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Chordality of clutters with vertex decomposable dual and ascent of clutters

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ABSTRACT

In this paper, we consider the generalization of chordal graphs to clutters proposed by Bigdeli, et al. (2017) in [2]. Assume that \mathcal{C} is a d -dimensional uniform clutter. It is known that if \mathcal{C} is chordal, then $I(\overline{\mathcal{C}})$ has a linear resolution over all fields. The converse has recently been rejected, but the following question which poses a weaker version of the converse is still open: “if $I(\overline{\mathcal{C}})$ has linear quotients, is \mathcal{C} necessarily chordal?”. Here, by introducing the concept of the ascent of a clutter, we split this question into two simpler questions and present some clues in support of an affirmative answer. In particular, we show that if $I(\overline{\mathcal{C}})$ is the Stanley-Reisner ideal of a simplicial complex with a vertex decomposable Alexander dual, then \mathcal{C} is chordal.

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1. Introduction

In this paper all rings are commutative with identity, K is a field and $S = K[x_1, \dots, x_n]$ is the polynomial ring in n indeterminates over K . Given an arbitrary ideal I of S , we can get a monomial ideal by taking its initial ideal or its generic initial

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ideal with respect to some monomial order. Many of the properties of I is similar or at least related to its (generic) initial ideal, especially if I is graded (see, for example, [8, Section 3.3 & Corollary 4.3.18]). Also if I is a monomial ideal of S then its polarization is a squarefree monomial ideal, again with algebraic properties similar to I (see [8, Section 1.6]). Therefore, if we know the algebraic properties of squarefree monomial ideals well, then we can understand many of the algebraic properties of much larger classes of ideals.

On the other hand, squarefree monomial ideals have a combinatorial nature and correspond to combinatorial objects such as simplicial complexes, graphs, clutters and posets. Many researchers have tried to characterize algebraic properties of squarefree monomial ideals, using their combinatorial counterparts, see for example [9] and Part III of [8] and their references. In this regard, two important objectives are to classify combinatorially squarefree monomial ideals which are Cohen-Macaulay or those which have a linear resolution (either over every field or over a fixed field). Indeed these two tasks are equivalent under taking Alexander dual.

A well-known theorem of Fröberg states that if I is a squarefree monomial ideal generated in degree two, then I has a linear resolution if and only if I is the edge ideal of the complement of a chordal graph. Motivated by this, many have tried to generalize the concept of chordality to clutters or simplicial complexes in such a way that at least one side of Fröberg's theorem stays true in degrees > 2 (see, for instance, [2,6,13]). One of the most promising such generalizations seems to be the concept of chordal clutters presented in [2]. In [2] it is shown that many other classes of “chordal clutters” defined by other researchers, including the class defined in [13] which we call W-chordal clutters, is strictly contained in the class defined in [2]. They also show that if a clutter is chordal, then the circuit ideal of its complement has a linear resolution over every field. Several clues were presented in [1,2] supporting the correctness of the converse (see [2, Question 1]). But recently a counterexample to the converse was presented by Eric Babson (see Example 2.4). Despite this it is still unknown whether the following statement which is a weaker version of the converse is true or not: “if the circuit ideal of the complement of a clutter has linear quotients, then that clutter is chordal.” Moreover, in [1], it is shown that many numerical invariants of the ideal corresponding to a chordal clutter can be combinatorially read off the clutter.

Here, after presenting a brief review of the main concepts and setting the notations, in Section 3 we prove that if the Alexander dual of the clique complex of a clutter \mathcal{C} is vertex decomposable, then \mathcal{C} is chordal. As clique complexes of W-chordal clutters are vertex decomposable, this generalizes the results of Subsection 3.1 of [2]. Then in Section 4, we define the notion of the ascent of a clutter and show how we can use this concept to divide the question “is \mathcal{C} chordal, given that the circuit ideal of the complement of \mathcal{C} has linear quotients?” into two simpler questions.

2. Preliminaries, notations and a counterexample

Algebraic background Suppose that I is a graded ideal of S considered with the standard grading. This grading induces a natural grading on $\text{Tor}_i^S(K, I)$. We denote the degree j part of $\text{Tor}_i^S(K, I)$ by $\text{Tor}_i^S(K, I)_j$ and its dimension over K is denoted by $\beta_{ij}^S(I) = \beta_{ij}(I)$. These β_{ij} 's are called the graded *Betti numbers* of I . If there is a $d \geq 0$ such that $\beta_{ij}(I) = 0$ for $j \neq i + d$, it is said that I has a *linear* (or *d-linear*) *resolution*. Also we say that I has *linear quotients* with respect to an ordered system of homogeneous generators f_1, \dots, f_t , if for each i the ideal $\langle f_1, \dots, f_{i-1} \rangle : f_i$ is generated by linear forms. It is known that if I is generated in degree d and has linear quotients with respect to some system of homogeneous generators, then I has a d -linear resolution (see [8, Proposition 8.2.1]).

Now assume that I is a squarefree monomial ideal, that is, I is generated by some squarefree monomials. Then I has a unique smallest generating set consisting of squarefree monomials, say x_{F_1}, \dots, x_{F_t} where $x_F = \prod_{i \in F} x_i$ for $F \subseteq [n] = \{1, \dots, n\}$. In this case, if I has linear quotients with respect to a permutation of this minimal system of generators, we simply say that I has linear quotients. Also a specific order of minimal generators of I , for which the aforementioned colon ideals are linear, is called an *admissible order*. According to [8, Corollary 8.2.4], x_{F_1}, \dots, x_{F_t} is an admissible order for the ideal they generate, if and only if for each i and all $j < i$, there is a $l \in F_j \setminus F_i$ and a $k < i$ such that $F_k \setminus F_i = \{l\}$. For more details on these algebraic concepts see [8].

Clutters A clutter \mathcal{C} on the vertex set $V = V(\mathcal{C})$ is a family of subsets of V which are pairwise incomparable under inclusion. We call the elements of \mathcal{C} *circuits*. For any subset F of V we set $\dim F = |F| - 1$. If all circuits of \mathcal{C} have the same dimension d , we say that \mathcal{C} is a *d-dimensional uniform clutter* or a *d-clutter* for short. If $v \in V(\mathcal{C})$, then by $\mathcal{C} - v$ we mean the clutter on $V(\mathcal{C}) \setminus \{v\}$ with circuits $\{F \in \mathcal{C} | v \notin F\}$. For simplicity, we write for example ab , Ex or Eab instead of $\{a, b\}$, $E \cup \{x\}$ or $E \cup \{a, b\}$ for $a, b, x \in V$ and $E \subseteq V$. If we assume that $V = [n]$, then $I(\mathcal{C})$ is defined as the ideal of S generated by $\{x_F | F \in \mathcal{C}\}$.

From now on, we always assume that \mathcal{C} is a d -clutter with $|V(\mathcal{C})| = n$. We call \mathcal{C} *complete*, when all d -dimensional subsets of $V = V(\mathcal{C})$ are in \mathcal{C} . A *clique* of \mathcal{C} is a subset A of V such that \mathcal{C}_A is complete, where $\mathcal{C}_A = \{F \in \mathcal{C} | F \subseteq A\}$ is the induced d -clutter on the vertex set A . The complement $\overline{\mathcal{C}}$ of \mathcal{C} is the d -clutter with the same vertex set as \mathcal{C} and circuits $\{F \subseteq V | \dim F = d, F \notin \mathcal{C}\}$.

The set $\{e \subseteq V | \dim e = d - 1, \exists F \in \mathcal{C} \ e \subseteq F\}$ is called the set of *maximal subcircuits* of \mathcal{C} and is denoted by $\mathcal{MS}(\mathcal{C})$. For a $(d - 1)$ -dimensional subset e of V , the *closed neighborhood* $N_{\mathcal{C}}[e]$ is defined as $e \cup \{v \in V | ev \in \mathcal{C}\}$. If $e \in \mathcal{MS}(\mathcal{C})$, then $\mathcal{C} - e$ means the clutter on the vertex set V with circuits $\{F \in \mathcal{C} | e \not\subseteq F\}$.

A maximal subcircuit e of \mathcal{C} is called a *simplicial maximal subcircuit*, when $N_{\mathcal{C}}[e]$ is a clique. We denote the set of all simplicial maximal subcircuits of \mathcal{C} by $\mathcal{SMS}(\mathcal{C})$. If for each $1 \leq i < t$ there is an $e_i \in \mathcal{SMS}(\mathcal{C}_{i-1})$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_i = \mathcal{C}_{i-1} - e_i$, such

that \mathcal{C}_t has no circuits, then \mathcal{C} is called *chordal* (see [2, Section 3]). In the case that $d = 1$ (that is, \mathcal{C} is graph) this notion coincides with the usual notion of chordal graphs. Theorem 3.3 of [2] states that if \mathcal{C} is chordal and not complete, then $I(\overline{\mathcal{C}})$ has a linear resolution over every field. There is an example showing that the converse is not true, that is, there is a non-chordal clutter \mathcal{C} with $I(\overline{\mathcal{C}})$ having a linear resolution over every field (see Example 2.4). But it is still unknown whether there is a non-chordal clutter \mathcal{C} with $I(\overline{\mathcal{C}})$ having linear quotients. For further reference, we label this statement:

(A) If $I(\overline{\mathcal{C}})$ has linear quotients, then \mathcal{C} is chordal.

Simplicial complexes A simplicial complex Δ on the vertex set $V = V(\Delta)$ is a family of subsets of V (called *faces* of Δ) such that if $A \subseteq B \in \Delta$, then $A \in \Delta$. The dimension of Δ is defined as $\dim \Delta = \max_{F \in \Delta} \dim F$. The set of maximal faces of Δ which are called *facets* is denoted by $\text{Facets}(\Delta)$. If $|\text{Facets}(\Delta)| = 1$, then Δ is called a *simplex*.

