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Bases for diagonally alternating harmonic polynomials of low degree

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ABSTRACT

Given a list of n cells $L = [(p_1, q_1), \dots, (p_n, q_n)]$ where $p_i, q_i \in \mathbf{Z}_{\geqslant 0}$, we let $\Delta_L = \det \| (p_j!)^{-1}(q_j!)^{-1} x_i^{p_j} y_i^{q_j} \|$. The space of diagonally alternating polynomials is spanned by $\{\Delta_L\}$ where L varies among all lists with n cells. For $a > 0$, the operators $E_a = \sum_{i=1}^n y_i \partial_{x_i}^a$ act on diagonally alternating polynomials. Haiman has shown that the space A_n of diagonally alternating harmonic polynomials is spanned by $\{E_\lambda \Delta_n\}$ where $\lambda = (\lambda_1, \dots, \lambda_\ell)$ varies among all partitions, $E_\lambda = E_{\lambda_1} \cdots E_{\lambda_\ell}$ and $\Delta_n = \det \| ((n-j)!)^{-1} x_i^{n-j} \|$. For $t = (t_m, \dots, t_1) \in \mathbf{Z}_{>0}^m$ with $t_m > \cdots > t_1 > 0$, we consider here the operator $F_t = \det \| E_{t_{m-j+1}+(j-i)} \|$. Our first result is to show that $F_t \Delta_L$ is a linear combination of $\Delta_{L'}$ where L' is obtained by *moving* $\ell(t) = m$ distinct cells of L in some determined fashion. This allows us to control the leading term of some elements of the form $F_{t_{(1)}} \cdots F_{t_{(r)}} \Delta_n$. We use this to describe explicit bases of some of the bihomogeneous components of $A_n = \bigoplus A_n^{k,l}$ where $A_n^{k,l} = \mathrm{Span}\{E_\lambda \Delta_n \colon \ \ell(\lambda) = l, \ |\lambda| = k\}$. More precisely, we give an explicit basis of $A_n^{k,l}$ whenever $k < n$. To this end, we introduce a new variation of Schensted insertion on a special class of tableaux. This produces a bijection between partitions and this new class of tableaux. The combinatorics of these tableaux T allow us to know exactly the leading term of $F_T \Delta_n$ where F_T is the operator corresponding to the columns of T, whenever n is greater than the weight of T.

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1. Introduction

The theory of Macdonald symmetric polynomials [4] is a very active and deep area of mathematics. At the heart of this theory lies the study of (q,t)-Catalan numbers and the space of diagonal harmonics introduced by Garsia, Haiman and collaborators (see [3] and references therein). The space over $\mathbb{Q}$ of diagonal harmonics in $2n$ variables is given by

$$H_n = \left\{ P \in \mathbb{Q}[X_n, Y_n]\colon \sum_{i=1}^{n} \partial_{x_i}^k \partial_{y_i}^h P = 0,\ h+k>0 \right\},$$

where $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$. One of the many fascinating properties of this space is that its dimension [2] is $(n+1)^{n-1}$.

The symmetric group $\mathcal{S}_n$ acts diagonally on $\mathbb{Q}[X_n, Y_n]$. That is, for $P \in \mathbb{Q}[X_n, Y_n]$, the action $\sigma \in \mathcal{S}_n$ is defined by $\sigma P = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$. Since the defining equations of H_n are all symmetric, it is clear that H_n is an $\mathcal{S}_n$-module. We can then consider the subspace of H_n consisting of alternating polynomials. That is

$$A_n = \left\{ P \in H_n\colon \sigma P = (-1)^{\ell(\sigma)} P,\ \forall \sigma \in \mathcal{S}_n \right\},$$

where $\ell(\sigma)$ denotes the length of σ. The space of polynomials $\mathbb{Q}[X_n, Y_n]$ is bigraded in $\mathbf{Z}^2_{\geqslant 0}$ using the total degree in the variables X_n and the total degree in the variables Y_n. Since the diagonal action of $\mathcal{S}_n$ on $\mathbb{Q}[X_n, Y_n]$ preserves both degrees in X_n and Y_n, we have that H_n is a bigraded $\mathcal{S}_n$-module and $A_n = \bigoplus_{k,l} A_n^{k,l}$ is an $\mathcal{S}_n$-submodule of H_n. Here $A_n^{k,l}$ consists of the bihomogeneous polynomials in A_n of total degree $\frac{n(n-1)}{2} - k$ in the variables X_n and total degree l in the variables Y_n. We have shifted the degree in X_n to simplify the formulation of our theorems. The polynomial

$$C_n(q,t) = q^{\frac{n(n-1)}{2}} \sum_{k,l} \dim\bigl(A_n^{k,l}\bigr) q^{-k} t^l,$$

is known as the (q,t)-Catalan polynomial [2,3]. In particular, the dimension of A_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Given a list of n cells, $L = [(p_1, q_1), \ldots, (p_n, q_n)]$ where $p_i, q_i \in \mathbf{Z}_{\geqslant 0}$, we let

$$\Delta_L = \det \left\| \frac{1}{p_j! q_j!} x_i^{p_j} y_i^{q_j} \right\|.$$

The space of all diagonally alternating polynomials in $\mathbb{Q}[X_n, Y_n]$ has a basis given by $\{\Delta_L\}$, where $L = L(D)$ varies among all sets $D \subset \mathbf{Z}^2_{\geqslant 0}$ of cardinality n and $L(D)$ is the elements of D given in a sorted list. For $a > 0$, the operators

$$E_a = \sum_{i=1}^{n} y_i \partial_{x_i}^a$$

act on diagonally alternating polynomials. Haiman [2] has shown that the space A_n is spanned by $\{E_\lambda \Delta_n\}$, where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ varies among all partitions, $E_\lambda = E_{\lambda_1} \cdots E_{\lambda_\ell}$ and $\Delta_n = \Delta_{[(0,0),(1,0),\ldots,(n-1,0)]}$. We have that

$$A_n^{k,l} = \operatorname{Span}\bigl\{ E_\lambda \Delta_n\colon \ell(\lambda) = l,\ |\lambda| = k \bigr\},$$

where $\ell(\lambda) = l$ denotes the number of parts of λ and $|\lambda| = \lambda_1 + \cdots + \lambda_l$.

For $t = (t_m, \ldots, t_1) \in \mathbf{Z}^m_{>0}$ with $t_m > \cdots > t_1 > 0$, we consider the operator $F_t = \det \| E_{t_{m-j+1}+(j-i)} \|$. Our first result (Theorem 4.6) is to show that $F_t \Delta_L$ is a linear combination of $\Delta_{L'}$, where L' is obtained by *moving* $\ell(t) = m$ distinct cells of L in some determined fashion. Given a column-strict Young tableau T we can associate to each column of T an F-operator as above and define F_T to be the operator obtained as the product of the F-operators corresponding to the columns of T. For $k < n$, we show (Corollary 8.3) that a basis of $A_n^{k,l}$ is given by $\{F_T\Delta_n\}$, where T runs over certain column-strict Young tableaux.

To this end, we introduce a new variation of Schensted insertion on a special class of tableaux (Section 6 and Section 7). This produces a bijection $\lambda \leftrightarrow T(\lambda)$ between partitions and this new class of tableaux (Section 8). The combinatorics of the tableaux $T(\lambda)$ allow us to know exactly the leading term of $F_{T(\lambda)}\Delta_n$ whenever n is larger than the weight of T. We believe that the insertion algorithm presented here may be of interest on its own: Corollary 8.3 is just one application of our construction. We point out that it is possible to get Corollary 8.3 more directly but this is less revealing for us.

We present two short sections to recall some facts about (q,t)-Catalan numbers (Section 2) and an ordering of the diagonally alternating polynomials (Section 3).

2. (q,t)-Catalan

In this section we recall some of the basic definitions related to (q,t)-Catalan numbers.

Definition 2.1. A *Dyck path* of length n is a lattice path from the point $(0,0)$ to the point (n,n) consisting of n north steps $(0,1)$ and n east steps $(1,0)$, that never cross the line $y=x$. The ith row of a Dyck path lies between the line $y=i-1$ and $y=i$.

We denote by DP_n, the set of all the Dyck paths of length n. Dyck paths of length n are in bijection with sequences $g=(g_0,\dots,g_{n-1})$ of n nonnegative integers satisfying the two conditions

$$\begin{cases} g_0 = 0, \\ 0 \leqslant g_{i+1} \leqslant g_i + 1, \quad \forall i < n-1. \end{cases} \tag{2.1}$$

The ith entry g_{i-1} of the sequence g corresponds to the number of complete lattice squares between the north step of the ith row of the Dyck path and the diagonal $y=x$. Such sequences are called Dyck sequences.

Definition 2.2. Given a Dyck path $c \in DP_n$, let $(g_0,\dots,g_{n-1})$ be its corresponding Dyck sequence. The *area* and *coarea* of the Dyck path are given by

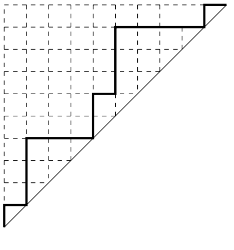
$$a(c) = \sum_{i=0}^{n-1} g_i \quad \text{and} \quad ca(c) = \sum_{i=0}^{n-1} i - g_i = \frac{n(n-1)}{2} - a(c)$$

respectively. The *bounce* statistic of the Dyck path is defined recursively as follows:

$$b(c) = b(g_0,\dots,g_{n-1}) = n-1-g_{n-1} + b(g_0,\dots,g_{n-2-g_{n-1}}),$$

where for the empty sequence $\epsilon = (\,)$, we let $b(\epsilon)=0$.

Example 2.3. The Dyck sequence $g=(0,0,1,2,0,1,1,2,3,0)$ corresponds to the following Dyck path c



The area and coarea of this Dyck path are $a(c)=1+2+1+1+2+3=10$ and $ca(c)=45-10=35$. The bounce statistic of c is given by

$$\begin{aligned} b(c) &= 9 + b(0,0,1,2,0,1,1,2,3) = 9 + 5 + b(0,0,1,2,0) = 9 + 5 + 4 + b(0,0,1,2) \\ &= 9 + 5 + 4 + 1 + b(0) = 9 + 5 + 4 + 1 + 0 = 19. \end{aligned}$$

Remark 2.4. A *partition* μ of $m \in \mathbf{Z}_{>0}$, denoted by $\mu \vdash m$, is a sequence $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ of positive integers in non-increasing order: $\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_l$ and $|\mu| := \sum_{i=1}^{l} \mu_i = m$. The coarea of a Dyck path corresponds to the size of the partition $\lambda^t = \mu = (\mu_1, \dots, \mu_{n-1})$ defined by $\mu_i = n - i - g_{n-i}$. Here λ^t denotes the transpose of the partition λ. In the Example 2.3, $\mu = (9, 5, 5, 5, 4, 4, 1, 1, 1)$ and $\lambda = \mu^t = (9, 6, 6, 6, 4, 1, 1, 1, 1)$ are partitions of size 35. The transpose here is the reflection of μ across the anti-diagonal.

