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Topology of eigenspace posets for imprimitive reflection groups



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ABSTRACT

This paper studies the poset of eigenspaces of elements of an imprimitive unitary reflection group, for a fixed eigenvalue, ordered by the reverse of inclusion. The study of this poset is suggested by the eigenspace theory of Springer and Lehrer. The posets are shown to be isomorphic to certain subposets of Dowling lattices (the “ d -divisible, k -evenly coloured Dowling lattices”). This enables us to prove that these posets are Cohen–Macaulay, and to determine the dimension of their top homology.

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1. Introduction

Let V be a complex vector space of finite dimension, and $G \subseteq GL(V)$ a unitary reflection group in V . Let $\mathcal{A}(G)$ be the set of reflecting hyperplanes of all reflections in G , and $\mathcal{L}(\mathcal{A}(G))$ the poset of intersections of the hyperplanes in $\mathcal{A}(G)$. This poset is closely connected to the topology of the hyperplane complement of $\mathcal{A}(G)$ (see [1,9,10,22,27,28]).

It is well-known that the poset $\mathcal{L}(\mathcal{A}(G))$ is a geometric lattice. Hence it is Cohen–Macaulay, and its reduced homology vanishes except in top dimension. The poset

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$\mathcal{L}(\mathcal{A}(G))$ is also known to coincide with the poset of fixed point subspaces (or 1-eigenspaces) of elements of G (see [19, Theorem 2.10]). Springer and Lehrer [24,25,31] developed a general theory of eigenspaces for unitary reflection groups. This paper is the second in a series (commencing with [19]) whose purpose is to study topological properties of generalisations of $\mathcal{L}(\mathcal{A}(G))$ for arbitrary eigenvalues. This paper concentrates on the case where G is the imprimitive unitary reflection group $G(r, p, n)$, but the definitions in the following paragraph remain valid for any unitary reflection group (indeed, for any finite subset of $\text{End}(V)$).

Namely, let ζ be a complex root of unity, and g be an element of G . Define $V(g, \zeta) \subseteq V$ to be the ζ -eigenspace of g . That is, $V(g, \zeta) := \{v \in V \mid gv = \zeta v\}$. Let $\mathcal{S}_\zeta^V(G)$ be the set $\{V(g, \zeta) \mid g \in G\}$, partially ordered by the reverse of inclusion. More generally, if $\gamma \in N_{GL(V)}(G)$ (the normaliser of G in $GL(V)$) and γG is a reflection coset, we may define $\mathcal{S}_\zeta^V(\gamma G)$ to be the set $\{V(x, \zeta) \mid x \in \gamma G\}$, partially ordered by the reverse of inclusion. This is a linear analogue of the poset of p -subgroups of a group G first studied by Quillen [30]. The study of this poset was first suggested by Lehrer (see [26, Appendix C, p. 270]).

The paper commences with a discussion of wreath products and the infinite family of imprimitive unitary reflection groups $G(r, p, n)$. Following this is a combinatorial section on Dowling lattices. In particular we introduce the “ d -divisible, k -evenly coloured” Dowling lattices, which to our knowledge have not been studied before.

An important result is Theorem 3.18, which describes an isomorphism between the poset $\mathcal{S}_\zeta^V(\gamma G)$, where G is an imprimitive unitary reflection group, and these sublattices of Dowling lattices.

From this isomorphism it follows, in the imprimitive case, that the poset $\mathcal{S}_\zeta^V(\gamma G)$ is Cohen–Macaulay (Corollary 3.25). This is perhaps the key result of this paper, which is used to prove a similar statement for arbitrary unitary reflection groups in [19].

Given the poset $\mathcal{S}_\zeta^V(\gamma G)$, remove the unique maximal element (which will be shown to always exist), and the unique minimal element (if it exists), and denote the resulting poset $\tilde{\mathcal{S}}_\zeta^V(\gamma G)$. This construction is necessary since any poset with a unique minimal or maximal element is contractible, so its homology is uninteresting. Since $\tilde{\mathcal{S}}_\zeta^V(\gamma G)$ is also Cohen–Macaulay when $G = G(r, p, n)$, its reduced homology vanishes except in top dimension. When G is an imprimitive reflection group, the posets $\mathcal{S}_\zeta^V(\gamma G)$ are in fact examples of exponential Dowling structures, a class of posets defined by Ehrenborg and Readdy [14] as generalisations of Stanley’s exponential structures [32]. This enables us to find generating functions for the Möbius function of these posets, and hence for the dimension of the top homology. This has applications to the representation theory of unitary reflection groups.

The notation and preliminary material for this paper can be found in [19, §2]. Rather than repeat material here, we refer the reader to that paper.

2. Shellability

A useful tool for establishing Cohen–Macaulayness of posets or simplicial complexes is shellability. The original theory applied only to pure complexes; a non-pure generalisation was given by Björner and Wachs [5]. The posets in this paper are all pure, but it is no more difficult to give the general definition. A good reference for this material is [35], and this treatment follows that source very closely.

Definition 2.1. (See [35, §3.1].) For each simplex F of a simplicial complex Δ , let $\langle F \rangle$ denote the subcomplex generated by F (i.e. $\langle F \rangle = \{G \in \Delta \mid G \subseteq F\}$). Then Δ is said to be *shellable* if its facets (maximal faces) can be arranged in linear order F_1, F_2, \dots, F_t in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle$ is pure and $(\dim F_k - 1)$ -dimensional for all $k = 2, \dots, t$. Such an ordering of facets is called a *shelling*.

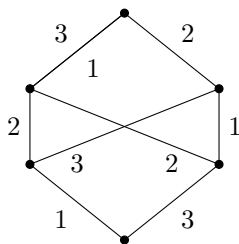
Recall that if Δ is a pure complex of dimension n , all facets have dimension n , so that $(\bigcup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle$ must have dimension $(n - 1)$ for all $k = 2, 3, \dots, t$. This definition carries over to posets if we replace simplices by chains. Unlike the property of being Cohen–Macaulay, shellability is not a topological property. As an example, there exist nonshellable triangulations of the 3-ball and the 3-sphere (see [35, §3.1] for more details).

In order to establish shellability of a simplicial complex, Björner and Wachs developed the notion of lexicographic shellability for bounded posets, first for pure posets [3,4] and later in the general situation [5,6]. There are two basic versions of lexicographic shellability – edge labelling (EL-) shellability and chain labelling (CL-) shellability. It is the second of these which concerns us here.

Definition 2.2. Let P be a bounded poset, and Λ any poset. Let $\mathcal{ME}(P)$ be the set of pairs $(c, x \prec y)$, where c is a maximal chain in P and $x \prec y$ is an edge along that chain. A *chain-edge labelling* of P is a map $\lambda: \mathcal{ME}(P) \rightarrow \Lambda$ which satisfies the following condition:

Suppose two maximal chains coincide along their bottom d edges, for some d . Then their labels coincide along those edges.

The following example of a chain-edge labelling is found in [4, Fig. 2.2] (and [35, Fig. 3.3.1]). In this case, $\Lambda = \mathbb{Z}$. Note that the top left edge has two labels – it has the label 3 if paired with the leftmost maximal chain, and 1 if paired with the other maximal chain which shares this edge.



Given a chain-edge labelling and a maximal chain $c = (\hat{0} \prec x_1 \prec \cdots \prec x_t \prec \hat{1})$ it is possible to form a word

$$\lambda(c) = \lambda(c, \hat{0} \prec x_1) \lambda(c, x_1 \prec x_2) \cdots \lambda(c, x_t \prec \hat{1}).$$

We order the maximal chains lexicographically by using the lexicographic order on the corresponding words.

To define chain-lexicographic labelling, it is necessary to restrict chain-edge labellings $\lambda : \mathcal{ME}(P) \rightarrow A$ to closed intervals $[x, y]$. However as seen above each edge may have more than one label, depending upon which maximal chain containing that edge we choose. Given the condition on the chain-edge labelling, each maximal chain r of $[\hat{0}, x]$ determines a unique restriction of λ to $\mathcal{ME}([x, y])$.

Definition 2.3. Let P be a bounded poset. Define a closed rooted interval $[x, y]_r$ to be a closed interval $[x, y]$, together with a maximal chain r of $[\hat{0}, x]$.

Definition 2.4. We say that c is *increasing* if the associated word $\lambda(c)$ is *strictly* increasing. That is, c is increasing if

$$\lambda(c, \hat{0} \prec x_1) < \lambda(c, x_1 \prec x_2) < \cdots < \lambda(c, x_t \prec \hat{1}) \quad \text{in } A.$$

Definition 2.5. Let P be a bounded poset. A *chain-lexicographic labelling* (CL-labelling) of P is a chain-edge labelling such that in each closed rooted interval $[x, y]_r$ of P , there is a unique strictly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. If a poset admits a CL-labelling it is said to be *CL-shellable*.

The chain-edge labelling in the figure above is a CL-labelling. The unique increasing maximal chain for the whole poset is the leftmost maximal chain.

The following implications between the concepts we have discussed are noted in [35]:

Proposition 2.6. (See [35, Lecture 4.1].) The following implications hold for a poset P :

$$\begin{aligned} (\text{pure}) \text{ CL-shellable} &\Rightarrow (\text{pure}) \text{ shellable} \\ &\Rightarrow \text{homotopy Cohen-Macaulay} \\ &\Rightarrow \text{Cohen-Macaulay over } \mathbb{Z} \\ &\Rightarrow \text{Cohen-Macaulay over } \mathbb{F} \text{ (for any field } \mathbb{F}). \end{aligned}$$

It is known that all these implications are strict. For a further discussion see [2, Appendix], [4, §9]. We note in passing that a similar chain of implications holds in the non-pure case, when each of the above properties is replaced by its non-pure analogue.

