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Upper bounds for the Stanley–Wilf limit of 1324 and other layered patterns

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ABSTRACT

We prove that the Stanley–Wilf limit of any layered permutation pattern of length ℓ is at most $4\ell^2$, and that the Stanley–Wilf limit of the pattern 1324 is at most 16. These bounds follow from a more general result showing that a permutation avoiding a pattern of a special form is a merge of two permutations, each of which avoids a smaller pattern.
We also conjecture that, for any $k \geqslant 0$, the set of 1324-avoiding permutations with k inversions contains at least as many permutations of length $n+1$ as those of length n. We show that if this is true then the Stanley–Wilf limit for 1324 is at most $e^{\pi\sqrt{2/3}} \simeq 13.001954$.

1. Introduction

For a permutation pattern τ, let $\mathcal{S}_n(\tau)$ be the set of permutations of length n avoiding τ, and let $S_n(\tau)$ be the cardinality of $\mathcal{S}_n(\tau)$. In 2004, Marcus and Tardos [17] proved the Stanley–Wilf conjecture, stating that, for any pattern τ, $S_n(\tau) < C^n$ for some constant C depending only on τ. The limit

$$L(\tau) = \lim_{n\to\infty} S_n(\tau)^{1/n}$$

is called the *Stanley–Wilf limit* for τ. Arratia [4] has shown that this limit exists for any pattern τ.

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Marcus and Tardos's original proof gives a general upper bound for the Stanley–Wilf limit of the form

$$L(\tau) \leqslant 15^{2\ell^4\binom{\ell^2}{\ell}},$$

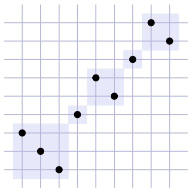
where $\ell = |\tau|$. This bound was later improved by Cibulka [12] to

$$L(\tau) \leqslant 2^{O(\ell \log \ell)}.$$

A result of Valtr presented in [15] shows that for any pattern τ of length ℓ we have $L(\tau) \geqslant (1 - o(1))\ell^2/e^3$ as $\ell \to \infty$.

For certain families of patterns, more precise estimates are available. An important example is given by the *layered patterns*. A permutation τ is layered if it is a concatenation of decreasing sequences, the letters of each sequence being smaller than the letters in the following sequences. An example is

321465798 =



whose layers are 321, 4, 65, 7, and 98. Bóna [8–10] has shown that $(\ell - 1)^2 \leqslant L(\tau) \leqslant 2^{O(\ell)}$ for any layered pattern τ of length ℓ.

A motivation for the study of Stanley–Wilf limits of layered patterns stems from the fact that these patterns appear to yield the largest values of $S_n(\tau)$ among the patterns τ of a given length. More precisely, computer enumeration of $S_n(\tau)$ for patterns τ of fixed size up to eight and small n suggests that $S_n(\tau)$ is maximized by a layered pattern τ. This supports the following conjecture.

Conjecture 1. *(See [11].) Among the patterns of a given length, the largest Stanley–Wilf limit is attained by a layered pattern.*

We remark that Bóna, just after Theorem 4.6 in [11], presents a stronger version of Conjecture 1 as a 'long-standing conjecture'. The stronger conjecture states that the maximum for $L(\tau)$ over all τ of a given length ℓ is attained by $1 \oplus 21 \oplus \cdots \oplus 21 \oplus 1$ or $1 \oplus 21 \oplus \cdots \oplus 21$, depending on whether ℓ is even or odd.

Two patterns τ and σ are *Wilf equivalent* if $S_n(\tau) = S_n(\sigma)$ for all n. If τ is of length three, then $S_n(\tau)$ is the nth Catalan number, and so $L(\tau) = 4$. For patterns of length four there are three Wilf (equivalence) classes, represented by 1234, 1342 and 1324. Regev [18] proved that $L(1234) = 9$ and, more generally, that $L(12\cdots\ell) = (\ell - 1)^2$. Bóna [6] proved that $L(1342) = 8$. In fact, exact formulas for $S_n(1234)$ and $S_n(1342)$ are known, the first one being a special case of such a result for the increasing pattern of any length, established by Gessel [14], the second one obtained by Bóna [6].

The last Wilf class of patterns of length 4, represented by the pattern 1324, has so far resisted all attempts at exact enumeration or exact asymptotic formulas. A lower bound was found by Albert et al. [1], who showed that $S_n(1324) > 9.47^n$. This bound disproved a conjecture of Arratia [4] that for any pattern τ of length ℓ, $L(\tau)$ is at most $(\ell - 1)^2$. Recently, Madras and Liu [16] studied random permutations avoiding 4231, using Markov chain Monte Carlo methods. Their conclusion is "... the conservative "subjective" 95% confidence interval of $[10.71, 11.83]$ on $L(4231)$". Of course, $L(1324) = L(4231)$.

As far as we know, the best published upper bound so far for $L(1324)$ is 288, proved by Bóna [9].[3]

[3] In an earlier version of the proof [7], Bóna claims that $L(1324) \leqslant 36$, but the argument appears flawed.

In this paper, we present a general method that allows us to bound the Stanley–Wilf limit of an arbitrary layered pattern, and which may also be used for non-layered patterns of a special form. In particular, for an arbitrary layered pattern τ of length ℓ, we prove the bound $L(\tau) \leqslant 4\ell^2$, improving Bóna's bound of $2^{O(\ell)}$. Our bound is sharp up to a multiplicative constant, since $(\ell-1)^2 \leqslant L(\tau)$. If Conjecture 1 holds, then this result has the following consequence, which we state as a separate conjecture.

