcal{B}$ is

$$\left[\theta_0, \theta_1, 0, \theta_2, 0, \dots, \theta_{D-1}, 0, \theta_D\right]^t. \tag{64}$$

Lemma 8.4. *For* $i \in \{0, D\}$,

$$t_0.E_iv_0 = \kappa_0 E_i v_0,$$

where κ_0 *is from* (54).

Proof. The matrix representing t_0 relative to $\mathcal{B}$ has the form (61). This matrix has the eigenvector (64) corresponding to the eigenvalue κ_0. Use this and linear algebra to get the result. □

Next consider the eigenvector $\epsilon_1^{-1}\hat{C}_0^+ + \hat{C}_1^-$ of t_0 for κ_0^{-1}. From the left in (50) for $i = 0$,

$$\hat{C}_0^+ = \tfrac{\epsilon_1}{\epsilon_1 - 1} v_1 + \tfrac{1}{\xi_1(1-\epsilon_1)} v_0^\perp = \sum_{r=0}^{D} \tfrac{\epsilon_1 \theta_r}{\epsilon_1 - 1} E_r v_0 + \sum_{s=0}^{D-2} \tfrac{1}{\xi_1(1-\epsilon_1)} E_s^\perp v_0^\perp, \tag{65}$$

where the last equality holds from $v_1 = Av_0 = \sum_{r=0}^{D} \theta_r E_r$ and $v_0^\perp = \sum_{s=0}^{D-2} E_s^\perp v_0^\perp$. Similarly, from the right in (50) for $i = 1$ we find

$$\hat{C}_1^- = \sum_{r=0}^{D} \tfrac{\theta_r}{1-\epsilon_1} E_r v_0 + \sum_{s=0}^{D-2} \tfrac{1}{\xi_1(\epsilon_1 - 1)} E_s^\perp v_0^\perp. \tag{66}$$

Using (65) and (66) we have $\epsilon_1^{-1}\hat{C}_0^+ + \hat{C}_1^- = \sum_{s=0}^{D-2} \frac{1}{\xi_1 \epsilon_1} E_s^\perp v_0^\perp$. Therefore the coordinate vector of $\epsilon_1^{-1}\hat{C}_0^+ + \hat{C}_1^-$ relative to $\mathcal{B}$ is

$$\left[0, 0, \tfrac{1}{\xi_1\epsilon_1}, 0, \tfrac{1}{\xi_1\epsilon_1}, 0, \tfrac{1}{\xi_1\epsilon_1}, \ldots, 0, \tfrac{1}{\xi_1\epsilon_1}, 0\right]^t. \tag{67}$$

Lemma 8.5. *The matrix representation of t_0 relative to $\mathcal{B}$ is*

$$\text{blockdiag}\Big([\kappa_0], [t_0(1)], [t_0(2)], \ldots, [t_0(D-1)], [\kappa_0]\Big),$$

where $[t_0(i)] = \text{diag}(\kappa_0, \kappa_0^{-1})$ for $1 \le i \le D-1$.

Proof. Let $[t_0]_{\mathcal{B}}$ denote the matrix representation of t_0 relative to $\mathcal{B}$. Since $[t_0]_{\mathcal{B}}$ has the form (61) and by Lemma 8.4 we denote $[t_0]_{\mathcal{B}}$ by blockdiag$([\kappa_0], [t_0(1)], \ldots, [t_0(D-1)], [\kappa_0])$, where the $[t_0(i)]$ is the matrix representing t_0 relative to $\{E_i v_0, E_{i-1}^\perp v_0^\perp\}$ for $1 \le i \le D-1$. Using linear algebra along with (64) and (67) we find the equation

$$[t_0(i)] \begin{bmatrix} \theta_i & 0 \\ 0 & \frac{1}{\xi_1\epsilon_1} \end{bmatrix} = \begin{bmatrix} \theta_i & 0 \\ 0 & \frac{1}{\xi_1\epsilon_1} \end{bmatrix} \begin{bmatrix} \kappa_0 & 0 \\ 0 & \kappa_0^{-1} \end{bmatrix}.$$

Evaluate $[t_0(i)]$ using the above equation. The result follows. □

We now find the matrix representation of t_1 relative to $\mathcal{B}$. We start with the following lemma.

Lemma 8.6. *For* $0 \leq i \leq D-1$,

$$t_1.(\hat{C}_i^- + \hat{C}_i^+) = \kappa_1(\hat{C}_i^- + \hat{C}_i^+),$$
$$t_1.(\tau_i\hat{C}_i^- + \hat{C}_i^+) = \kappa_1^{-1}(\tau_i\hat{C}_i^- + \hat{C}_i^+),$$

where κ_1 *is from* (54) *and*

$$\tau_i = \frac{s^*(1-r_1q^{i+1})(1-r_2q^{i+1})}{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}.$$

Proof. It is similar to Lemma 8.3. Use Theorem 6.5(v) and the fact that $\{\hat{C}_i^- + \hat{C}_i^+\}_{i=0}^{D-1}$ is a basis for $M\hat{C}$ and $\{\tau_i\hat{C}_i^- + \hat{C}_i^+\}_{i=0}^{D-1}$ is a basis for $M\hat{C}^\perp$. □

Consider the eigenvector $\hat{C}_0^- + \hat{C}_0^+$ of t_1 for κ_1. Since $\hat{C}_0^- = v_0 = \sum_{r=0}^D E_r v_0$, by this and (65) we have

$$\hat{C}_0^- + \hat{C}_0^+ = v_0 + \tfrac{\epsilon_1}{\epsilon_1-1}v_1 + \tfrac{1}{\xi_1(1-\epsilon_1)}v_0^\perp = \sum_{r=0}^{D}\left(1+\tfrac{\epsilon_1\theta_r}{\epsilon_1-1}\right)E_rv_0 + \sum_{s=0}^{D-2}\tfrac{1}{\xi_1(1-\epsilon_1)}E_s^\perp v_0^\perp.$$

One routinely checks that $1+\frac{\epsilon_1\theta_D}{\epsilon_1-1}=0$, and so the coordinate vector of $\hat{C}_0^- + \hat{C}_0^+$ relative to $\mathcal{B}$ is

$$\left[1+\tfrac{\epsilon_1\theta_0}{\epsilon_1-1}, 1+\tfrac{\epsilon_1\theta_1}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, 1+\tfrac{\epsilon_1\theta_2}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, \dots, 1+\tfrac{\epsilon_1\theta_{D-1}}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, 0\right]^t. \qquad (68)$$

Similarly, for the eigenvector $\tau_0\hat{C}_0^- + \hat{C}_0^+$ of t_1 for k_1^{-1} we find

$$\tau_0\hat{C}_0^- + \hat{C}_0^+ = \sum_{r=0}^{D}\left(\tau_0+\tfrac{\epsilon_1\theta_r}{\epsilon_1-1}\right)E_rv_0 + \sum_{s=0}^{D-2}\tfrac{1}{\xi_1(1-\epsilon_1)}E_s^\perp v_0^\perp.$$

One routinely checks that $\tau_0+\frac{\epsilon_1\theta_0}{\epsilon_1-1}=0$, and so the coordinate vector of $\tau_0\hat{C}_0^- + \hat{C}_0^+$ relative to $\mathcal{B}$ is

$$\left[0, \tau_0+\tfrac{\epsilon_1\theta_1}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, \tau_0+\tfrac{\epsilon_1\theta_2}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, \dots, \tau_0+\tfrac{\epsilon_1\theta_{D-1}}{\epsilon_1-1}, \tfrac{1}{\xi_1(1-\epsilon_1)}, \tau_0+\tfrac{\epsilon_1\theta_D}{\epsilon_1-1}\right]^t. \qquad (69)$$