Let

$$\widetilde{C}_n(q,t) = q^{\frac{n(n-1)}{2}} C_n\big(q^{-1}, t\big) = \sum_{k,l} \dim\big(A_n^{k,l}\big) q^k t^l. \tag{2.2}$$

A result by Garsia and Haglund [1,3] gives that

$$C_n(q,t) = \sum_{c \in DP_n} q^{a(c)} t^{b(c)}.$$

In particular,

$$\widetilde{C}_n(q,t) = \sum_{c \in DP_n} q^{ca(c)} t^{b(c)}. \tag{2.3}$$

3. Sorting and ordering of diagonally alternating polynomials

In this section, we give a basis of diagonally alternating polynomials. Using an order on this basis we define a notion of leading term for any diagonally alternating polynomial.

Given a set of n distinct cells $D = \{(p_1, q_1), \dots, (p_n, q_n)\} \subset \mathbf{Z}^2$ we say that the cells are *sorted* if for all $i < j$ we have that $q_i < q_j$ or ($q_i = q_j$ and $p_i < p_j$). We let $L(D) = [(p_1, q_1), \dots, (p_n, q_n)] \in (\mathbf{Z}^2)^n$ denote the sorted list. On the other hand, if we are given a list of n cells $L = [(p_1, q_1), \dots, (p_n, q_n)] \in (\mathbf{Z}^2)^n$ and if all $p_i, q_i \geqslant 0$, we let

$$\Delta_L = \det \left\| \frac{1}{p_j! q_j!} x_i^{p_j} y_i^{q_j} \right\|.$$

Otherwise we let $\Delta_L = 0$. Notice that if the cells of L are not distinct, then we also get $\Delta_L = 0$. We call a list of n cells $L = [(p_1, q_1), \dots, (p_n, q_n)] \in (\mathbf{Z}^2)^n$ a *lattice diagram*.

For a lattice diagram $L = [(p_1, q_1), \dots, (p_n, q_n)]$, let $\bar{L} = L(\{(p_1, q_1), \dots, (p_n, q_n)\})$. That is $\bar{L}$ is the list L sorted. In particular we have

$$\Delta_{\bar{L}} = \pm \Delta_L,$$

where the sign is determined by the sign of the permutation that reorders L into $\bar{L}$. A basis of diagonally alternating polynomials in $\mathbb{Q}[X_n, Y_n]$ is given by the set

$$\big\{\Delta_{L(D)}\colon\ D \subset \mathbf{Z}^2_{\geqslant 0} \text{ and } |D| = n\big\}. \tag{3.1}$$

Two sorted lattice diagrams $L(D) = [(p_1, q_1), \dots, (p_n, q_n)]$ and $L(D') = [(p'_1, q'_1), \dots, (p'_n, q'_n)]$ can be compared using the following *lexicographic* order:

$$L(D) \prec L\big(D'\big) \quad \Leftrightarrow \quad \exists i \begin{cases} (p_s, q_s) = (p'_s, q'_s), & i + 1 \leqslant s \leqslant n, \\ (p_i, q_i) < (p'_i, q'_i). \end{cases} \tag{3.2}$$

Given a diagonally alternating polynomial $f(X_n; Y_n) = a_1 \Delta_{L(D_1)} + a_2 \Delta_{L(D_2)} + \dots + a_r \Delta_{L(D_r)}$ with all $a_i \neq 0$, we define the *leading diagram* of $f(X_n; Y_n)$ to be $\Delta_{L(D_k)} \neq 0$, where $L(D_k) \succ L(D_i)$ for all $i \neq k$ and $1 \leqslant i \leqslant r$.

4. *F*-operators

In this section we introduce the operator F for a column and show its basic properties. A *composition* a of n, denoted by $a \models n$, is an ordered sequence of positive integers $a = (a_1, a_2, \dots, a_k)$ such that $|a| := \sum_i a_i = n$. For $a \models n$ we let $E_a = E_{a_1} E_{a_2} \cdots E_{a_k}$. Let S_k denote the symmetric group on k elements and let

$$\begin{aligned} \alpha : S_k &\longmapsto \mathbf{Z}^k, \\ w &\longmapsto \alpha(w) = \big(\alpha_1(w), \alpha_2(w), \dots, \alpha_k(w)\big), \end{aligned}$$

where $\alpha_i(w) = i - w(i)$.

Remark 4.1. For any $w \in S_k$, we have that $\sum_{i=1}^k \alpha_i(w) = \sum_{i=1}^k (i - w(i)) = 0$. This implies that for any $t = (t_k, t_{k-1}, \dots, t_1) \in \mathbf{Z}^k_{>0}$ and $w \in S_k$,

$$|t| = \sum_{i=1}^k t_{k-i+1} = \sum_{i=1}^k \big(t_{k-i+1} + \alpha_i(w)\big) = \big|t + \alpha(w)\big|.$$

If $t = (t_k, t_{k-1}, \dots, t_1) \in \mathbf{Z}^k_{>0}$ satisfies $t_k > t_{k-1} > \cdots > t_1$, then

$$t_{k-i+1} \geqslant t_{k-i} + 1 \geqslant \cdots \geqslant t_1 + (k-i)$$

for all $1 \leqslant i \leqslant k$. Since $\alpha_i(w) = i - w(i) \geqslant i - k$, we have that

$$t_{k-i+1} + \alpha_i(w) \geqslant t_1 + (k-i) + i - k \geqslant t_1 > 0$$

for all $1 \leqslant i \leqslant k$. This shows that $t + \alpha(w)$ is a composition of $|t|$.

Definition 4.2. Given $t = (t_k, t_{k-1}, \dots, t_1) \in \mathbf{Z}^k_{>0}$ with $t_k > t_{k-1} > \cdots > t_1 > 0$, let

$$F_{\begin{array}{|c|}\hline t_k \\ \hline \vdots \\ \hline t_1 \\ \hline \end{array}} = \det(E_{t_{k-j+1}+(j-i)}) = \sum_{w \in S_k} (-1)^{l(w)} E_{t+\alpha(w)}.$$

Here $l(w) = Card\{(i,j) \mid i < j,\ \ w(i) > w(j)\}$. From Remark 4.1 the operator F is well defined and takes a homogeneous polynomial of bidegree (r,s) to a homogeneous polynomial of bidegree $(r - t_1 - \cdots - t_k, s + k)$.

Lemma 4.3. *Given $t = (t_k, t_{k-1}, \dots, t_1) \in \mathbf{Z}^k_{>0}$ with $t_k > t_{k-1} > \cdots > t_1$, if for some i we have $t_i = t_{i-1} + 1$, then $F_t = 0$.*

Proof. If $t_i = t_{i-1} + 1$ for some $1 < i \leqslant k$, then the $(k-i+1)$th and $(k-i+2)$th columns in the determinant of F are the same. $\square$

Therefore, we shall assume that $t_i \geqslant t_{i-1} + 2$ in the definition of F-operators. The following lemma is useful for our purpose.

Lemma 4.4. *Let $L = [(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n)]$ be a lattice diagram. We have*

$$E_j \Delta_L = \sum_{i=1}^n \Delta_{E_j^i(L)},$$

where

$$E_j^i(L) = \big[(p_1, q_1), \dots, (p_i - j, q_i + 1), \dots, (p_n, q_n)\big]. \tag{4.1}$$

Proof. Recall that the determinant $\Delta_L = c \cdot \mathrm{Alt}(x_1^{p_1} y_1^{q_1} \cdots x_n^{p_n} y_n^{q_n})$ where c is a constant and Alt denotes the alternating sum over the symmetric group. For any symmetric operator Ψ, we have that $\mathrm{Alt} \circ \Psi = \Psi \circ \mathrm{Alt}$. The lemma follows using $\Psi = E_j$. $\square$

We can also generalize this result to the case where there are several E_i's acting consecutively on Δ_L.

Lemma 4.5. *Let $L = [(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)]$ and $a = (a_k, a_{k-1}, \ldots, a_1)$ a composition. We have:*

$$E_a \Delta_L = E_{a_k} E_{a_{k-1}} \cdots E_{a_1} \Delta_L = \sum_{f:\{1,\ldots,k\} \to \{1,\ldots,n\}} \Delta_{E_a^f(L)},$$

where $E_a^f(L) = E_{a_k}^{f(k)} E_{a_{k-1}}^{f(k-1)} \cdots E_{a_1}^{f(1)}(L)$ as in Eq. (4.1).

Combining the definition of the operator F with Lemma 4.5 gives

$$F_{\begin{array}{|c|}\hline t_k \\ \hline \vdots \\ \hline t_1 \\ \hline \end{array}} \Delta_L = \sum_{\substack{f:\{1,\ldots,k\} \to \{1,\ldots,n\} \\ w \in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)}.$$

As the following theorem shows, many terms in this sum cancel.

Theorem 4.6. *For $t = (t_k, t_{k-1}, \ldots, t_1) \in \mathbf{Z}^k_{>0}$ where $t_i \geqslant t_{i-1} + 2$ for all $2 \leqslant i \leqslant k$, and $L = [(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)]$, we have*

$$F_{\begin{array}{|c|}\hline t_k \\ \hline \vdots \\ \hline t_1 \\ \hline \end{array}} \Delta_L = \sum_{\substack{f:\{1,\ldots,k\} \to \{1,\ldots,n\} \\ \text{injective} \\ w \in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)}. \tag{4.2}$$

For f injective, we can explicitly describe $E^f_{t+\alpha(w)}(L) = [(p'_1, q'_1), (p'_2, q'_2), \ldots, (p'_n, q'_n)]$

$$(p'_s, q'_s) = \begin{cases} (p_s - t_i - \alpha_{k-i+1}(w), q_s + 1), & \text{if } s = f(i), \\ (p_s, q_s), & \text{otherwise.} \end{cases}$$

Proof. The left-hand side of equality (4.2) can be written into sums as follows:

$$\begin{aligned} F_{\begin{array}{|c|}\hline t_k \\ \hline \vdots \\ \hline t_1 \\ \hline \end{array}} \Delta_L &= \sum_{\substack{f:\{1,\ldots,k\} \to \{1,\ldots,n\} \\ w \in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)} \\ &= \sum_{\substack{f:\{1,\ldots,k\} \to \{1,\ldots,n\} \\ \text{non-injective} \\ w \in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)} \\ &\quad + \sum_{\substack{f:\{1,\ldots,k\} \to \{1,\ldots,n\} \\ \text{injective} \\ w \in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)}. \end{aligned}$$