We now present a technical tool for establishing CL-shellability, called *recursive atom ordering*. It is equivalent to CL-shellability, but does not involve edge labellings.

Definition 2.7. A bounded poset P is said to admit a recursive atom ordering if its length $l(P)$ is 1, or if $l(P) > 1$ and there is an ordering a_1, a_2, \dots, a_t of the atoms of P that satisfies

- (i) For all $j = 1, 2, \dots, t$ the interval $[a_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}]$ that belong to $[a_i, \hat{1}]$, for some $i < j$, come first.
- (ii) For all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}]$ such that $a_k < z \leq y$.

A *recursive coatom ordering* is a recursive atom ordering of the dual poset P^* .

A key theorem relating recursive atom orderings to shellability, which will be used later is the following:

Theorem 2.8. (See [4, 35].) *A bounded poset P is CL-shellable if and only if P admits a recursive atom ordering.*

In particular, it is known (see [35, Theorem 4.2.7]) that every geometric lattice admits a recursive atom ordering, and is therefore CL-shellable.

3. The groups $G(r, p, n)$

3.1. Wreath products

The following treatment is taken from [26, §3.2].

Given a group G acting on the set $\Omega := \{1, 2, \dots, n\}$ and any group H , let B denote the direct product of n copies of H , and define an action of G on B as follows. If $g \in G$ and $h = (h_1, h_2, \dots, h_n) \in B$, define

$$g \cdot h := (h_{g^{-1}(1)}, h_{g^{-1}(2)}, \dots, h_{g^{-1}(n)}).$$

The *wreath product* H by G , denoted $H \wr G$, is the semidirect product of B by G . Write elements of $H \wr G$ as $(h; g)$, with multiplication given by

$$(h; g)(h'; g') = (hg \cdot h'; gg'), \quad \text{for all } g \in G, h \in H.$$

If H and G are finite groups, the order of $H \wr G$ is $|H|^n |G|$.

Assume now that H is the cyclic group μ_m of complex m -th roots of unity, for some $m \in \mathbb{N}$. Let V be a complex vector space of dimension n , equipped with positive definite hermitian form $(-, -)$, and orthonormal basis e_1, e_2, \dots, e_n . The elements of B can be represented as diagonal transformations of V . That is, if $h = (h_1, h_2, \dots, h_n) \in B$, set

$$he_i := h_i e_i \quad \text{for } 1 \leq i \leq n.$$

Clearly B is a subgroup of $U(V)$.

G also acts on V , by permutation of the basis vectors:

$$ge_i := e_{g(i)} \quad \text{for } 1 \leq i \leq n.$$

The actions of $B = (\mu_m)^n$ and G combine to give an action of $\mu_m \wr G$ on V :

$$(h; g)e_i := h_{g(i)}e_{g(i)} \quad \text{for } 1 \leq i \leq n.$$

The matrices representing these linear transformations (with respect to the basis $\{e_1, \dots, e_n\}$) are monomial (one nonzero entry in each row and each column) and this representation of $\mu_r \wr G$ is known as the *standard monomial representation*. Note that $\mu_r \wr G$ is in fact represented as a subgroup of $U(V)$.

With $B = (\mu_r)^n$ and p a divisor of r , define the group $A(r, p, n)$ as follows:

$$A(r, p, n) := \{(\theta_1, \theta_2, \dots, \theta_n) \in B \mid (\theta_1 \theta_2 \dots \theta_n)^{\frac{r}{p}} = 1\}.$$

The group $G(r, p, n)$ is defined to be the semidirect product of $A(r, p, n)$ by the symmetric group $\text{Sym}(n)$. The group $G(r, p, n)$ is a normal subgroup of index p in $\mu_r \wr \text{Sym}(n)$, so it can be represented as a group of monomial linear transformations in the standard monomial representation. The order of $G(r, p, n)$ is $r^n n! / p$.

The group $G(r, p, n)$ is defined to be the group of all $n \times n$ monomial matrices whose non-zero entries are complex r -th roots of unity such that the product of the non-zero entries is an r/p -th root of unity.

The following results about the groups $G(r, p, n)$ are standard:

Proposition 3.1. (See [8, Proposition 2.1.6], [12, Theorem 2.4, Remark 2.5], [26, Proposition 2.10, Theorems 2.14 and 2.15].) Let V be a complex vector space of dimension n , equipped with positive definite hermitian form, and denote the unitary group of the form by $U(V)$. Let r , p and n be positive integers such that p divides r . Then:

- (i) $G(r, p, n)$ is a unitary reflection group.
- (ii) Suppose $n \geq 2$. If (r_1, p_1, n) and (r_2, p_2, n) are triples of positive integers with $p_1 \mid r_1$ and $p_2 \mid r_2$ then $G(r_1, p_1, n)$ and $G(r_2, p_2, n)$ are conjugate in $U(V)$ if and only if $(r_1, p_1, n) = (r_2, p_2, n)$ or $\{(r_1, p_1, n), (r_2, p_2, n)\} = \{(2, 1, 2), (4, 4, 2)\}$.
- (iii) $G(r, p, n)$ is an irreducible unitary reflection group except when $(r, p, n) = (1, 1, n)$ and $n > 1$ or $(r, p, n) = (2, 2, 2)$. The group $G(1, 1, n)$ is the symmetric group on n letters, which splits as a sum of irreducible representations of dimensions 1 and $n - 1$. The group $G(2, 2, 2)$ is the Klein four group.

3.2. Dowling lattices

When G is an imprimitive reflection group, the posets $\mathcal{S}_\zeta^V(\gamma G)$ turn out to be isomorphic to subposets of the well-known *Dowling lattices*. This section provides an introduction to Dowling lattices. This treatment follows [15] closely, with an adaptation from [18].

Let $\pi = B_1 \mid \dots \mid B_k$ denote a partition of a set S . This means that S is the disjoint union of the B_i , which are called *blocks* of π . Write $|\pi|$ for the number of blocks in π .

Let G be a finite group. A G -prepartition is a triple (I, π, γ) where $I \subseteq [n] = \{1, \dots, n\}$, $\pi = B_1 \mid \dots \mid B_k$ is a partition of $[n] \setminus I$ and γ is a map $B_1 \cup B_2 \cup \dots \cup B_k \rightarrow G$. The subset I is called the *zero block* of the prepartition. For $i = 1, 2, \dots, k$ the restriction of γ to B_i is denoted γ_i . The pair (B_i, γ_i) , $i = 1, 2, \dots, k$ is called a *nonzero block*, and can be considered as a subset $\{(a, \gamma_i(a)) \mid a \in B_i\}$ of $[n] \times G$. Thus, γ may be thought of as a labelling of the elements of the nonzero blocks by elements of G . Note that the zero block is allowed to be empty, but that nonzero blocks are, by definition, nonempty.

There is a right action of G of $[n] \times G$ given by $(i, h) \cdot g = (i, hg)$. This action extends to an action on the set of all subsets of $[n] \times G$ by $S \cdot g = \{x \cdot g \mid x \in S\}$. There is an equivalence relation on the power set of $[n] \times G$ given by setting two subsets S_1 and S_2 of $[n] \times G$ to be equivalent if they lie in the same G -orbit under this action.

There is also an equivalence relation on the set of G -prepartitions. Two G -prepartitions are equivalent if their zero blocks are equal and there is a one-to-one correspondence between their nonzero blocks such that corresponding blocks are equivalent.

Define a G -partition to be an equivalence class of G -prepartitions. If (I, π, γ) is a member of this equivalence class, the corresponding G -partition is denoted $(\overline{I}, \pi, \gamma)$. The equivalence class of a nonzero block (B_i, γ_i) of a G -prepartition is called a *nonzero G -block* of the G -partition, and is denoted $\overline{(B_i, \gamma_i)}$. The zero block I of the G -prepartition is called the *zero G -block* of the G -partition. For ease of notation we write $\bar{I} = I$.

The elements of the Dowling lattice $Q_n(G)$ are the G -partitions. The partial order is determined by the covering relation described as follows. First define a *merge* of two G -blocks. The merge of the zero G -block I with a nonzero G -block $\overline{(B_i, \gamma_i)}$ is the zero block $I \cup B_i \subseteq [n]$. There are $|G|$ ways to merge two nonzero blocks $\overline{(B_i, \gamma_i)}$ and $\overline{(B_j, \gamma_j)}$, each of which has the form $\overline{(B_i, \gamma_i) \cup ((B_j, \gamma_j) \cdot g)}$ for some $g \in G$ where the union is of two subsets of $[n] \times G$. Note that once equivalence class representatives are fixed for the G -blocks being merged, the merge is defined uniquely by g . The covering relation can now be simply defined: in $Q_n(G)$, the G -partition $(\overline{I}, \pi, \gamma)$ is covered by $(\overline{I'}, \pi', \gamma')$ if $(\overline{I'}, \pi', \gamma')$ can be obtained from $(\overline{I}, \pi, \gamma)$ by merging two G -blocks of $(\overline{I}, \pi, \gamma)$. The minimum element $\hat{0}$ of $Q_n(G)$ is the G -partition in which $I = \emptyset$, and π partitions the set $\{1, \dots, n\}$ into singleton subsets (so there is no choice for γ , up to equivalence). The maximum element $\hat{1}$ of $Q_n(G)$ is the G -partition in which $I = \{1, \dots, n\}$, π is empty, and γ is trivial.