Conjecture 2. *For any pattern τ of length ℓ, we have $L(\tau) \leqslant 4\ell^2$.*

For some specific patterns, we are able to give better estimates. Notably, we are able to show that $L(1324) \leqslant 16$. These results appear in Section 2.

In Section 3, we investigate an approach that may lead to a further improvement of our bounds, based on the analysis of pattern-avoiding permutations with a restricted number of inversions. For a pattern τ, let $\mathcal{S}_n^k(\tau)$ be the set of 1324-avoiding permutations of length n with exactly k inversions, and let $S_n^k(\tau)$ be its cardinality. We conjecture that for every n and k, we have $S_n^k(1324) \leqslant S_{n+1}^k(1324)$. We prove that if this conjecture holds then

$$L(1324) \leqslant e^{\pi\sqrt{2/3}} \simeq 13.001954.$$

In the last section, we extend our considerations to more general patterns. We conjecture that the inequality $S_n^k(\tau) \leqslant S_{n+1}^k(\tau)$ is valid for any pattern τ other than the increasing patterns. As an indirect support of this conjecture, we describe how the asymptotic behavior of $S_n^k(\tau)$ for k fixed and n going to infinity depends on the structure of τ.

2. The Stanley–Wilf limit of 1324 is at most 16

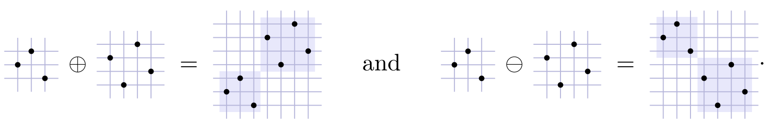
Let us begin by recalling some standard notions related to permutation patterns. Two sequences of integers $a_1 \cdots a_k$ and $b_1 \cdots b_k$ are *order-isomorphic* if for every $i, j \in \{1, \ldots, k\}$ we have $a_i < a_j \Leftrightarrow b_i < b_j$. We let $\mathcal{S}_n$ be the set of permutations of the letters $\{1, 2, \ldots, n\}$. For a permutation $\pi \in \mathcal{S}_n$ and a set $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, we let $\pi[I]$ denote the permutation in $\mathcal{S}_k$ order-isomorphic to the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$. A permutation $\pi \in \mathcal{S}_n$ *contains* a permutation $\sigma \in \mathcal{S}_k$ if $\pi[I] = \sigma$ for some I. In such context, σ is often called *a pattern*.

We say that a permutation $\pi \in \mathcal{S}_n$ is a *merge* of two permutations $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_{n-k}$ if there are two disjoint sets I and J such that $I \cup J = \{1, \ldots, n\}$, $\pi[I] = \sigma$ and $\pi[J] = \tau$. For example, 3175624 is a merge of $\sigma = 132$ and $\tau = 1423$ with $I = \{1, 3, 4\}$ and $J = \{2, 5, 6, 7\}$.

For a pair of permutations $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_\ell$, their *direct sum*, denoted by $\sigma \oplus \tau$, and their *skew sum*, denoted by $\sigma \ominus \tau$, are defined by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leqslant k, \\ \tau(i-k)+k & \text{if } i > k \end{cases} \quad \text{and} \quad (\sigma \ominus \tau)(i) = \begin{cases} \sigma(i)+\ell & \text{if } i \leqslant k, \\ \tau(i-k) & \text{if } i > k. \end{cases}$$

For example, $231 \oplus 3142 = 2316475$ and $231 \ominus 3142 = 6753142$, or in pictures



A permutation is *decomposable* if it can be written as a direct sum of two nonempty permutations, otherwise it is *indecomposable*. Every permutation π can be uniquely written as a direct sum (possibly with a single summand) of the form $\pi = \alpha_1 \oplus \cdots \oplus \alpha_m$, where each summand α_i is indecomposable. The summands $\alpha_1, \ldots, \alpha_m$ are the *components* of π. For example, the permutation 31425786 is decomposed as $31425786 = 3142 \oplus 1 \oplus 231$, which means that it has three components, corresponding to 3142, 5 and 786.

The key tool in our approach is the next proposition, which shows that a permutation avoiding a pattern of a particular kind is a merge of two permutations, each of them avoiding a smaller pattern.

Theorem 3. *Let σ, τ, and ρ be three (possibly empty) permutations. Then every permutation avoiding $\sigma \oplus (\tau \ominus 1) \oplus \rho$ is a merge of a permutation avoiding $\sigma \oplus (\tau \ominus 1)$ and a permutation avoiding $(\tau \ominus 1) \oplus \rho$.*

Proof. We may assume that σ and ρ are nonempty, otherwise the lemma holds trivially. Let $\pi = \pi_1 \cdots \pi_n$ be a permutation. Successively color the π_i, in the order $\pi_1, \pi_2, \ldots, \pi_n$, red or blue according to the following rule:

> If coloring π_i red completes a red occurrence of $\sigma \oplus (\tau \ominus 1)$, or if there already is a blue element smaller than π_i, then color π_i blue; otherwise color π_i red.