By definition, $t + \alpha(w) = (t_k + 1 - w(1), t_{k-1} + 2 - w(2), \ldots, t_1 + k - w(k))$ and thus $E^f_{t+\alpha(w)}(L) = E^{f(k)}_{t_k+1-w(1)} E^{f(k-1)}_{t_{k-1}+2-w(2)} \cdots E^{f(1)}_{t_1+k-w(k)}(L)$. If f is non-injective, we can always find a pair (i, j) such that $f(i) = f(j)$ where $k \geqslant i > j \geqslant 1$. The operator $E^f_{t+\alpha(w)}(L)$ related to such an f is

$$E^{f(k)}_{t_k+1-w(1)} \cdots E^{f(i)}_{t_i+(k-i+1)-w(k-i+1)} \cdots E^{f(j)}_{t_j+(k-j+1)-w(k-j+1)} \cdots E^{f(1)}_{t_1+k-w(k)}(L). \tag{4.3}$$

Let $\bar{w} = w(k-i+1, k-j+1)$, and notice that $\alpha(\bar{w}) = (1-w(1), \ldots, k-i+1-w(k-j+1), \ldots, k-j+1-w(k-i+1), \ldots, k-w(k))$. We then have that $E^f_{t+\alpha(\bar{w})}(L)$ is equal to

$$E^{f(k)}_{t_k+1-w(1)} \cdots E^{f(i)}_{t_i+(k-i+1)-w(k-j+1)} \cdots E^{f(j)}_{t_j+(k-j+1)-w(k-i+1)} \cdots E^{f(1)}_{t_1+k-w(k)}(L). \tag{4.4}$$

From Eqs. (4.3) and (4.4), we can see that the only difference between $E^f_{t+\alpha(w)}(L)$ and $E^f_{t+\alpha(\bar{w})}(L)$ are the E-operators related to i and j. Since $f(i) = f(j)$, we have

$$E^{f(i)}_{t_i+(k-i+1)-w(k-i+1)} E^{f(j)}_{t_j+(k-j+1)-w(k-j+1)}(L) = E^{f(i)}_{t_i+(k-i+1)-w(k-j+1)} E^{f(j)}_{t_j+(k-j+1)-w(k-i+1)}(L).$$

Indeed, this only changes the cell $(p_{f(i)}, q_{f(i)})$ in L into

$$\bigl(p_{f(i)} - t_i - t_j - (k-i+1) - (k-j+1) + w(k-i+1) + w(k-j+1), q_{f(i)} + 2\bigr).$$

Since the E-operators commute, we conclude that $E^f_{t+\alpha(w)}(L) = E^f_{t+\alpha(\bar{w})}(L)$.

For any f non-injective, we pick the lexicographically unique pair (i_f, j_f) such that $f(i_f) = f(j_f)$ and $i_f > j_f \geqslant 1$. We have

$$\begin{aligned}
&\sum_{\substack{f:\{1,\ldots,k\}\to\{1,\ldots,n\} \\ \text{non-injective} \\ w\in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)} \\
&= \sum_{\substack{f:\{1,\ldots,k\}\to\{1,\ldots,n\} \\ \text{non-injective} \\ w\in S_k;\ l(w)\ \text{even}}} (-1)^{l(w)} \bigl(\Delta_{E^f_{t+\alpha(w)}(L)} - \Delta_{E^f_{t+\alpha(w(k-i_f+1,k-j_f+1))}(L)}\bigr) \\
&= 0.
\end{aligned}$$

This implies the theorem. □

Remark 4.7. For $L = L(D)$ a sorted lattice diagram, let $f^o : \{1, \ldots, k\} \to \{1, \ldots, n\}$ be defined by $f^o(i) = i$. We have

$$\begin{aligned}
F_t \Delta_L &= \Delta_{E^{f^o}_t(L)} + \sum_{\substack{(f,w)\neq(f^o,\mathrm{id}) \\ f\ \text{injective} \\ w\in S_k}} (-1)^{l(w)} \Delta_{E^f_{t+\alpha(w)}(L)} \\
&= \Delta_{E^{f^o}_t(L)} + (\text{lower terms}).
\end{aligned}$$

In particular, if $\Delta_{E^{f^o}_t(L)} \neq 0$, then it is the leading diagram of $F_t \Delta_L$. To see this, we show the following claim. For any $(f, w) \neq (f^o, \mathrm{id})$, we have that

$$E^f_{t+\alpha(w)}(L) \prec E^{f^o}_t(L) \tag{4.5}$$

in the order defined in Eq. (3.2). Also if $L_1 \prec L_2$, then for any f we have

$$E^f_k(L_1) \prec E^f_k(L_2). \tag{4.6}$$

First consider the case where $w \neq \mathrm{id}$. Let s be the largest integer such that $\alpha_s(w) \neq 0$. We must have that $\alpha_s(w) = s - w(s) > 0$ since $\alpha_{s+1}(w) = \cdots = \alpha_k(w) = 0$. We have $t_{k-i+1} + \alpha_i(w) = t_{k-i+1}$ for all $s+1 \leqslant i \leqslant k$, and $t_{k-s+1} + \alpha_s(w) > t_{k-s+1}$. This gives

$$E^{f^o}_t(L) \succ E^{f^o}_{t+\alpha(w)}(L).$$

In the case where $w = \mathrm{id}$, we have $\alpha(\mathrm{id}) = (0, \ldots, 0)$. In particular, for $f \neq f^0$,

$$E^{f^o}_{t+\alpha(w)}(L) \succ E^f_{t+\alpha(w)}(L).$$

The Eq. (4.5) follows by transitivity. Eq. (4.6) is clear.

Example 4.8. Let $t = (3, 1)$ and $L = [(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)]$. Applying F_t we get

$$F_{\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}} \Delta_L = \Delta_{[(0,0),(1,0),(2,0),(0,1),(3,1)]} - 3\Delta_{[(0,0),(1,0),(2,0),(1,1),(2,1)]}$$
$$- 3\Delta_{[(0,0),(1,0),(4,0),(0,1),(1,1)]} + \Delta_{[(0,0),(2,0),(3,0),(0,1),(1,1)]}$$
$$+ 2\Delta_{[(0,0),(1,0),(3,0),(0,1),(2,1)]}$$

Clearly $\Delta_{[(0,0),(1,0),(2,0),(0,1),(3,1)]}$ is the leading diagram and it is obtained by $E_3^2 E_1^1([(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)])$.

5. *F*-operators on the space of alternating diagonal harmonics

In this section we define the operator F_T associated with a column-strict Young tableau T. We also construct an explicit basis of $A_n^{k,l}$ for $l \leqslant 2$. This allows us to better understand for which T the elements $F_T \Delta_n$ are linearly independent.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, let $D_\mu = \{(i, j) \in \mathbf{Z}^2_{>0}\colon\ 1 \leqslant j \leqslant \mu_i,\ \forall 1 \leqslant i \leqslant l\}$. A *tableau* T is a function $T : D_\mu \to \mathbf{Z}_{>0}$. Here the partition μ is called the *shape* of T. By convention, set $T(i, j) = \infty$ if $(i, j) \notin D_\mu$. The *transpose* of μ is the partition μ^t where D_{μ^t} is the transpose of D_μ. A *column-strict tableau* is a tableau T such that for all $(i, j) \in D_\mu$, we have $T(i, j) < T(i + 1, j)$ and $T(i, j) \leqslant T(i, j + 1)$. We usually let $t_{i,j}(T) = T(i, j)$ or just $t_{i,j}$ if it is unambiguous. We graphically put $t_{i,j}$ in a box at position (i, j) in D_μ. For example,

$$T = \begin{array}{|c|c|cc}\hline 4 & 5 & \multicolumn{1}{c}{} & \\\hline 2 & 3 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4}\\\hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2}\\\hline\end{array},$$

is column-strict and $t_{2,3} = 3$. Given a column-strict tableau T we denote its shape by $\mu(T) = (\mu_1(T), \mu_2(T), \ldots, \mu_l(T))$ and row sum by $s(T) = (s_1(T), s_2(T), \ldots, s_l(T))$, where $s_i(T) = t_{i,1} + \cdots + t_{i,\mu_i}$. We may simply use $\mu = (\mu_1, \ldots, \mu_l)$ and $s = (s_1, \ldots, s_l)$ if the tableau is clear. We see that $(s_1, s_2, \ldots, s_l)$ forms a composition of $|s| = s_1 + s_2 + \cdots + s_l$, which is the sum of all numbers in the tableau T. Let T_j denote the jth column in the tableau T for all $1 \leqslant j \leqslant \mu_1$. We define the F-operator corresponding to T by:

$$F_T := F_{T_{\mu_1}} F_{T_{\mu_1 - 1}} \cdots F_{T_2} F_{T_1}. \tag{5.1}$$

Since T is column-strict, F_T is well defined. Now we use it to generate sets of polynomials which are linearly independent.

Theorem 5.1. *For $1 \leqslant k \leqslant n - 1$, let $\mathcal{T}_k = \{\boxed{k}\}$. The set*

$$B_{\mathcal{T}_k} = \{F_{\boxed{k}} \Delta_n\}$$

is a basis of the space $A_n^{k,1}$ and $\dim A_n^{k,1} = 1$. For all other values of k, $\dim A_n^{k,1} = 0$.

Proof. Notice that the leading term of $F_k \Delta_n$ is $\Delta_{[(0,0),(1,0),\ldots,(n-1-k,1)]} \neq 0$ for $1 \leqslant k \leqslant n - 1$. It is clear that the set $B_{\mathcal{T}_k} = \{F_{\boxed{k}} \Delta_n\} \neq \{0\}$ is linearly independent. Combining Eq. (2.3) and Eq. (2.2), we see that $\dim A_n^{k,1}$ is given by the number of Dyck paths with bounce equal to 1 and coarea equal to k. We then claim that $\dim A_n^{k,1} = 1$ for $1 \leqslant k \leqslant n - 1$ and zero otherwise. This is because the only way to get $b(c) = 1$ is if the Dyck sequence of c is of the form

$$(0, 1, 2, \ldots, n - k - 1, n - k - 1, \ldots, n - 3, n - 2)$$

for $1 \leqslant k \leqslant n - 1$. $\square$

When $j \geqslant i+2$, we have $F_{\begin{array}{|c|}\hline j\\\hline i\\\hline\end{array}} = E_{j,i} - E_{j-1,i+1}$. For $j = i$ or $i+1$, we have $F_{\begin{array}{|c|c|}\hline i & j\\\hline\end{array}} = F_{\begin{array}{|c|}\hline i\\\hline\end{array}} F_{\begin{array}{|c|}\hline j\\\hline\end{array}} = E_i E_j$.

Theorem 5.2. *For $2 \leqslant k \leqslant 2n-2$, the set of polynomials $\{F_T \Delta_n\}$ where*

$$T = \begin{cases} \begin{array}{|c|c|}\hline i & i\\\hline\end{array} & i \leqslant n-2,\\[2ex] \begin{array}{|c|c|}\hline i & i+1\\\hline\end{array} & i \leqslant n-3,\\[2ex] \begin{array}{|c|}\hline j\\\hline i\\\hline\end{array} & \begin{array}{l} i+2 \leqslant j \leqslant n-2,\\ 1 \leqslant i \leqslant n-4,\end{array} \end{cases}$$

and $|s(T)| = k$ forms a basis of $A_n^{k,2}$. For all other values of k, $\dim A_n^{k,2} = 0$.