Lemma 8.7. *We have*

$$t_1.E_0v_0 = \kappa_1E_0v_0, \qquad t_1.E_Dv_0 = \kappa_1^{-1}E_Dv_0.$$

Proof. The matrix representing t_1 relative to $\mathcal{B}$ has the form (61). Use this and linear algebra together with (68), (69). The result follows. □

Lemma 8.8. *The matrix representation of t_1 relative to $\mathcal{B}$ is*

$$\mathrm{blockdiag}\Big([\kappa_1], [t_1(1)], [t_1(2)], \ldots, [t_1(D-1)], [\kappa_1^{-1}]\Big),$$

where for $1 \leq i \leq D-1$

$$[t_1(i)] = \begin{bmatrix} \kappa_1^{-1}\left(\frac{\kappa_1^2-\tau_0}{1-\tau_0} - \theta_i \frac{\epsilon_1(\kappa_1^2-1)}{(1-\epsilon_1)(1-\tau_0)}\right) & \frac{\kappa_1^{-1}\xi_1(1-\kappa_1^2)(\epsilon_1\theta_i-1+\epsilon_1)(\epsilon_1\theta_i-\tau_0+\tau_0\epsilon_1)}{(1-\epsilon_1)(1-\tau_0)} \\ \frac{\kappa_1^{-1}(\kappa_1^2-1)}{\xi_1(1-\epsilon_1)(1-\tau_0)} & \kappa_1^{-1}\left(\theta_i \frac{\epsilon_1(\kappa_1^2-1)}{(1-\epsilon_1)(1-\tau_0)} - \frac{k_1^2\tau_0-1}{1-\tau_0}\right) \end{bmatrix}.$$

Proof. Similar to the proof of Lemma 8.5. □

Based on our discussion so far, we find the matrix representation of $\mathbf{Y}$ relative to $\mathcal{B}$.

Lemma 8.9. *The matrix representing* $\mathbf{Y}$ *relative to* $\mathcal{B}$ *is*

$$\mathrm{blockdiag}\Big([(sq)^{1/2}], [\mathbf{Y}(1)], [\mathbf{Y}(2)], \ldots [\mathbf{Y}(D-1)], [q^{-D}(sq)^{-1/2}]\Big),$$

where for $1 \leq i \leq D-1$

$$[\mathbf{Y}(i)] = \begin{bmatrix} \kappa_0\kappa_1^{-1}\left(\frac{\kappa_1^2-\tau_0}{1-\tau_0} - \theta_i \frac{\epsilon_1(\kappa_1^2-1)}{(1-\epsilon_1)(1-\tau_0)}\right) & \kappa_0\kappa_1^{-1}\frac{\xi_1(1-\kappa_1^2)(\epsilon_1\theta_i-1+\epsilon_1)(\epsilon_1\theta_i-\tau_0+\tau_0\epsilon_1)}{(1-\epsilon_1)(1-\tau_0)} \\ \kappa_0^{-1}\kappa_1^{-1}\frac{(\kappa_1^2-1)}{\xi_1(1-\epsilon_1)(1-\tau_0)} & \kappa_0^{-1}\kappa_1^{-1}\left(\theta_i \frac{\epsilon_1(\kappa_1^2-1)}{(1-\epsilon_1)(1-\tau_0)} - \frac{k_1^2\tau_0-1}{1-\tau_0}\right) \end{bmatrix}. \tag{70}$$

Proof. Recall $\mathbf{Y} = t_0 t_1$. The matrix representing $\mathbf{Y}$ relative to $\mathcal{B}$ is the product of the matrix representing t_0 relative to $\mathcal{B}$ and the matrix representing t_1 relative to $\mathcal{B}$. By Lemma 8.5 and Lemma 8.8 the result follows. □

Lemma 8.10. *The eigenvalues of* $\mathbf{Y}$ *are*

$$\begin{array}{llllll} (sq)^{1/2}, & q(sq)^{1/2}, & q^2(sq)^{1/2}, & \ldots, & q^{D-1}(sq)^{1/2}, & \\ & q^{-1}(sq)^{-1/2}, & q^{-2}(sq)^{-1/2}, & \ldots, & q^{1-D}(sq)^{-1/2}, & q^{-D}(sq)^{-1/2}. \end{array}$$

Therefore $\mathbf{Y}$ *is multiplicity-free.*

Proof. The eigenvalues of $\mathbf{Y}$ are routinely obtained from Lemma 8.9. The last assertion follows from assumption (ii) in Example 4.3. □

For notational convenience we abbreviate

$$\lambda_r = \begin{cases} q^r(sq)^{1/2} & (r = 0, 1, 2, \ldots, D-1), \\ q^r(sq)^{-1/2} & (r = -1, -2, \ldots, -D). \end{cases}$$

Note that $\lambda_i^{-1} = \lambda_{-i}$ for $i = \pm 1, \pm 2, \ldots, \pm(D-1)$ and $\ell_j = \lambda_j + \lambda_{-j}$ for $0 \leq j \leq D$. We will find an eigenvector of $\mathbf{Y}$ associated with $\lambda_r(-D \leq r \leq D-1)$, and express as a linear combination of elements of $\mathcal{B}$.

Lemma 8.11. *With the above notation, there exist an eigenvector* $\overline{\mathbf{y}}_i$ *of* $\mathbf{Y}$ *associated with the eigenvalue* λ_i *such that for* $i \in \{0, D\}$

$$\overline{\mathbf{y}}_0 = E_0 v_0, \qquad \overline{\mathbf{y}}_{-D} = E_D v_0,$$

and for $1 \leq i \leq D-1$

$$\overline{\mathbf{y}}_{-i} = \omega_{-i} E_i v_0 + E_{i-1}^{\perp} v_0^{\perp}, \qquad \overline{\mathbf{y}}_i = \omega_i E_i v_0 + E_{i-1}^{\perp} v_0^{\perp},$$

where

$$\omega_{-i} = m\left(q^{-i} + \frac{\theta_i}{h(q^{-D}-1)} + \frac{r_1 r_2(1-r_1 q)(1-r_2 q) - (r_1 - s^* q)(r_2 - s^* q)}{(r_1 r_2 - s^*)(1 - s^* q^2)}\right),$$

$$\omega_i = m\left(sq^{i+1} + \frac{\theta_i}{h(q^{-D}-1)} + \frac{r_1 r_2(1-r_1 q)(1-r_2 q) - (r_1 - s^* q)(r_2 - s^* q)}{(r_1 r_2 - s^*)(1 - s^* q^2)}\right),$$

and $m = \frac{s^*(1-q^{-D})(1-s^*q^2)(1-s^*q^3)}{(r_1 - s^* q)(r_2 - s^* q)}$.