There is a left action of $G \wr \text{Sym}(n)$ on $Q_n(G)$ defined as follows. Suppose $(i, a) \in [n] \times G$, $g = (g_1, \dots, g_n) \in G^n$ and $\sigma \in \text{Sym}(n)$. Thus $(g; \sigma) \in G \wr \text{Sym}(n)$. Define

$$(g; \sigma) \cdot (i, a) = (\sigma(i), g_{\sigma(i)}a).$$

This induces an action of $G \wr \text{Sym}(n)$ on subsets of $[n] \times G$ and therefore on G -prepartitions, by letting an element $(g; \sigma) \in G \wr \text{Sym}(n)$ act on each nonzero block viewed as a subset of $[n] \times G$, and on the zero block $I = \{i_1, i_2, \dots, i_k\}$ by $(g; \sigma) \cdot \{i_1, i_2, \dots, i_k\} = \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}$. Notice that the left action of $G \wr \text{Sym}(n)$ on nonzero blocks commutes with the right action of G . Therefore the left action of $G \wr \text{Sym}(n)$ on G -prepartitions respects the equivalence relation on G -prepartitions, and hence induces an action on G -partitions. This action preserves the partial order on $Q_n(G)$.

The Dowling lattice $Q_n(G)$ is ranked, with rank function given by

$$\text{rk}(\overline{(I, \pi, \gamma)}) = n - |\pi|.$$

Thus the rank of the whole poset $Q_n(G)$ is n . It is known (see [13, Theorem 3]) that $Q_n(G)$ is a geometric lattice, and therefore Cohen–Macaulay.

Now specialise to the case where G is the cyclic group μ_r , identified with $\mathbb{Z}/r\mathbb{Z}$. Let $Q_n(r)$ denote the Dowling lattice $Q_n(\mu_r)$. When G is the trivial group μ_1 the Dowling lattice $Q_n(1)$ is the ordinary partition lattice Π_{n+1} on $n+1$ elements, and $\mu_1 \wr \text{Sym}(n)$ is just the symmetric group $\text{Sym}(n)$. The action of $\mu_1 \wr \text{Sym}(n)$ on $Q_n(1)$ is isomorphic to the restriction of $\text{Sym}(n+1)$ on Π_{n+1} to $\text{Sym}(n)$. When $G = \mu_2$, the Dowling lattice $Q_n(2)$ is the signed partition lattice Π_n^B . The wreath product $\mu_2 \wr \text{Sym}(n)$ is the hyperoctahedral group. The lattice $Q_n(1)$ is the intersection lattice for the Coxeter group of type A_n , while $Q_n(2)$ is the intersection lattice for the Coxeter group of type B_n .

It is well known (see Proposition 3.13) that the Dowling lattice $Q_n(r)$ is the intersection lattice for the imprimitive reflection group $G(r, 1, n)$.

Several subposets of $Q_n(r)$ occur in the work which follows:

Definition 3.2. Let $J \subseteq \{0, 1, \dots, n\}$. The poset $Q_n(r, J)$ is the subposet of $Q_n(r)$ whose elements $\overline{(I, \pi, \gamma)}$ satisfy $|I| \notin J \subseteq \{0, 1, \dots, n\}$, together with the unique minimal element of $Q_n(r)$.

Note that this definition applies more generally to the Dowling lattice $Q_n(G)$.

Definition 3.3. The poset $Q_n(r, d)$ is the subposet of $Q_n(r)$ whose nonzero blocks have length divisible by d , together with the unique minimal element of $Q_n(r)$ (the ‘ d -divisible Dowling lattice’). As a matter of convention we allow the case $d = 1$, in which case $Q_n(r, d) = Q_n(r)$.

Definition 3.4. The poset $Q_n(r, d, k)$ is the subposet of $Q_n(r, d)$ consisting of elements $(\overline{I, \pi, \gamma})$ such that for each nonzero block $(\overline{B_i, \gamma_i})$, $|\{a \in B_i \mid \gamma(a) \equiv j \pmod{k}\}|$ is independent of j , together with the minimum element of $Q_n(r, d)$ (the ‘ k -evenly coloured, d -divisible Dowling lattice’). Note that we require $k \mid r$ and will also assume that $k \mid d$.

Definition 3.5. Let $J \subseteq \{0, 1, \dots, n\}$. The poset $Q_n(r, d, k, J)$ is the subposet of $Q_n(r, d, k)$ whose elements $(\overline{I, \pi, \gamma})$ satisfy $|I| \notin J \subseteq \{0, 1, \dots, n\}$, together with the unique minimal element of $Q_n(r, d, k)$.

We would like to show that these posets are Cohen–Macaulay over \mathbb{Z} . It is well-known (see Proposition 2.6, Theorem 2.8) that it suffices to show that the posets in question have a recursive atom ordering.

The following proposition regarding the topological properties of some of these posets appears in [18]. Note that the result looks slightly different because we have defined Dowling lattices differently.

Proposition 3.6. (See [18, Proposition 1.6].) *The poset $Q_n(r, d)$ is a pure lattice with a recursive atom ordering. Its rank function is*

$$\text{rk}(\overline{(I, \pi, \gamma)}) = \begin{cases} 0 & \text{if } \overline{(I, \pi, \gamma)} = \hat{0} \\ \lfloor \frac{n}{d} \rfloor + 1 - |\pi| & \text{otherwise.} \end{cases}$$

Proof. A proof is given for completeness, using the definition of the Dowling lattice given here, rather than that in [18].

The poset $Q_n(r, d) \setminus \{\hat{0}\}$ is an upper order ideal of $Q_n(r)$, so $Q_n(r, d)$ is a lattice. The atoms of $Q_n(r, d)$ are those $(\overline{I, \pi, \gamma})$ such that $|B_i| = d$ for each block B_i of π , and $|\pi| = \lfloor \frac{n}{d} \rfloor$. These all have rank $(n - \lfloor \frac{n}{d} \rfloor)$ as elements of $Q_n(r)$, so $Q_n(r, d)$ is pure with the required rank function. To find a recursive atom ordering, note that for any element $(\overline{I, \pi, \gamma}) \in Q_n(r, d) \setminus \{\hat{0}\}$, the principal upper order ideal $[(\overline{I, \pi, \gamma}), \hat{1}]$ is isomorphic to $Q_{|\pi|}(r)$, which is a geometric lattice. Thus $[(\overline{I, \pi, \gamma}), \hat{1}]$ is CL-shellable, and hence admits a recursive atom ordering. Hence (see Definition 2.7) it suffices to check that the atoms of $Q_n(r, d)$ can be ordered a_1, a_2, \dots, a_t in such a way that for all $i < j$ and $y \in Q_n(r, d) \setminus \{\hat{0}\}$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}]$ such that $a_k < z \leq y$.

Such an ordering can be described as follows (see [17, Proposition 1.6] and [35, Exercise 4.3.6(a)]). Let $a = (\overline{I, \pi, \gamma})$ be an atom of $Q_n(r, d)$. This atom is characterised by its zero block I of size $n - d\lfloor \frac{n}{d} \rfloor$ and $\lfloor \frac{n}{d} \rfloor$ nonzero blocks $(B_1, \gamma_1), \dots, (B_{\lfloor \frac{n}{d} \rfloor}, \gamma_{\lfloor \frac{n}{d} \rfloor})$, where each subset B_i of $[n] \setminus I$ has d elements. List the elements of I and of each block in increasing order, ignoring the labels determined by γ . Call these lists $\bar{I}, \bar{B}_1, \dots, \bar{B}_{\lfloor \frac{n}{d} \rfloor}$. Assume that the ordering of $\bar{B}_1, \dots, \bar{B}_{\lfloor \frac{n}{d} \rfloor}$ themselves is determined by the smallest element in each of these lists. Form the word $\alpha = \bar{I} \mid \bar{B}_1 \mid \dots \mid \bar{B}_{\lfloor \frac{n}{d} \rfloor}$. This process associates a word of length n to each atom a .

As an example, suppose $n = 8$, $r = 2$, $d = 3$. Since there are only two possible labellings for each element, we can use the presence or absence of a bar ($-$) above an

element to indicate this labelling. Use a vertical line ($|$) to separate nonzero blocks, and a double line ($\|$) to separate the zero block from the nonzero blocks. Thus, a typical atom a might be $a = 146 | \bar{2}78 \| 35$. According to the above definition, the word α associated with a is $\alpha = 35146278$.

Now order the atoms of $Q_n(r, d)$ by lexicographic order of these words. It is possible that different atoms may produce the same word. For example, the atom $a' = 146 | \bar{2}78 \| 35$ also has the word $\alpha = 35146278$ associated to it. Within atoms with the same word, choose an arbitrary ordering of all G -blocks of size d with underlying set B , for each d -element subset B of $[n]$. Apply this ordering first to (B_1, γ_1) , then to (B_2, γ_2) , and so on. Thus the dependence of the ordering on γ is arbitrary.