Note that the first element, π_1, will always be colored red. Further, the red elements clearly avoid $\sigma \oplus (\tau \ominus 1)$. We claim that if π avoids $\sigma \oplus (\tau \ominus 1) \oplus \rho$ then the blue elements avoid $(\tau \ominus 1) \oplus \rho$, and we proceed by proving the contrapositive statement. Assume that there is a blue occurrence of $(\tau \ominus 1) \oplus \rho$. Let τ_B, 1_B, and ρ_B be the three sets of blue elements corresponding to the three parts τ, 1 and ρ forming the occurrence. In particular, 1_B contains a single element, which will be denoted by π_t.

Fix a blue element π_s such that $s \leqslant t$, $\pi_s \leqslant \pi_t$, and π_s is as small as possible with these properties. This means that π_s was colored blue for the reason that coloring it red would have completed a red occurrence of $\sigma \oplus (\tau \ominus 1)$. Therefore, π_s is the rightmost element of an occurrence of $\sigma \oplus (\tau \ominus 1)$, and all other elements of this occurrence are red. Let σ_R and τ_R be the sets of elements corresponding to σ and τ in this occurrence.

We now distinguish two cases depending on the relative position of τ_B and σ_R. If all the elements of σ_R precede all the elements of τ_B (including the case $\tau = \emptyset$), then $\sigma_R \cup \tau_B \cup 1_B \cup \rho_B$ forms an occurrence of $\sigma \oplus (\tau \ominus 1) \oplus \rho$. This is because each element of σ_R is smaller than π_s, which, in turn, is at most as large as π_t.

Suppose now that at least one element of σ_R is to the right of the leftmost element of τ_B. Then all the elements of τ_R are to the right of the leftmost element of τ_B. Consequently, all the elements of τ_R are smaller than the leftmost element of τ_B, otherwise they would be blue. Therefore all elements of τ_R are smaller than any element of ρ_B, and $\sigma_R \cup \tau_R \cup \{\pi_s\} \cup \rho_B$ is an occurrence of $\sigma \oplus (\tau \ominus 1) \oplus \rho$. $\square$

We remark that the special case $\tau = \emptyset$ in Theorem 3 corresponds to an argument by Bóna [11]. For our purposes, a more important special case corresponds to $\sigma = \tau = \rho = 1$, which gives a representation of any 1324-avoiding permutation as a merge of a 132-avoiding permutation and a 213-avoiding permutation. For instance, coloring the 1324-avoiding permutation 364251 we find that the red and blue elements are 3621 and 45, respectively.

To apply Theorem 3, we need an estimate on the number of permutations obtainable by merging two permutations from given *permutation classes*. A permutation class is a set of permutations $\mathcal{C}$ that is down-closed for the containment relation, that is, if $\tau \in \mathcal{C}$ and τ contains σ, then $\sigma \in \mathcal{C}$. The *growth rate* of $\mathcal{C}$ is defined as $\limsup_{n\to\infty} |\mathcal{C} \cap \mathcal{S}_n|^{1/n}$. As pointed out by Arratia [4], if $\mathcal{C} = \mathcal{S}(\tau)$ is a principal class, that is, $\mathcal{C}$ is the set of permutations avoiding a single pattern τ, then the lim sup is actually a limit, and the growth rate of $\mathcal{C}$ is the Stanley–Wilf limit $L(\tau)$ of τ.

The following theorem is due to Albert et al. [2]. For completeness we include a short proof.

Theorem 4. *(See Theorem 4 in [2].) Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be three permutation classes with growth rates α, β and γ, respectively. If every permutation of $\mathcal{C}$ can be expressed as a merge of a permutation from $\mathcal{A}$ and a permutation from $\mathcal{B}$, then*

$$\sqrt{\gamma} \leqslant \sqrt{\alpha} + \sqrt{\beta}.$$

Proof. Let a_n, b_n and c_n be the numbers of permutations of length n in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. For every $\varepsilon > 0$, we may fix a constant K such that $a_n \leqslant K\alpha^n(1+\varepsilon)^n$ and $b_n \leqslant K\beta^n(1+\varepsilon)^n$ for each n.

There are at most $\binom{n}{k}^2$ possibilities to merge a given permutation of length k with a given permutation of length $n-k$, because we get to choose k positions and k values to be covered by the first permutation. We thus have

$$\begin{aligned} c_n &\leqslant \sum_{k=0}^{n} \binom{n}{k}^2 a_k b_{n-k} \leqslant K^2(1+\varepsilon)^n \sum_{k=0}^{n} \binom{n}{k}^2 \alpha^k \beta^{n-k} \\ &\leqslant K^2(1+\varepsilon)^n \sum_{k=0}^{n} \left(\binom{n}{k} \sqrt{\alpha}^k \sqrt{\beta}^{n-k} \right)^2 \\ &\leqslant K^2(1+\varepsilon)^n \left(\sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k \sqrt{\beta}^{n-k} \right)^2 \\ &\leqslant K^2(1+\varepsilon)^n (\sqrt{\alpha}+\sqrt{\beta})^{2n}, \end{aligned}$$

which implies that γ is at most $(\sqrt{\alpha}+\sqrt{\beta})^2$, as claimed. □

Taking $\sigma = \tau = \rho = 1$ in Theorem 3, using the fact that $L(132) = L(213) = 4$, and applying Theorem 4, we get the following result.

Corollary 5. *The Stanley–Wilf limit of* 1324 *is at most* 16.

We may apply Theorems 3 and 4 to get an upper bound for the Stanley–Wilf limit of any layered pattern. Let $\oplus^n 1$ denote the identity permutation $12\cdots n$, and let $\ominus^n 1$ denote its reverse $n\cdots 21$.