Proof. Note that $\mathbf{A}$ and $\mathbf{Y}$ share common eigenvectors. Without loss of generality one can choose $\overline{\mathbf{y}}_0 = E_0 v_0$ and $\overline{\mathbf{y}}_{-D} = E_D v_0$. Let $1 \leq i \leq D-1$. Since $\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}$, each of $\overline{\mathbf{y}}_i, \overline{\mathbf{y}}_{-i}$ is an eigenvector of $\mathbf{A}$ associated with ℓ_i. So $\overline{\mathbf{y}}_i, \overline{\mathbf{y}}_{-i} \in W_{\ell_i}$. Let $[\overline{\mathbf{y}}_i]$ (resp. $[\overline{\mathbf{y}}_{-i}]$) denote the coordinate vector of $\overline{\mathbf{y}}_i$ (resp. $\overline{\mathbf{y}}_{-i}$) relative to $\{E_i v_0, E_{i-1}^{\perp} v_0^{\perp}\}$. It suffices to find the vectors $[\overline{\mathbf{y}}_i]$ and $[\overline{\mathbf{y}}_{-i}]$. By Lemma 8.9 we have

$$[\mathbf{Y}(i)][\overline{\mathbf{y}}_i] = \lambda_i[\overline{\mathbf{y}}_i] \qquad \text{and} \qquad [\mathbf{Y}(i)][\overline{\mathbf{y}}_{-i}] = \lambda_{-i}[\overline{\mathbf{y}}_{-i}].$$

Evaluate the above equations using (70) and simplify the vectors $[\overline{\mathbf{y}}_i], [\overline{\mathbf{y}}_{-i}]$. The result follows. $\square$

We normalize the vectors $\{\overline{\mathbf{y}}_i\}_{i=-D}^{D-1}$ so that the sum of these vectors is equal to v_0. For $1 \leq i \leq D-1$ set

$$\mathbf{y}_{-i} := -\frac{q^i}{m(sq^{2i+1}-1)}\overline{\mathbf{y}}_{-i} = -\frac{q^i \omega_{-i}}{m(sq^{2i+1}-1)} E_i v_0 - \frac{q^i}{m(sq^{2i+1}-1)} E_{i-1}^{\perp} v_0^{\perp}, \tag{71}$$

$$\mathbf{y}_i := \frac{q^i}{m(sq^{2i+1}-1)}\overline{\mathbf{y}}_i = \frac{q^i \omega_i}{m(sq^{2i+1}-1)} E_i v_0 + \frac{q^i}{m(sq^{2i+1}-1)} E_{i-1}^{\perp} v_0^{\perp}. \tag{72}$$

Set

$$\mathbf{y}_0 := \overline{\mathbf{y}}_0 = E_0 v_0, \qquad \mathbf{y}_{-D} := \overline{\mathbf{y}}_{-D} = E_D v_0. \tag{73}$$

Theorem 8.12. *With reference to the notation (71)–(73), each* $\mathbf{y}_i$ *is an eigenvector of* $\mathbf{Y}$ *associated with* λ_i*, and*

$$\sum_{i=-D}^{D-1} \mathbf{y}_i = v_0. \tag{74}$$

Proof. It suffices to show (74). Evaluate the right-hand side in (74) using Lemma 8.11 together with (71)–(73). Then

$$\begin{aligned}\sum_{i=-D}^{D-1} \mathbf{y}_i &= \mathbf{y}_0 + \sum_{i=1}^{D-1}(\mathbf{y}_i + \mathbf{y}_{-i}) + \mathbf{y}_{-D} = E_0 v_0 + \sum_{i=1}^{D-1} \frac{q^i(\overline{\mathbf{y}}_i - \overline{\mathbf{y}}_{-i})}{m(sq^{2i+1}-1)} + E_D v_0 \\ &= \sum_{i=0}^{D} E_i v_0 = v_0,\end{aligned}$$

as required. □

For the rest of this paper we fix the eigenvectors $\{\mathbf{y}_i\}_{i=-D}^{D-1}$ of $\mathbf{Y}$ as in Theorem 8.12. Let W_{λ_i} denote the eigenspace of $\mathbf{Y}$ for λ_i. Observe that $\mathbf{W} = \bigoplus_{i=-D}^{D-1} W_{\lambda_i}$, an orthogonal direct sum. Recall the Hermitian inner product $\langle \cdot, \cdot \rangle_V$ from the first paragraph in Section 3. We find the norm of $\mathbf{y}_i$ for $-D \le i \le D-1$.

Proposition 8.13. *For* $1 \le i \le D-1$*,*

$$\begin{aligned}\|\mathbf{y}_{-i}\|^2 &= \frac{q^{2i}\omega_{-i}^2}{m^2(sq^{2i+1}-1)^2} k_i^* \nu^{-1} + \frac{q^{2i}}{m^2(sq^{2i+1}-1)^2} k_{i-1}^{*\perp} \nu^{\perp -1} \|v_0^\perp\|^2, \\ \|\mathbf{y}_i\|^2 &= \frac{q^{2i}\omega_i^2}{m^2(sq^{2i+1}-1)^2} k_i^* \nu^{-1} + \frac{q^{2i}}{m^2(sq^{2i+1}-1)^2} k_{i-1}^{*\perp} \nu^{\perp -1} \|v_0^\perp\|^2,\end{aligned}$$

where k_i^**,* $k_i^{*\perp}$*,* ν*,* $\nu^\perp$ *are from Lemma 5.9,* m *is from Lemma 8.11, and*

$$\|v_0^\perp\|^2 = \frac{s^*(1-q^D)(1-q^{1-D})(1-s^*q^2)(1-s^*q^3)(1-r_1q)(1-r_2q)}{q^D r_1 r_2 (1-s^*q/r_1)(1-s^*q/r_2)}. \tag{75}$$

Moreover,

$$\|\mathbf{y}_0\|^2 = \nu^{-1}, \qquad \|\mathbf{y}_{-D}\|^2 = k_D^* \nu^{-1}.$$

Proof. Recall from [32, Theorem 15.3] that

$$\langle E_i v_0, E_j v_0 \rangle_V = \delta_{ij} k_i^* \nu^{-1} \|v_0\|^2 \qquad (0 \le i, j \le D-1). \tag{76}$$

Let $1 \le i \le D-1$. We first find $\|\mathbf{y}_{-i}\|^2$. By (71), we have

$$\begin{aligned}\|\mathbf{y}_{-i}\|^2 &= \Big\langle -\frac{q^i\omega_{-i}}{m(sq^{2i+1}-1)}E_iv_0 - \frac{q^i}{m(sq^{2i+1}-1)}E_{i-1}^{\perp}v_0^{\perp}, \\ &\qquad -\frac{q^i\omega_{-i}}{m(sq^{2i+1}-1)}E_iv_0 - \frac{q^i}{m(sq^{2i+1}-1)}E_{i-1}^{\perp}v_0^{\perp}\Big\rangle_V \\ &= \frac{q^{2i}\omega_{-i}^2}{m^2(sq^{2i+1}-1)^2}\|E_iv_0\|^2 + \frac{q^{2i}}{m^2(sq^{2i+1}-1)^2}\|E_{i-1}^{\perp}v_0^{\perp}\|^2 \\ &= \frac{q^{2i}\omega_{-i}^2}{m^2(sq^{2i+1}-1)^2}k_i^*\nu^{-1}\|v_0\|^2 + \frac{q^{2i}}{m^2(sq^{2i+1}-1)^2}k_{i-1}^{*\perp}\nu^{\perp-1}\|v_0^{\perp}\|^2,\end{aligned}$$

where the last equation is from (76). Recall that $\|v_0\|^2 = 1$. The line (75) is obtained from [19, Lemma 6.14] together with Lemma 5.8. Similarly we obtain the norm of $\mathbf{y}_i$. □

9. Orthogonality for $\varepsilon_i^{\pm}$

Recall the nonsymmetric Laurent polynomials $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ from Definition 7.2 and recall the subspace L of $\mathbb{C}[y, y^{-1}]$ spanned by $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ from Remark 7.9. In this section we define a bilinear form on L that satisfies the orthogonality relations for $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$. Recall the bilinear form $\langle \cdot, \cdot \rangle_V$ and the $\hat{H}_q$-module $\mathbf{W}(s, s^*, r_1, r_2, D; q)$.