Now suppose that $a_j = (\overline{I}, \pi, \gamma)$, $y = (\overline{I'}, \pi', \gamma')$, and that $a_j < y$. Write the nonzero blocks of a_j as $(B_1, \gamma_1), \dots, (B_{\lfloor \frac{n}{d} \rfloor}, \gamma_{\lfloor \frac{n}{d} \rfloor})$, and those of y as $(B'_1, \gamma'_1), \dots, (B'_k, \gamma'_l)$ for some l . Suppose that there is no k such that there is a common cover z of both a_k and a_j , and that $z \leq y$. It will suffice to show that a_j is the earliest atom which is less than y . Suppose B'_i is any part of π' . Then $B'_i = B_{s_1} \cup B_{s_2} \cup \dots \cup B_{s_t}$ for some $s_1 < s_2 < \dots < s_t$. Suppose that for some q , $a = \max(B_{s_q}) > \min(B_{s_{q+1}}) = b$. Consider the atom $a_k = (\overline{I''}, \pi'', \gamma'')$ defined by

$$\begin{aligned} I'' &= I \\ B''_{s_q} &= B_{s_q} + b - a \quad (\text{that is, include } b \text{ in } B''_{s_q} \text{ but exclude } a) \\ B''_{s_{q+1}} &= B_{s_{q+1}} + a - b \\ B''_j &= B_j \quad \text{for } j \neq s_q, s_{q+1} \\ \gamma'' &= \gamma \end{aligned}$$

Then a_k occurs earlier in the ordering than a_j , and their join z is a cover of both a_k and a_j which is $\leq y$. This contradicts the assumption on a_j . As a result, it must be the case that $\max(B_{s_i}) < \min(B_{s_{i+1}})$. Hence the parts of π contained in B'_i are simply those obtained by ordering the elements of B'_i and dividing the list into d -element subsets. The labelling on each of these parts is determined by the labelling on B'_i . A similar argument shows that I must consist of the smallest $n - d\lfloor \frac{n}{d} \rfloor$ elements of I' , and that the parts of π contained in I' can be found by listing the remaining elements of I' in ascending order, and dividing this list into d -element subsets. The labelling on each of these parts is determined by choosing the smallest element in the (fixed, but arbitrary) ordering of all G -blocks of size d , for each d -element subset B of $[n]$. From the construction, it is clear that this a_j is the smallest atom less than or equal to y . This completes the proof. \square

A result similar to [Proposition 3.6](#) holds for the poset $Q_n(r, d, k)$:

Proposition 3.7. *The poset $Q_n(r, d, k)$ is a pure lattice with a recursive atom ordering. Its rank function is*

$$\text{rk}(\overline{(I, \pi, \gamma)}) = \begin{cases} 0 & \text{if } \overline{(I, \pi, \gamma)} = \hat{0} \\ \lfloor \frac{n}{d} \rfloor + 1 - |\pi| & \text{otherwise.} \end{cases}$$

Proof. $Q_n(r, d, k) \setminus \{\hat{0}\}$ is an upper order ideal of $Q_n(r, d)$, generated by those atoms of $Q_n(r, d)$ which are k -evenly coloured. Hence $Q_n(r, d, k)$ is a lattice. The atoms of $Q_n(r, d, k)$ are precisely the k -evenly coloured atoms of $Q_n(r, d)$. These all have the same rank, and so $Q_n(r, d, k)$ is pure, with the same rank function as $Q_n(r, d)$.

To see that $Q_n(r, d, k)$ has a recursive atom ordering, use the same lexicographic ordering described in Proposition 3.6. As in the proof of Proposition 3.6, suppose that $a_j = \overline{(I, \pi, \gamma)}$, $y = \overline{(I', \pi', \gamma')}$, and that $a_j < y$. Write the nonzero blocks of a_j as $(B_1, \gamma_1), \dots, (B_{\lfloor \frac{n}{d} \rfloor}, \gamma_{\lfloor \frac{n}{d} \rfloor})$, and those of y as $(B'_1, \gamma'_1), \dots, (B'_k, \gamma'_l)$ for some l .

Suppose that there is no k such that there is a common cover z of both a_k and a_j , and that $z \leq y$. It suffices to show that a_j is the earliest atom which is less than y . Suppose B'_i is a part of π' , and that $B'_i = B_{s_1} \cup \dots \cup B_{s_t}$ for some $s_1 < s_2 < \dots < s_t$. A similar argument to that used in Proposition 3.6 shows that the parts B_{s_j} of π contained in B'_i are obtained as follows. Suppose $B'_i = C_1 \cup \dots \cup C_k$, where for $1 \leq j \leq k$ the elements in C_j all have the same colouring. Order each of the sets C_i by lexicographic order. Then B_{s_1} contains the smallest $\frac{d}{k}$ elements of each of the sets C_i , B_{s_2} contains the next smallest $\frac{d}{k}$ elements of each of the C_i , and so on. The labelling on each of these parts is determined by the labelling on B'_i . Similarly, the zero block I consists of the smallest $n - d\lfloor \frac{n}{d} \rfloor$ elements of I' , and the parts of π contained in I' can be found by listing the remaining elements of I' in ascending order, and dividing this list into d -element subsets. The labelling on each of these parts is determined by choosing the smallest element in the (fixed, but arbitrary) ordering of all G -blocks of side d , for each d -element subset B of $[n]$. The argument is the same as in the proof of Proposition 3.6, except that we can only perform the switching algorithm within elements of the same colour in each block B'_i of π' , and we perform this algorithm for each colour separately. It is clear that a_j is the smallest atom less than or equal to y . \square

In addition, in the case $n \equiv 0 \pmod{d}$:

Proposition 3.8. *If $n \equiv 0 \pmod{d}$ then $Q_n(r, d, k, \{0\})$ is a pure lattice with recursive atom ordering. Its rank function is given by*

$$\text{rk}(\overline{(I, \pi, \gamma)}) = \begin{cases} 0 & \text{if } \overline{(I, \pi, \gamma)} = \hat{0} \\ \lfloor \frac{n}{d} \rfloor - |\pi| & \text{otherwise.} \end{cases}$$

Proof. The atoms of $Q_n(r, d, k, \{0\})$ consist of $(\lfloor \frac{n}{d} \rfloor - 1)$ k -evenly coloured nonzero blocks of length d , and a zero block also of length d . $Q_n(r, d, k, \{0\}) \setminus \{\hat{0}\}$ is an upper order ideal of $Q_n(r)$ generated by these atoms, and therefore $Q_n(r, d, k, \{0\})$ is a lattice. The rank function is one less than that of $Q_n(r, d, k)$ as no atom of $Q_n(r, d, k)$ is in $Q_n(r, d, k, \{0\})$. The removal of elements of $Q_n(r, d, k)$ with zero block of size 0 has does not alter the

proof of the existence of a recursive atom ordering in [Proposition 3.7](#), so the same proof indicated above applies here. \square

Finally,

Proposition 3.9. *If $n \equiv 1 \pmod{d}$ then $Q_n(r, d, k, \{1\})$ is a pure lattice with recursive atom ordering. Its rank function is given by*

$$\text{rk}(\overline{(I, \pi, \gamma)}) = \begin{cases} 0 & \text{if } \overline{(I, \pi, \gamma)} = \hat{0} \\ \lfloor \frac{n}{d} \rfloor - |\pi| & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as for [Proposition 3.8](#). \square

Since the posets $Q_n(G)$, $Q_n(r)$, $Q_n(r, d)$, $Q_n(r, d, k)$, $Q_n(r, d, k, \{0\})$, $Q_n(r, d, k, \{1\})$ all admit a recursive atom ordering, they are all Cohen–Macaulay over \mathbb{Z} (see [Proposition 2.6](#) and [Theorem 2.8](#)). Thus the proper part of each of these lattices is Cohen–Macaulay (by [\[19, Corollary 2.3\]](#)). So for each poset Q in this list, $\tilde{H}_i(\bar{Q}, \mathbb{Z}) = 0$ unless $i = l(\bar{Q})$.

The study of the action of the symmetric group on the partition lattice was initiated by Stanley [\[33\]](#), with the famous result that $\tilde{H}_{n-3}(\overline{\Pi}_n, \mathbb{C}) \simeq \varepsilon_n \otimes \text{Ind}_{\mu_n}^{\text{Sym}(n)}(\psi)$, where ψ is a faithful character of the cyclic group μ_n generated by an n -cycle, and ε_n is the sign representation. Sundaram [\[34\]](#) used Whitney homology to provide a conceptual proof of Stanley’s result (which had relied on earlier computations by Hanlon). See also the paper [\[23\]](#).

For Dowling lattices in general, Hanlon [\[16\]](#) calculated the characters of the representations of $G \wr \text{Sym}(n)$ on $Q_n(G)$. Henderson [\[18\]](#) adapted Joyal’s theory of tensor species to understand the representation of $G \wr \text{Sym}(n)$ on several subposets of $Q_n(G)$, including (as a special case) $Q_n(r, d)$.

3.3. Reflection arrangements for $G(r, p, n)$

This section describes the hyperplane arrangements for the groups $G(r, p, n)$. The notation of [\[29\]](#) is used throughout. Let $V = \mathbb{C}^n$, and let ζ be an r -th root of unity.