Proof. We first show linear independence. When $2i \leqslant n-1$, the leading diagram of $E_i E_i \Delta_n$ is

$$\Delta_{[(0,0),(1,0),\ldots,(n-2,0),(n-1-2i,2)]} \neq 0.$$

When $2i+1 \leqslant n-1$, the leading diagram of $E_i E_{i+1} \Delta_n$ is

$$\Delta_{[(0,0),(1,0),\ldots,(n-2,0),(n-2i-2,2)]} \neq 0.$$

When $2i \geqslant n$ and $i \leqslant n-2$, the leading diagram of $E_i E_i \Delta_n$ is

$$\Delta_{[(0,0),(1,0),\ldots,(n-3,0),(n-2-i,1),(n-1-i,1)]} \neq 0$$

and when $2i \geqslant n$ and $i \leqslant n-3$, the leading diagram of $E_i E_{i+1} \Delta_n$ is

$$\Delta_{[(0,0),(1,0),\ldots,(n-3,0),(n-3-i,1),(n-1-i,1)]} \neq 0.$$

Now when $i+2 \leqslant j \leqslant n-2$ and $1 \leqslant i \leqslant n-4$, using Remark 4.7, the leading diagram of $F_{\begin{array}{|c|}\hline j\\\hline i\\\hline\end{array}} \Delta_n$ is

$$\Delta_{[(0,0),(1,0),\ldots,(n-3,0),(n-2-j,1),(n-1-i,1)]} \neq 0.$$

In each of the above cases, $F_T \Delta_n$ has a different leading diagram for different T, which implies that $\{F_T \Delta_n\}$ is linearly independent.

The only way to get $b(c) = 2$ in Eq. (2.3) is if the Dyck sequence of c is of the form $(g_0, g_1, \ldots, g_{n-1})$ where $g_{n-1} = n-3$ and $g_0 = 0$. As in Remark 2.4 we consider the partition $\lambda = \mu^t$. The restriction on the Dyck sequence gives us that λ has exactly two non-zero parts and the largest part is less than $n-2$. That is $\lambda = (j, i)$, $j \leqslant n-2$ and $i + j = k$. In particular, the coarea of such a path has to be $2 \leqslant k \leqslant 2n-2$. The case $j = i$ or $i+1$ corresponds to the tableaux T of shape (2) and when $j \geqslant i+2$ it corresponds to the tableaux T of shape $(1,1)$. The dimension of $A_n^{k,2}$ has exactly the desired cardinality. For all other values of k, $\dim A_n^{k,2} = 0$. □

6. Definition and some properties of framed tableaux

In order to generalize the results of the previous section, we need to study a special kind of Young tableau. In Theorem 5.1 and 5.2, a basis of $A_n^{k,l}$ for $l \leqslant 2$ is obtained from a set of the form $\{F_T \Delta_n\}$ where T are well chosen. It suggests that for T with small shape, the rows must weakly increase with small differences and columns should have jump greater than or equal to 2. In the light of this observation, we introduce a new kind of tableau which allow us to generalize the result for larger l.

Definition 6.1. Given $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ and $s = (s_1, s_2, \ldots, s_l)$, we say that (μ, s) satisfies the *framing condition* if

1. $s_i \geqslant (2i-1)\mu_i$ and
2. $s_{i+1} \geqslant s_i + 2\mu_i$ for all $1 \leqslant i \leqslant l-1$ such that $\mu_{i+1} = \mu_i$.

Our goal is to build a new kind of tableau $T_{(\mu,s)}$ with shape μ and row sum s. The following definition is useful for our algorithms.

Definition 6.2. Given an integer $c \in \mathbf{Z}_{>0}$, there is a unique way to decompose it into m positive integers $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m$ such that $c = c_1 + \cdots + c_m$ and $0 \leqslant c_j - c_i \leqslant 1$ for all $1 \leqslant i < j \leqslant m$. We say that $\text{B-comp}(c, m) = (c_1, c_1, \ldots, c_m)$ is the *balance composition* of c.

Given (μ, s) satisfying the framing condition in Definition 6.1, we give a procedure that constructs a unique column-strict tableau of shape μ and row sum s. We call the procedure *framing* and the resulting tableau a *framed tableau*. By convention let $\mu_{l+1} = 0$.

Fram$(\mu = (\mu_1, \mu_2, \ldots, \mu_l), s = (s_1, s_2, \ldots, s_l))$

  $(t_{l,1}, t_{l,2}, \ldots, t_{l,\mu_l}) := \text{B-comp}(s_l, \mu_l)$

  For $i = l - 1$ **Downto** 1 **Do**

    $k := l;\ \ a := s_i;\ \ b := \mu_i$

    While $k \geqslant i$ **Do**

      $(r_{i,\mu_{k+1}+1}, \ldots, r_{i,\mu_i}) = \text{B-comp}(a, b)$

      If $r_{i,j} \leqslant t_{i+1,j} - 2,\ \ \forall \mu_{k+1} + 1 \leqslant j \leqslant \mu_k$

        Then $t_{i,j} := r_{i,j},\ \ \forall \mu_{k+1} + 1 \leqslant j \leqslant \mu_k$

        Else $t_{i,j} := t_{i+1,j} - 2,\ \ \forall \mu_{k+1} + 1 \leqslant j \leqslant \mu_k$

      $a := a - (t_{i,\mu_{k+1}+1} + \cdots + t_{i,\mu_k});\ \ b := b - (\mu_k - \mu_{k+1});\ \ k := k - 1;$

  Output $T = [t_{i,j}]$.

We write $T = \text{Fram}(\mu, s)$.

Remark 6.3. The framing procedure is well defined and gives a unique framed tableau $\text{Fram}(\mu, s)$ for each (μ, s) satisfying the framing condition. The top row is clearly unique. Suppose we perform the procedure properly and uniquely up to row $i + 1$. For row i, if $\mu_i > \mu_{i+1}$, then the procedure works well. If $\mu_i = \mu_{i+1}$, then the framing condition gives that $s_i + 2\mu_i \leqslant s_{i+1}$, which guarantees that the procedure produces a unique tableau.

Proposition 6.4. *The framing procedure is an injection from the (μ, s) which satisfies the framing condition for column-strict tableaux. We call* framed tableaux *the subset of tableaux in the image of* Fram.

Example 6.5. For a given $s = (22, 18, 24, 14)$ and $\mu = (8, 5, 4, 2)$, we construct the corresponding framed tableau $\text{Fram}(\mu, s)$ with the above procedure:

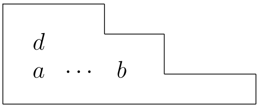
$$
\begin{array}{|c|c|}\hline 7 & 7 \\ \hline \end{array}
\ \rightarrow\
\begin{array}{cccc} 7 & 7 & & \\ 5 & 5 & 7 & 7 \end{array}
\ \rightarrow\
\begin{array}{ccccc} 7 & 7 & & & \\ 5 & 5 & 7 & 7 & \\ 3 & 3 & 4 & 4 & 4 \end{array}
\ \rightarrow\
\begin{array}{cccccccc} 7 & 7 & & & & & & \\ 5 & 5 & 7 & 7 & & & & \\ 3 & 3 & 4 & 4 & 4 & & & \\ 1 & 1 & 2 & 2 & 2 & 4 & 5 & 5 \end{array}.
$$

We have the following properties for framed tableaux.

Lemma 6.6. *A framed tableau T of shape μ, is a column-strict Young tableau of shape μ satisfying the following properties*:

1. *Any two numbers in the same column differ by at least* 2.
2. *For any $a \leqslant b$ in the same row of T we have $b - a \leqslant 1$, unless there is a number d above a such that $d = a + 2$.*

To illustrate this, consider the following picture:



Normally, a and b need to satisfy the condition $b - a \leqslant 1$. However if $d = a + 2$, then there is no restriction on $b - a$. To prove Lemma 6.6, we need the following auxiliary result.

Lemma 6.7. *Suppose that we have two sequences of integers $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{n+m}$ satisfying $c_j - c_i \leqslant 1$, for all $1 \leqslant i < j \leqslant n$ and $d_j - d_i \leqslant 1$, for all $1 \leqslant i < j \leqslant n + m$. If there is a k such that $c_k - d_k \leqslant 1$, then $c_j - d_j \leqslant 2$, for all $1 \leqslant j \leqslant n$.*

Proof. For $1 \leqslant j < k$, we have $c_j = c_k$ or $c_k - 1$ and, $d_j = d_k$ or $d_k - 1$. Hence $c_j - d_j \leqslant c_k - (d_k - 1) \leqslant 2$. For $k \leqslant j < n$, we have $c_j = c_k$ or $c_k + 1$ and, $d_j = d_k$ or $d_k + 1$. Hence $c_j - d_j \leqslant c_k + 1 - d_k \leqslant 2$. $\square$

Proof of Lemma 6.6. It is clear that the row l of $T_{(s,\mu)}$ given by B-comp$(s_l, \mu_l) = (t_{l,1}, t_{l,2}, \ldots, t_{l,\mu_l})$ satisfies properties 1 and 2. By induction, suppose that up to row $i+1$, properties 1 and 2 are satisfied. Moreover, we assume (by induction) that for $i + 1 \leqslant k' \leqslant l$, we have

3. $t_{k',j_2} - t_{k',j_1} \leqslant 1$, for all $\mu_{k'+1} + 1 \leqslant j_1 < j_2 \leqslant \mu_{k'}$.

Recall here that we let $\mu_{l+1} = 0$. For row i, we consider the while loop of the framing procedure. The properties 1,2 and 3, certainly hold whenever $t_{i+1,j} \geqslant r_{i,j} + 2$, for $1 \leqslant j \leqslant \mu_{i+1}$. If at one point, for $i \leqslant k \leqslant l$, there is $\mu_{k+1} + 1 \leqslant j_0 \leqslant \mu_k$ such that $t_{i+1,j_0} - r_{i,j_0} \leqslant 1$, then by Lemma 6.7 we have $t_{i+1,j} - r_{i,j} \leqslant 2$ for all $\mu_{k+1} + 1 \leqslant j \leqslant \mu_k$. The framing procedure sets all $t_{i,j} := t_{i+1,j} - 2$, for $\mu_{k+1} + 1 \leqslant j \leqslant \mu_k$. When we compare $a := a - (t_{i,\mu_{k+1}+1} + \cdots + t_{i,\mu_k})$ with $a' := a - (r_{i,\mu_{k+1}+1} + \cdots + r_{i,\mu_k})$, we obtain $a' \leqslant a$. Hence B-comp$(a', b) \leqslant$ B-comp(a, b) component-wise. This implies that the row is weakly increasing. Properties 1, 2 and 3 also hold in this case. $\square$

Example 6.8. The following are framed tableaux of shape $(3, 2, 1)$:

$$\begin{array}{|c|cc}\hline 6 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{4} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|cc}\hline 6 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} \\ \hline\end{array}, \quad \begin{array}{|c|cc}\hline 6 & \multicolumn{2}{|c}{} \\ \hline 4 & \multicolumn{1}{c|}{4} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|cc}\hline 5 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{6} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|cc}\hline 5 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{7} & \\ \hline 1 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline\end{array}.$$

The following is not a framed tableau: $\begin{array}{|c|c|c}\hline 4 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{6} \\ \hline\end{array}$, since the difference between 1 and 6 is greater than 1, but the number above 1 is 4, which is not exactly 2 more than 1.