If all facets of Δ have the same dimension, we say that Δ is *pure*. In this case $\text{Facets}(\Delta)$ is a d -dimensional uniform clutter. Also if \mathcal{D} is a clutter, then $\langle \mathcal{D} \rangle$ denotes the simplicial complex Δ with $\text{Facets}(\Delta) = \mathcal{D}$. It should be clear that $\langle \text{Facets}(\Delta) \rangle = \Delta$ and $\text{Facets}(\langle \mathcal{D} \rangle) = \mathcal{D}$, for any simplicial complex Δ and any (not necessarily uniform) clutter \mathcal{D} . Another simplicial complex associated to a d -clutter \mathcal{C} is the *clique complex* $\Delta(\mathcal{C})$ of \mathcal{C} defined as the family of all subsets L of $V(\mathcal{C})$ with the property that L is a clique in \mathcal{C} . Note that all subsets of $V(\mathcal{C})$ with size $\leq d$ are cliques by assumption.

For a face F of Δ , we define $\text{link}_\Delta F = \{G \setminus F \mid F \subseteq G \in \Delta\}$, which is a simplicial complex on the vertex set $V \setminus F$. Also if $v \in V$, $\Delta - v$ is the simplicial complex on the vertex set $V \setminus \{v\}$ with faces $\{F \in \Delta \mid v \notin F\}$.

Assuming that $V(\Delta) = [n]$, the ideal of S generated by $\{x_F \mid F \text{ is a minimal non-face of } \Delta\}$ is called the *Stanley-Reisner ideal* of Δ and is denoted by I_Δ . When $\frac{S}{I_\Delta}$ is a Cohen-Macaulay ring, Δ is said to be Cohen-Macaulay over K .

Let A be a commutative ring with identity and $-1 \leq d \leq \dim \Delta$ and denote by $\tilde{C}_d(\Delta) = \tilde{C}_d(\Delta, A)$ the free A -module whose basis is the set of all d -dimensional faces of Δ . Consider the A -homomorphism $\partial_d : \tilde{C}_d(\Delta) \rightarrow \tilde{C}_{d-1}(\Delta)$ defined by

$$\partial_d(\{v_0, \dots, v_d\}) = \sum_{i=0}^d (-1)^i \{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_d\},$$

where $v_0 < \dots < v_d$ for a fixed total order $<$ on $V(\Delta)$. Then $(\tilde{C}_\bullet, \partial_\bullet)$ is a complex of free A -modules and A -homomorphisms called the *augmented oriented chain complex* of Δ over A . We denote the i -th homology of this complex by $\tilde{H}_i(\Delta; A)$.

By the *Alexander dual* of a simplicial complex Δ we mean $\Delta^\vee = \{V(\Delta) \setminus F \mid F \subseteq V(\Delta), F \notin \Delta\}$ and also we set $\mathcal{C}^\vee = \{V(\mathcal{C}) \setminus F \mid F \in \overline{\mathcal{C}}\}$. Then it follows from the Eagon-Reiner theorem ([8, Theorem 8.1.9]) and the lemma below that $I(\overline{\mathcal{C}})$ has a linear resolution over K , if and only if $\langle \mathcal{C}^\vee \rangle$ is Cohen-Macaulay over K . For more details on

simplicial complexes and related algebraic concepts the reader is referred to [8]. We frequently use the following lemma in the sequel without any further mention.

Lemma 2.1 ([9, Lemma 1.1]). *Let \mathcal{C} be a d -clutter. Then*

- (i) $I(\overline{\mathcal{C}}) = I_{\Delta(\mathcal{C})}$;
- (ii) $\langle \mathcal{C}^\vee \rangle = (\Delta(\mathcal{C}))^\vee$.

When working with both clutters and simplicial complexes, one should notice the differences between the similar concepts and notations defined for these objects. For example, we say that a d -dimensional simplicial complex Δ is i -complete to mean that Δ has all possible faces of dimension i , where $i \leq d$. But when we are talking about a d -clutter \mathcal{C} , there is no need to mention the dimension and say that \mathcal{C} is i -complete, since we must have $i = d$ which is implicit in \mathcal{C} and it is completely meaningless to say that \mathcal{C} is i -complete for $i < d$. Therefore we just say that \mathcal{C} is complete. Another difference arises from the concept of Alexander dual which is illustrated in the following example.

Example 2.2. Let $\mathcal{C} = \{125, 235, 345\}$ be a 2-clutter on $[5]$, $\Gamma = \langle \mathcal{C} \rangle$ and $\Delta = \Delta(\mathcal{C})$. Note that $14 \notin \Gamma$ because it is not contained in any facet of Γ . But by definition, 14 is a clique of \mathcal{C} and hence $14 \in \Delta = \langle \{125, 235, 345, 13, 14, 24\} \rangle$. Now for example $245 \in \overline{\mathcal{C}}$ and hence $[5] \setminus 245 = 13 \in \mathcal{C}^\vee$. Indeed, $\mathcal{C}^\vee = \{13, 15, 23, 24, 25, 35, 45\}$, which by (2.1) is equal to the set of facets of Δ^\vee . But as $14 \notin \Gamma$, we see that $[5] \setminus 14 = 235 \in \Gamma^\vee$. In fact, $\langle \mathcal{C} \rangle^\vee = \Gamma^\vee = \langle \{235, 245, 135\} \rangle \neq \langle \mathcal{C}^\vee \rangle$.

Recall that $\Delta_W = \{F \in \Delta \mid F \subseteq W\}$ where $W \subseteq V$.

Theorem 2.3 (Fröberg [7]). *The ideal I_Δ has a t -linear resolution over K , if and only if $\tilde{H}_i(\Delta_W; K) = 0$ for all $W \subseteq V(\Delta)$ and $i \neq t - 2$.*

Note that in the case that $\Delta = \Delta(\mathcal{C})$, then since all possible faces of dimension less than d are in Δ , it follows that $\tilde{H}_i(\Delta_W; K) = 0$ for all $W \subseteq V$ and $i < d - 1$. Hence $I(\overline{\mathcal{C}})$ has a $(d + 1)$ -linear resolution over K , if and only if $\tilde{H}_i(\Delta_W; K) = 0$ for all $W \subseteq V(\Delta)$ and $i \geq d$. Using this we can state an example of a non-chordal clutter \mathcal{C} with $I(\overline{\mathcal{C}})$ having a linear resolution over every field. This example is due to Eric Babson who visited IASBS as a lecturer in the first Research School on Commutative Algebra and Algebraic Geometry in 2017.

Example 2.4 (Babson). Let \mathcal{C} be the 2-clutter shown in Fig. 1 which is a triangulation of the dunce hat. Then $\Delta = \Delta(\mathcal{C})$ is obtained by adding the missing edges (1-dimensional subsets of vertices) to $\Delta' = \langle \mathcal{C} \rangle$, since \mathcal{C} has no cliques on more than 3 vertices. Therefore, $\tilde{H}_t(\Delta; K) = \tilde{H}_t(\Delta'; K)$ for every $t \geq 2$. It is known that the dunce hat is a contractible space and hence $0 = \tilde{H}_t(\Delta'; K)$ for all $t \geq 2$. Also it is easy to verify that for any $W \subsetneq [8]$, \mathcal{C}_W is chordal (indeed, any proper subclutter of \mathcal{C} has an edge which is contained in

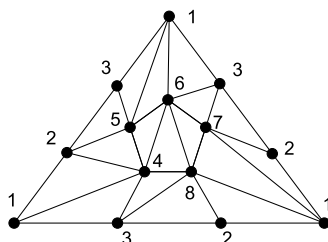


Fig. 1. A triangulation of the dunce hat; the circuits are the triangles.

exactly one triangle and hence is simplicial) and thus has a linear resolution over every field. Therefore, $\tilde{H}_t(\Delta_W; K) = \tilde{H}_t(\Delta(\mathcal{C}_W); K) = 0$ for all $t \geq 2$. Consequently, by Föberg's theorem $I_\Delta = I(\overline{\mathcal{C}})$ has a linear resolution over every field.

3. Clutters with vertex decomposable dual are chordal

A vertex v of a nonempty simplicial complex Δ is called a *shedding vertex*, when no face (or equivalently facet) of $\text{link}_\Delta(v)$ is a facet of $\Delta - v$. Recall that a nonempty simplicial complex Δ is called *vertex decomposable*, when either it is a simplex or there is a shedding vertex $v \in V(\Delta)$ such that both $\text{link}_\Delta v$ and $\Delta - v$ are vertex decomposable. This concept was first introduced in [10] in connection with the Hirsch conjecture which has applications in the analysis of the simplex method in linear programming.

Lemma 3.1. *Suppose that Δ is a pure d -dimensional simplicial complex. Then v is a shedding vertex if and only if $\Delta - v$ is pure with $\dim(\Delta - v) = d$.*

Proof. (\Rightarrow): Assume that $\Delta - v$ is not pure or $\dim(\Delta - v) \neq d$. Since every facet of $\Delta - v$ has dimension $d - 1$ or d , there is a facet F of $\Delta - v$ such that $\dim F = d - 1$. As $F \in \Delta$ and Δ is pure, we see that $F \subsetneq F'$ for some $F' \in \text{Facets}(\Delta)$. Hence $F' = Fv \in \Delta$ and $F \in \text{link}_\Delta(v)$, which contradicts the shedding property for v .

(\Leftarrow): Every face of $\text{link}_\Delta(v)$ has dimension $\leq d - 1$ and by assumption $\Delta - v$ is pure and of dimension d . Thus $\text{link}_\Delta(v) \cap \text{Facets}(\Delta - v) = \emptyset$ and v is a shedding vertex. \square

Example 3.2. Suppose that Γ is as in Example 2.2. By choosing 1 and then 4 as the shedding vertex, we see that Γ is vertex decomposable. Here $\Gamma - 5 = \langle 12, 23, 34 \rangle$ is pure with dimension $1 = \dim(\Gamma) - 1$ and $\Gamma - 2 = \langle 15, 345 \rangle$ is not pure. So by the previous lemma, neither 5 is a shedding vertex of Γ nor 2. Also Γ^\vee which is indeed isomorphic to Γ is vertex decomposable. Now assume that $\Delta = \langle 12, 34 \rangle$ which is a 1-dimensional simplicial complex. Then $\Delta - 1 = \langle 2, 34 \rangle$ is not pure, so 1 is not a shedding vertex of Δ . Similarly we see that Δ has no shedding vertex and is not vertex decomposable.