Let $\alpha(\ell_1, \ell_2, \ldots, \ell_m)$ denote the Stanley–Wilf limit of the generic layered permutation $(\ominus^{\ell_1} 1) \oplus (\ominus^{\ell_2} 1) \oplus \cdots \oplus (\ominus^{\ell_m} 1)$.

Theorem 6. *For any integer $m \geqslant 2$ and positive integers $\ell_1, \ldots, \ell_m$, we have*

$$\alpha(\ell_1, \ldots, \ell_m) \leqslant (2\ell - \ell_1 - \ell_m - m + 1)^2,$$

where $\ell = \ell_1 + \cdots + \ell_m$.

Proof. We proceed by induction on m. Let $m = 2$. By a result of Backelin, West and Xin [5], we know that for arbitrary σ, the pattern $(\oplus^n 1) \oplus \sigma$ is Wilf equivalent to the pattern $(\ominus^n 1) \oplus \sigma$. Moreover, from Regev's [18] result we know that $\oplus^k 1$ has Stanley–Wilf limit $(k-1)^2$. Thus $\alpha(\ell_1, \ell_2) = (\ell_1 + \ell_2 - 1)^2$.

Assume now that $m \geqslant 3$. Combining Theorems 3 and 4, we see that

$$\begin{aligned} \sqrt{\alpha(\ell_1, \ldots, \ell_m)} &\leqslant \sqrt{\alpha(\ell_1, \ell_2)} + \sqrt{\alpha(\ell_2, \ldots, \ell_m)} \\ &\leqslant (\ell_1 + \ell_2 - 1) + \big(2(\ell_2 + \cdots + \ell_m) - \ell_2 - \ell_m - m + 2\big), \end{aligned}$$

which gives the desired bound. □

Corollary 7. *A layered permutation of length ℓ has Stanley–Wilf limit at most $4\ell^2$.*

As we pointed out in the introduction, any layered pattern of length ℓ has Stanley–Wilf limit at least $(\ell - 1)^2$, so the quadratic bound in the previous corollary is best possible.

3. On 1324-avoiding permutations with a fixed number of inversions

An *inversion* in a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is a pair (i, j) such that $1 \leqslant i < j \leqslant n$ and $\pi_i > \pi_j$. The number of inversions in π is denoted $\textsc{inv}(\pi)$. In this section we will consider the distribution of inversions over 1324-avoiding permutations. We will show that a certain conjectured property of this distribution implies an improved upper bound for $L(1324)$. Recall that $S_n^k(\tau)$ is the number of τ-avoiding permutations of length n with k inversions.

To illustrate our approach, and to introduce tools we use later, we will first derive an upper bound on $L(132)$ (even though we know that $L(132) = 4$). Here are the first few rows of the distribution of inversions over $\mathcal{S}(132)$, where the kth entry in the nth row is the number $S_n^k(132)$, and n starts at 1, k at 0:

1																
1	1															
1	1	2	1													
1	1	2	3	3	3	1										
1	1	2	3	5	5	7	7	6	4	1						
1	1	2	3	5	7	9	11	14	16	16	17	14	10	5	1	
1	1	2	3	5	7	11	13	18	22	28	32	37	40	44	43	...
1	1	2	3	5	7	11	15	20	26	34	42	53	63	73	85	...

We make two observations: (1) the columns are weakly increasing when read from top to bottom; (2) each column is eventually constant, as shown by the grayed area. If we can prove this and give a formula for the eventual value $c(k)$ of the kth column, then we can bound $S_n(132)$ by $\sum_{k \leqslant \binom{n}{2}} c(k)$. For instance, $S_4(132) \leqslant 1+1+2+3+5+7+11 = 30$.

To prove that the columns are weakly increasing is easy: the map $\pi \mapsto \pi \oplus 1$ from $\mathcal{S}_{n-1}(132)$ to $\mathcal{S}_n(132)$ is injective and inversion-preserving. Our goal is to show that each column is eventually constant and to find the formula for the eventual value of kth column.

Lemma 8. *Let $\pi \in \mathcal{S}_n$ and let c be the number of components of π. Then*

$$\mathrm{INV}(\pi) \geqslant n - c.$$

Proof. We use induction on n. The case $n = 1$ is trivial. Assume $n > 1$ and write π as the sum of its components $\pi = \alpha_1 \oplus \cdots \oplus \alpha_c$. Note that if (i, j) is an inversion in $\pi = \pi_1\pi_2\cdots\pi_n$ then π_i and π_j must belong to the same component of π. Thus, if $c > 1$, we have

$$\begin{aligned}\mathrm{INV}(\pi) &= \mathrm{INV}(\alpha_1) + \cdots + \mathrm{INV}(\alpha_c)\\ &\geqslant |\alpha_1| - 1 + \cdots + |\alpha_c| - 1 \quad \text{by induction}\\ &= n - c.\end{aligned}$$