Definition 3.10. Define the hyperplanes

$$\begin{aligned} H_{i,j}^\zeta &:= \{(v_1, \dots, v_n) \in V \mid v_i = \zeta v_j\} \quad \text{and} \\ H_k &:= \{(v_1, \dots, v_n) \in V \mid v_k = 0\}. \end{aligned}$$

Definition 3.11. Let

$$\begin{aligned} \mathcal{A}_n^0(r) &:= \{H_{i,j}^\zeta \mid 1 \leq i < j \leq n, \zeta \in \mu_r\} \quad \text{and} \\ \mathcal{A}_n(r) &:= \mathcal{A}_n^0(r) \cup \{H_k \mid 1 \leq k \leq n\}. \end{aligned}$$

Proposition 3.12. *There is an isomorphism $\mathcal{L}(\mathcal{A}_n(r)) \cong Q_n(r)$.*

Proof. Define a map $\rho : Q_n(r) \rightarrow \mathcal{L}(\mathcal{A}_n(r))$ as follows. Suppose $(\overline{I, \pi, \gamma}) \in Q_n(r)$, with $\pi = B_1 \mid \dots \mid B_k$. Let $\rho(\overline{I, \pi, \gamma})$ be the subspace defined by the equations

$$z_i = \begin{cases} \gamma(i)^{-1} \gamma(j) z_j & \text{if } i, j \text{ lie in the same block } B_s (1 \leq s \leq k) \\ 0 & \text{if } i \in I. \end{cases}$$

It is clear that ρ is an isomorphism. \square

The following result is well-known:

Proposition 3.13. (See [29, §6.4].) *Suppose r, p and n are positive integers such that $p \mid r$. Let G be the unitary reflection group $G(r, p, n)$ and $\mathcal{A}(G(r, p, n))$ the corresponding arrangement of reflecting hyperplanes. Then*

$$\mathcal{A}(G(r, p, n)) = \begin{cases} \mathcal{A}_n(r) & \text{if } r \neq p \\ \mathcal{A}_n^0(r) & \text{if } r = p. \end{cases}$$

Corollary 3.14. *Let G be the unitary reflection group $G(r, p, n)$ acting on a complex vector space V of dimension n . Then*

$$\mathcal{S}_1^V(G(r, p, n)) \cong \begin{cases} Q_n(r) & \text{if } r \neq p, r \neq 1 \\ Q_n(r, \{1\}) & \text{if } r = p \neq 1 \\ \Pi_n & \text{if } r = p = 1. \end{cases}$$

Furthermore, this isomorphism is G -equivariant with respect to the actions described in [19, §2.9] and Section 3.2.

Proof. For the isomorphism, the case $r = 1$ is well-known. The case $r = 2$ is discussed in [7, 21], and the general case follows the same argument. The G -equivariance is clear from the explicit definition of each of the actions. \square

Recall the definition of the Möbius function of a poset:

Definition 3.15. Let P be a locally finite poset. The *Möbius function* of P is a function $\mu_P : P \times P \rightarrow \mathbb{Z}$ defined recursively as follows:

$$\begin{aligned} \mu_P(x, x) &= 1 \quad \text{for all } x \in P, \\ \mu_P(x, y) &= - \sum_{x \leq z < y} \mu_P(x, z) \quad \text{for all } x < y \in P, \\ \mu_P(x, y) &= 0 \quad \text{otherwise.} \end{aligned}$$

If P is any poset, define \hat{P} to be the poset obtained from P by adjoining a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ (see [19, §2.2]).

The following lemma is the reason for our interest in the Möbius function:

Lemma 3.16. *If a poset P is Cohen–Macaulay over \mathbb{Z} and the rank of P is n , then*

$$\mu(\hat{P}) = \text{rank}(\tilde{H}_n(P, \mathbb{Z})).$$

Proof. This follows immediately from the Euler–Poincaré formula and the definition of a Cohen–Macaulay poset. \square

Proposition 3.17. (See [29, §6.4].)

- (i) *The Möbius function of $\mathcal{A}_n(r) \cong Q_n(r)$ is $(-1)^n(r+1)(2r+1)\cdots((n-1)r+1)$.*
- (ii) *The Möbius function of $\mathcal{A}_n^0(r) \cong Q_n(r, \{1\})$ is*

$$(-1)^n(r+1)(2r+1)\cdots((n-2)r+1)((n-1)(r-1)).$$

- (iii) *The Möbius function of Π_n is $(-1)^{n-1}(n-1)!$.*

3.4. Posets of eigenspaces for the groups $G(r, p, n)$

Recall from the Introduction that if γG is a reflection coset in $V = \mathbb{C}^n$, and $\zeta \in \mathbb{C}^\times$ is a complex root of unity then the poset $\mathcal{S}_\zeta^V(\gamma G)$ is defined to be the set $\{V(g, \zeta) \mid g \in \gamma G\}$, partially ordered by reverse inclusion. There is a close connection between the poset $\mathcal{S}_\zeta^V(\gamma G)$, where G is the reflection group $G(r, p, n)$, and subposets of the Dowling lattices defined in Section 3.2.

Suppose $G = G(r, p, n)$ acts on $V = \mathbb{C}^n$, and that γG is a reflection coset in V . By [11, Proposition 3.13], up to scalar multiples either $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$, where $e \in \mathbb{N}$, $e \mid p$ and ξ_m denotes a primitive m -th root of unity, or γ falls into one of three exceptional cases (see also [26, Table D.5, p. 278]). The first of these possibilities is considered in the following theorem. Note that it is sufficient to state the results for the case where this scalar multiple is 1, since if α is a scalar matrix then $\mathcal{S}_\zeta^V(\alpha\gamma G) = \mathcal{S}_{\alpha^{-1}\zeta}^V(\gamma G)$.

Theorem 3.18. *Let G be the group $G(r, p, n)$ acting on $V = \mathbb{C}^n$, γG a reflection coset in V with $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$ as above, and ζ a primitive m -th root of unity, for some positive integer m . Let $d = m/\gcd(m, r)$. Then*

- (i) *If $d > 1$, $n \equiv 0 \pmod{d}$ and $(\zeta^n \xi_{\frac{er}{p}}^{-1})^{r/p} \neq 1$, then $\mathcal{S}_\zeta^V(\gamma G) \cong Q_n(dr, d, d, \{0\}) \setminus \{\hat{0}\}$.*
- (ii) *If $d > 1$ and the conditions of (i) are not satisfied, then $\mathcal{S}_\zeta^V(\gamma G) \cong Q_n(dr, d, d) \setminus \{\hat{0}\}$.*
- (iii) *If $d = 1$ and $(\zeta^n \xi_{\frac{er}{p}}^{-1})^{r/p} \neq 1$, then $\mathcal{S}_\zeta^V(\gamma G) \cong Q_n(r, \{0\}) \setminus \{\hat{0}\}$.*
- (iv) *If $d = 1$, $r = p \neq 1$, and $\zeta^n = \xi_e$, then $\mathcal{S}_\zeta^V(\gamma G) \cong Q_n(r, \{1\})$.*

- (v) If $d = 1$, $r = p = 1$, then $m = 1$ and $\mathcal{S}_\zeta^V(\gamma G) \cong Q_{n-1}(1) \cong \Pi_n$.
 (vi) If $d = 1$ and the conditions of (iii)–(v) are not satisfied, then $\mathcal{S}_\zeta^V(\gamma G) \cong Q_n(r)$.

Proof. The proof of this theorem occupies the next several pages. Before embarking on the proof, we provide an outline. The first step is to describe the eigenspace $V(x, \zeta)$ for $x \in \gamma G$. This is Lemma 3.19. Next, we define a map $\tau : \mathcal{S}_\zeta^V(\gamma G) \rightarrow Q_n(dr)$ which associates to each eigenspace $V(x, \zeta)$ an element of the Dowling lattice $Q_n(dr)$. It will be clear by construction that $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d)$. In fact (Lemma 3.20) it is true that $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d)$. Up to this point the proof is case free. The next series of lemmas (until Lemma 3.24) provide restrictions on the size of the image of τ . These are case dependent. Finally it is shown that the image of τ is in fact as large as possible, subject to the restrictions given in the previous lemmas.

We first describe explicitly the map $\Gamma : \gamma G \rightarrow \mathcal{S}_\zeta^V(\gamma G)$ which sends $x \in \gamma G$ to $V(x, \zeta)$. Recalling that γ is a diagonal matrix, it makes sense to write $x = (\Omega, \sigma)$, where $\Omega = (\omega^{i_1}, \omega^{i_2}, \dots, \omega^{i_n}) \in (\mu_r)^n$, ω is a primitive r -th root of unity, and $\sigma \in \text{Sym}(n)$. Note that since $G = G(r, p, n)$ and $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$, it is equivalent to say that $(\omega^{\sum_{s=1}^n i_s} \xi_{\frac{er}{p}}^{-1})^{r/p} = 1$. Let $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, and suppose that $z \in V(x, \zeta)$.

According to the action described in Section 3.1, if $z \in V(x, \zeta)$ then for each $j \in \{1, \dots, n\}$, $\zeta z_j = \omega^{i_j} z_{\sigma^{-1}(j)}$. Equivalently, $z_j = \zeta^{-1} \omega^{i_j} z_{\sigma^{-1}(j)}$.

Now suppose that (j_1, j_2, \dots, j_k) is a cycle of σ (so that $\sigma(j_1) = j_2$ etc.). Then

$$z_{j_a} = \zeta^{-1} \omega^{i_{j_a}} z_{j_{a-1}}, \quad 1 \leq a \leq k \quad (3.1)$$

where we use cyclic notation, i.e. $j_a = j_{a+k}$. Combining these equations, $z_{j_1} = \zeta^{-k} \omega^{\sum_{s=1}^k i_{j_s}} z_{j_1}$.