It is well-known that if Δ is pure and vertex decomposable, then it is shellable and Cohen-Macaulay (see [10]), hence I_{Δ^\vee} has a linear resolution and in fact linear quotients.

So if \mathcal{C} is the clutter with $I(\overline{\mathcal{C}}) = I_{\Delta^\vee}$ and if statement (A) is true, \mathcal{C} should be chordal. In this section, we prove that this is indeed the case. By (2.1), in the above situation we have $\Delta = (\Delta(\mathcal{C}))^\vee = \langle \mathcal{C}^\vee \rangle$. First we need some lemmas. In the sequel, we assume that \mathcal{C} is a d -clutter and $|V(\mathcal{C})| = n$. Also recall that the *pure i -skeleton* of a simplicial complex Δ , is the simplicial complex whose facets are i -dimensional faces of Δ .

Lemma 3.3. *Let \mathcal{C} be a d -clutter. Suppose that $\Delta = \langle \mathcal{C} \rangle$ and $\Gamma = \langle \mathcal{C}^\vee \rangle$. Also assume that $v \in V(\Gamma)$ ($= V(\Delta)$) and $\Gamma - v$ is pure of dimension $= \dim(\Gamma)$ (that is, v is a shedding vertex of Γ) and set $\mathcal{D} = \text{Facets}(\text{link}_\Delta(v))$. Then*

- (i) $\text{link}_\Delta(v)$ is the pure $(d-1)$ -skeleton of $(\Gamma - v)^\vee$;
- (ii) $\langle \mathcal{D}^\vee \rangle = \Gamma - v$;
- (iii) $e \in \mathcal{SMS}(\mathcal{D})$ if and only if $ev \in \mathcal{SMS}(\mathcal{C})$.

Proof. (i) Suppose that $F \in (\Gamma - v)^\vee$ and $\dim F = d-1$. Then $A = V(\Gamma - v) \setminus F \notin \Gamma - v$ and hence $A \notin \Gamma$ and $A \notin \mathcal{C}^\vee$. But $\dim(A) = (n-1) - \dim(F) - 2 = n-d-2 = \dim(\Gamma) = \dim(\mathcal{C}^\vee)$. Thus $Fv = V(\Delta) \setminus A \in \mathcal{C}$, that is, F is a facet of $\text{link}_\Delta(v)$. So the pure $(d-1)$ -skeleton of $(\Gamma - v)^\vee$ is contained in $\text{link}_\Delta(v)$. The proof of the reverse inclusion is similar.

(ii) Noting that $(\text{Facets}(\Gamma - v))^\vee$ is exactly the set of $(d-1)$ -dimensional faces of $(\Gamma - v)^\vee$, we see that part (i) indeed states $\mathcal{D} = (\text{Facets}(\Gamma - v))^\vee$, which is equivalent to (ii).

(iii) Assume that $e \in \mathcal{MS}(\mathcal{D})$. Then for $x \in V(\mathcal{D}) \setminus e$ we have $ex \in \mathcal{D} \Leftrightarrow exv \in \text{Facets}(\Delta) = \mathcal{C}$. Therefore, $N_{\mathcal{D}}[e] = N_{\mathcal{C}}[ev] \setminus \{v\}$. If $ev \in \mathcal{SMS}(\mathcal{C})$ and A is a $(d-1)$ -dimensional subset of $N_{\mathcal{D}}[e]$, then $Av \subseteq N_{\mathcal{C}}[ev]$ which is a clique and hence $Av \in \mathcal{C}$, that is, $A \in \mathcal{D}$. So $N_{\mathcal{D}}[e]$ is a clique and $e \in \mathcal{SMS}(\mathcal{C})$.

Conversely, assume that $e \in \mathcal{SMS}(\mathcal{D})$ and A is a d -dimensional subset of $N_{\mathcal{C}}[ev]$. We must show $A \in \mathcal{C}$. If $v \in A$, then $A \setminus \{v\} \subseteq N_{\mathcal{D}}[e]$ and hence $A \setminus \{v\} \in \mathcal{D}$, which means $A \in \mathcal{C}$. Thus suppose $v \notin A$. If $A \notin \mathcal{C}$, then $v \in B = V(\mathcal{C}) \setminus A \in \mathcal{C}^\vee = \text{Facets}(\Gamma)$. Consequently, $B \setminus \{v\} \in \Gamma - v$ and there is a $F \in \text{Facets}(\Gamma - v) = \mathcal{D}^\vee$ containing $B \setminus \{v\}$. So $A' = V(\mathcal{D}) \setminus F \notin \mathcal{D}$. Since $\dim(\Gamma - v) = \dim \Gamma = n-d-2 = |F| - 1$ and $V(\mathcal{D}) = V(\mathcal{C}) \setminus \{v\}$, we see that $|A'| = d$. Also $A' \subseteq A \subseteq N_{\mathcal{C}}[ev] \setminus \{v\} = N_{\mathcal{D}}[e]$, and as e is simplicial, we get $A' \in \mathcal{D}$, a contradiction from which the result follows. \square

Example 3.4. Suppose that \mathcal{C} is a 2-clutter on [6] with circuits 123, 124, 134, 234, 345, 346, 126 and Δ , Γ and \mathcal{D} are defined as in (3.3). Then

$$\begin{aligned}\Gamma &= \langle \mathcal{C}^\vee \rangle = \langle 136, 146, 236, 246, 346, 135, 235, 145, 245, 123, 124, 134, 234 \rangle \text{ and} \\ \Gamma - 6 &= \langle 135, 235, 145, 245, 123, 124, 134, 234 \rangle.\end{aligned}$$

So 6 is a shedding vertex of Γ . Note that $\text{Facets}(\Gamma - 6)$ is a 3-clutter on [5], thus $(\Gamma - 6)^\vee = \langle 12, 34 \rangle$ which is exactly $\text{link}_\Delta(6)$. Also $\mathcal{D} = \{12, 34\}$ is indeed a graph and $\mathcal{SMS}(\mathcal{D}) =$

$\{1, 2, 3, 4\}$. As claimed in the previous lemma, simplicial maximal subcircuits of \mathcal{C} which contain the vertex 6 are 16, 26, 36, 46.

Lemma 3.5. *Let \mathcal{C} be a d -clutter. Assume that $\Delta = \langle \mathcal{C} \rangle$ and $\Gamma = \langle \mathcal{C}^\vee \rangle$. Let v be a shedding vertex of Γ and $v \in e \in \mathcal{MS}(\mathcal{C})$. Then v is a shedding vertex of $\langle (\mathcal{C} - e)^\vee \rangle = (\Delta(\mathcal{C} - e))^\vee$. Furthermore, if $\Delta' = \langle \mathcal{C} - e \rangle$, then $\text{Facets}(\text{link}_{\Delta'} v) = \text{Facets}(\text{link}_\Delta v) - e'$, where $e' = e \setminus \{v\}$.*

Proof. Let $\Gamma' = \langle (\mathcal{C} - e)^\vee \rangle$ and suppose that $F \in \text{Facets}(\text{link}_{\Gamma'} v)$. We have to show that F is not a facet of $\Gamma' - v$. Note that $\mathcal{C}^\vee \subseteq (\mathcal{C} - e)^\vee$ and hence $\Gamma - v \subseteq \Gamma' - v$. Therefore, it suffices to show that there is a $G \in \text{Facets}(\Gamma) = \mathcal{C}^\vee$ with $v \notin G$ such that $F \subsetneq G$. We know that $Fv \in \text{Facets}(\Gamma') = (\mathcal{C} - e)^\vee$, so $V \setminus Fv \notin \mathcal{C} - e$ where $V = V(\mathcal{C} - e) = V(\mathcal{C})$. As $v \in e$, $e \not\subseteq V \setminus Fv$ and thus $V \setminus Fv \notin \mathcal{C}$. It follows that $Fv \in \Gamma$, that is, $F \in \text{link}_\Gamma v$. No face of $\text{link}_\Gamma v$ is a facet of $\Gamma - v$, because v is a shedding vertex of Γ . Consequently, F is strictly contained in the facet G of $\Gamma - v$, as claimed. The proof of the “furthermore” statement, which follows from definitions, is left to the reader. \square

Example 3.6. Let $\mathcal{C}, \Delta, \Gamma$ and $v = 6$ be as in Example 3.4 and set $e = 26$, $\Gamma' = \langle (\mathcal{C} - e)^\vee \rangle$ and $\Delta' = \langle \mathcal{C} - e \rangle$. Then $\mathcal{C} - e = \mathcal{C} \setminus \{126\}$ and $(\mathcal{C} - e)^\vee = \mathcal{C}^\vee \cup \{345\}$ and the facets of Γ' (resp. $\Gamma' - 6$) are obtained by adding the face 345 to the set of facets of Γ (resp. $\Gamma - 6$). Thus $\Gamma' - 6$ is pure and of dimension 2 and hence 6 is still a shedding vertex in Γ' . Also the only facet of $\text{link}_{\Delta'}(6)$ which does not contain the vertex $2 = e \setminus \{v\}$ is 34 and $\text{link}_{\Delta'}(6) = \langle 34 \rangle$.

Suppose that $v \in V(\mathcal{C})$ has not appeared in any circuit of \mathcal{C} . Then chordality (and many other properties) of \mathcal{C} and $\mathcal{C} - v$ are equivalent, although $V(\mathcal{C}) \neq V(\mathcal{C} - v)$. In this case, we misuse the notation and write $\mathcal{C} = \mathcal{C} - v$.