Lemma 9.1. *Let the antiautomorphism $\dagger$ be as in Lemma 6.2. For $h \in \hat{H}_q$ and $u, v \in \mathbf{W}(s, s^*, r_1, r_2, D; q)$,*

$$\langle h.u, v\rangle_V = \langle u, h^{\dagger}.v\rangle_V.$$

Proof. Let $0 \le i, j \le D-1$. Routinely check

$$\langle t_n^{\delta}.\hat{C}_i^{\sigma}, \hat{C}_j^{\tau}\rangle_V = \langle \hat{C}_i^{\sigma}, t_n^{\delta\dagger}.\hat{C}_j^{\tau}\rangle_V,$$

for $n \in \mathbb{I}$ and $\sigma, \tau, \delta \in \{+, -\}$. Since $\{t_n^{\pm 1}\}_{n\in\mathbb{I}}$ generates $\hat{H}_q$, the result follows. □

Lemma 9.2. *For $-D \le i \le D-1$, let $\mathbf{y}_i$ be an eigenvector of $\mathbf{Y}$ for the eigenvalue λ_i as in Theorem 8.12. For $L_1, L_2 \in L$,*

$$\langle L_1[\mathbf{Y}].v_0, L_2[\mathbf{Y}].v_0\rangle_V = \sum_{i=-D}^{D-1} L_1[\lambda_i]\overline{L_2[\lambda_i]}\|\mathbf{y}_i\|^2,$$

where the norm $\|\mathbf{y}_i\|^2$ is given by Proposition 8.13.

Proof. Compute

$$\begin{aligned}\langle L_1[\mathbf{Y}].v_0, L_2[\mathbf{Y}].v_0\rangle_V &= \langle L_1[\mathbf{Y}]\sum_i \mathbf{y}_i, L_2[\mathbf{Y}]\sum_j \mathbf{y}_j\rangle_V \qquad \text{(by line (74))}\\ &= \sum_{i,j}\langle L_1[\mathbf{Y}]\mathbf{y}_i, L_2[\mathbf{Y}]\mathbf{y}_j\rangle_V\\ &= \sum_{i,j}\langle L_1[\lambda_i]\mathbf{y}_i, L_2[\lambda_j]\mathbf{y}_j\rangle_V\\ &= \sum_{i,j} L_1[\lambda_i]\overline{L_2[\lambda_j]}\langle \mathbf{y}_i, \mathbf{y}_j\rangle_V\\ &= \sum_i L_1[\lambda_i]\overline{L_2[\lambda_i]}\|\mathbf{y}_i\|^2,\end{aligned}$$

where the last equation follows since $\{\mathbf{y}_i\}_{i=-D}^{D-1}$ is an orthogonal basis for $\mathbf{W}$. □

Motivated by Lemma 9.2 we define a bilinear form $\langle\cdot,\cdot\rangle_L : L \times L \to \mathbb{C}$ as follows. For $L_1, L_2 \in L$

$$\langle L_1[y], L_2[y]\rangle_L := \sum_{i=-D}^{D-1} L_1[\lambda_i]\overline{L_2[\lambda_i]}\|\mathbf{y}_i\|^2, \tag{77}$$

where $\|\mathbf{y}_i\|^2$ is from Proposition 8.13.

Lemma 9.3. *Let $L_1, L_2 \in \mathbb{C}[y, y^{-1}]$. With reference to the form (77),*

$$\langle yL_1[y], L_2[y]\rangle_L = \langle L_1[y], yL_2[y]\rangle_L. \tag{78}$$

Proof. On the left-hand side in (78) put $L_1'[y] = yL_1[y]$. Then by Lemma 9.2

$$\langle L_1'[y], L_2[y]\rangle_L = \langle L_1'[\mathbf{Y}]v_0, L_2[\mathbf{Y}]v_0\rangle_V = \langle \mathbf{Y}L_1[\mathbf{Y}]v_0, L_2[\mathbf{Y}]v_0\rangle_V.$$

By Lemma 9.1 and $\mathbf{Y} = \mathbf{Y}^\dagger$ from (53), it follows $\langle \mathbf{Y}L_1[\mathbf{Y}]v_0, L_2[\mathbf{Y}]v_0\rangle_V = \langle L_1[\mathbf{Y}]v_0, \mathbf{Y}L_2[\mathbf{Y}]v_0\rangle_V$. But the right-hand side in (78) is equal to $\langle L_1[\mathbf{Y}]v_0, \mathbf{Y}L_2[\mathbf{Y}]v_0\rangle_V$. The result follows. □

We now show that the Laurent polynomials $\varepsilon_i^+, \varepsilon_i^-$ satisfy the orthogonality relation with respect to the bilinear form (77).

Theorem 9.4. *Let $\varepsilon_i^+, \varepsilon_i^-$ be the Laurent polynomials in Definition 7.2. For $-D \le r \le D-1$, let $\mathbf{y}_r$ be an eigenvector of $\mathbf{Y}$ for the eigenvalue λ_r as in Theorem 8.12. Then for $\sigma, \tau \in \{+, -\}$,*

$$\sum_{r=-D}^{D-1} \varepsilon_i^\sigma[\lambda_r]\overline{\varepsilon_j^\tau[\lambda_r]}\|\mathbf{y}_r\|^2 = \delta_{\sigma,\tau}\delta_{i,j}\|\hat{C}_i^\sigma\|^2,$$

where $\|\mathbf{y}_r\|^2$ *is given in Proposition 8.13 and* $\|\hat{C}_i^\sigma\|^2$ *is given in Lemma 5.8.*

Proof. Using Lemma 9.2 and Corollary 7.4 we find

$$\sum_{r=-D}^{D-1} \varepsilon_i^\sigma[\lambda_r]\overline{\varepsilon_j^\tau[\lambda_r]}\|\mathbf{y}_r\|^2 = \langle \varepsilon_i^\sigma[\mathbf{Y}]v_0, \varepsilon_j^\tau[\mathbf{Y}]v_0\rangle_V = \langle \hat{C}_i^\sigma, \hat{C}_j^\tau\rangle_V.$$

By Lemma 5.8 the result follows. □

Recall $\|\hat{C}_i^\pm\|^2$ from Lemma 5.8 and the scalars a, b, c, d from Definition 5.1. Using (33) evaluate $\|\hat{C}_i^\pm\|^2$ in terms of a, b, c, d. Then by Theorem 9.4 it follows

$$\begin{aligned}\langle \varepsilon_i^\sigma, \varepsilon_j^\tau\rangle_L &= \delta_{\sigma,\tau}\delta_{i,j}\|\hat{C}_i^\sigma\|^2 \\ &= \begin{cases} \delta_{\sigma,\tau}\delta_{i,j}\dfrac{(abq, ac, ad, abcd; q)_i}{a^{2i}(q, bc, bd, cd; q)_i} & \text{if } \sigma = -, \\ \delta_{\sigma,\tau}\delta_{i,j}(-1)\dfrac{b(1-ac)(1-ad)}{a(1-bc)(1-bd)}\dfrac{(abq, acq, adq, abcd; q)_i}{a^{2i}(q, bcq, bdq, cd; q)_i} & \text{if } \sigma = +. \end{cases}\end{aligned} \tag{79}$$

10. The algebra $\hat{H}_{q^{-1}}$

Recall from Section 6 that $\hat{H}_q$ is the universal DAHA of type $(C_1^\vee, C_1)$. In this section we change q by q^{-1} and discuss the algebra $\hat{H}_{q^{-1}}$ and its module. Recall from Section 2 that $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}(a, b, c, d; q)$ is the DAHA of type $(C_1^\vee, C_1)$. We will compare $\hat{H}_{q^{-1}}$ and $\tilde{\mathfrak{H}}$ shortly.