These equations have nontrivial solution if and only if $\zeta^k = \omega^{\sum_{s=1}^k i_{j_s}}$.

So if nontrivial solutions exist, ζ^k must be an r -th root of unity (since ω is defined as an r -th root of unity). That is, $(\zeta^k)^r = 1$, and kr is a multiple of m . Thus k must be a multiple of $d = m/\text{gcd}(m, r)$.

If the system of equations (3.1) has a nontrivial solution, call it a d -cycle. We have therefore proved the following lemma:

Lemma 3.19. Suppose G , γ and x are as above. The equations defining $V(x, \zeta)$ consist of some set of d -cycles involving disjoint sets of variables, together with equations $z_{i_s} = 0$ whenever z_{i_s} does not appear in a d -cycle.

This lemma determines the map Γ completely.

The next step is to associate an element $(I, \pi, \gamma) \in Q_n(dr)$ with each $V(x, \zeta)$ ($x \in \gamma G$). Note that for each $s \in \{1, \dots, k\}$, the complex number $\zeta^{-1} \omega^{i_{j_s}}$ is a (dr) -th root of unity. There is a natural way to associate an element $(I, \pi, \gamma) \in Q_n(dr)$ with such a $V(x, \zeta)$. This map is the inverse of the one denoted ρ in Proposition 3.12, stated there for the

special case $m = 1$, $\gamma = \text{Id}$. Write this map as $\tau : \mathcal{S}_\zeta^V(\gamma G) \rightarrow Q_n(dr)$. We describe τ explicitly here.

To define $\tau(V(x, \zeta))$ for $x \in \gamma G$, the first task is to describe π . If (j_1, j_2, \dots, j_k) is a cycle of σ , k is a multiple of d , and $\zeta^k = \omega^{\sum_{s=1}^k i_{j_s}}$, then $\{j_1, j_2, \dots, j_k\}$ forms a nonzero block of π . Otherwise, j_1, j_2, \dots, j_k are all in the zero block of π .

Now let $B_i = \{j_1, j_2, \dots, j_k\}$ be a nonzero block of π . Define $\gamma_i : B_i \rightarrow \mu_{dr}$ by:

$$\gamma(j_a) = \zeta^{-(a-1)} \omega^{(\sum_{s=2}^a i_{j_s})}, \quad 1 \leq a \leq k \quad (3.2)$$

where we use cyclic notation, i.e. $j_a = j_{a+k}$.

Do this for each nonzero block B_i . Starting the cycle with a different element has the effect of shifting the labels by a constant factor, so the equivalence class $\overline{(I, \pi, \gamma)}$ is independent of the ordering of the cycles. This defines the map $\tau : \mathcal{S}_\zeta^V(\gamma G) \rightarrow Q_n(dr)$.

It is clear from the definition that τ is injective. Thus it remains to determine the size of the image.

To start, it is true by construction that $\overline{(I, \pi, \gamma)} \in Q_n(dr, d)$.

Recall (Definition 3.4) that $Q_n(r, d, k)$ is the ‘ d -divisible, k -evenly coloured’ Dowling lattice.

Lemma 3.20. $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d)$.

Proof of Lemma 3.20. Setting $r = \gcd(m, r)q$, note that every element of \mathbb{Z}_{dr} can be written uniquely in the form $aq + bd$ ($0 \leq a < d$, $0 \leq b < r$). Every element of μ_{dr} can therefore be uniquely written in the form $\zeta^a \omega^b$ ($0 \leq a < d$, $0 \leq b < r$) via the bijection $\zeta^a \omega^b \leftrightarrow aq + bd$. Identifying μ_{dr} with \mathbb{Z}_{dr} in this way, the elements $\zeta^a \omega^b$ (a fixed, $0 \leq a < d$, $0 \leq b < r$) all lie in the same congruence class modulo d . In other words, the congruence class modulo d of $\zeta^a \omega^b$ depends only on a . By construction the labels in each block $\overline{(B_i, \gamma_i)}$ are therefore d -evenly coloured, and so $\overline{(I, \pi, \gamma)} \in Q_n(dr, d, d)$. \square

Thus far everything has been case free. We now consider restrictions to the size of the image of τ in the different cases.

Lemma 3.21. If $d \neq 1$, then $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d) \setminus \{\hat{0}\}$.

Proof of Lemma 3.21. This is clear, since if $d \neq 1$ the unique minimal element $\{\hat{0}\}$ is added artificially, in order to make $Q_n(dr, d, d)$ a lattice. \square

Note that this lemma applies to both cases (i) and (ii). In particular if the conditions of (ii) are satisfied then $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d) \setminus \{\hat{0}\}$ as claimed.

Recall (Definition 3.5) that $Q_n(r, d, k, J)$ is the subposet of $Q_n(r, d, k)$ consisting of elements $\overline{(I, \pi, \gamma)}$ whose zero block I satisfies $|I| \notin J \subseteq \{0, 1, \dots, n\}$, together with the unique minimal element $\{\hat{0}\}$ of $Q_n(r, d, k)$.

Lemma 3.22. *If $n \equiv 0 \pmod{d}$ and $(\zeta^n \xi_{e_r}^{-1})^{r/p} \neq 1$, then $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d, \{0\})$.*

Proof of Lemma 3.22. Suppose that the conditions of the lemma hold, and that $(\overline{I, \pi, \gamma}) = \tau(V(x, \zeta))$, for some $x \in \gamma G$. As above, write $x = (\Omega, \sigma)$, where $\Omega = (\omega^{i_1}, \omega^{i_2}, \dots, \omega^{i_n}) \in (\mu_r)^n$, ω is a primitive r -th root of unity, and $\sigma \in \text{Sym}(n)$. As remarked previously, since $G = G(r, p, n)$ and $\gamma = \text{diag}(\xi_{e_r}^{e_r}, 1, \dots, 1)$, it follows that $(\omega^{\sum_{s=1}^n i_s} \xi_{e_r}^{-1})^{r/p} = 1$. If $j = (j_1, j_2, \dots, j_k)$ is a cycle of σ which contributes a nonzero block to $(\overline{I, \pi, \gamma})$, it follows by the argument preceding Lemma 3.19 that the cycle-product $\omega^{\sum_{s=1}^k i_{j_s}} = \zeta^k$. Considering all nonzero blocks together, $\omega^{\sum_{s \notin I} i_s} = \zeta^{n-|I|}$. Hence if $|I| = 0$, it follows that $\omega^{\sum_{s=1}^n i_s} = \zeta^n$. Hence $(\zeta^n \xi_{e_r}^{-1})^{r/p} = 1$, which contradicts the hypothesis of the lemma. Thus if $(\zeta^n \xi_{e_r}^{-1})^{r/p} \neq 1$, $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d, \{0\})$ as claimed. \square

Lemma 3.22 applies to both cases (i) and (iii). Combining Lemmas 3.21 and 3.22, it follows that if the conditions of (i) are satisfied then $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d, \{0\}) \setminus \{\hat{0}\}$. If the conditions of (iii) are satisfied then $Q_n(dr, d, d, \{0\}) = Q_n(r, 1, 1, \{0\}) = Q_n(r, \{0\})$. Further, the unique minimal element $\{\hat{0}\}$ of $Q_n(r, \{0\})$ does not correspond to an eigenspace, since it has zero block of size zero. Hence $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(r, \{0\}) \setminus \{\hat{0}\}$ as claimed.

Lemma 3.23. *If $n \equiv 1 \pmod{d}$, $r = p$, and $\zeta^n = \xi_e$, then $m \mid r$ and $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d, \{1\}) = Q_n(r, 1, 1, \{1\}) = Q_n(r, \{1\})$.*

Proof of Lemma 3.23. Adopt the notation of the previous lemma. To show that $m \mid r$ under the assumptions of the lemma, first note that $\zeta^{nr} = (\zeta^n)^r = (\xi_e)^r = 1$, since $e \mid p$ by definition, and in this case $r = p$. Also since $n \equiv 1 \pmod{d}$, we can write $n - 1 = jd$ for some $j \in \mathbb{Z}$. Hence $\zeta^{(n-1)r} = \zeta^{(jd)r} = \zeta^{jmr/\gcd(m, r)} = (\zeta^m)^{jr/\gcd(m, r)} = 1$, noting that $jr/\gcd(m, r) \in \mathbb{Z}$. Hence $\zeta^r = \zeta^{nr} \zeta^{-(n-1)r} = 1$. Since ζ is by definition a primitive m -th root of unity, it follows that $m \mid r$. In particular, $d = 1$.

For the second part of the lemma, note that as in the proof of the previous lemma, $\omega^{\sum_{s \notin I} i_s} = \zeta^{n-|I|}$. Suppose, by way of contradiction, that $|I| = 1$, and in particular that $I = \{t\}$ for some $t \in [n]$. Then $\omega^{\sum_{s \neq t} i_s} = \zeta^{n-1}$. Since $r = p$ we know that $\omega^{\sum_{s=1}^n i_s} = \xi_e$, so therefore $\omega^{i_t} = \zeta^{-(n-1)} \xi_e$. Since by assumption $\zeta^n = \xi_e$, the conclusion is that $\zeta = \omega^{i_t}$.