Lemma 3.7. *Let \mathcal{C} be a d -clutter. Assume that $\Delta = \langle \mathcal{C} \rangle$, v is a shedding vertex of $\Gamma = \langle \mathcal{C}^\vee \rangle$ and $\mathcal{D} = \text{Facets}(\text{link}_\Delta v)$. If \mathcal{D} is chordal, then there is a sequence e_1, \dots, e_t with $e_i \in \mathcal{SMS}(\mathcal{C}_{i-1})$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_i = \mathcal{C}_{i-1} - e_i$, such that $\mathcal{C}_t = \mathcal{C} - v$.*

Proof. We prove the statement by induction on $|\mathcal{C}|$. If $\mathcal{D} = \emptyset$, then $\mathcal{C} = \mathcal{C} - v$ and the claim holds trivially. So assume $\mathcal{D} \neq \emptyset$ and $e' \in \mathcal{SMS}(\mathcal{D})$ be such that $\mathcal{D} - e'$ is chordal. By (3.3), $e = e'v \in \mathcal{SMS}(\mathcal{C})$. Let $\Delta' = \langle \mathcal{C} - e \rangle$ and $\mathcal{D}' = \text{Facets}(\text{link}_{\Delta'} v)$, then according to (3.5), $\mathcal{D}' = \mathcal{D} - e'$ is chordal and v is a shedding vertex of $\langle (\mathcal{C} - e)^\vee \rangle$. Thus by applying the induction hypothesis on $\mathcal{C} - e$, the assertion follows. \square

Example 3.8. Let's use the notations of Examples 3.4 and 3.6. We saw that $\mathcal{D} = \{12, 34\}$, which is a chordal graph with $2 \in \mathcal{SMS}(\mathcal{D})$ and $4 \in \mathcal{SMS}(\mathcal{D} - 2)$. Now we see that $26 \in \mathcal{SMS}(\mathcal{C})$ and $46 \in \mathcal{SMS}(\mathcal{C} - 26)$ and in $\mathcal{C} - 26 - 46$ the vertex 6 has not appeared in any circuit. In our notations, this means that $\mathcal{C} - 26 - 46 = \mathcal{C} - 6$, as asserted in the previous lemma.

Lemma 3.9. *Suppose that \mathcal{C} is obtained from a complete clutter by deleting exactly one circuit. Then \mathcal{C} is chordal.*

Proof. Let $\mathcal{C} = \mathcal{C}_0 - F$, where \mathcal{C}_0 is a complete clutter and $F \in \mathcal{C}_0$. By [2, Corollary 3.11], \mathcal{C}_0 is chordal and there is a sequence of simplicial maximal subcircuits e_1, \dots, e_t which sends \mathcal{C}_0 to the empty clutter. By symmetry of \mathcal{C}_0 , e_1 can be any maximal subcircuit of \mathcal{C}_0 and we can assume $e_1 \subseteq F$. Hence $\mathcal{C} - e = \mathcal{C}_0 - e$ is chordal and it is easy to see that $e \in \text{SMS}(\mathcal{C})$, that is, \mathcal{C} is also chordal. \square

Now we are ready to state the main result of this section.

Theorem 3.10. *Assume that \mathcal{C} is a d -clutter and $\langle \mathcal{C}^\vee \rangle$ is vertex decomposable. Then \mathcal{C} is chordal.*

Proof. We use induction on the number of vertices of \mathcal{C} . Let $\Delta = \mathcal{C}$ and $\Gamma = \langle \mathcal{C}^\vee \rangle$. If Γ is a simplex, then \mathcal{C} is obtained from a complete clutter by deleting one circuit and by (3.9) is chordal. Thus assume that Γ is not a simplex. So there is a shedding vertex v of Γ such that both $\Gamma - v$ and $\text{link}_\Gamma v$ are vertex decomposable. Setting $\mathcal{D} = \text{Facets}(\text{link}_\Delta v)$, it follows from (3.3) that $\langle \mathcal{D}^\vee \rangle = \Gamma - v$ and is vertex decomposable. Therefore, \mathcal{D} is chordal by induction hypothesis and by (3.7), there is a sequence e_1, \dots, e_t with $e_i \in \text{SMS}(\mathcal{C}_{i-1})$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_i = \mathcal{C}_{i-1} - e_i$, such that $\mathcal{C}_t = \mathcal{C} - v$. Consequently, we just need to show that $\mathcal{C} - v$ is chordal. For this we show that $\langle (\mathcal{C} - v)^\vee \rangle = \text{link}_\Gamma v$, which is vertex decomposable and the result follows by the induction hypothesis:

$$\begin{aligned} F \in \text{Facets}(\text{link}_\Gamma v) &\Leftrightarrow Fv \in \text{Facets}(\Gamma) = \mathcal{C}^\vee \Leftrightarrow V \setminus Fv \notin \mathcal{C} \\ &\Leftrightarrow (V \setminus \{v\}) \setminus F \notin \mathcal{C} - v \Leftrightarrow F \in (\mathcal{C} - v)^\vee. \quad \square \end{aligned}$$

Example 3.11. With notations as in Example 3.8, we saw that $\mathcal{C} - 26 - 46 = \mathcal{C} - 6$ and $26 \in \text{SMS}(\mathcal{C})$ and $46 \in \text{SMS}(\mathcal{C} - 26)$. Now in $\langle (\mathcal{C} - 6)^\vee \rangle = \langle 13, 14, 23, 24, 34 \rangle$ the vertices 1, 2, 3, 4 are shedding vertices. Applying (3.7) with $\mathcal{C} - 6$ instead of \mathcal{C} and with $v = 1$, we find for example $12 \in \text{SMS}(\mathcal{C} - 6)$ and $13 \in \text{SMS}(\mathcal{C} - 6 - 12)$ with $\mathcal{D} = (\mathcal{C} - 6) - 12 - 13 = \mathcal{C} - 6 - 1 = \{234, 345\}$. Applying (3.7) again we can find say $23 \in \text{SMS}(\mathcal{D})$ and finally $35 \in \text{SMS}(\mathcal{D} - 23)$ to get that \mathcal{C} is chordal.

In [13], a vertex of a not necessarily uniform clutter, \mathcal{D} is called a *simplicial vertex* if for every $e_1, e_2 \in \mathcal{D}$ with $v \in e_1, e_2$, there is an $e_3 \in \mathcal{D}$ such that $e_3 \subseteq (e_1 \cup e_2) \setminus \{v\}$. Also the *contraction* \mathcal{D}/v is defined as the clutter of minimal sets of $\{e \setminus \{v\} | e \in \mathcal{D}\}$. Now if every clutter obtained from \mathcal{D} by a sequence of deletions or contractions of vertices has a simplicial vertex, then Woodroffe in [13] calls \mathcal{D} chordal and we call it *W-chordal*. In [2, Corollary 3.7] it is proved that every W-chordal clutter which is uniform is chordal. As a corollary of the above theorem, we can get the following slightly stronger version of Corollary 3.7 of [2].

Corollary 3.12. *Suppose that \mathcal{D} is a (not necessarily uniform) W -chordal clutter, $d = \min \{|F| \mid F \in \mathcal{D}\}$ and $\mathcal{C} = \{F \in \mathcal{D} \mid |F| = d\}$. Then \mathcal{C} is chordal.*

Proof. Theorem 6.9 of [13] states that the Alexander dual of the independence complex of $\bar{\mathcal{C}}$ (for the definition of independence complex, see [13, p. 3]) is vertex decomposable. But independence complex of $\bar{\mathcal{C}}$ is exactly $\Delta(\mathcal{C})$ and hence the result follows from Theorem 3.10. \square

4. Chordality and ascent of clutters

In [5,6], the concept of chorded simplicial complexes is defined and it is proved that I_Δ has a $(d+1)$ -linear resolution over a field of characteristic 2, if and only if Δ is chorded and it is the clique complex of a d -clutter. We briefly recall this concept. Let Δ be a simplicial complex. We say that Δ is d -path connected, when it is pure and for each pair F and G of d -faces of Δ there is a sequence F_0, \dots, F_k of d -faces of Δ , with $F_0 = F$, $F_k = G$ and $|F_i \cap F_{i-1}| = d$. Also a pure d -dimensional simplicial complex Δ is called a d -cycle, when Δ is d -path connected and every $(d-1)$ -face of Δ is contained in an even number of d -faces of Δ . A d -cycle Δ is called *face-minimal*, if there is no d -cycle on a strict subset of the d -faces of Δ . Finally, we say that a pure d -dimensional simplicial complex Δ is d -chorded, when for every face-minimal d -cycle $\Omega \subseteq \Delta$ which is not d -complete, there exists a family $\{\Omega_1, \dots, \Omega_k\}$ of $2 \leq k$ d -cycles with each $\Omega_i \subseteq \Delta$ such that

- (a) $\Omega \subseteq \cup_{i=1}^k \Omega_i$,
- (b) each d -face of Ω is contained in an odd number of Ω_i 's,
- (c) each d -face in $\cup_{i=1}^k \Omega_i \setminus \Omega$ is contained in an even number of Ω_i 's,
- (d) $V(\Omega_i) \subsetneq V(\Omega)$.

We refer the reader to [6, Section 3], for several examples of these concepts.

Suppose that Δ is the clique complex of a d -clutter. Connon and Faridi, first in [6] prove that if I_Δ has a $(d+1)$ -linear resolution over a field of characteristic 2, then Δ is d -chorded. Then in [5, Theorem 18], the same authors show that I_Δ has a $(d+1)$ -linear resolution over a field of characteristic 2, if and only if Δ is d -chorded, and moreover $\Delta^{[m]}$ is m -chorded for all $m > d$ ($\Delta^{[m]}$ is the pure m -skeleton of Δ). Now if $I_\Delta = I(\bar{\mathcal{C}})$ for a d -clutter \mathcal{C} , then $\Delta = \Delta(\mathcal{C})$ and hence $\text{Facets}(\Delta^{[m]})$ is just the set of cliques of \mathcal{C} with dimension m . Therefore, the aforementioned results show that to study when $I(\bar{\mathcal{C}})$ has a linear resolution, it may be useful to consider the higher dimensional clutters $\text{Facets}(\Delta(\mathcal{C})^{[m]})$. Motivated by this observation, we define the ascent of a clutter as follows.