Lemma 10.1. *There exists a $\mathbb{C}$-algebra isomorphism $\eta_1 : \hat{H}_{q^{-1}} \to \hat{H}_q$ that sends*

$$t_0 \mapsto t_0^{-1}, \qquad t_1 \mapsto t_0 t_1^{-1} t_0^{-1}, \qquad t_2 \mapsto t_3^{-1} t_2^{-1} t_3, \qquad t_3 \mapsto t_3^{-1}.$$

Moreover, $\eta_1^2 = 1$.

Proof. Use Definition 6.1. □

Lemma 10.2. [34, Lemma 16.8] *There is a surjective $\mathbb{C}$-algebra homomorphism $\eta_2 : \hat{H}_q \to \tilde{\mathfrak{H}}$ that sends*

$$\begin{aligned} &t_0 \mapsto -(ab)^{-1/2}T_1, \quad t_1 \mapsto -(ab)^{1/2}T_1^{-1}Z^{-1}, \\ &t_2 \mapsto -q^{-1}(cd)^{1/2}ZT_0^{-1}, \quad t_3 \mapsto -q^{1/2}(cd)^{-1/2}T_0. \end{aligned}$$

Referring to Lemma 10.1 and Lemma 10.2 we define the map $\xi : \hat{H}_{q^{-1}} \to \tilde{\mathfrak{H}}$ to be the composition $\xi = \eta_2\eta_1$. Observe that this map is surjective and sends

$$\begin{aligned} t_0 &\mapsto -(ab)^{1/2}T_1^{-1}, \\ t_1 &\mapsto -(ab)^{-1/2}T_1 Z, \\ t_2 &\mapsto -q(cd)^{-1/2}Z^{-1}T_0, \\ t_3 &\mapsto -q^{-1/2}(cd)^{1/2}T_0^{-1}. \end{aligned} \tag{80}$$

Lemma 10.3. *Recall $Y = T_1 T_0 \in \tilde{\mathfrak{H}}$. Referring to the map ξ in (80), for $\mathbf{X}, \mathbf{Y} \in \hat{H}_{q^{-1}}$*

$$\begin{aligned} \mathbf{X}^\xi &= q^{-1/2}(abcd)^{1/2}Y^{-1}, & (\mathbf{X}^{-1})^\xi &= q^{1/2}(abcd)^{-1/2}Y, \\ \mathbf{Y}^\xi &= Z, & (\mathbf{Y}^{-1})^\xi &= Z^{-1}. \end{aligned}$$

Proof. Recall $\mathbf{X} = t_3 t_0$ and $\mathbf{Y} = t_0 t_1$. Use definition of ξ. $\square$

In Section 6 we discussed the $\hat{H}_q$-module $\mathbf{W}(s, s^*, r_1, r_2, D; q)$. We consider an $\hat{H}_{q^{-1}}$-module on $\mathbf{W}$, denoted by $\mathbf{W}(s', s^{*\prime}, r_1', r_2', D'; q^{-1})$. Note that $D = D'$. By Proposition 4.7,

$$\mathbf{W}(s', s^{*\prime}, r_1', r_2', D'; q^{-1}) = \mathbf{W}(s^{-1}, s^{*-1}, r_1^{-1}, r_2^{-1}, D; q^{-1}).$$

Recall the scalars a, b, c, d from Definition 5.1. In the q^{-1}-Racah version, by (33) the scalars a, b, c, d become a^{-1}, b^{-1}, c^{-1}, d^{-1}, respectively. By Lemma 6.6, we can describe a structure of an $\hat{H}_{q^{-1}}$-module $\mathbf{W}(a^{-1}, b^{-1}, c^{-1}, d^{-1}; q^{-1})$. For the rest of the paper, we denote $\mathbf{W}_{q^{-1}} := \mathbf{W}(a^{-1}, b^{-1}, c^{-1}, d^{-1}; q^{-1})$.

Lemma 10.4. *On the $\hat{H}_{q^{-1}}$-module $\mathbf{W}_{q^{-1}}$ the action of $\mathbf{X}^{-1}$ on $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ is as follows.*

$$\begin{cases} \mathbf{X}^{-1}.\hat{C}_i^- = q^{i-\frac{1}{2}}(abcd)^{1/2}\hat{C}_i^- & \textit{for} \quad i = 0, 1, \ldots, D-1, \\ \mathbf{X}^{-1}.\hat{C}_{i-1}^+ = q^{-i+\frac{1}{2}}(abcd)^{-1/2}\hat{C}_{i-1}^+ & \textit{for} \quad i = 1, 2, \ldots, D. \end{cases}$$

Proof. Use Lemma 6.7. $\square$

Recall $\varepsilon_i^- = \varepsilon_i^-[y; a, b, c, d \mid q]$ and $\varepsilon_i^+ = \varepsilon_i^+[y; a, b, c, d \mid q]$ from Proposition 7.8. We describe these polynomials in q^{-1}-version

$$\varepsilon_i^\sigma[y; q^{-1}] = \varepsilon_i^\sigma[y; a^{-1}, b^{-1}, c^{-1}, d^{-1} \mid q^{-1}],$$

where $0 \le i \le D-1$ and $\sigma \in \{+, -\}$. To this end we need a few preliminary lemmas.

Lemma 10.5. *For $n = 0, 1, 2\ldots$ and a nonzero scalar $x \in \mathbb{C}$,*

$$(x^{-1}; q^{-1})_n = (-1)^n x^{-n} q^{-\frac{n(n-1)}{2}} (x; q)_n.$$

Proof. Use the definition (1) to compute $(x^{-1}; q^{-1})_n$. □

Lemma 10.6. *Let the monic polynomials $P_i[y; a, b, c, d \mid q]$ be as in (60). Then*

$$P_i[y; a, b, c, d \mid q] = P_i[y; a^{-1}, b^{-1}, c^{-1}, d^{-1} \mid q^{-1}].$$

Proof. Evaluate $P_i[y; a^{-1}, b^{-1}, c^{-1}, d^{-1} \mid q^{-1}]$ using (60) and Lemma 10.5. □

For notational convenience, for $1 \le i \le D-1$ we define the Laurent polynomials $Q_i = Q_i[y; a, b, c, d \mid q]$ by

$$Q_i := a^{-1} b^{-1} y^{-1} (1 - ay)(1 - by) P^{\perp}_{i-1}, \tag{81}$$

where we recall $P^{\perp}_i = P_i[y; aq, bq, c, d \mid q]$ from (60); cf. (3).