From the definition and properties of framed tableaux the following lemmas can be obtained immediately:

Lemma 6.9. *Suppose T is a framed tableau. If we add or subtract some constant k from each number in T and if all entries remain positive, then we get a framed tableau and we denote this by $T \pm k$.*

Lemma 6.10. *Suppose T is a framed tableau. If we delete the bottom row in T, then the remaining tableau is still a framed tableau.*

Lemma 6.11. *For a framed tableau T, suppose the ith and jth columns ($i < j$) have the same height. We list the entries of each column, from bottom to top, as $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ respectively. Then $b_k - a_k \leqslant 1$, for all $1 \leqslant k \leqslant n$.*

Proof. Since the ith column and the jth column ($i < j$) have the same height in the framed tableau T, by the framing procedure we know that there exists a k such that $\mu_{k+1} + 1 \leqslant i < j \leqslant \mu_k$. We have $b_n - a_n \leqslant 1$ since both entries are parts of a balance composition. By induction, assume $b_s - a_s \leqslant 1$ for $1 \leqslant k < s \leqslant n$. For k, either the entries a_k and b_k are parts of a balance composition, or they satisfy $a_k = a_{k+1} - 2$ and $b_k = b_{k+1} - 2$. By the induction hypothesis we have in the latter case $b_k - a_k = b_{k+1} - a_{k+1} \leqslant 1$. $\square$

Remark 6.12. The contrapositive of this lemma gives the following: suppose columns i and j are $a_1, \dots, a_n$ and $b_1, \dots, b_m$ respectively, and there exists some k such that $b_k - a_k \geqslant 2$, then $n > m$. That is column i is strictly higher than column j. We use this to detect the structure of framed tableaux.

7. Insertion and Taquin

The goal of this section is to describe the procedures that give a bijection between partitions $\lambda \vdash n$ and framed tableaux with row sum $s = (s_1, \dots, s_l) \models n$. For this purpose we need a procedure similar to Schensted's algorithm, with some additional straightening steps to get a framed tableau.

Given a framed tableau $T = [t_{i,j}]$ of shape μ and $0 < x \leqslant t_{1,1}$, we define a procedure to insert x into T and denote the resulting framed tableau by $T \leftarrow x$. The algorithm is done in three steps. First we do an insertion that gives an auxiliary tableau Y. The tableau Y determines a shape $\mu' = \mu(Y)$. We use Y in the second step to determine a row sum s' such that (μ', s') satisfies the framing condition of Definition 6.1. Finally, $T \leftarrow x$ is given by $\mathrm{Fram}(\mu', s')$. In our pseudocode, a loop of the form "**For** ... **To** ... **While** A **Do** ..." is a standard **For** loop that stops as soon as A is false.

Recall that for a framed tableau $T = [t_{i,j}]$ of shape $\mu = \mu(T) = (\mu_1, \dots, \mu_l)$ we assume that $\mu_{l+1} = 0$ and $t_{i,j} = \infty$ for $j > \mu_i$.

T ← x

Step 1: auxiliary insertion, getting Y and μ'

$i := 1$; $x_0 := x$

While $t_{i,\mu_i} \geqslant x + 2$ **do**

  $j := \mathrm{Min}\{j : t_{i,j} \geqslant x + 2\}$

  $(t_{i,1}, \dots, t_{i,\mu_i}) := \mathrm{Sort}(t_{i,1}, \dots, t_{i,j-1}, x, t_{i,j+1}, \dots, t_{i,\mu_i})$

  $x := t_{i,j}$; $i := i + 1$

$(t_{i,1}, \dots, t_{i,\mu_i+1}) := \mathrm{Sort}(t_{i,1}, \dots, t_{i,\mu_i}, x)$

$Y := [t_{i,j}]$; $\mu' := \mu(Y)$; $l' := \mathrm{length}(\mu')$

Step 2: finding the new row sum s'

$x := x_0$; $(s_1, \dots, s_{l'}) := s(Y)$

For $i = 2$ **To** l' **Do** $d_i := 0$

For $k = 1$ **To** $l' - 1$ **While** $t_{k,\mu'_k} \geqslant x + 2$ **Do**

  For $j = 1$ **To** μ'_{k+1} **Do**

    If $t_{k,j} = x$ **Then** $t_{k+1,j} := x + 2$

    If $t_{k,\mu'_k} > x + 2$ and $t_{k,j} = x + 1$ **Then** $t_{k+1,j} := x + 3$

  $\bar{s}_{k+1} := t_{k+1,1} + t_{k+1,2} + \cdots + t_{k+1,\mu'_{k+1}}$

  $d_{k+1} := s_{k+1} - \bar{s}_{k+1}$

  $x := x + 2$

$s' := (s_1 + d_2, s_2 + d_3 - d_2, \dots, s_{l'-1} + d_{l'} - d_{l'-1}, s_{l'} - d_{l'})$

Step 3:

Output $\mathrm{Fram}(\mu', s')$

We show in Section 8 that the $\mathbf{T} \leftarrow \mathbf{x}$ algorithm is well defined for $0 < x \leqslant t_{1,1}$ and produces a framed tableau. We give a short example to better demonstrate the steps.

Example 7.1. Let $x = 1$ for

$$T = \begin{array}{|c|cccc}\hline 4 & \multicolumn{1}{|c}{} & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{6} \\ \hline\end{array}. \quad \text{In Step 1, we get} \quad Y = \begin{array}{|c|c|cc}\hline 4 & 5 & \multicolumn{1}{|c}{} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{6} \\ \hline\end{array}.$$

Notice that the resultant tableau Y from step 1 may not be a framed tableau. We need to straighten Y to get a framed tableau. We get $\mu' = (4,2)$ and $s(Y) = (15,9)$. The second loop in Step 2 sets $d_2 = 1 + 1 = 2$ and in the end $s' = (15 + 2, 9 - 2) = (17,7)$. The pair (μ', s') satisfies the framing condition so we can apply the framing procedure and get

$$T \leftarrow 1 = \mathrm{Fram}(17,7) = \begin{array}{|c|c|cc}\hline 3 & 4 & \multicolumn{1}{|c}{} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{7} \\ \hline\end{array}.$$

Most entries in $\mathbf{T} \leftarrow \mathbf{x}$ might be different from those in T. But we remark that all the entries with value equal to x or $x+1$ in the tableau Y in Step 1 still remain unchanged in $T \leftarrow x$, and x is the smallest entry in $\mathbf{T} \leftarrow \mathbf{x}$. This fact is important and allows us to introduce an inverse procedure. This is done by playing a variation of Jeu de Taquin. Again this is done in three steps. We start with a framed tableau $T = [t_{i,j}]$ of shape $\mu = \mu(T) = (\mu_1, \ldots, \mu_l)$ and assume that $x = t_{1,1}$. We denote by ${}_xT$ the framed tableau we obtain by removing x from T with the following procedure:

$_{\mathbf{x}}\mathbf{T}$

Step 1: Jeu de Taquin to get Y and μ'

$i := 1;\ j := 1$

$x := t_{1,1}$

While $t_{i+1,j} < \infty$ or $t_{i,j+1} < \infty$ **Do**

  If $t_{i+1,j} \geqslant t_{i,j+1} + 2$ **Then** $t_{i,j} := t_{i,j+1};\ j := j+1$

  If $t_{i+1,j} \leqslant t_{i,j+1} + 1$

    Then $t_{i,j} := t_{i+1,j};\ \ (t_{i,1}, \ldots, t_{i,\mu_i}) := \mathrm{Sort}(t_{i,1}, \ldots, t_{i,\mu_i});\ i := i+1$

$t_{i,j} := \infty;$

$Y := [t_{i,j}];\ \ \mu' := \mu(Y);\ l' := \mathrm{length}(\mu')$

Step 2: row sum s'

$(s_1, \ldots, s_{l'}) := s(Y)$

For $i = 2$ **To** l' **Do** $d_i := 0$

For $k = 1$ **To** $l' - 1$ **While** $t_{k,\mu'_k} \geqslant x + 2$ and $t_{k,1} \leqslant x + 1$ **Do**

  For $j = 1$ **To** μ'_{k+1} **Do**

    If $t_{k,j} = x$ **Then** $t_{k+1,j} := x + 2$

    If $t_{k,\mu'_k} > x + 2$ and $t_{k,j} = x + 1$ **Then** $t_{k+1,j} := x + 3$

  $\bar{s}_{k+1} := t_{k+1,1} + t_{k+1,2} + \cdots + t_{k+1,\mu'_{k+1}}$

  $d_{k+1} := s_{k+1} - \bar{s}_{k+1}$

  $x := x + 2$

$s' := (s_1 + d_2, s_2 + d_3 - d_2, \ldots, s_{l'-1} + d_{l'} - d_{l'-1}, s_{l'} - d_{l'})$

Step 3:

Output $\mathrm{Fram}(\mu', s')$

Again, we show in the next section that this algorithm works and is well defined. We give here a short example to better show the steps.

Example 7.2. Given a framed tableau $T = \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 1 & 1 \\ \hline \end{array}$, we remove $x = 1$ from T in the first step of ${}_xT$. We use a dot to indicate the position of the cell as we perform the Jeu de Taquin.

$$\begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline \bullet & 1 \\ \hline \end{array} \to \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & 5 \\ \hline 1 & \bullet \\ \hline \end{array} \to \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \hline 4 & \bullet \\ \hline 1 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 1 & 5 \\ \hline \end{array} = Y.$$

Again, Y may not be a framed tableau. We have $\mu' = \mu(Y) = (2, 1, 1)$ and $s(Y) = (6, 4, 6)$. The second loop in Step 2 sets $d_2 = 1$ and $d_3 = 0$. In the end $s' = (6 + 1, 4 + 0 - 1, 6 - 0) = (7, 3, 6)$. The pair (μ', s') satisfies the framing condition so we can apply the framing procedure and get

$${}_xT = \mathrm{Fram}\big(\mu', s'\big) = \begin{array}{|c|c|}\cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 6 \\ \hline \end{array}.$$

8. 1 − 1 Correspondence between partitions and framed tableaux

In this section we construct a $1 - 1$ correspondence between partitions and framed tableaux. Given a partition $(\lambda_1, \ldots, \lambda_k) \vdash l$, we get a framed tableau as follows:

$$\emptyset \leftarrow \lambda := \big(\cdots\big((\emptyset \leftarrow \lambda_1) \leftarrow \lambda_2\big)\cdots \leftarrow \lambda_k\big). \tag{8.1}$$

On the other hand, given a framed tableau T, we get a partition $\lambda(T)$ by recording the numbers removed each time with

$${}_{x_k}\big(\cdots {}_{x_2}({}_{x_1}T)\cdots\big) = \emptyset. \tag{8.2}$$

Then $\lambda(T) := (x_k, \ldots, x_2, x_1) \vdash |s(T)|$. This is not the shape of T that we denoted by $\mu(T)$. We prove here that these two maps are inverse to each other, and thus there is a bijection between partitions and framed tableaux. First, we give a lemma to reduce the number of cases we have to consider.