Definition 4.1. Let \mathcal{C} be a d -clutter. We call the family of all $(d+1)$ -dimensional cliques of \mathcal{C} (that is, cliques on $d+2$ vertices), the ascent of \mathcal{C} and denote it by \mathcal{C}^+ . In other words, $\mathcal{C}^+ = \text{Facets}(\Delta(\mathcal{C})^{[d+1]})$.

Example 4.2. Let \mathcal{C} be as in Example 2.4. Then $\mathcal{D} = \mathcal{MS}(\mathcal{C})$ is a 1-clutter (that is, a graph) on [8]. The circuits of \mathcal{D} are the edges in Fig. 1. Now \mathcal{D}^+ is the set of cliques of size 3 of \mathcal{D} , in particular, all of the triangles in Fig. 1 are in \mathcal{D}^+ . Thus $\mathcal{C} \subseteq \mathcal{D}^+$. Note that all of the edges 12, 13, 23, 24, 34 are in \mathcal{D} , so 123 and 234 are in \mathcal{D}^+ and $\mathcal{D}^+ \neq \mathcal{C}$. This shows that \mathcal{C} is not of the form $(\mathcal{C}')^+$ for some 1-clutter \mathcal{C}' (else \mathcal{C}' should contain $\mathcal{MS}(\mathcal{C}) = \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}'^+$, a contradiction). Now \mathcal{C}^+ means the set of all 4-subsets of [8] such as A , with the property that all 3-subsets of A are in \mathcal{C} . But there is no such 4-subset of [8] and hence $\mathcal{C}^+ = \emptyset$. Moreover, one could check that for example

$$\begin{aligned}\mathcal{D}^+ &\supsetneq \mathcal{C} \cup \{123, 23a, 12a, 13a | a = 4, 5, 7, 8\} \\ \mathcal{D}^{++} &= (\mathcal{D}^+)^+ \supsetneq \{123a, 136a | a = 4, 5, 7, 8\} \\ \mathcal{D}^{+++} &= \{123ab, 136ab | ab = 45, 78, 48\}, \quad \mathcal{D}^{++++} = \emptyset\end{aligned}$$

Our first result, considers how the property of having a linear resolution behaves under ascension. Recall that throughout the paper, we assume that \mathcal{C} is a d -clutter on vertex set V with $|V| = n$ and K is an arbitrary field.

Proposition 4.3. Let $\Delta = \Delta(\mathcal{C})$. Then $I(\overline{\mathcal{C}})$ has a linear resolution over K , if and only if $I(\overline{\mathcal{C}^+})$ has a linear resolution over K and $\tilde{H}_d(\Delta_W; K) = 0$ for all $W \subseteq V$.

Proof. Note that if $\Delta' = \Delta(\mathcal{C}^+)$, then Δ'_W and Δ_W differ only in d -dimensional faces and hence $\tilde{H}_t(\Delta'_W; K) = \tilde{H}_t(\Delta_W; K)$ for each $t > d$ and $W \subseteq V$. Thus the result follows directly from Fröberg's theorem (2.3). Note that $\Delta^{[d]} = \langle \mathcal{C} \rangle$, thus Δ is d -chorded if and only if $\langle \mathcal{C} \rangle$ is so. \square

The previous simple result shows why the concept of ascent of a clutter can be useful. For example, we show that a main theorem of [5] can simply follow (4.3) and the results of [6]. First we state the needed results of [6] as a lemma.

Lemma 4.4. Let \mathcal{C} be a d -clutter. Suppose that K is a field of characteristic 2 and $\Delta = \Delta(\mathcal{C})$. Then $\tilde{H}_d(\Delta_W; K) = 0$ for all $W \subseteq V$ if and only if $\langle \mathcal{C} \rangle$ is d -chorded.

Proof. (\Leftarrow) is [6, Proposition 5.8] and the proof for (\Rightarrow) is the proof of part 2 of [6, Theorem 6.1]. \square

We inductively define a *CF-chordal* clutter. We consider \emptyset , CF-chordal and we say that a non empty d -clutter \mathcal{C} is CF-chordal, when $\langle \mathcal{C} \rangle$ is d -chorded and \mathcal{C}^+ is CF-chordal. It is easy to check that \mathcal{C} is CF-chordal if and only if $\Delta(\mathcal{C})^{[m]}$ is m -chorded for all $m \geq d$.

In [5] simplicial complexes with this property are called *chorded*. Thus the next result is indeed Theorem 18 of [5], which is one of the two main theorems of that paper. Before stating this result, it should be mentioned that an example of a clutter \mathcal{C} with $\langle \mathcal{C} \rangle$ d -chorded but $\langle \mathcal{C}^+ \rangle$ not $(d+1)$ -chorded, is presented in [5, Example 16]. This example shows that $\langle \mathcal{C} \rangle$ can be d -chorded while \mathcal{C} is not CF-chordal.

Corollary 4.5 ([5, Theorem 18]). *Let \mathcal{C} be a d -clutter. Suppose that K is a field of characteristic 2. Then $I(\bar{\mathcal{C}})$ has a linear resolution over K , if and only if \mathcal{C} is CF-chordal.*

Proof. This follows from (4.4) and (4.3) and a simple induction on $n - d$. \square

As the following result shows, similar to having a linear resolution, chordality is also preserved under ascension.

Theorem 4.6. *Let \mathcal{C} be a d -clutter. If \mathcal{C} is chordal, then so is \mathcal{C}^+ .*

Proof. We use induction on $|\mathcal{C}|$. The result is clear for $|\mathcal{C}| = 0, 1$, where $\mathcal{C}^+ = \emptyset$. Choose $e \in \text{SMS}(\mathcal{C})$ such that $\mathcal{C} - e$ is chordal. If e is contained in only one circuit, then it is not contained in any clique on more than $d+1$ vertices. Thus $\mathcal{C}^+ = (\mathcal{C} - e)^+$ and the claim follows the induction hypothesis. So we assume that $N_{\mathcal{C}}[e]$ is a clique on at least $d+2$ vertices. Let F be any $d+1$ subset of $N_{\mathcal{C}}[e]$ with $e \subseteq F$. If $v \in N_{\mathcal{C}}[F] \setminus F$, then Fv is a clique in \mathcal{C} and hence $ev \in \mathcal{C}$. This shows that $N_{\mathcal{C}^+}[F] \subseteq N_{\mathcal{C}}[e]$. Let $\{v_1, \dots, v_t\} = N_{\mathcal{C}}[e] \setminus e$, $F_i = ev_i$ and $\mathcal{C}_i = \mathcal{C} - F_1 - \dots - F_i$. Note that $\mathcal{C}_i^+ = \mathcal{C}^+ - F_1 - \dots - F_i$. We show that $F_i \in \text{SMS}(\mathcal{C}_{i-1}^+)$, if $F_i \in \text{MS}(\mathcal{C}_{i-1}^+)$. Then as $\mathcal{C}_t = \mathcal{C} - e$ is chordal it follows by the induction hypothesis that \mathcal{C}_t^+ and hence \mathcal{C}^+ are chordal.

Suppose that $G \subseteq N_{\mathcal{C}_{i-1}^+}[F_i]$ and $|G| = d+2$. Then by the above argument $G \subseteq N_{\mathcal{C}}[e]$ and hence $G \in \mathcal{C}^+$. If $v_j \in G$ for some $j < i$, then $F_i v_j \in \mathcal{C}_{i-1}^+$. But $F_j = ev_j \subseteq F_i v_j$ and $F_j \notin \mathcal{C}_{i-1}$, a contradiction. Therefore, no $v_j \in G$ for $j < i$ and hence no $F_j \subseteq G$ for such a j . Thus G is a clique in \mathcal{C}_{i-1} , that is, $G \in \mathcal{C}_{i-1}^+$ and hence $F_i \in \text{SMS}(\mathcal{C}_{i-1}^+)$. \square

Example 4.7. Suppose that \mathcal{C} is the 2-clutter shown in Fig. 2. The circuits of \mathcal{C} are the faces of the hollow octahedron and the four triangles shown in a darker color. It is not hard to check that \mathcal{C} is chordal, for example, the sequence 12, 14, 64, 62, 13, 23, 63, 43, is a sequence of consecutive simplicial maximal subcircuits, by deletion of which, we reach to the empty clutter. Hence $\mathcal{C}^+ = \{1235, 1345, 2356, 3456\}$ is also chordal by (4.6). Indeed, all maximal subcircuits of \mathcal{C}^+ are simplicial except for the four darker ones.

Next we state a theorem that shows some connections between deleting elements of $\text{SMS}(\mathcal{C}^+)$ and having a linear resolution.

Theorem 4.8. *Let \mathcal{C} be a d -clutter. Suppose that $\mathcal{C}^+ \neq \emptyset$. Then, considering the following statements, we have (i) \Rightarrow (ii) \Rightarrow (iii). Moreover, if $\text{SMS}(\mathcal{C}^+) \neq \emptyset$, then ((i), (ii) and (iii)) are equivalent.*

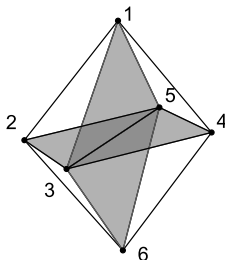


Fig. 2. A chordal 2-clutter.

- (i) There is a $F \in \mathcal{MS}(\mathcal{C}^+)$ such that $I(\overline{\mathcal{C} - F})$ has a linear resolution over K and also $I(\overline{\mathcal{C}^+})$ and $I(\overline{\mathcal{C} - v})$ have linear resolutions over K for each $v \in V$.
- (ii) $I(\overline{\mathcal{C}})$ has a linear resolution over K .
- (iii) For all $F \in \mathcal{SMS}(\mathcal{C}^+)$, $I(\overline{\mathcal{C} - F})$ has a linear resolution over K and also $I(\overline{\mathcal{C}^+})$ and $I(\overline{\mathcal{C} - v})$ have linear resolutions over K for each $v \in V$.