Proposition 10.7. *Let the scalars a, b, c, d be as in Definition 5.1. Recall the polynomials $P_i = P_i[y; a, b, c, d \mid q]$ from (60) and $Q_i = Q_i[y; a, b, c, d \mid q]$ from (81). Then the Laurent polynomials $\varepsilon^{\pm}_i[y; q^{-1}]$ are described as follows: for $1 \le i \le D-1$*

$$\begin{aligned}
\varepsilon^{+}_{i-1}[y; q^{-1}] &= \frac{ab(1 - q^i)(1 - cdq^{i-1})}{(ab - 1)} \frac{(abcd; q)_{2i-1}}{a^i (q, bc, bd, cd; q)_i} \left(P_i - Q_i\right), \\
\varepsilon^{-}_i[y; q^{-1}] &= \frac{(1 - abq^i)(1 - abcdq^{i-1})}{(1 - ab)} \frac{(abcd; q)_{2i-1}}{a^i (q, bc, bd, cd; q)_i} \\
&\quad \times \left(P_i - \frac{ab(1 - q^i)(1 - cdq^{i-1})}{(1 - abq^i)(1 - abcdq^{i-1})} Q_i\right).
\end{aligned}$$

Moreover,

$$\varepsilon^{-}_0[y; q^{-1}] = 1, \qquad \varepsilon^{+}_{D-1}[y; q^{-1}] = \frac{(abcd; q)_{2D-1}}{a^D (q, bc, bd; q)_D (cd; q)_{D-1}} P_D.$$

Proof. In Proposition 7.8, replace a, b, c, d, q by $a^{-1}, b^{-1}, c^{-1}, d^{-1}, q^{-1}$. Evaluate this using Lemma 10.5 and Lemma 10.6 and simplify the result. Note that $y(1 - a^{-1}y^{-1})(1 - b^{-1}qy^{-1}) = a^{-1}b^{-1}y^{-1}(1 - ay)(1 - by)$. The results routinely follow. □

We finish this section with some comments.

Lemma 10.8. *Referring the basis* $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ *for* $\mathbf{W}_{q^{-1}}$*, the following hold. For* $0 \le i \le D-1$*,*

$$\|\hat{C}_i^-\|^2 = \frac{1-abq^i}{1-ab}\frac{(ab, ac, ad, abcd; q)_i}{a^{2i}(q, bc, bd, cd; q)_i},$$

$$\|\hat{C}_i^+\|^2 = \frac{ab(1-q^{i+1})(1-cdq^i)}{(ab-1)(1-abcdq^i)}\frac{(ab, ac, ad, abcd; q)_{i+1}}{a^{2i+2}(q, bc, bd, cd; q)_{i+1}}.$$

Proof. In line (79) we expressed $\|\hat{C}_i^{\pm}\|^2$ in terms of the scalars a, b, c, d, q. Replace these scalars by $a^{-1}, b^{-1}, c^{-1}, d^{-1}, q^{-1}$, respectively. Evaluate this using Lemma 10.5 and simplify it. The result routinely follows. □

Note 10.9. In Remark 7.9 we discussed the $\hat{H}_q$-module $L = L(a, b, c, d; q)$. We recall from (77) that $\langle \cdot, \cdot \rangle_L$ is the bilinear form on L. Consider the $\hat{H}_{q^{-1}}$-module $L(a^{-1}, b^{-1}, c^{-1}, d^{-1}; q^{-1})$. Abbreviate $L_{q^{-1}} := L(a^{-1}, b^{-1}, c^{-1}, d^{-1}; q^{-1})$. Observe that the Laurent polynomials $\varepsilon^+[y; q^{-1}]$, $\varepsilon^-[y; q^{-1}]$ in Proposition 10.7 form a basis for $L_{q^{-1}}$. By (79) and Lemma 10.8, the bilinear form $\langle \cdot, \cdot \rangle_{L_{q^{-1}}}$ satisfies

$$\begin{aligned}&\langle \varepsilon_i^\sigma[y; q^{-1}], \varepsilon_j^\tau[y; q^{-1}]\rangle_{L_{q^{-1}}} \\ &= \begin{cases} \delta_{\sigma,\tau}\delta_{i,j}\dfrac{1-abq^i}{1-ab}\dfrac{(ab, ac, ad, abcd; q)_i}{a^{2i}(q, bc, bd, cd; q)_i} & \text{if } \sigma = -, \\ \delta_{\sigma,\tau}\delta_{i,j}\dfrac{ab(1-q^{i+1})(1-cdq^i)}{(ab-1)(1-abcdq^i)}\dfrac{(ab, ac, ad, abcd; q)_{i+1}}{a^{2i+2}(q, bc, bd, cd; q)_{i+1}} & \text{if } \sigma = +. \end{cases}\end{aligned} \tag{82}$$

11. Nonsymmetric Askey–Wilson polynomials and $\varepsilon_i^{\pm}$

We continue to work with the algebra $\hat{H}_{q^{-1}}$ in Section 10. Throughout this section we let the scalars a, b, c, d be as in Definition 5.1. Recall the nonsymmetric Laurent polynomials $\varepsilon_i^+ = \varepsilon_i^+[y; q^{-1}]$, $\varepsilon_i^- = \varepsilon_i^-[y; q^{-1}]$ from Proposition 10.7. Referring to this proposition, we make a definition of the Laurent polynomials E_i $(-D \le i \le D-1)$ which is a natural normalization of $\varepsilon_i^+, \varepsilon_i^-$.

Definition 11.1. Recall the Laurent polynomial sequences P_i from (60) and Q_i from (81). With reference to Proposition 10.7, we define

$$E_{-i} := P_i - Q_i \qquad (1 \le i \le D-1),$$

$$E_i := P_i - \frac{ab(1-q^i)(1-cdq^{i-1})}{(1-abq^i)(1-abcdq^{i-1})}Q_i \qquad (1 \le i \le D-1),$$

and $E_0 := 1$ and $E_{-D} := P_D$; cf. Definition 2.1.

By Proposition 10.7 and Definition 11.1 one can readily find that for $1 \le i \le D-1$,

$$E_{-i} = \frac{(ab-1)}{ab(1-q^i)(1-cdq^{i-1})} \frac{a^i(q,bc,bd,cd;q)_i}{(abcd;q)_{2i-1}} \varepsilon_{i-1}^+, \tag{83}$$

$$E_i = \frac{(1-ab)}{(1-abq^i)(1-abcdq^{i-1})} \frac{a^i(q,bc,bd,cd;q)_i}{(abcd;q)_{2i-1}} \varepsilon_i^-. \tag{84}$$

Moreover,

$$E_0 = \varepsilon_0^-, \qquad E_{-D} = \frac{a^D(q,bc,bd;q)_D(cd;q)_{D-1}}{(abcd;q)_{2D-1}} \varepsilon_{D-1}^+. \tag{85}$$

Recall from Note 10.9 that $L_{q^{-1}}$ is the $\hat{H}_{q^{-1}}$-module and $\langle \cdot, \cdot \rangle_{L_{q^{-1}}}$ is the bilinear form on $L_{q^{-1}}$. By a comment in Note 10.9 and (83)–(85) the Laurent polynomials E_i in Definition 11.1 form a basis for $L_{q^{-1}}$. Observe that E_i are orthogonal with respect to $\langle \cdot, \cdot \rangle_{L_{q^{-1}}}$. In the following proposition, we give a discrete version of Lemma 2.2.