Assume then that $c = 1$ and let i be the position of n in π, that is, $\pi_i = n$. Also, let $\sigma \in \mathcal{S}_{n-1}$ be the permutation obtained by removing n from π. Although σ can be any permutation, the number i has some restrictions. Obviously, $i \leqslant |\sigma| = n-1$, since if $i = n$, then n would constitute a component of its own, contradicting the assumption that $c = 1$. More generally, if we decompose σ into its components

$$\sigma = \beta_1 \oplus \cdots \oplus \beta_d,$$

then we see that $i \leqslant |\beta_1|$. Thus

$$\begin{aligned}\mathrm{INV}(\pi) &= \mathrm{INV}(\sigma) + n - i\\ &\geqslant |\sigma| - d + n - i \quad \text{by induction}\\ &= n - 1 - i + n - d \quad \text{since } |\sigma| = n-1\\ &\geqslant n - 1 - i + |\beta_1| \quad \text{since } n - d \geqslant |\beta_1|\\ &\geqslant n - 1 \quad \text{since } i \leqslant |\beta_1|,\end{aligned}$$

as claimed. $\square$

Definition 9. The *inversion table* of a permutation $\pi = \pi_1 \cdots \pi_n$ is the sequence $b_1 b_2 \cdots b_n$, where b_i is the number of letters in π to the right of π_i that are smaller than π_i.

For example, the inversion table of 352614 is 231200. Clearly, the number of inversions in a permutation equals the sum of the entries in the inversion table. It is also easy to see that the map taking a permutation to its inversion table is a bijection. In other words, a permutation can be reconstructed from this table.

Lemma 10. *A permutation avoids the pattern* 132 *if and only if its inversion table is weakly decreasing.*

Proof. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation with inversion table $b_1 b_2 \cdots b_n$. Suppose that $b_i < b_{i+1}$ for some i. Then we must have $\pi_i < \pi_{i+1}$, and the number of letters to the right of π_{i+1} that are smaller than π_{i+1} is greater than the number of such letters that are smaller than π_i. Thus there is a $j > i+1$ such that $\pi_i < \pi_j < \pi_{i+1}$. But then $\pi_i\pi_{i+1}\pi_j$ form the pattern 132.

Conversely, assume that $\pi_i\pi_j\pi_k$ is an occurrence of 132 in π, and assume that the occurrence has been chosen in such a way that the index i is as large as possible. This choice implies that π_{i+1} is greater than π_k, and consequently, $b_{i+1} > b_i$. $\square$

A *partition* of an integer k is a weakly decreasing sequence of positive integers whose sum is k. By dropping the trailing zeros from the inversion table of a permutation $\pi \in \mathcal{S}_n^k(132)$ we obtain a partition λ of k. We then say that λ *represents* π. For instance, the inversion table of $\pi = 65723148$ is 5441100, so π is represented by the integer partition $5+4+4+1+1$. Two distinct 132-avoiding permutations of the same size are represented by distinct integer partitions. On the other hand, a permutation $\pi \in \mathcal{S}_n(132)$ is represented by the same partition as $\pi \oplus 1$.

In any 132-avoiding permutation π, only the first component may have size greater than 1, so π has a decomposition of the form $\sigma \oplus 1 \oplus 1 \oplus \cdots \oplus 1$ where σ is an indecomposable permutation represented by the same partition as π. It is easy to see that a partition λ of an integer k represents a unique indecomposable permutation σ, and by Lemma 8, σ has size at most $k+1$. Consequently, for every $n \geqslant k+1$, λ represents a unique permutation π of size n. This yields the following result.

Proposition 11. *For every $k < n$, we have $S_n^k(132) = p(k)$, where $p(k)$ is the number of integer partitions of k.*

The following rather elementary upper bound for $p(k)$ can, for example, be found in [3, pp. 316–318].

Lemma 12. *Let $p(k)$ be the number of integer partitions of k. For $k > 0$ we have*

$$p(k) < \rho^{\sqrt{k}},$$

where $\rho = e^{\pi\sqrt{\frac{2}{3}}} \simeq 13.001954$.

Letting $m = \binom{n}{2}$, we thus have

$$\begin{aligned} S_n(132) &= \sum_{k=0}^{m} S_n^k(132) \leqslant (m+1)S_{m+1}^m \\ &= (m+1)p(m+1) < (m+1)e^{\pi\sqrt{\frac{2(m+1)}{3}}}, \end{aligned}$$

and

$$\begin{aligned} L(132) &= \lim_{n\to\infty} \big(S_n(132)\big)^{1/n} \leqslant \lim_{n\to\infty} \big(n^2/2 + O(n)\big)^{1/n} e^{\frac{\pi}{n}\sqrt{\frac{2}{3}(\frac{n^2}{2}+O(n))}} \\ &= e^{\frac{\pi}{\sqrt{3}}} \simeq 6.1337. \end{aligned}$$

We have now seen that counting 132-avoiding permutations with relatively few inversions leads to an upper bound for $L(132)$. Can we similarly bound $L(1324)$? These are the first few rows of the distribution of inversions over $\mathcal{S}(1324)$, starting at $n=1$ and where k starts at 0 in each row:

1													
1	1												
1	2	2	1										
1	2	5	6	5	3	1							
1	2	5	10	16	20	20	15	9	4	1			
1	2	5	10	20	32	51	67	79	80	68	49	29	...
1	2	5	10	20	36	61	96	148	208	268	321	351	...
1	2	5	10	20	36	65	106	171	262	397	568	784	...
1	2	5	10	20	36	65	110	181	286	443	664	985	...
1	2	5	10	20	36	65	110	185	296	467	714	1077	...
1	2	5	10	20	36	65	110	185	300	477	738	1127	...
1	2	5	10	20	36	65	110	185	300	481	748	1151	...