Proof. (i) \Rightarrow (ii): Set $\Delta = \Delta(\mathcal{C})$. According to (4.3), we have to show that $\tilde{H}_d(\Delta_W; K) = 0$ for all $W \subseteq V$. If $W \subsetneq V$, say $v \in V \setminus W$, then $\Delta_W = \Delta(\mathcal{C}_W) = \Delta((\mathcal{C} - v)_W)$. Thus by Fröberg's theorem, as $I(\overline{\mathcal{C} - v})$ has a linear resolution over K , it follows that $\tilde{H}_d(\Delta_W; K) = 0$. Consequently it remains to show that $\tilde{H}_d(\Delta; K) = 0$.

Assume that $x = \sum_{i=1}^l a_i F_i \in \tilde{C}_d(\Delta; K)$ with $\partial_d(x) = 0$ where $0 \neq a_i \in K$ and $F_i \in \mathcal{C}$. We have to show that $x \in \text{Im } \partial_{d+1}$. Let $F \in \mathcal{MS}(\mathcal{C}^+)$ be such that $I(\overline{\mathcal{C} - F})$ has a linear resolution over K . Assume that for some i , we have $F_i = F$. As $F \in \mathcal{MS}(\mathcal{C}^+)$, there is a $G \in \mathcal{C}^+$ such that $F \subseteq G$. Now $y = \partial_{d+1}(a_i G)$ has a term $\pm a_i F_i$. Hence in one of the two elements $x \mp y$, the coefficient of F is zero. Call this element z . Then $\partial_d(z) = \partial_d(x) \mp \partial_d(\partial_{d+1}(a_i G)) = 0$. Thus we can assume that for no i , $F_i = F$. Then $x \in \tilde{C}_d(\Delta(\mathcal{C} - F); K)$ and as $I(\overline{\mathcal{C} - F})$ has a linear resolution, $\tilde{H}_d(\Delta(\mathcal{C} - F); K) = 0$. So there are G_i 's in $(\mathcal{C} - F)^+ \subseteq \mathcal{C}^+$ and b_j 's in K such that $x = \partial_{d+1}(\sum b_j G_j) \in \text{Im } \partial_{d+1}$.

(ii) \Rightarrow (iii): As $\Delta(\mathcal{C} - v)_W = \Delta(\mathcal{C})_{W \setminus \{v\}}$ and by Fröberg's theorem, $I(\overline{\mathcal{C} - v})$ has a linear resolution over K for each $v \in V$. According to (4.3), $I(\overline{\mathcal{C}^+})$ also has a linear resolution over K . Now let $F \in \mathcal{SMS}(\mathcal{C}^+)$ and also assume that $V = [n]$. Let $L = \langle x_F \rangle$ and $I = I(\overline{\mathcal{C}})$. We have to show $I(\overline{\mathcal{C} - F}) = I + L$ has a linear resolution over K .

First we compute the minimal generating set of $I \cap L$. Suppose that u is one of the minimal generators of the squarefree monomial ideal $I \cap L$. Then as $x_F | u$, we should have $u = x_{F \cup A}$ for some $A \subseteq [n] \setminus F$. Since $u \in I$, there is a $G_0 \in \overline{\mathcal{C}}$ such that $G_0 \subseteq F \cup A$. Assume $|A| > 1$ and $v \in A$. Then for no $G \in \overline{\mathcal{C}}$ we have $G \subseteq Fv$, else $x_{Fv} \in I \cap L$ which contradicts u being a minimal generator. This means that Fv is a clique of \mathcal{C} and hence $v \in N_{\mathcal{C}^+}[F]$. Since $v \in A$ was arbitrary, we have $F \cup A \subseteq N_{\mathcal{C}^+}[F]$ which is a clique, because F is simplicial. But this contradicts $G_0 \subseteq F \cup A$ and it follows that $|A| = 1$, say $A = \{a\}$. As $G_0 \subseteq Fa$, we get $a \notin N_{\mathcal{C}^+}[F]$. On the other hand, for an arbitrary $a \in B = [n] \setminus N_{\mathcal{C}^+}[F]$, it is easy to see that $x_{Fa} \in I \cap L$. Consequently, $I \cap L = \langle x_{Fa} | a \in B \rangle = x_F \langle x_a | a \in B \rangle$.

Since multiplying in x_F is an S -isomorphism of degree $d + 1$ from $\langle x_a | a \in B \rangle$ to $I \cap L$ and by [8, Corollary 7.4.2], $I \cap L$ has a $(d + 2)$ -linear resolution over K . Also I and L have $(d + 1)$ -linear resolutions over K . Now consider the exact sequence

$$0 \rightarrow I \cap L \rightarrow I \oplus L \rightarrow I + L \rightarrow 0.$$

Writing the $(i + j)$ -th degree part of the long exact sequence of $\text{Tor}^S(K, -)$ applied on the above sequence, we get the exact sequence

$$\cdots \rightarrow \text{Tor}_i^S(K, I \oplus L)_{i+j} \rightarrow \text{Tor}_i^S(K, I + L)_{i+j} \rightarrow \text{Tor}_{i-1}^S(K, I \cap L)_{i-1+(j+1)} \rightarrow \cdots.$$

If $j \neq d + 1$ then both flanking terms are zero and hence the middle term is also zero. This means that $I + L$ has a $(d + 1)$ -linear resolution, as required. \square

In the following example, we show how the above theorem can be used.

Example 4.9. Suppose that \mathcal{C} is as in Example 2.4. Let $\mathcal{D} = \mathcal{C} \cup \{278\}$. Then $\mathcal{D}^+ = \{1278\}$, $278 \in \mathcal{SMS}(\mathcal{D}^+)$ and \mathcal{D}^+ is chordal. Hence $I(\overline{\mathcal{D}^+})$ has a linear resolution over every field. Also it is easy to verify that for any $W \subsetneq [8]$, \mathcal{D}_W is chordal (indeed, if $W \neq \{1, 2, 7, 8\}$, then either \mathcal{D}_W is empty or it has an edge which is contained in exactly one triangle and hence is simplicial and if $W = \{1, 2, 7, 8\}$, \mathcal{D}_W is complete) and thus has a linear resolution over every field. Thus, as $I(\overline{\mathcal{D} - 278}) = I(\overline{\mathcal{C}})$ has a linear resolution over any field (see Example 2.4) and according to (4.8), we get that $I(\overline{\mathcal{D}})$ has a linear resolution over every field.

Note that in (4.8)(iii), against part (i), we cannot replace $F \in \mathcal{SMS}(\mathcal{C}^+)$ with $F \in \mathcal{MS}(\mathcal{C}^+)$. For example, if \mathcal{C} is as in Example 4.7, then $\langle \mathcal{C} - 135 \rangle$ is not 2-chorded, and hence $I(\overline{\mathcal{C} - 135})$ has not a linear resolution over \mathbb{Z}_2 , although \mathcal{C} is chordal and $I(\overline{\mathcal{C}})$ has a linear resolution over every field.

Also note that (4.8)(iii) does not necessarily imply (4.8)(i), since we may have a clutter \mathcal{C} satisfying (4.8)(iii), but with $\mathcal{SMS}(\mathcal{C}^+) = \emptyset$. Although, it should be mentioned that the author could not find such an example. Thus we raise the following question.

Question 4.10. Are the three statements of (4.8) equivalent for a d -clutter \mathcal{C} with a non-empty ascent?

Remark 4.11. In the proof of (i) \Rightarrow (ii) of (4.8), we proved that if there exists $F \in \mathcal{MS}(\mathcal{C}^+)$ such that $I(\overline{\mathcal{C} - F})$ has a linear resolution over K , then $\tilde{H}_d(\Delta(\mathcal{C}); K) = 0$.

The following is also a corollary to the proof of (4.8). Recall that $\beta_i(I) = \sum_j \beta_{ij}(I)$.

Corollary 4.12. Let \mathcal{C} be a d -clutter. Suppose that $I = I(\overline{\mathcal{C}})$ has a linear resolution over K , $F \in \mathcal{SMS}(\mathcal{C}^+)$ and $J = I(\overline{\mathcal{C} - F})$. Then J has a linear resolution over K with Betti numbers $\beta_i(J) = \beta_i(I) + \binom{t}{i}$, where $t = n - |\mathcal{N}_{\mathcal{C}^+}[F]|$.

Proof. Suppose that $L = \langle x_F \rangle$ and $B = [n] \setminus N_{\mathcal{C}^+}[F]$ as in the proof of (ii) \Rightarrow (iii). So $t = |B|$ and if $L' = \langle x_a | a \in B \rangle$, then $I \cap L = x_F L'$ and $\beta_i(I \cap L) = \beta_i(L')$ which by [8, Corollary 7.4.2] is equal to $\sum_{k=1}^t \binom{k-1}{i} = \sum_{k=i}^{t-1} \binom{k}{i} = \binom{t}{i}$. Also $\beta_0(L) = 1$ and $\beta_i(L) = 0$ for all $i > 0$. Thus by taking \dim_K of the long exact sequence of $\text{Tor}_i^S(K, -)$ in the proof of (4.8), we get $\beta_i(J) = \beta_i(I + L) = \beta_i(I) + \beta_i(L) + \beta_{i-1}(I \cap L)$, where the last term is assumed to be zero if $i = 0$. \square

If in (4.8) we restrict to the case that $\text{char} K = 2$ and by using (4.5), we get the following.

Corollary 4.13. *Let \mathcal{C} be a d -clutter. Suppose that $\mathcal{C}^+ \neq \emptyset$.*

- (i) *If \mathcal{C}^+ is CF-chordal and there is a $F \in \mathcal{MS}(\mathcal{C}^+)$ such that $\mathcal{C} - F$ is CF-chordal and $\mathcal{C} - v$ is CF-chordal for all $v \in V(\mathcal{C})$, then \mathcal{C} is CF-chordal.*
- (ii) *If \mathcal{C} is CF-chordal, then for all $F \in \mathcal{SMS}(\mathcal{C}^+)$ and all $v \in V(\mathcal{C})$, $\mathcal{C} - F$ and $\mathcal{C} - v$ are CF-chordal.*

Moreover, if $\mathcal{SMS}(\mathcal{C}^+) \neq \emptyset$, then the converses of both (i) and (ii) hold.