Proposition 11.2. (cf. Lemma 2.2) *For* $1 \le i \le D-1$

$$\langle E_{-i}, E_{-i} \rangle_{L_{q^{-1}}} = \frac{(ab-1)(1-abcdq^{2i-1})}{ab(1-q^i)(1-cdq^{i-1})} \frac{(q,ab,ac,ad,bc,bd,cd;q)_i}{(abcd;q)_{2i}(abcdq^{i-1};q)_i}, \tag{86}$$

$$\langle E_i, E_i \rangle_{L_{q^{-1}}} = \frac{(1-ab)(1-abcdq^{2i-1})}{(1-abq^i)(1-abcdq^{i-1})} \frac{(q,ab,ac,ad,bc,bd,cd;q)_i}{(abcd;q)_{2i}(abcdq^{i-1};q)_i}. \tag{87}$$

Moreover,

$$\langle E_0, E_0 \rangle_{L_{q^{-1}}} = 1, \qquad \langle E_{-D}, E_{-D} \rangle_{L_{q^{-1}}} = \frac{ab(1-q^D)}{(ab-1)} \frac{(q,ab,ac,ad,bc,bd;q)_D(cd;q)_{D-1}}{(abcd;q)_{2D-1}(abcdq^{D-1};q)_D}. \tag{88}$$

Proof. We first show (86). Evaluate the left-hand side of (86) using (83) and (82). Simplify the result to get the right-hand side of (86). Lines (87), (88) are similarly obtained using (84), (85) together with (82). □

From Proposition 11.2, we can view that the Laurent polynomials E_i in Definition 11.1 are a discrete analogue of the nonsymmetric Askey–Wilson polynomials in Definition 2.1. We further describe a discrete analogue of the eigenspace of Y of the basic representation in Section 2. Consider the $\hat{H}_{q^{-1}}$-module $L_{q^{-1}}$. By construction each basis element E_i is the eigenvector of the action of $\mathbf{X}$ on $L_{q^{-1}}$. With reference to Lemma 10.4, we visualize the eigenspaces of $\mathbf{X}^{-1}$ on $L_{q^{-1}}$ as follows. Let $\eta = a^{-1/2}b^{-1/2}c^{-1/2}d^{-1/2}$.

Note that each white node represents the eigenspace of $\mathbf{X}^{-1}$ corresponding the eigenvalue $q^{-i+\frac{1}{2}}\eta$ and the eigenvector E_{-i} for $i = 1, 2, \ldots, D$, and each black node represents the eigenspace of $\mathbf{X}^{-1}$ corresponding the eigenvalue $q^{i-\frac{1}{2}}\eta^{-1}$ and the eigenvector E_i for

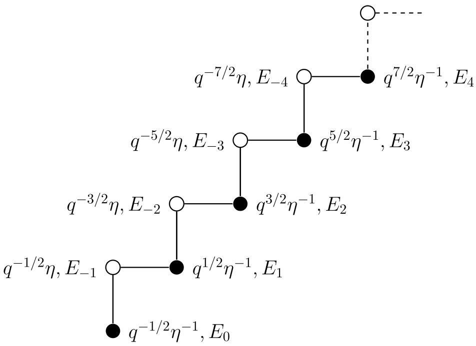

Fig. 3. The eigenspaces of $\mathbf{X}^{-1}$.

$i = 0, 1, 2, \ldots, D-1$. Compare Fig. 3 with Fig. 1 in Section 2. Then one can see that Fig. 3 coincides with Fig. 1 up to $(D-1)$-th horizontal edge. This is very natural since $\mathbf{X}^{-1}$ corresponds to $q^{1/2}\eta Y$ by Lemma 10.3.

12. Conclusion

In this paper we have studied certain Laurent polynomials, which are naturally obtained from a Q-polynomial distance-regular graph Γ of q-Racah type that contains a Delsarte clique. Using an irreducible module of the universal DAHA $\hat{H}_q$ and Γ, we proved the orthogonality relations for those polynomials. Consequently, we showed that the above Laurent polynomials are a finite/combinatorial analogue of the nonsymmetric Askey–Wilson polynomials.

As we mentioned earlier in Section 1, the theorem of D. Leonard [20] (cf. [3, Section III.5]) characterized the terminating branch of the Askey scheme [16] of basic hypergeometric orthogonal polynomials by the duality property of Γ, which has had a significant impact on the theory of orthogonal polynomials. According to this theorem, the q-Racah polynomials are the most general self-dual orthogonal polynomials in the above branch. Our results in the present paper can be thought of as a nonsymmetric version of the q-Racah polynomials in the situation of Leonard's theorem. We are planning to apply our results to the study of nonsymmetric version of other types of orthogonal polynomials in the terminating branch of the Askey scheme, such as Krawtchouk polynomials, using Γ of the corresponding type that contains a Delsarte clique. This will give a characterization of a nonsymmetric case of Leonard's theorem.

13. Appendix

In this Appendix we describe the $\hat{H}_q$-module structure $\mathbf{W}(s, s^*, r_1, r_2, D; q)$ twisted via ρ (see §6) and display the action of $\mathbf{Y}$ on this module explicitly. Recall the scalars s, s^*, r_1, r_2, D, q from Note 4.6.

13.1. An $\hat{H}_q$-module in terms of the scalars s, s^*, r_1, r_2, D, q

Definition 13.1. (cf. [19, Definition 11.1]) We define some matrices as follows.

(a) For $1 \le i \le D-1$, the (2×2)-matrix $t_0(i)$ is

$$\begin{bmatrix} \frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^{D/2}} & \frac{q^{D/2}(q^{i-D}-1)(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \\ \frac{(1-q^i)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} & \frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} + \frac{1}{q^{D/2}} \end{bmatrix}$$

and

$$t_0(0) = \left[\ \frac{1}{q^{D/2}}\ \right], \qquad t_0(D) = \left[\ \frac{1}{q^{D/2}}\ \right].$$

(b) For $0 \le i \le D-1$, the (2×2)-matrix $t_1(i)$ is

$$\begin{bmatrix} \frac{1}{(s^*r_1r_2)^{1/2}}\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^*\right) & -\left(\frac{s^*}{r_1r_2}\right)^{1/2}\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \\ \frac{1}{(s^*r_1r_2)^{1/2}}\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \left(\frac{s^*}{r_1r_2}\right)^{1/2}\left(1-\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right) \end{bmatrix}.$$

(c) $0 \le i \le D-1$, the (2×2)-matrix $t_2(i)$ is

$$\begin{bmatrix} \frac{1}{q^{i+1}(r_1r_2)^{1/2}}\left(1-\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right) & \frac{s^*q^{i+1}}{(r_1r_2)^{1/2}}\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \\ -\frac{1}{s^*q^{i+1}(r_1r_2)^{1/2}}\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \frac{q^{i+1}}{(r_1r_2)^{1/2}}\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^*\right) \end{bmatrix}.$$

(d) For $1 \le i \le D-1$, the (2×2)-matrix $t_3(i)$ is

$$\begin{bmatrix} \frac{1}{q^i(s^*q)^{1/2}}\left(\frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} + \frac{1}{q^{D/2}}\right) & \frac{1}{q^i(s^*q)^{1/2}}\left(\frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}}\right) \\ q^i(s^*q)^{1/2}\left(\frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})}\right) & q^i(s^*q)^{1/2}\left(\frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^{D/2}}\right) \end{bmatrix}$$

and

$$t_3(0) = \left[\ (s^*q^{D+1})^{1/2}\ \right], \qquad t_3(D) = \left[\ \frac{1}{(s^*q^{D+1})^{1/2}}\ \right].$$

With reference to Definition 13.1, we define a $2D\times 2D$ block diagonal matrix $\mathcal{T}_n (n \in \mathbb{I})$ as follows.

$$\mathcal{T}_0 := \text{blockdiag}\Big(t_0(0), t_0(1), \ldots, t_0(D-1), t_0(D)\Big),$$
$$\mathcal{T}_1 := \text{blockdiag}\Big(t_1(0), t_1(1), \ldots, t_1(D-1)\Big),$$

$$\mathcal{T}_2 := \text{blockdiag}\Big(t_2(0), t_2(1), \ldots, t_2(D-1)\Big),$$

$$\mathcal{T}_3 := \text{blockdiag}\Big(t_0(0), t_0(1), \ldots, t_0(D-1), t_0(D)\Big).$$

One checks that (i) each $\mathcal{T}_n (n \in \mathbb{I})$ is invertible; (ii) $\mathcal{T}_n + \mathcal{T}_n^{-1} = (\kappa_n + \kappa_n)I$, where κ_n is from (54); (iii) $\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3 = q^{-1/2}I$. By this and Definition 6.1, there exists an $\hat{H}_q$-module structure on $\mathbf{W}$ such that for $n \in \mathbb{I}$ the matrix $\mathcal{T}_n$ represents the generator t_n relative to the basis $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ [19, Proposition 11.10].