Lemma 8.1. *Given $T = [t_{i,j}]$ and $0 < x + 1 \leqslant t_{1,1}$, we have*

$$T \leftarrow (x+1) = \big((T - x) \leftarrow 1\big) + x,$$
$${}_{x+1}T = \big({}_1(T - x)\big) + x.$$

Proof. The result of $T \leftarrow (x + 1)$ is determined by the differences between x and the $t_{i,j}$'s. It is clear that $\mu(T \leftarrow (x+1)) = \mu((T - x) \leftarrow 1)$. Furthermore, the d_i's in the procedure $T \leftarrow (x+1)$ and $(T - x) \leftarrow 1$ are the same for all $2 \leqslant i \leqslant l$. Thus we have that $s(T \leftarrow (x+1)) = s(((T - x) \leftarrow 1) + x)$. In both cases, we produce the same framed tableau. The argument for the second equality is similar. □

Theorem 8.2. *Let $T = [t_{i,j}]$ and $0 < x \leqslant t_{1,1}$.*

(a) *The procedures $T \leftarrow x$ and ${}_xT$ are well defined and inverse to each other.*
(b) *The maps defined by (8.1) and (8.2) give a bijection $\lambda \leftrightarrow T$ between $\lambda \vdash k$ with l parts and the framed tableaux T such that $\mu(T) \vdash l$ and $s(T) \models k$.*

Proof. Part (b) follows directly from Part (a). We show (a) case by case. From Lemma 8.1 it is sufficient to consider only the cases of inserting and removing 1 from any given framed tableau. Let $T : D_\mu \to \mathbf{Z}_{>0}$ be a framed tableau with shape μ and row sum $s = (s_1, s_2, \ldots, s_l)$:

$$\begin{array}{|lll}
\vdots & & \\
t_{2,1} & t_{2,2} & \\
t_{1,1} & t_{1,2} & \cdots \\ \hline
\end{array}$$

Case 1. Assume that $t_{1,1} \geqslant 3$ and let $m = \mu_2^t$. In Step 1 of $T \leftarrow 1$, we obtain

$$\begin{array}{lll}
t_{l,1} & & \\
\vdots & & \\
t_{m,1} & & \\
t_{m-1,1} & t_{m,2} & \\
\vdots & \vdots & \\
t_{1,1} & t_{2,2} & \\
1 & t_{1,2} & \cdots
\end{array}$$

Let $[y_{i,j}] = Y$. We have that $s(Y) = (s_1 - t_{1,1} + 1, s_2 - t_{2,1} + t_{1,1}, \ldots, s_l - t_{l,1} + t_{l-1,1}, t_{l,1})$ and $\mu' = \mu(Y) = (\mu_1, \ldots, \mu_l, 1)$. In Step 2 of $T \leftarrow 1$, as k varies from row 1 to m, we have $x = 2k - 1$. Since $t_{1,1} \geqslant 3$ we have $y_{k,\mu_k} \geqslant t_{k,1} \geqslant t_{1,1} + 2k - 2 \geqslant 2k + 1 = x + 2$. The entries in the first column are sequentially changed to $y_{k+1,1} := 2k + 1$ since the entry in row k is $y_{k,1} = 2k - 1 = x$. No other entries are changed since for $j \geqslant 2$ we have $y_{k,j} \geqslant t_{k,1} \geqslant 2k + 1 > x + 1$. The loop stops after $k = m$ since for row $m + 1$, we have $x = 2m + 1$ and at that moment, $y_{m+1,\mu'_{m+1}} = 2m + 1 = x \ngeqslant x + 2$. We then have that $d_2 = t_{1,1} - 3, \ldots, d_{m+1} = t_{m,1} - (2m + 1)$, $d_{m+2} = \cdots = d_l = 0$. Thus the new row sum is $s' = (s_1 - 2, s_2 - 2, \ldots, s_m - 2, 2m + 1, t_{m+1,1}, \ldots, t_{l,1})$. Since (μ, s) satisfies the framing condition of Definition 6.1 it is easy to check that (μ', s') also satisfies the framing condition. We can thus compute $\mathrm{Fram}(\mu', s')$ in Step 3 of $T \leftarrow 1$. Since $\mu'_{m+1} = 1$ and $s'_{m+1} = 2m + 1$, we must have that the first entry of each row $1 \leqslant k \leqslant m + 1$ of $\mathrm{Fram}(\mu', s')$ is $2k - 1$. We obtain

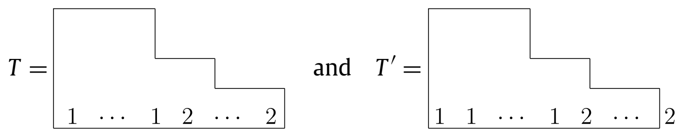
$$T' = T \leftarrow 1 = \mathrm{Fram}\big(\mu', s'\big) = \begin{array}{lll}
t_{l,1} & & \\
\vdots & & \\
2m+1 & & \\
2m-1 & t'_{m,2} & \\
\vdots & \vdots & \\
3 & t'_{2,2} & \\
1 & t'_{1,2} & \cdots
\end{array}$$

For $1 \leqslant k \leqslant m$, we have $t'_{k,2} + \cdots + t'_{k,\mu_k} = s'_k - 2k + 1 = s_k - 2k - 1 \geqslant s_k - t_{k,1} = t_{k,2} + \cdots + t_{k,\mu_k}$. This implies $t'_{k,j} \geqslant t_{k,j}$ for all $1 \leqslant k \leqslant m$ and $j \geqslant 2$. Now we want to show ${}_1T' = T$. In Step 1 of ${}_1T'$, we get

$$Y = \begin{array}{|lll}
t_{l,1} & & \\
\vdots & & \\
5 & t'_{2,2} & \\
3 & t'_{1,2} & \cdots \\ \hline
\end{array}$$

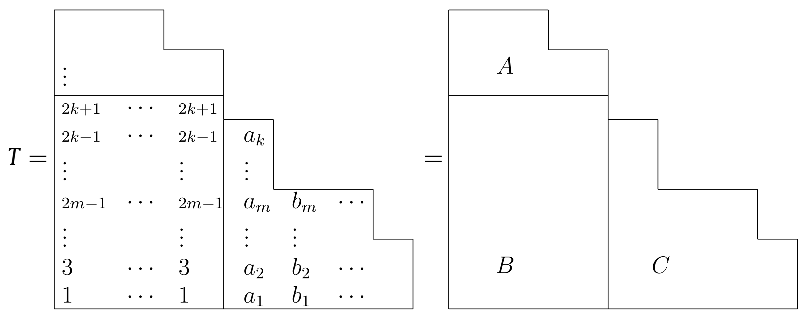
We now get that $\mu' = \mu(Y) = \mu$ and $s(Y) = (s_1, \ldots, s_l)$. Let $[y_{i,j}] = Y$. Since $y_{1,1} = 3 \nleqslant 2 = x + 1$, we do not do any loops in Step 2. Clearly (μ, s) satisfies the framing condition and ${}_1T' = \mathrm{Fram}(\mu, s) = T$.

Case 2. Row $k=1$ of T contains only 1's or 2's or both. Let



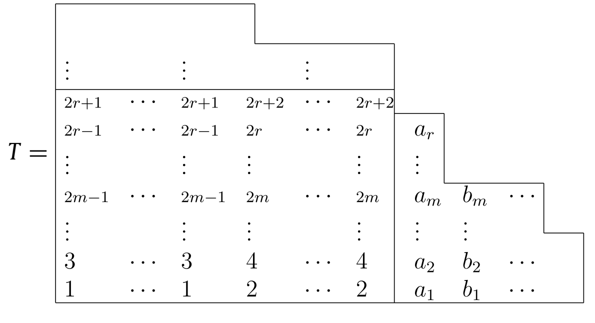
In Step 1 of $T \leftarrow 1$, we obtain $Y = T'$. Nothing happens in Step 2 since for $k=1$ we have $t_{1,\mu'_1} \leqslant 2 \ngeqslant 3 = x+2$. Hence $s' = s(Y) = s(T')$ and $\mu' = \mu(Y) = \mu(T')$. In the procedure $\mathrm{Fram}(\mu', s')$ it is clear that the entries in the row $k>1$ will be the same as in T. For $k=1$, the balanced composition $(1,1,\ldots,1,2,\ldots,2)$ will not change as all entries will be at least two less the entry directly above. Hence $T \leftarrow 1 = \mathrm{Fram}(\mu', s') = T'$. For the inverse procedure, $Y = T$ in Step 1 of ${}_1T'$. Again nothing happens in Step 2 since $t_{1,\mu_1} \leqslant 2 \ngeqslant 3$. Hence ${}_1T' = \mathrm{Fram}(\mu, s) = T$.

Case 3. Row $k=1$ of T only contains 1's and numbers greater than or equal to 3. From Remark 6.12, since $a_1 \geqslant 3$, the shape of T must be as follows



We use A, B, C to denote the corresponding portion of T. Notice that C has the same structure as in Case 1. When we insert 1 in Step 1 of $T \leftarrow 1$, the tableau Y is obtained by inserting 1 in C and the first column of C is shifted up. In Step 2, as in Case 1, the loop runs for $k=1$ to m. All the entries in the portion B of the tableau are set back to their current values, hence left unchanged. Only the entries in C are affected. In conclusion, this case reduces to Case 1. The same argument applies for the reverse procedure where the loop in Step 2 may run but no entries will be changed.