In the rest of the paper, we study if we can get some results similar to (4.8) for having linear quotients or being chordal instead of having a linear resolution. For having linear quotients we have:

Theorem 4.14. *Let \mathcal{C} be a d -clutter. Suppose that $I(\overline{\mathcal{C}})$ has linear quotients. Then $I(\overline{\mathcal{C}^+})$, $I(\overline{\mathcal{C} - F})$ and $I(\overline{\mathcal{C} - v})$ have linear quotients for each $F \in \mathcal{SMS}(\mathcal{C}^+)$ and $v \in V(\mathcal{C})$.*

Proof. If we denote the ideal generated by all squarefree monomials in I with degree t by $I_{[t]}$, then it is easy to see that $I(\mathcal{C}^+) = I(\mathcal{C})_{[d+2]}$. Hence the fact that $I(\mathcal{C}^+)$ has linear quotients follows from [12, Corollary 2.11]. But for the convenience of the readers, we present a direct shorter proof for this. Assume that x_{F_1}, \dots, x_{F_t} is an admissible order of $I(\overline{\mathcal{C}})$. Then $\overline{\mathcal{C}^+} = \{F_i v | v \in V \setminus F_i, 1 \leq i \leq t\}$. For each $1 \leq i \leq t$, let $\mathcal{C}_i = \{F_i v | v \in V \setminus F_i\} \setminus (\cup_{j < i} \mathcal{C}_j)$ and suppose that $\mathcal{C}_i = \{F_{i1}, \dots, F_{ir_i}\}$. We prove that $F_{11}, \dots, F_{1r_1}, \dots, F_{t1}, \dots, F_{tr_t}$ corresponds to an admissible order for $I(\overline{\mathcal{C}^+})$.

Consider two circuits $F_{ij}, F_{i'j'}$ of $\overline{\mathcal{C}^+}$ with $i \leq i'$. If $i = i'$, then $|F_{ij} \setminus F_{i'j'}| = 1$ and hence the admissibility condition holds trivially for them. Thus we assume $i < i'$. Therefore by assumption there is a $l \in F_i \setminus F_{i'}$ and a $k < i'$ such that $F_k \setminus F_{i'} = \{l\}$. Let $F_{i'j'} \setminus F_{i'} = \{v\}$. Note that as $F_{i'j'} \in \mathcal{C}_{i'}$, there is no $j < i'$ with $F_{i'j'} \in \mathcal{C}_j$ by the definition of \mathcal{C}_i 's. So $v \neq l$, else $F_{i'j'} = F_k \cup (F_{i'} \setminus F_k)$ and as $|F_{i'} \setminus F_k| = 1$, we have $F_{i'j'} \in \mathcal{C}_j$ for some $j \leq k$, a contradiction. Consequently, $v \notin F_k$ and $F_k v$ which is a circuit of \mathcal{C}^+ , should appear in some \mathcal{C}_j with $j \leq k$. Noting that $F_k v \setminus F_{i'j'} = \{l\}$, the proof is concluded.

Now assume that $F \in \mathcal{SMS}(\mathcal{C}^+)$. We show that if we add x_F to an admissible order of $I(\overline{\mathcal{C}})$, we get an admissible order of $I(\overline{\mathcal{C} - F}) = I(\overline{\mathcal{C}}) + \langle x_F \rangle$. We just need to show if

$G \in \overline{\mathcal{C}}$, then there is a $G' \in \overline{\mathcal{C}}$ and an $l \in G$ such that $G \setminus F = \{l\}$. Since $N_{\mathcal{C}^+}[F]$ is a clique of \mathcal{C} , $G \not\subseteq N_{\mathcal{C}^+}[F]$, say $l \in G \setminus N_{\mathcal{C}^+}[F]$. Then $Fl \notin \mathcal{C}^+$ and hence there is a $G' \subseteq Fl$ with $G' \in \overline{\mathcal{C}}$. So $G' \setminus F = \{l\}$, as required.

Finally, note that if x_{F_1}, \dots, x_{F_t} is an admissible order for $I(\overline{\mathcal{C}})$, then by deleting x_{F_i} 's with $v \in F_i$, we get an admissible order for $I(\overline{\mathcal{C} - v})$ and $I(\overline{\mathcal{C} - v})$ has linear quotients. \square

Example 4.15. Let \mathcal{C} be the clutter of Example 4.7. It is not hard to check that the following is an admissible order for $I(\overline{\mathcal{C}})$: 162, 163, 164, 165, 124, 624, 245, 234. Thus by the previous result, $I(\overline{\mathcal{C}^+})$ has linear quotients. Indeed, by the proof of (4.14), we get the following admissible order for $I(\overline{\mathcal{C}^+})$: 1623, 1624, 1625, 1634, 1635, 1645, 1243, 1245, 6243, 6245, 2453. Also note that the assumption that F is simplicial, is crucial in (4.14). For example, $I(\overline{\mathcal{C} - 135})$ has not a linear resolution and hence has not linear quotients.

We do not know whether the converse of the above theorem is correct or if a statement similar to (i) \Rightarrow (ii) of (4.8) holds for having linear quotients.

Next consider the “chordal version” of (4.8), that is, consider the statements obtained by replacing “ $I(\overline{\mathcal{D}})$ has a linear resolution over K ” with “ \mathcal{D} is chordal” in the three parts of (4.8), where the replacement occurs for all clutters \mathcal{D} appearing in these assertions. Then clearly (iii) \Rightarrow (i) holds for the chordal version. Also (4.6) shows that part of (ii) \Rightarrow (iii) is true for chordality. We will also show that if \mathcal{C} is chordal and $F \in \mathcal{SMS}(\mathcal{C}^+)$, then $\langle \mathcal{C} - F \rangle$ is d -chorded, which is weaker than being chordal. But before proving this, we utilize (4.14) to show that if the chordal version of (iii) \Rightarrow (ii) (or (i) \Rightarrow (ii)) holds, then we can reduce proving statement (A) to verifying it only for clutters with empty ascent.

In what follows, by a *free maximal subcircuit* of \mathcal{C} , we mean a maximal subcircuit which is contained in exactly one circuit of \mathcal{C} . In the particular case that $\dim \mathcal{C} = 1$, that is \mathcal{C} is a graph, free maximal subcircuits are exactly leaves (or free vertices) of the graph. Also a simplicial complex Δ is called *extendably shellable*, if any shelling of a subcomplex of Δ could be continued to a shelling of Δ . To see a brief literature review of this concept and some related results consult [3,4].

Corollary 4.16. Consider the following statements and also statement (A) of Section 2 on a uniform clutter \mathcal{C} .

- (B) If $\emptyset \neq \mathcal{C}^+$ is chordal and $\mathcal{C} - F$ and $\mathcal{C} - v$ are chordal for each $F \in \mathcal{SMS}(\mathcal{C}^+)$ and $v \in V(\mathcal{C})$, then $\mathcal{SMS}(\mathcal{C}) \neq \emptyset$.
- (C) If $\mathcal{C}^+ = \emptyset$ and $I(\overline{\mathcal{C}})$ has linear quotients, then \mathcal{C} has a free maximal subcircuit.
- (D) Simon’s Conjecture ([11, Conjecture 4.2.1]): Every d -skeleton of a simplex is extendably shellable.

Then: (B) + (C) \Rightarrow (A) \Rightarrow (C) + (D).

Proof. Assume that (B) and (C) hold. We claim that if $I(\overline{\mathcal{C}})$ has linear quotients, then $\mathcal{SMS}(\mathcal{C}) \neq \emptyset$. Then it follows from [9, Theorem 2.1] that (A) is correct. To prove the claim, we use induction on $(n - d, |\mathcal{C}|)$ considered with lexicographical order. If $\mathcal{C}^+ = \emptyset$, then as every free maximal subcircuit is simplicial, the claim holds by (C). If $\mathcal{C}^+ \neq \emptyset$, then applying (4.14) and using the induction hypothesis, we see that \mathcal{C}^+ , $\mathcal{C} - F$ and $\mathcal{C} - v$ are chordal for every $F \in \mathcal{SMS}(\mathcal{C}^+)$ and $v \in V(\mathcal{C})$. Consequently, the claim follows from (B).

Now suppose that statement (A) is correct. If $\mathcal{C}^+ = \emptyset$, then $|N_{\mathcal{C}}[e]| = d + 1$ for every $e \in \mathcal{SMS}(\mathcal{C})$ and hence every simplicial maximal subcircuit of \mathcal{C} is a free maximal subcircuit. Therefore, (C) holds as a special case of (A). Finally (A) \Rightarrow (D) follows from Theorem 2.5 and Corollary 3.7 of [3]. \square

The “chordal version” of (ii) \Rightarrow (iii) in (4.8), partly states that if \mathcal{C} is chordal, then for every $F \in \mathcal{SMS}(\mathcal{C}^+)$, $\mathcal{C} - F$ should be chordal. The author could neither prove nor reject this in the general case, but we prove a weaker result. In particular, we will show in the case that $\dim \mathcal{C} = 1$, this is true. We need the following lemmas which present an equivalent condition for being d -chorded.