13.2. The action of $\mathbf{Y}$

Referring to Section 13.1, we display the action of $\mathbf{Y}$ on $\{\hat{C}_i^{\pm}\}_{i-1}^{D}$. For this section, we abbreviate $\mathbf{W}_q := \mathbf{W}(s, s^*, r_1, r_2, D; q)$.

Lemma 13.2. (cf. [19, Lemma 12.2]) *On* $\mathbf{W}_q$,

(a) *For* $0 \le i \le D-1$, *the action* $\mathbf{Y}.\hat{C}_i^-$ *is given as a linear combination with the following terms and coefficients:*

term	coefficient
$\hat{C}_{i-1}^+$	$\left(\frac{q^D}{s^*r_1r_2}\right)^{1/2}\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}}+s^*\right)\left(\frac{(q^{i-D}-1)(1-s^*q^{i+1})}{1-s^*q^{2i+1}}\right)$
$\hat{C}_i^-$	$\left(\frac{1}{q^Ds^*r_1r_2}\right)^{1/2}\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}}+s^*\right)\left(\frac{(q^i-1)(1-s^*q^{D+i+1})}{1-s^*q^{2i+1}}+1\right)$
$\hat{C}_i^+$	$\left(\frac{q^D}{s^*r_1r_2}\right)^{1/2}\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}}\right)\left(\frac{(1-q^{i+1-D})(1-s^*q^{i+2})}{1-s^*q^{2i+3}}+\frac{1}{q^D}\right)$
$\hat{C}_{i+1}^-$	$\left(\frac{1}{s^*r_1r_2q^D}\right)^{1/2}\frac{(1-q^{i+1})(1-s^*q^{D+i+2})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(1-s^*q^{2i+2})(1-s^*q^{2i+3})}$

,

where $\hat{C}_{-1}^+ = 0$ *and* $\hat{C}_D^- = 0$.

(b) *For* $0 \le i \le D-1$, *the action* $\mathbf{Y}.\hat{C}_i^+$ *is given as a linear combination with the following terms and coefficients:*

term	coefficient
$\hat{C}_{i-1}^+$	$\left(\frac{s^*q^D}{r_1r_2}\right)^{1/2}\frac{(1-q^{i-D})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})}$
$\hat{C}_i^-$	$-\left(\frac{s^*}{r_1r_2q^D}\right)^{1/2}\left(\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right)\left(\frac{(q^i-1)(1-s^*q^{i+D+1})}{1-s^*q^{2i+1}}+1\right)$
$\hat{C}_i^+$	$\left(\frac{s^*q^D}{r_1r_2}\right)^{1/2}\left(1-\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right)\left(\frac{(1-q^{i+1-D})(1-s^*q^{i+2})}{1-s^*q^{2i+3}}+\frac{1}{q^D}\right)$
$\hat{C}_{i+1}^-$	$\left(\frac{s^*}{r_1r_2q^D}\right)^{1/2}\left(1-\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right)\left(\frac{(1-q^{i+1})(1-s^*q^{D+i+2})}{1-s^*q^{2i+3}}\right)$

,

where $\hat{C}_{-1}^+ = 0$ *and* $\hat{C}_D^- = 0$.

Lemma 13.3. (cf. [19, Lemma 12.3]) *On* $\mathbf{W}_q$,

(a) *For* $0 \le i \le D-1$, *the action* $\mathbf{Y}^{-1}.\hat{C}_i^-$ *is given as a linear combination with the following terms and coefficients:*

term	coefficient
$\hat{C}_{i-1}^-$	$\left(\frac{s^*q^D}{r_1r_2}\right)^{1/2} \frac{(1-q^{i-D})(1-s^*q^{i+1})(1-r_1q^i)(1-r_2q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})}$
$\hat{C}_{i-1}^+$	$\left(\frac{q^D}{s^*r_1r_2}\right)^{1/2} \left(\frac{(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}}\right)\left(\frac{(r_1-s^*q^i)(r_2-s^*q^i)}{1-s^*q^{2i}} + s^*\right)$
$\hat{C}_i^-$	$\left(\frac{s^*q^D}{r_1r_2}\right)^{1/2} \left(\frac{(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^D}\right)\left(1 - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right)$
$\hat{C}_i^+$	$-\left(\frac{q^D}{s^*r_1r_2}\right)^{1/2} \left(\frac{(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^D}\right)\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}}\right)$

where $\hat{C}_{-1}^- = 0$ *and* $\hat{C}_{-1}^+ = 0$.

(b) *For* $0 \le i \le D-1$, *the action* $\mathbf{Y}^{-1}.\hat{C}_i^+$ *is given as a linear combination with the following terms and coefficients:*

term	coefficient
$\hat{C}_i^-$	$\left(\frac{q^D}{s^*r_1r_2}\right)^{1/2} \left(\frac{(q^{i+1}-1)(1-s^*q^{D+i+2})}{1-s^*q^{2i+3}} + 1\right)\left(\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}}\right)$
$\hat{C}_i^+$	$\left(\frac{1}{s^*r_1r_2q^D}\right)^{1/2} \left(\frac{(q^{i+1}-1)(1-s^*q^{D+i+2})}{1-s^*q^{2i+3}} + 1\right)\left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^*\right)$
$\hat{C}_{i+1}^-$	$\left(\frac{s^*}{r_1r_2q^D}\right)^{1/2} \left(\frac{(1-q^{i+1})(1-s^*q^{D+i+2})}{1-s^*q^{2i+3}}\right)\left(\frac{(1-r_1q^{i+2})(1-r_2q^{i+2})}{1-s^*q^{2i+4}} - 1\right)$
$\hat{C}_{i+1}^+$	$\left(\frac{1}{s^*r_1r_2q^D}\right)^{1/2} \frac{(1-q^{i+1})(1-s^*q^{D+i+2})(r_1-s^*q^{i+2})(r_2-s^*q^{i+2})}{(1-s^*q^{2i+3})(1-s^*q^{2i+4})}$

where $\hat{C}_{-1}^- = 0$ *and* $\hat{C}_{-1}^+ = 0$.

Acknowledgments

The author would like to thank Paul Terwilliger and Hajime Tanaka for many helpful discussions and comments. In particular, Hajime Tanaka gave valuable ideas and suggestions for Sections 8, 9. The author also thanks the two anonymous referees for careful reading and helpful comments. This work is supported by the JSPS KAKENHI; grant numbers: $26 \cdot 04019$.

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