Case 4. Row $k=1$ of T contains 2's, and possibly some 1's, together with numbers greater than or equal to 4. Again from Remark 6.12, since $a_1 \geqslant 4$, the shape of T must be as follows:



In Step 1 of $T \leftarrow 1$ we get

$$Y = \begin{array}{|ccccccccc}
\hline
 & & & & & & & & \\
\vdots & & \vdots & & & \vdots & & & \\
\hline
2r{+}1 & \cdots & 2r{+}1 & 2r{+}2 & \cdots & 2r{+}2 & a_r & & \\
2r{-}1 & \cdots & 2r{-}1 & 2r & \cdots & 2r & a_{r-1} & & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \\
2m{+}1 & \cdots & 2m{+}1 & 2m{+}2 & \cdots & 2m{+}2 & a_m & & \\
2m{-}1 & \cdots & 2m{-}1 & 2m & \cdots & 2m & a_{m-1} & b_m & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
3 & \cdots & 3 & 4 & \cdots & 4 & a_1 & b_2 & \cdots \\
1 & \cdots & 1 & 1 & \cdots & 2 & 2 & b_1 & \cdots \\
\hline
\end{array}$$

We have that

$$\begin{aligned}
s(Y) &= \big(s_1(Y), s_2(Y), \ldots, s_l(Y)\big) \\
&= (s_1 - a_1 + 1, s_2 - a_2 + a_1, \ldots, s_r - a_r + a_{r-1}, s_{r+1} + a_r, s_{r+2} \ldots, s_l)
\end{aligned}$$

and $\mu' = \mu(Y) = (\mu_1, \ldots, \mu_r, \mu_{r+1} + 1, \mu_{r+2}, \ldots, \mu_l)$. Let $[y_{i,j}] = Y$ and let c be the index of the column with the a_i's. That is $y_{1,c} = 2$ and $y_{i,c} = a_{i-1}$ for $2 \leqslant i \leqslant r+1$. Also let $c' < c$ be the index of the column where 1 is inserted. That is $y_{1,c'} = 1$ and $y_{i,c'} = 2i$ for $2 \leqslant i \leqslant r+1$. In Step 2 of $T \leftarrow 1$, as k varies from 1 to m, we have $x = 2k-1$ and since $b_1 \geqslant 4$ we have $y_{k,\mu_k} \geqslant b_k \geqslant b_1 + 2k - 2 \geqslant 2k+2 > x+2$. The entries in column c are sequentially changed to $y_{k+1,c} := 2k+2$ since the entry in row k is $y_{k,c} = 2k = x+1$. Also the entries in column c' are sequentially changed to $y_{k+1,c'} := 2k+1$ since the entry in row k is $y_{k,c'} = 2k-1 = x$. The other entries in columns $1 \leqslant j \leqslant c'-1$ and $c'+1 \leqslant j \leqslant c-1$ are set back to their current values, hence left unchanged. No other entries are changed since for $j \geqslant c+1$ we have $y_{k,j} \geqslant b_k \geqslant 2k+2 > x+1$. The loop stops after $k = m$ since for row $m+1$, we have $x = 2m+1$ and at that moment, $y_{m+1,\mu'_{m+1}} = 2m+2 = x+1 \ngeqslant x+2$. We then have that $d_2 = a_1 - 3$, $\ldots$, $d_{m+1} = a_m - (2m+1)$, $d_{m+2} = \cdots = d_l = 0$. Thus the new row sum is

$$\begin{aligned}
s' = (&s_1 - 2, \ldots, s_m - 2, s_{m+1} + 2m + 1 - a_{m+1}, s_{m+2} + a_{m+1} - a_{m+2}, \ldots, \\
&s_r + a_{r-1} - a_r, s_{r+1} + a_r, s_{r+2}, \ldots, s_l).
\end{aligned} \tag{8.3}$$

We claim that (μ', s') satisfies the framing condition of Definition 6.1. In the insertion algorithm every $y_{i,j}$ should satisfy $y_{i,j} \geqslant 2i-1$, and the value change in step 2 still guarantees $s_i(Y) - d_i \geqslant (2i-1)\mu'_i$. These imply $s'_i \geqslant (2i-1)\mu'_i$ for all $1 \leqslant i \leqslant l$. For $1 \leqslant i \leqslant m-1$ and $\mu'_{i+1} = \mu'_i$, we have $s'_{i+1} - s'_i = (s_{i+1} - 2) - (s_i - 2) = s_{i+1} - s_i \geqslant 2\mu_i = 2\mu'_i$. For $m+1 \leqslant i \leqslant r$, since $s_{i+1} - a_{i+1} - (s_i - a_i) \geqslant 2(\mu_i - 1) = 2(\mu'_i - 1)$, we get $s'_{i+1} - s'_i = s_{i+1} + a_i - a_{i+1} - (s_i + a_{i-1} - a_i) \geqslant 2(\mu'_i - 1) + 2 = 2\mu'_i$. Remember that (μ, s) satisfies the framing condition. Thus together with the case when $i \geqslant r+2$, $s'_i = s_i$, $\mu'_i = \mu_i$, we obtain that (μ', s') also satisfies the framing condition. We can now compute $\mathrm{Fram}(\mu', s')$ in Step 3 of $T \leftarrow 1$. Notice that row $m+1$ of Y after Step 2 contains only $2m+1$'s and $2m+2$'s and it is already balanced. Moreover the entries in the row just above will be at least two more. This implies that row

$m+1$ of $\mathrm{Fram}(\mu', s')$ contains only $2m+1$'s and $2m+2$'s, which determines uniquely the numbers below and preserves all the 1's and 2's in $T \leftarrow 1$. We get

$$T' = T \leftarrow 1 = \begin{array}{llllll|lll}
 & & & & & & & & \\
\vdots & & \vdots & & \vdots & & & & \\
t'_{r+1,1} & \cdots & & & \cdots & & a'_r & & \\
t'_{r,1} & \cdots & & & \cdots & & a'_{r-1} & & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \\
2m+1 & \cdots & 2m+1 & 2m+1 & \cdots & 2m+2 & 2m+2 & & \\
2m-1 & \cdots & 2m-1 & 2m-1 & \cdots & 2m & 2m & b'_m & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
3 & \cdots & 3 & 3 & \cdots & 4 & 4 & b'_2 & \cdots \\
1 & \cdots & 1 & 1 & \cdots & 2 & 2 & b'_1 & \cdots
\end{array} \tag{8.4}$$

where $b'_i \geqslant b_i \geqslant 2i+2$, for $1 \leqslant i \leqslant m$. Also $t'_{m+2,j} \geqslant 2m+4$ for $c' \leqslant j \leqslant c$ and $a'_{m+1} \geqslant 2m+4$.

Now, we want to show ${}_1T' = T$. In Step 1 of ${}_1T'$, the dotted box $(i,j) = (1,1)$ travels right on the first line to $(1, c')$, then up the column to $(m+1, c')$, then right along row $m+1$ to $(m+1, c)$ and up that column to the end. We get

$$Y = \begin{array}{llllll|lll}
 & & & & & & & & \\
\vdots & & \vdots & & \vdots & & & & \\
t'_{r+1,1} & \cdots & & & \cdots & & & & \\
t'_{r,1} & \cdots & & & \cdots & & a'_r & & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \\
2m+1 & \cdots & 2m+1 & 2m+2 & \cdots & 2m+2 & a'_{m+1} & & \\
2m-1 & \cdots & 2m-1 & 2m & \cdots & 2m & 2m+1 & b'_m & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
3 & \cdots & 3 & 4 & \cdots & 4 & 5 & b'_2 & \cdots \\
1 & \cdots & 1 & 2 & \cdots & 2 & 3 & b'_1 & \cdots
\end{array} \tag{8.5}$$

We get that $\mu' = \mu(Y) = \mu$. Let $[y_{i,j}] = Y$ and we have that $y_{1,1} \leqslant 2$. In Step 2, for $k = 1$ to m, we have $x = 2k-1$ and the conditions $y_{k,\mu'_k} \geqslant b'_k \geqslant b_k \geqslant 2k+2 > x+2$ and $y_{k,1} \leqslant 2k \leqslant x+1$ hold. The loop sets $y_{k+1,j}$ in column $1 \leqslant j \leqslant c-1$ to the same values, so no change occurs here. That is $d_2 = \cdots = d_{m+1} = 0$. Now for $k = m+1$ to r we have that conditions $y_{k,\mu'_k} = a'_k \geqslant a'_{m+1} + 2(k-m-1) \geqslant 2k+2 > x+2$ and $y_{k,1} = 2k-1 < x+1$ hold. The loop sets $y_{k+1,j} = 2k+1$ for $1 \leqslant j \leqslant c'-1$ and $y_{k+1,j} = 2k+2$ for $c' \leqslant j \leqslant c-1$. The loop stops after $k = r$ since $y_{r+1,\mu_{r+1}} = 2r+2 = x+1 \ngeqslant x+2$. We have $d_{r+2} = \cdots = d_l = 0$ and for $m+1 \leqslant k \leqslant r$ we have

$$d_{k+1} = t'_{k+1,1} + \cdots + t'_{k+1,c-1} - \bigl(c'-1\bigr)(2k+1) - \bigl(c-c'\bigr)(2k+2) = a_k - a'_k.$$

The second equality follows from comparing the $k+1$th entry of $s(T')$ in Eq. (8.3) with the row sum of $s(T')$ in the framed tableau in Eq. (8.4). We also remind the reader that from the start, row $k+1$ of T is such that $s_{k+1} - a_{k+1} = (c'-1)(2k+1) + (c-c')(2k+2)$. The row sum $s(Y)$ for Y in Eq. (8.5) is obtained from $s(T')$ in Eq. (8.3). We have

$$\begin{aligned} s(Y) = (&s_1, \ldots, s_m, s_{m+1} + a'_{m+1} - a_{m+1}, s_{m+2} + a_{m+1} - a_{m+2} - a'_{m+1} + a'_{m+2}, \ldots, \\ &s_r + a_{r-1} - a_r - a'_{r-1} + a'_r, s_{r+1} + a_r - a'_r, s_{r+2}, \ldots, s_l). \end{aligned}$$

The expression for s' at the end of Step 2 in the procedure ${}_1T'$ then gives us $s' = s = s(T)$ and then ${}_1T' = \mathrm{Fram}(\mu, s) = T$.