Again, it seems as though (1) the columns are increasing when read from top to bottom; (2) each column is eventually constant. Unfortunately, we have not been able to show that the columns are increasing, but we conjecture that they are. The following has been verified for $n \leqslant 15$ and all k by exhaustive computer computation:

Conjecture 13 *(Increasing columns). For all non-negative integers n and k, we have $S_n^k(1324) \leqslant S_{n+1}^k(1324)$.*

We are, however, able to say what the fixed sequence is. First we need some lemmas.

Lemma 14. *If $\pi \in S_n^k(1324)$ and $k < n-1$, then*

$$\pi = \sigma \oplus 1 \oplus \cdots \oplus 1 \oplus \tau$$

for some nonempty permutations $\sigma \in \mathcal{S}(132)$ and $\tau \in \mathcal{S}(213)$.

Proof. Let c be the number of components in π, and write $\pi = \alpha_1 \oplus \cdots \oplus \alpha_c$. Since $k < n-1$ it follows from Lemma 8 that $c \geqslant 2$. There can be no inversions in π except within the first component and within the last component, since otherwise we would have an occurrence of 1324; thus $\alpha_2 = \cdots = \alpha_{c-1} = 1$. Since the letters in the first component have a larger letter to their right, the first component must avoid 132 (so that π avoids 1324). Likewise, the last component must avoid 213. □

Let $\mathcal{P}(m)$ be the set of partitions of an integer m, and let $\mathcal{Q}(m)$ be the set

$$\bigl\{(\lambda, \mu)\colon\ \lambda \in \mathcal{P}(i),\ \mu \in \mathcal{P}(j),\ i+j=m\bigr\}.$$

Proposition 15. *For $k < n-1$, there is a one-to-one correspondence between $\mathcal{S}_n^k(1324)$ and $\mathcal{Q}(k)$.*

Proof. Fix $k < n-1$ and choose $\pi \in \mathcal{S}_n^k(1324)$. Let c be the number of components in π, and write $\pi = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_c$, where each α_i is nonempty. By Lemma 14, we know that $\alpha_1 \in \mathcal{S}(132)$, $\alpha_c \in \mathcal{S}(213)$, and $\alpha_2 = \cdots = \alpha_{c-1} = 1$. Let $\sigma = \alpha_1$ and $\tau = \alpha_c$. Let $i = \textsc{inv}(\sigma)$, and $j = \textsc{inv}(\tau)$. Note that $i + j = k$.

Let ℓ be the length of τ and let τ' be the reverse-complement of τ, that is, $\tau'_i = \ell + i - \tau_{\ell+1-i}$ for $i = 1, \ldots, \ell$. Then τ' is an indecomposable 132-avoiding partition with j inversions. Let $\lambda \in \mathcal{P}(i)$ and $\mu \in \mathcal{P}(j)$ be the partitions representing σ and τ', respectively. We then let $(\lambda, \mu) \in \mathcal{Q}(k)$ be the image of π.

To see that this is a bijection, choose a pair $(\lambda, \mu) \in \mathcal{Q}(k)$. Let i be the sum of λ, and j the sum of μ. Let σ and τ' be the unique indecomposable 132-avoiding permutations represented by λ and μ, respectively. Let τ be the reverse-complement of τ'. We have $n \geqslant k+2 = (i+1)+(j+1) \geqslant |\sigma|+|\tau|$. We can therefore construct a permutation $\pi = \sigma \oplus (\oplus^{n-|\sigma|-|\tau|}1) \oplus \tau$, which is the preimage of (λ, μ). $\square$

Lemma 16. *Let ρ be as in Lemma 12. Then, for $k > 0$,*

$$\left|\mathcal{Q}(k)\right| < (k+1)\rho^{\sqrt{2k}}.$$

Proof. We have

$$\begin{aligned}
\left|\mathcal{Q}(k)\right| &= \sum_{i=0}^{k} p(i)p(k-i) < \sum_{i=0}^{k} \rho^{\sqrt{i}+\sqrt{k-i}} \quad \text{by Lemma 12} \\
&\leqslant \sum_{i=0}^{k} \rho^{\sqrt{2k}} \quad \text{since } \sqrt{i}+\sqrt{k-i} \leqslant \sqrt{2k} \\
&= (k+1)\rho^{\sqrt{2k}},
\end{aligned}$$

as claimed $\square$

Theorem 17. *If Conjecture 13 is true, then the Stanley–Wilf limit for 1324 is at most $\rho = e^{\pi\sqrt{\frac{2}{3}}} \simeq 13.001954$.*

Proof. With $m = \binom{n}{2}$ we have

$$\begin{aligned}
S_n(1324) &= \sum_{k=0}^{m} S_n^k(1324) \leqslant \sum_{k=0}^{m} S_{m+2}^k(1324) \quad \text{by Conjecture 13} \\
&= \sum_{k=0}^{m} \left|\mathcal{Q}(k)\right| \quad \text{by Proposition 15} \\
&< \sum_{k=0}^{m} (k+1)\rho^{\sqrt{2k}} \quad \text{by Lemma 16} \\
&\leqslant (m+1)(m+1)\rho^{\sqrt{2m}} \\
&= \frac{1}{4}\left(n^2-n+2\right)^2 \rho^{n\sqrt{1-1/n}}.
\end{aligned}$$

On taking the nth root and letting $n \to \infty$, the result follows. $\square$

4. Generalizations

We have seen that the conjectured inequality $S_n^k(1324) \leqslant S_{n+1}^k(1324)$ implies an estimate on $L(1324)$. Let us now focus on the behavior of $S_n^k(\tau)$ for general patterns τ. Let us say that a pattern τ is INV-*monotone* if for every n and every k, we have the inequality $S_n^k(\tau) \leqslant S_{n+1}^k(\tau)$.