Lemma 4.17. *The set of facets of a d -cycle is a disjoint union of the set of facets of a family of face-minimal d -cycles.*

Proof. Suppose that Ω is a d -cycle. We use induction on $|\text{Facets}(\Omega)|$. If Ω is face-minimal, then the result is trivial. Thus we can assume that there is a d -cycle Ω' whose facets form a strict subset of $\text{Facets}(\Omega)$. Let $\Gamma = \langle \text{Facets}(\Omega) \setminus \text{Facets}(\Omega') \rangle$. Note that each $(d - 1)$ -face of Γ is contained in an even number of facets of Ω and an even number of facets of Ω' . Hence each $(d - 1)$ -face of Γ is contained in an even number of facets of Γ . Suppose that $\Gamma_1, \dots, \Gamma_t$ are d -path connected components of Γ , that is, the maximal pure d -dimensional subcomplexes of Γ which are d -path connected. Note that if $i \neq j$, then Γ_i and Γ_j do not share a $(d - 1)$ -face, else $\Gamma_i \cup \Gamma_j$ is d -path connected, contradicting maximality of both Γ_i and Γ_j . Consequently, for every i , each $(d - 1)$ -face of Γ_i is contained in an even number of d -faces of Γ_i and Γ_i is a d -cycle. Noting that $\text{Facets}(\Gamma_i)$ are mutually disjoint, and by applying induction hypothesis on Γ_i 's, it follows that $\text{Facets}(\Gamma)$ is a disjoint union of facets of a family \mathfrak{A} of face-minimal d -cycles. So $\text{Facets}(\Omega)$ is the disjoint union of facets of the family $\mathfrak{A} \cup \{\Omega'\}$ of face-minimal d -cycles, as required. \square

Lemma 4.18. *Let \mathcal{C} be a d -clutter. The simplicial complex $\langle \mathcal{C} \rangle$ is d -chorded if and only if the facets of every d -cycle Ω of $\langle \mathcal{C} \rangle$ is the symmetric difference of a family of complete subclutters of \mathcal{C} , each on a $(d + 2)$ -subset of $V(\Omega)$.*

Proof. (\Rightarrow): By induction on $|V(\Omega)|$. Because of (4.17), we can assume that Ω is face-minimal. If Ω is d -complete, then by face-minimality $|V(\Omega)| = d + 2$ and we are done. Thus we assume that Ω is not d -complete. Suppose that Ω_i 's are d -cycles in $\langle \mathcal{C} \rangle$ satisfying (a)–(d) in the definition of a d -chorded complex. Let $\mathcal{D}_i = \text{Facets}(\Omega_i)$. Since each

d -face in $\cup_{i=1}^k \mathcal{D}_i \setminus \text{Facets}(\Omega)$ appears in an even number of \mathcal{D}_i 's by (c), such d -faces are not in the symmetric difference of \mathcal{D}_i 's. Also by (a) and (b), we see that each element of $\text{Facets}(\Omega_i)$ is in an odd number of \mathcal{D}_i 's and hence is in their symmetric difference. Therefore, $\text{Facets}(\Omega) = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_k$, where \oplus denotes symmetric difference. Also by (d) of the definition of a d -chorded complex, each $\langle \mathcal{D}_i \rangle$ is a cycle on an smaller number of vertices in $V(\Omega)$. Thus by applying the induction hypothesis on $\langle \mathcal{D}_i \rangle$'s we get a decomposition of Ω as the symmetric difference of a set of complete subclutters of \mathcal{C} on $(d+2)$ -subsets of $V(\Omega)$.

(\Leftarrow): Suppose that Ω is a face-minimal cycle of $\langle \mathcal{C} \rangle$ which is not d -complete. Then $\text{Facets}(\Omega) = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_t$ where \mathcal{D}_i 's are complete subclutters of \mathcal{C} with size $d+2$ and $V(\mathcal{D}_i) \subseteq V(\Omega)$. If we set $\Omega_i = \langle \mathcal{D}_i \rangle$, then Ω_i 's clearly satisfy (a)–(d) of the definition of a d -chorded clutter. This means that $\langle \mathcal{C} \rangle$ is d -chorded. \square

Theorem 4.19. *Let \mathcal{C} be a d -clutter. Suppose that $\langle \mathcal{C} \rangle$ is d -chorded (for example, if \mathcal{C} is chordal) and $F \in \mathcal{SMS}(\mathcal{C}^+)$. Then $\langle \mathcal{C} - F \rangle$ is d -chorded.*

Proof. Suppose that Ω is a d -cycle of $\mathcal{C} - F$ and $\mathcal{D} = \text{Facets}(\Omega)$. Since $\langle \mathcal{C} \rangle$ is d -chorded and according to (4.18), there are complete subclutters $\mathcal{D}_1, \dots, \mathcal{D}_t$ of \mathcal{C} on $(d+2)$ -subsets of $V(\mathcal{D})$ such that $\mathcal{D} = \oplus_{i=1}^t \mathcal{D}_i$. We choose \mathcal{D}_i 's in a way that $m = |\{i | F \in \mathcal{D}_i\}|$ is minimum possible. Assume that $m \geq 1$. Because $F \notin \mathcal{D}$, m is at least 2. Thus there are two \mathcal{D}_i , say $\mathcal{D}_1, \mathcal{D}_2$, containing F . Therefore there exist $v_1 \neq v_2 \in V(\mathcal{D})$ such that $V(\mathcal{D}_i) = Fv_i$ for $i = 1, 2$. Let $W = Fv_1v_2$ and $\mathcal{C}' = \mathcal{C}_W$. Then since $F \in \mathcal{SMS}(\mathcal{C}^+)$ and $W \subseteq N_{\mathcal{C}^+}(F)$, \mathcal{C}' is complete. Now if $\mathcal{D}' = \mathcal{D}_1 \oplus \mathcal{D}_2$, then $\langle \mathcal{D}' \rangle$ is a d -cycle in $\langle \mathcal{C}' - F \rangle$. By (3.9), $\mathcal{C}' - F$ is chordal and hence has a linear resolution over every field. Thus by [6, Theorem 6.1] $\langle \mathcal{C}' - F \rangle$ is d -chorded. Consequently according to (4.18), $\mathcal{D}' = \mathcal{D}'_1 \oplus \cdots \oplus \mathcal{D}'_{t'}$, where \mathcal{D}'_i 's are complete subclutters of $\mathcal{C}' - F$ (and hence \mathcal{C}) on $(d+2)$ -subsets of $V(\mathcal{D}') \subseteq V(\mathcal{D})$. By replacing $\oplus_{i=1}^{t'} \mathcal{D}'_i$ instead of $\mathcal{D}_1 \oplus \mathcal{D}_2$ in the decomposition of \mathcal{D} , we get a decomposition with less terms containing F . This contradicts the choice of the decomposition of \mathcal{D} and hence $m = 0$. Therefore, the decomposition $\mathcal{D} = \oplus_{i=1}^t \mathcal{D}_i$ is in $\mathcal{C} - F$ and the result follows by (4.18). \square

It should be mentioned that the previous result is not correct for arbitrary $F \in \mathcal{MS}(\mathcal{C}^+)$. This can be seen by noting that $I(\mathcal{C} - 135)$ is not 2-chorded, where \mathcal{C} is as in Example 4.7.

If \mathcal{C} is a graph, we call a $F \in \mathcal{SMS}(\mathcal{C}^+)$ a *simplicial edge* of \mathcal{C} .

Corollary 4.20. *Suppose that \mathcal{C} is a chordal graph and F is a simplicial edge of \mathcal{C} . Then $\mathcal{C} - F$ is chordal. Hence there is an ordering $F_1, \dots, F_t, F_{t+1}, \dots, F_m$ of edges of \mathcal{C} such that for $1 \leq i \leq t$, F_i is a simplicial edge of the chordal graph $\mathcal{C}_i = \mathcal{C} - F_1 - \cdots - F_{i-1}$ and for $t < i \leq m$, \mathcal{C}_i is a tree and F_i is a leaf edge of \mathcal{C}_i .*

Proof. The first statement is just (4.19) in the case that $\dim \mathcal{C} = 1$. Now if \mathcal{C} is chordal and $\mathcal{C}^+ \neq \emptyset$, then $\mathcal{SMS}(\mathcal{C}^+) \neq \emptyset$ by (4.6). So starting with \mathcal{C} , we can delete simplicial

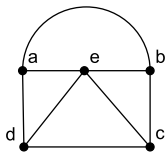


Fig. 3. A non-chordal graph.

edges until we reach a chordal graph \mathcal{C}' with $(\mathcal{C}')^+ = \emptyset$. But a chordal graph without any cliques on more than two vertices is a tree and hence we can delete leaf edges from \mathcal{C}' until there is no more edges and the statement is established. \square

Let \mathcal{C} be the graph in Fig. 3. Then \mathcal{C}^+ and $\mathcal{C} - F$ are chordal and also $I(\overline{\mathcal{C}^+})$ and $I(\overline{\mathcal{C} - F})$ have linear quotients for each $F \in \mathcal{SMS}(\mathcal{C}^+)$. Despite this \mathcal{C} is not chordal, since $\mathcal{C} - e$ is a cycle of length 5. Thus the converses of (4.19) and (4.6) do not hold. Also edges of \mathcal{C} can be ordered as in (4.20), so the converse of (4.20) is not true, either.

Clearly Statement (C) holds when $\dim \mathcal{C} = 1$. Using the concept of d -cycles, we show that (C) holds when $\dim \mathcal{C}^\vee \leq 1$ or equivalently, if $n - d \leq 3$. As in [9], by a CF -tree we mean a d -clutter \mathcal{C} with the property that $\langle \mathcal{C} \rangle$ has no d -cycles.

Proposition 4.21. *Assume that \mathcal{C} is a d -clutter on n vertices with $n \leq d+3$. If $\mathcal{C}^+ = \emptyset$ and $I(\overline{\mathcal{C}})$ has a linear resolution over every field, then \mathcal{C} is chordal. In particular, statement (C) of (4.16) holds for \mathcal{C} .*

Proof. Suppose that $\mathcal{C}^+ = \emptyset$ and $I(\overline{\mathcal{C}})$ has a linear resolution over every field. According to [6, Theorem 6.1], $\langle \mathcal{C} \rangle$ is d -chorded. But since \mathcal{C} has no cliques of size $d+2$, it follows from (4.18) that $\langle \mathcal{C} \rangle$ has no d -cycles, that is, \mathcal{C} is a CF -tree. Now the result follows from [9, Corollary 3.7]. \square

The proof of Corollary 3.7 of [9] uses Alexander dual. We end this paper mentioning that more generally, one can get a statement equivalent to (C) by passing to the Alexander dual of $\langle \mathcal{C} \rangle$. Indeed, by arguments quite similar to [9, Theorem 3.6] one can see that (C) holds for all d -clutters on n vertices, if and only if statement (ii) of [9, Theorem 3.6] holds, when we replace “Cohen-Macaulay over \mathbb{Z}_2 ” in that statement with “shellable”.

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