Furthermore, since t is the only element of I , it follows that (t) is a singleton cycle of σ . By (3.1), and the fact that $t \in I$, the equation $z_t = \zeta^{-1} \omega^{i_t} z$ has no nontrivial solution. This is a contradiction, since $\zeta = \omega^{i_t}$. Hence the assumption that $|I| = 1$ is invalid, and $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(dr, d, d, \{1\})$, as claimed. Since $m = r$, it follows that $d = 1$, and thus $Q_n(dr, d, d, \{1\}) = Q_n(r, 1, 1, \{1\}) = Q_n(r, \{1\})$, completing the proof of Lemma 3.23. \square

Lemma 3.23 applies to case (iv). In particular if the conditions of (iv) are satisfied then $\mathcal{S}_\zeta^V(\gamma G) \subseteq Q_n(r, \{1\})$. Note that **Lemma 3.23** shows that the case $J = \{1\}$ only arises when $d = 1$. Hence there is no corresponding case for $d > 1$.

Lemma 3.24. *If $d = 1$, $r = p = 1$, then $m = 1$ and $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(1, [n]) \cong \Pi_n \cong Q_{n-1}(1)$.*

Proof of Lemma 3.24. First, it is clear that if $d = 1$ then $m \mid r$, and so $m = 1$. Since $m = 1$, $\zeta = 1$ also. If the conditions of the lemma are satisfied, then so are the conditions of **Lemma 3.23**. Hence $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(r, \{1\}) = Q_n(1, \{1\})$. However since $r = 1$, $\omega = 1$. Hence the system of equations (3.1) always has a nonzero solution. Thus I cannot contain any elements at all, from which it follows that $\mathcal{S}_\zeta^V(\gamma G) \subseteq Q_n(1, [n])$. It is clear that $Q_n(1, [n]) \cong \Pi_n \cong Q_{n-1}(1)$. \square

Hence under the conditions of case (v), $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq \Pi_n$. Finally, under the conditions of case (vi), $Q_n(dr, d, d) = Q_n(r, 1, 1) = Q_n(r)$, and hence by **Lemma 3.20**, $\tau(\mathcal{S}_\zeta^V(\gamma G)) \subseteq Q_n(r)$.

It remains to show that, in each of cases (i)–(vi) of **Theorem 3.18**, that there are no further restrictions on the image of τ . First note that if $m = 1$, $\gamma = \text{Id}$ the statement reduces to **Corollary 3.14**. In this case, $Q_n(r, 1, 1) = Q_n(r)$ and $Q_n(r, 1, 1, \{1\}) = Q(r, \{1\})$. Also if $r = p = m = 1$ then $\gamma = \text{Id}$ is the only possibility since $er/p = 1$, so (v) is true.

Given an element $(I, \pi, \gamma) \in Q_n(dr, d, d)$, we must find some $x = (\Omega, \sigma) \in \gamma G(r, p, n)$ such that $\tau(V(x, \zeta)) = (I, \pi, \gamma)$. Recall that $\pi = B_1 \mid \cdots \mid B_k$ is a partition of $[n] \setminus I$. Suppose $B_l = \{j_1, \dots, j_{k_l}\}$ is a nonzero block of π . We may assume that the map γ_i has the form described in Eq. (3.2), reordering the j_i if necessary:

$$\gamma(j_a) = \zeta^{-(a-1)} \omega^{\sum_{s=2}^a i_{js}}, \quad 1 \leq a \leq k$$

where cyclic notation is used, as before.

Define $(j_1 \dots, j_{k_l})$ to be a cycle of σ , and $\Omega_{j_s} := \omega^{i_{js}} (2 \leq s \leq k_l)$ while $\Omega_{j_1} := \zeta^{k_l} \omega^{-\sum_{s=2}^{k_l} i_{js}}$. Do this for each nonzero block B_l ($1 \leq l \leq k$). If $|I| = 0$ then this data completely specifies x .

Given $x = (\Omega, \sigma) \in \gamma G(r, p, n)$, note that for each nonzero block $B_l = \{j_1, \dots, j_k\}$, ($1 \leq l \leq k$) we know that $\zeta^{k_l} = \prod_{i=1}^l \Omega_{j_{k_l}}$. Taking the product of expressions for all B_l , ($1 \leq l \leq k$) gives $\zeta^{n-|I|} = \prod_{j \notin I} \Omega_j$. As $x = (\Omega, \sigma) \in \gamma G(r, p, n)$, it is also true that $(\prod_{i=1}^n \Omega_i \xi_{\frac{er}{p}}^{-1})^{r/p} = 1$. Combining these two results gives

$$\left(\left(\prod_{j \in I} \Omega_j \right) \zeta^{n-|I|} \xi_{\frac{er}{p}}^{-1} \right)^{r/p} = 1. \quad (3.3)$$

Thus if $|I| = 0$ it follows that $x \in \gamma G(r, p, n) \iff (\zeta^n \xi_{\frac{er}{p}}^{-1})^{r/p} = 1$. Note that this equation implies the condition in parts (i) and (iii) of [Theorem 3.18](#). If $|I| \neq 0$ let $I = \{i_1, \dots, i_q\}$. Then [Eq. \(3.3\)](#) becomes

$$\left(\prod_{j=1}^q \Omega_{i_j} \right)^{r/p} = (\zeta^{-(n-q)} \xi_{\frac{er}{p}})^{r/p}.$$

Now for any cycle $\{j_1, \dots, j_l\}$ in I it must be true that $\zeta^l \neq \prod_{k=1}^l \Omega_{j_k}$. It is left to show that in the remaining cases, we can choose Ω_{i_t} , $1 \leq t \leq q$ in such a way that this condition is satisfied.

First suppose $m \neq 1$. Define (i_t) ($1 \leq t \leq q$) to be singleton cycles of σ . We must choose $\Omega_{i_t} \neq \zeta$. If $r \neq p$ there is no problem doing this – define $\Omega_{i_t} := 1 \neq \zeta$ ($1 \leq t \leq q-1$). From [Eq. \(3.3\)](#), $(\Omega_{i_q})^{r/p} = (\zeta^{-(n-q)} \xi_{\frac{er}{p}})^{r/p}$. There are then r/p possible choices for Ω_{i_q} , and at least one of these is not equal to ζ , as required.

If $r = p$, then since $m \neq 1$, define $\Omega_{i_t} := 1 \neq \zeta$ ($1 \leq t \leq q-1$). If $|I| = 1$ then as above we are forced to define $\Omega_{i_q} := \zeta^{-(n-1)} \xi_e$. This is not equal to ζ unless $\zeta^n = \xi_{\frac{er}{p}} = \xi_e$. This is the condition for part (iv) of the theorem.

If $r = p$ and $|I| > 1$, we make two constructions, at least one of which will give the desired result. First, let (i_t) ($1 \leq t \leq q$) be singleton cycles of σ and define $\Omega_{i_t} := 1 \neq \zeta$ ($1 \leq t \leq q-1$), $\Omega_{i_q} := \zeta^{-(n-q)} \xi_e$. Alternatively, let (i_t) ($1 \leq t \leq q-2$) be singleton cycles of σ , and (i_{q-1}, i_q) a cycle of length 2. Define $\Omega_{i_t} := 1 \neq \zeta$ ($1 \leq t \leq q-1$), $\Omega_{i_q} := \zeta^{-(n-q)} \xi_e$ as before. Since $m \neq 1$, at least one of ζ and ζ^2 is not equal to $\zeta^{-(n-q)} \xi_e$, in which case the equations which contribute to the zero block of $(\overline{I}, \pi, \gamma)$ do indeed have only the trivial solution. This completes the proof when $m \neq 1$.

Now suppose $m = 1$. If $\xi_{\frac{er}{p}} \neq 1$, let (i_1, i_2, \dots, i_q) be a cycle of σ of length q and set $\Omega_{i_t} := 1$ ($1 \leq t \leq q-1$), $\Omega_{i_q} := \xi_{\frac{er}{p}}$. Since $1 = \zeta^q \neq \xi_{\frac{er}{p}}$, the relevant equations have only the trivial solution, as required. If $\xi_{\frac{er}{p}} = 1$ then $\gamma G = G$, and the problem reduces to [Corollary 3.14](#).

In each of the above cases it is easy to verify that $\tau(V(g, \zeta)) = (\overline{I}, \pi, \gamma)$, as required. This completes the proof of [Theorem 3.18](#). \square

Corollary 3.25. *The poset $\mathcal{S}_{\zeta}^V(\gamma G(r, p, n))$ is Cohen–Macaulay over \mathbb{Z} , and over any field \mathbb{F} .*

Proof. Each of the subposets of Dowling lattices which appear in [Theorem 3.18](#) admit a recursive atom ordering, by the results in [Section 3.2](#). The corollary now follows immediately from [Proposition 2.6](#) and [Theorem 2.8](#) in the case where γ acts diagonally.

The three exceptional cases (see [\[26, Table D.5, p. 278\]](#)) can be treated by direct computation. A detailed proof can be found in [\[20, Remark 4.2.9\]](#). \square

Corollary 3.26. *If $x, y \in \gamma G(r, p, n)$ and $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$ then for some $z \in \gamma G(r, p, n)$, $V(x, \zeta) \cap V(y, \zeta) = V(z, \zeta)$.*

Proof. Let $M = V(x, \zeta) \cap V(y, \zeta)$. The defining equations for M are those for $V(x, \zeta)$ together with those for $V(y, \zeta)$. Using the isomorphism established in the proof of [Theorem 3.18](#), combining these two systems of equations corresponds precisely to taking joins in Dowling lattices. Hence M is itself an eigenspace, as required. \square

By a direct computation, we could verify this corollary for all reflection cosets associated to imprimitive groups (including the exceptional cosets). In fact, this corollary holds for general reflection cosets, not just the imprimitive ones. This is proven in [\[19, Corollary 3.2\]](#).