Recall that $\oplus^{\ell}1$ is the identity pattern $12\cdots\ell$. Let us first observe that a pattern of this form cannot be INV-monotone.

Lemma 18. *For any k and for any n large enough, we have $S_n^k(\oplus^{\ell}1) = 0$. In particular, $\oplus^{\ell}1$ is not* INV-*monotone for any $\ell \geqslant 2$.*

Proof. For $n > (\ell-1)(k+1)$, a well-known result of Erdős and Szekeres [13] guarantees that any permutation of length n has either an increasing subsequence of length ℓ or a decreasing subsequence of length $k+2$. Therefore, there can be no $\oplus^{\ell}1$-avoiding permutations of length n with k inversions. $\square$

On the other hand, some patterns are INV-monotone for trivial reasons, as shown by the next lemma.

Lemma 19. *Let $\tau = \tau_1\tau_2\cdots\tau_\ell$ be a pattern such that $\tau_1 > 1$ or $\tau_\ell < \ell$. Then τ is* INV*-monotone.*

Proof. Suppose that $\tau_1 > 1$. It is plain that $\pi \mapsto 1 \oplus \pi$ is an injection from $\mathcal{S}_n^k(\tau)$ into $\mathcal{S}_{n+1}^k(\tau)$, demonstrating that $S_n^k(\tau) \leqslant S_{n+1}^k(\tau)$. The other case is symmetric. $\square$

We are not able to characterize the INV-monotone patterns. Based on numerical evidence obtained for patterns of small size, we make the following conjecture, which generalizes Conjecture 13. This has been verified for all patterns of lengths at most 6, and for all $n \leqslant 10$.

Conjecture 20. *Any pattern τ that is not an identity pattern is* INV*-monotone.*

Another source of support for Conjecture 20 comes from our analysis of the asymptotic behavior of $S_n^k(\tau)$ as n tends to infinity. To state the results precisely, we need some definitions.

A *Fibonacci permutation* is a permutation π that can be written as a direct sum $\pi = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_m$ where each α_i is equal to 1 or to 21. In other words, a Fibonacci permutation is a layered permutation whose every layer has size at most 2.

Proposition 21. *Let τ be a Fibonacci pattern with $r \geqslant 1$ inversions. For every $k \geqslant r$, there is a polynomial P of degree $r-1$ and an integer $n_0 = n_0(k,\tau)$, such that $S_n^k(\tau) = P(n)$ for all $n \geqslant n_0$.*

Proof. We first observe that an arbitrary permutation π can be uniquely expressed as a direct sum (possibly involving a single summand) of the form

$$\pi = \alpha_0 \oplus \beta_1 \oplus \alpha_1 \oplus \beta_2 \oplus \cdots \oplus \beta_m \oplus \alpha_m,$$

where $m \geqslant 0$ is an integer, each α_i is a (possibly empty) identity permutation, and each β_i is an indecomposable permutation of size at least two. For instance, if $\pi = 124365$, we have $\alpha_0 = 12$, $\alpha_1 = \alpha_2 = \emptyset$, and $\beta_1 = \beta_2 = 21$. We will call the sequence $(\beta_1,\dots,\beta_m)$ *the core* of π, and the sequence $(\alpha_0,\dots,\alpha_m)$ *the padding* of π. The sequence of integers $(|\alpha_0|,\dots,|\alpha_m|)$ will be referred to as the *padding profile* of π. Of course, the padding is uniquely determined by its profile, and the permutation π is uniquely determined by its core and its padding profile.

Let τ be a Fibonacci pattern with r inversions. Note that this is equivalent to saying that τ is a permutation whose core consists of r copies of 21. Let ℓ be the length of τ. Let us fix an integer $k \geqslant r$ and focus on the values of $S_n^k(\tau)$ as a function of n.

Note that the core of a permutation π with $\mathrm{INV}(\pi) = k$ can have at most k components. Moreover, each component of the core of π has size at most $k+1$, otherwise π would have more than k inversions by Lemma 8. In particular, the permutations with k inversions have only a finite number of distinct cores. Define $\mathcal{S}^k(\tau) = \bigcup_{n\geqslant 1} \mathcal{S}_n^k(\tau)$. Let C be the set of all the distinct cores formed by members of $\mathcal{S}^k(\tau)$. Let $\mathcal{S}_n^{[c]}(\tau)$ be the set of permutations from $\mathcal{S}_n^k(\tau)$ whose core is equal to c, and let $S_n^{[c]}(\tau)$ be its cardinality. Clearly, $S_n^k(\tau) = \sum_{c\in C} S_n^{[c]}(\tau)$.

To prove our proposition, it is enough to prove the following three claims:

1. There is a constant $\gamma = \gamma(k,\tau)$ such that $S_n^k(\tau) \geqslant \gamma n^{r-1}$ for every n.
2. For every $c \in C$, there is a constant $\delta = \delta(k,c,\tau)$ such that $S_n^{[c]}(\tau) \leqslant \delta n^{r-1}$.
3. For every $c \in C$, there is a polynomial P_c and a constant n_c such that $S_n^{[c]}(\tau) = P_c(n)$ for every $n \geqslant n_c$.