Case 5. Row $k=1$ of T contains 2's and 3's, possibly some 1's and possibly some numbers greater than or equal to 4. Depending on the numbers appearing in the first row of T, we have

$$T = \begin{array}{|llllll|} \vdots & \vdots & & & & \\ 3\cdots 3 & 4\cdots 4 & \cdots & & \cdots & \\ 1\cdots 1 & 2\cdots 2 & 3\cdots 3 & c\cdots & & \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|llll|} \vdots & \vdots & & \\ 3\cdots 3 & c\cdots c' & \cdots & \\ 1\cdots 1 & 2\cdots 2 & 3\cdots 3 & \\ \hline \end{array}$$

where $c, c' \geqslant 4$. If there is no c in the first row, then from Lemma 6.6, we are not forced to have 4 above the 2's in the first row and there is thus no restrictions on the numbers above those 2's. We use induction on the length l of T to prove this case. For $l(T)=1$, it is easy to check that all the procedures are well defined and ${}_1(T \leftarrow 1) = T$. Assume that the result is true up to $l(T)=n$, and $T \leftarrow 1$ preserves the added 1 and all the 1's and 2's in T. That is the 1's and 2's of Y in Step 1 of $T \leftarrow 1$ are left unchanged in the remaining steps. This was the situation in Cases 1–4 above. For $l(T)=n+1$, let R denote the first row of T and T_2 denote the remaining tableau. That is T_2 consists of rows 2 and up of T. From Lemma 6.10 we know that T_2 is a framed tableau of length n. In Step 1 of $T \leftarrow 1$, to get Y, we insert 1 in R. We then have that 3 is bumped up and inserted in T_2. Denote by Y_2 the result of Step 1 of $T_2 \leftarrow 3$. Clearly Y_2 is also the tableau we get from rows 2 and up of Y. We have

$$T = \begin{array}{|l|} T_2 \\ \hline R \\ \hline \end{array} \quad \text{and} \quad Y = \begin{array}{|l|} Y_2 \\ \hline 1\,R' \\ \hline \end{array}$$

It is important to remark that the number of 1's in the first row of Y is exactly the number of 3's in the first row of Y_2. In Step 2 of $T \leftarrow 1$, for $k=1$ we have $y_{1,\mu_1} \geqslant 3 \geqslant x+2$. In the case when there are numbers $c \geqslant 4$ in the first row of T, we must have 4 above each 2 in the first row. The first loop of Step 2 just sets all values 3 and 4 back to the same values. Hence $d_2 = 0$ in this case. If there are only 1's, 2's and 3's in the first row of T, then there is no restriction above the 2's. But in this case, we have $y_{1,\mu_1} = 3 \ngeqslant x+2$ and no number above the 2's changes. Hence in all cases $d_2 = 0$. The remaining loops of Step 2 of $T \leftarrow 1$ are identical to Step 2 for $T_2 \leftarrow 3$. By the induction hypothesis and Lemma 8.1, $T_2 \leftarrow 3$ is well defined and gives a framed tableau $T_2' = T_2 \leftarrow 3$ such that all the 3's and 4's in the first line are the same as Y_2. The shape $\mu' = \mu(Y) = (\mu_1, \mu_2', \dots, \mu_{l'}')$ where $(\mu_2', \dots, \mu_{l'}') = \mu(Y_2) = \mu(T_2')$. Also $s' = (s_1 - 2, s_2', \dots, s_{l'}')$ where $(s_2', \dots, s_{l'}') = s(T_2')$. It is clear, by definition, that $(\mu(T_2'), s(T_2'))$ satisfies the framing condition. In fact, since the smallest entry of T_2 is 3, we also have that $T_2 - 2$ is a framed tableau. This implies that $s_i' \geqslant (2i-1)\mu_j'$ for $2 \leqslant i \leqslant l'$. Clearly, $s_1 - 2 \geqslant \mu_1$, so we only need to verify Condition 2 of Definition 6.1 for $i=1$. If $\mu_1 > \mu_2'$, then there is nothing to check. By Cases 1–4 and by induction, we remark that $s_1' \geqslant s_1 - 2$. This implies that for T_2', we have $s_2' \geqslant s_2 - 2$. Hence if $\mu_1 = \mu_2 = \mu_2'$, then $s_2' \geqslant s_2 - 2 \geqslant s_1 + 2\mu_1 - 2 = (s_1 - 2) + 2\mu_1$. We are left to consider the case where $\mu_1 = \mu_2' = \mu_2 + 1$. This may only happen if all the entries in the second row of T are only 3's and 4's and in this case

$$T \leftarrow 1 = \begin{array}{|lll|} \vdots & \vdots & \\ 3\cdots 3 & 4\cdots 4 & \\ 1\cdots 1 & 2\cdots 2 & 3 \\ \hline \end{array} \leftarrow 1 = \begin{array}{|lll|} \vdots & \vdots & \\ 3\cdots 3 & 3\cdots 4 & 4 \\ 1\cdots 1 & 1\cdots 2 & 2 \\ \hline \end{array}$$

By induction, the entries in the second row are 3's and 4's. Clearly $s_2' \geqslant (s_1 - 2) + 2\mu_1$. We have that in all cases (μ', s') satisfy the framing condition and we get a well defined framed tableau $T' = \mathrm{Fram}(\mu', s') = T \leftarrow 1$. All the 1's and 2's in the first row of Y are preserved in T'.

Now we consider the procedure ${}_1T'$. Let T_2' be the framed tableau formed by rows 2 and up of T'. In Step 1, we get a tableau Y with a 1 replaced by a 3 in the first row of T', and a tableau Y_2 in rows 2 and up of Y. Again it is clear that Y_2 is the same as the one obtained in Step 1 of ${}_3T_2'$. In Step 2, for $k=1$, we have $y_{1,1} \leqslant 2 = x+1$ and $y_{1,\mu_1} \geqslant 3 = x+2$. The same argument as above shows that $d_2 = 0$. The remaining loops of Step 2 of ${}_1T'$ are the same as Step 2 in ${}_3T'_2$. By the induction hypothesis and Lemma 8.1, we know that ${}_3T'_2 = T_2$ is well defined and gives rows 2 and up of T. The first row sum of Y is now s_1, so at the end of Step 2 we have $s' = s(T)$. Also for $\mu' = \mu(Y)$, we clearly have $\mu'_1 = \mu_1$ and by the induction hypothesis $\mu'_i = \mu_i$ for $i \geqslant 2$. Hence we get $T = \mathrm{Fram}(\mu, s) =_1 T'$. This proves Case 5.

Let $\mathcal{F}_{k,l} = \{T \text{ framed tableau} : \mu(T) \vdash l, s(T) \models k\}$ and let $\mathcal{P}_{k,l} = \{\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k\}$. So far, we have that $T =_x (T \leftarrow x)$ for all $T \in \mathcal{F}_{k,l}$ and x. This implies that the map $(T, x) \mapsto (T \leftarrow x)$ is injective. We have an injection $\mathcal{P}_{k,l} \hookrightarrow \mathcal{F}_{k,l}$ defined by $\lambda \mapsto (\emptyset \leftarrow \lambda)$. Let us pick $n > k$ and consider $\{F_T \Delta_n : T \in \mathcal{F}_{k,l}\} \subset A_n^{k,l}$. For $T \in \mathcal{F}_{k,l}$, let $(\mu_1, \ldots, \mu_r) = \mu(T)$, $(s_1, \ldots, s_r) = s(T)$ and F_T defined as in Eq. (5.1). Iterating Remark 4.7, we get

$$\begin{aligned} F_T \Delta_n &= F_{T_{\mu_1}} F_{T_{\mu_1 - 1}} \cdots F_{T_2} F_{T_1} \Delta_n \\ &= F_{T_{\mu_1}} F_{T_{\mu_1 - 1}} \cdots F_{T_2} (\Delta_{E_{T_1}^{f_1^o}[(0,0),(1,0),\ldots,(n-1,0)]} + \text{lower terms}) \\ &= F_{T_{\mu_1}} F_{T_{\mu_1 - 1}} \cdots F_{T_3} (\Delta_{E_{T_2}^{f_2^o} E_{T_1}^{f_1^o}[(0,0),(1,0),\ldots,(n-1,0)]} + \text{lower terms}) = \cdots \\ &= \Delta_{E_{T_{\mu_1}}^{f_{\mu_1}^o} \circ \cdots \circ E_{T_1}^{f_1^o}[(0,0),(1,0),\ldots,(n-1,0)]} + \text{lower terms}, \end{aligned} \tag{8.6}$$

where $f_j^o : \{1, \ldots, \mu_j^t\} \to \{1, \ldots, n\}$, $1 \leqslant j \leqslant \mu_1$ are defined by $f_j^o(i) = i$ for all $1 \leqslant i \leqslant \mu_j^t$. So we have

$$\begin{aligned} &E_{T_{\mu_1}}^{f_{\mu_1}^o} \circ \cdots \circ E_{T_1}^{f_1^o}\big[(0,0), (1,0), \ldots, (n-1,0)\big] \\ &\quad = \big[(0,0), \ldots, (n-r-s_r, \mu_r), \ldots, (n-1-s_1, \mu_1)\big]. \end{aligned}$$

Since $n > k$, thus we have $\Delta_{[(0,0),\ldots,(n-r-s_r,\mu_r),\ldots,(n-1-s_1,\mu_1)]} \neq 0$, which gives the leading diagram of $F_T \Delta_n$. Proposition 6.4 gives us that for different framed tableaux T we get different pairs (μ, s), hence different leading terms for $F_T \Delta_n$. This gives us that the set $\{F_T \Delta_n\colon T \in \mathcal{F}_{k,l}\} \subset A_n^{k,l}$ is linearly independent. Recall that the dimension of $A_n^{k,l}$ is the coefficient of $q^k t^l$ in $\widetilde{C}_n(q,t)$. We claim that this coefficient is equal to $|\mathcal{P}_{k,l}|$. Indeed for $k < n$, we have that any partition $\lambda \in \mathcal{P}_{k,l}$ satisfy $\lambda_1 = k - \lambda_2 - \cdots - \lambda_l \leqslant k - l + 1 < n - l + 1$. For $k < n$, if we consider $\mu = \lambda^t \in \mathcal{P}_{k,l}$ as in Remark 2.4, then we have a bijection between $\lambda \in \mathcal{P}_{k,l}$ and the Catalan paths with coarea equal to k and a single bounce l. This gives

$$|\mathcal{P}_{k,l}| \leqslant |\mathcal{F}_{k,l}| \leqslant \dim A_n^{k,l} = |\mathcal{P}_{k,l}|,$$

and we must have equality everywhere. This shows that the map $(T, x) \mapsto (T \leftarrow x)$ must be surjective. Hence ${}_xT$ is well defined everywhere and inverse to $T \leftarrow x$. $\square$

The computation in Eq. (8.6) shows the following:

Corollary 8.3. *Let $\widetilde{C}_{n,k,l}$ be the coefficients of $q^k t^l$ in $\widetilde{C}_n(q,t)$. We have:*

1. *If $k < n$, then $\widetilde{C}_{n,k,l}$ is the number of partitions of k into l parts;*
2. *There exists a natural map $\lambda \mapsto F_{\emptyset \leftarrow \lambda}$ between partitions and F-operators such that if $|\lambda| < n$, then the set of polynomials $\{F_{\emptyset \leftarrow \lambda} \Delta_n\colon \lambda \in \mathcal{P}_{k,l}\}$ forms a basis of the space $A_n^{k,l}$.*

Remark 8.4. When $k \geqslant n$ the leading diagram for $F_{\emptyset \leftarrow \lambda}$ is not necessarily given by Remark 4.7. This complicates the investigation of finding a basis for those cases. We were successful in finding bases for any k and $l = 3$, but the analyses is much more complicated.

Remark 8.5. We presented the work here with the perspective of finding a basis for $A_n^{k,l}$. But the combinatorics of the bijection between partitions and framed tableaux via $T \leftarrow x$ and ${}_xT$ could be very interesting in their own right and have different applications.

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