To prove the first claim, we consider the set of all permutations with core $c = (\beta_1, \beta_2, \dots, \beta_{r-1})$, where $\beta_1 = 1 \ominus (\oplus^{k-r+2}1) = (k-r+3)12\cdots(k-r+2)$, and $\beta_2 = \beta_3 = \cdots = \beta_{r-1} = 21$. Note that any permutation with core c has exactly k inversions and avoids τ. The padding profile of such a permutation is a sequence of r non-negative numbers whose sum is $n - \sum_{i=1}^{r-1} |\beta_i| = n - k - r + 1$. The number of such sequences is $\binom{n-k}{r-1}$, which gives the claimed bound.

To prove the second claim, fix a core $c = (\beta_1, \dots, \beta_m) \in C$, and define $t = \sum_{i=1}^{m} |\beta_i|$. Assume that $m \geqslant r$, otherwise there are only $O(n^{r-1})$ permutations with core c and the claim is trivial. Let π be a permutation with core c, and let $a = (a_0, \dots, a_m)$ be the padding profile of π. Observe that if a has more than r integers greater than ℓ, then π must contain τ. Thus, $S_n^{[c]}(\tau)$ can be bounded from above by the number of all the padding profiles of sum $n-t$ and with at most r components greater than ℓ. The number of such padding profiles may be bounded from above by $\binom{m+1}{r}\ell^{m+1-r}\binom{n-t}{r-1}$, proving the second claim.

To prove the last claim, we reduce it to a known property of down-sets of integer compositions. Let $\mathbb{N}_0^d$ be the set of d-tuples of non-negative integers. Fix a core $c = (\beta_1, \dots, \beta_m) \in C$. Let $a(\pi)$ denote the padding profile of a permutation π. Define the sets $A_n = \{a(\pi)\colon\ \pi \in \mathcal{S}_n^{[c]}\}$ and $A = \bigcup_{n \geqslant 0} A_n$. Define a partial order $\leqslant$ on $\mathbb{N}_0^{m+1}$ by putting $(a_0, \dots, a_m) \leqslant (b_0, \dots, b_m)$ if for every $i \in \{0, \dots, m\}$ we have $a_i \leqslant b_i$. Note that for two permutations σ and π with core c, σ is contained in π if and only if $a(\sigma) \leqslant a(\pi)$. In particular, the set A is a down-set of $\mathbb{N}_0^{m+1}$, that is, if a belongs to A and $b \leqslant a$, then b belongs to A as well. To complete the proof, we use the following fact, due to Stanley [19,20].

Proposition 22 *(Stanley). For every d, if D is a down-set in $\mathbb{N}_0^d$ and $D(n)$ is the cardinality of the set $\{(a_1, \dots, a_d) \in D\colon\ a_1 + \cdots + a_d = n\}$, then there is a polynomial P such that $D(n) = P(n)$ for all n sufficiently large.*

From this fact, we directly obtain that $|A_n|$ is eventually equal to a polynomial, and therefore $S_n^{[c]}(\tau)$ is eventually equal to a polynomial as well. □

Let $P(n)$ be the polynomial from Proposition 21. We note that if $\tau = \oplus^\ell 1$, then $P(n)$ is the zero polynomial by Lemma 18; if $\tau = 132$, then $P(n) = p(k)$ by Proposition 11; and if $\tau = 1324$, then $P(n) = |\Omega(k)|$ by Proposition 15. It would be interesting to know what $P(n)$ is for other Fibonacci patterns.

The conclusion of Proposition 21 cannot be extended to non-Fibonacci patterns, as shown by the next proposition.

Proposition 23. *Let τ be a non-Fibonacci permutation. For every k there exists a polynomial P of degree k and an integer n_0 such that for every $n \geqslant n_0$, $S_n^k(\tau) = P(n)$. Moreover, $P(n) = n^k/k! + O(n^{k-1})$.*

Proof. We can show that $S_n^k(\tau)$ is eventually equal to a polynomial P by the same argument as we used in the proof of Proposition 21. It is therefore enough to provide upper and lower bounds for $S_n^k(\tau)$ of the form $\frac{n^k}{k!} + O(n^{k-1})$. To get the upper bound, note that the number of all permutations of length n with k inversions is at most $\binom{n+k-1}{k}$, as seen by encoding a permutation by its inversion table. For the lower bound, note that τ is not contained in any Fibonacci permutation, and the number of Fibonacci permutations of length n with k inversions is precisely $\binom{n-k}{k}$. □

Propositions 21 and 23 imply that for any pattern τ, any k and any n large enough, we have $S_n^k(\tau) \leqslant S_{n+1}^k(\tau)$, which corresponds to an 'asymptotic version' of Conjecture 20. The two propositions also imply a sharp dichotomy between Fibonacci and non-Fibonacci patterns, in the sense of the next corollary.

Corollary 24. *Let S_n^k be the number of all permutations of size n with k inversions. Let τ be a pattern with r inversions. Define*

$$Q(k,\tau)=\lim_{n\to\infty}\frac{S_n^k(\tau)}{S_n^k}$$

as the asymptotic probability that a large permutation with k inversions avoids τ. If τ is a Fibonacci pattern and $k \geqslant r$, then $Q(k,\tau)=0$. In all other cases $Q(k,\tau)=1$.

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