3.5. Dimensions of representations for the imprimitive groups

The posets $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ are all Cohen–Macaulay over \mathbb{Z} , and over any field \mathbb{F} ([Corollary 3.25](#)). However all of these posets have a unique maximal element, and possibly a unique minimal element as well, so their homology is uninteresting – see the remarks in the Introduction, p. 2. Hence our interest lies in the poset $\tilde{\mathcal{S}}_\zeta^V(\gamma G(r, p, n))$, which (recall from the Introduction) is obtained from $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ by removing the unique maximal element, and the unique minimal element if it exists.

Remark 3.27. It is possible to say precisely when the full space V is the unique minimal element of $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$. Assume once more that $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$.

The full space

$$\begin{aligned} V \in \mathcal{S}_\zeta^V(\gamma G(r, p, n)) &\Leftrightarrow \text{diag}(\zeta, \zeta, \dots, \zeta) \in \gamma G(r, p, n) \\ &\Leftrightarrow \zeta^n = \xi_{\frac{er}{p}} \xi_{\frac{r}{p}} \text{ (this notation defined in Section 3.4)} \\ &\quad \text{and } \zeta \text{ is an } r\text{-th root of unity} \\ &\Leftrightarrow (\zeta^n \xi_{\frac{er}{p}}^{-1})^{r/p} = 1 \text{ and } d = 1. \end{aligned}$$

Where homology is taken over a field \mathbb{F} , we would like information about the homology representation of G on the unique non-vanishing reduced homology of $\tilde{\mathcal{S}}_\zeta^V(\gamma G(r, p, n))$. At a minimum, we would like to know the dimension of such a representation. Recall that \hat{P} is defined to be the poset obtained from P by adjoining a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$.

Proposition 3.28. *Let G be the group $G(r, p, n)$ acting on $V = \mathbb{C}^n$, γG a reflection coset in V , and ζ a primitive m -th root of unity. Then $\dim \tilde{H}_{\text{top}}(\tilde{\mathcal{S}}_\zeta^V(\gamma G(r, p, n))) = \mu(\widehat{\tilde{\mathcal{S}}_\zeta^V(\gamma G(r, p, n))})$.*

Proof. This is a special case of [Lemma 3.16](#). \square

[Theorem 3.18](#) can be reformulated for the poset $\widehat{\tilde{\mathcal{S}}_\zeta^V(\gamma G(r, p, n))}$. The case $n = 1$ is trivial as if V is one-dimensional, the poset $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ is either the one-element poset

or the two-element totally ordered poset according to whether or not $V \in \mathcal{S}_\zeta^V(\gamma G(r, p, n))$ (see Remark 3.27). In the following proposition, assume that $n > 1$:

Proposition 3.29. *Let G be the group $G(r, p, n)$ acting on $V = \mathbb{C}^n$, γG a reflection coset in V with $\gamma = \text{diag}(\xi_{\frac{er}{p}}, 1, \dots, 1)$, and ζ a primitive m -th root of unity. Let $d = m/\gcd(m, r)$, and suppose $n > 1$. Then*

- (i) *If $n \equiv 0 \pmod{d}$ and $(\zeta^n \xi_{\frac{er}{p}}^{-1})^{r/p} \neq 1$, then $\widehat{\mathcal{S}_\zeta^V(\gamma G)} \cong Q_n(dr, d, d, \{0\})$.*
- (ii) *If $d = 1$, $r = p \neq 1$, and $\zeta^n = \xi_e$, then $\widehat{\mathcal{S}_\zeta^V(\gamma G)} \cong Q_n(r, \{1\})$.*
- (iii) *If $d = 1$, $r = p = 1$, then $m = 1$ and $\widehat{\mathcal{S}_\zeta^V(\gamma G)} \cong Q_{n-1}(1) \cong \Pi_n$.*
- (iv) *In all other cases, $\widehat{\mathcal{S}_\zeta^V(\gamma G)} \cong Q_n(dr, d, d)$.*

Proof. The poset $\widehat{\mathcal{S}_\zeta^V(\gamma G(r, p, n))}$ is isomorphic to $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ if the latter has a unique minimal element, and is equal to $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ with a unique minimal element adjoined otherwise. In the statement of Theorem 3.18, $\widehat{\mathcal{S}_\zeta^V(\gamma G(r, p, n))}$ does not have a unique minimal element in cases (i), (ii) and (iii) when $n > 1$, while it does in the remaining cases. The result now follows immediately. \square

Hence in order to compute the dimension of the unique non-vanishing reduced homology of $\widehat{\mathcal{S}_\zeta^V(\gamma G(r, p, n))}$ in all cases of Proposition 3.29, it will suffice to compute the Möbius function of the posets $Q_n(dr, d, d)$, $Q_n(dr, d, d, \{0\})$, and $Q_n(r, \{1\})$. The last of these posets is isomorphic to $\mathcal{L}(G(r, r, n))$ (Corollary 3.14), so its Möbius function is known (Proposition 3.17). The following section deals with the first two cases in question.

4. Exponential Dowling structures

An explicit formula for the dimension of the representation of G on the unique non-vanishing reduced homology of $\widehat{\mathcal{S}_\zeta^V(\gamma G(r, p, n))}$ (or equivalently, the Möbius function of $\widehat{\mathcal{S}_\zeta^V(\gamma G(r, p, n))}$) is not known in all cases. However generating functions for these Möbius functions can be found. The key results in this section are Corollary 4.10 and Corollary 4.17. The posets $\mathcal{S}_\zeta^V(\gamma G(r, p, n))$ are examples of *exponential Dowling structures*. This class of posets was introduced by Ehrenborg and Readdy in [14] as a generalisation of the *exponential structures* first described by Stanley [32]. Exponential structures are a general family of posets modelled on the partition lattice Π_n . They allow the study of generating functions associated to a sequence of posets $\mathbf{R} = \{R_1, R_2, \dots\}$ of the form $\sum_{n=0}^\infty f(n)x^n/n!M(n)$ in a mechanical way. The integers $M(n)$ have a combinatorial meaning – they are the number of minimal elements in the poset R_n .

Definition 4.1. (See [14, Definition 3.1].) An *exponential structure* $\mathbf{R} = (R_1, R_2, \dots)$ is a sequence of posets such that:

- (E1) The poset R_n has a unique maximal element $\hat{1}$ and every maximal chain in R_n contains n elements.
- (E2) For an element $x \in R_n$ of rank k , the interval $[x, \hat{1}]$ is isomorphic to the partition lattice on $n - k$ elements, Π_{n-k} .
- (E3) The lower order ideal generated by $x \in R_n$ is isomorphic to $R_1^{a_1} \times R_2^{a_2} \times \dots \times R_n^{a_n}$. Define the *type* of x to be the n -tuple (a_1, a_2, \dots, a_n) .
- (E4) The poset R_n has $M(n)$ minimal elements. The sequence $(M(1), M(2), \dots)$ is called the *denominator sequence* of \mathbf{R} .

The generalisation to Dowling lattices due to Ehrenborg and Readdy is as follows.

Definition 4.2. (See [14, Definition 3.2].) Let G' be a finite group. An *exponential Dowling structure* $\mathbf{S} = (S_0, S_1, \dots)$ associated to an exponential structure $\mathbf{R} = (R_1, R_2, \dots)$ is a sequence of posets such that:

- (D1) The poset S_n has unique maximal element $\hat{1}$ and every maximal chain in S_n contains $n + 1$ elements.
- (D2) For an element $x \in S_n$, $[x, \hat{1}]$ is isomorphic to the Dowling lattice $Q_{n-\text{rk}(x)}(G')$.
- (D3) Each element in S_n has a type $(b; a_1, \dots, a_n)$ assigned such that the lower order ideal generated by x is isomorphic to $S_b \times R_1^{a_1} \times \dots \times R_n^{a_n}$.
- (D4) The poset S_n has $N(n)$ minimal elements. The sequence $(N(0), N(1), \dots)$ is called the *denominator sequence* of \mathbf{S} .

Note that S_0 is the poset with one element and that $N(0) = 1$.

Remark 4.3. In this paper, we consider the special case where G' is a cyclic group.

The utility of knowing that a sequence is an exponential (Dowling) structure lies in the fact that Möbius functions are easy to compute. Recall (Definition 3.15) that for a bounded poset P , we define $\mu(P) := \mu(\hat{0}, \hat{1})$.

Theorem 4.4. (See [14, Corollary 3.8].) Let $\mathbf{S} = (S_0, S_1, \dots)$ be an exponential Dowling structure with denominator sequence $(N(0), N(1), \dots)$ and associated exponential structure $\mathbf{R} = (R_1, R_2, \dots)$ with denominator sequence $(M(1), M(2), \dots)$. Then the Möbius function of the posets $R_n \cup \{\hat{0}\}$, respectively $S_n \cup \{\hat{0}\}$, has the generating function:

$$\sum_{n \geq 1} \mu(R_n \cup \{\hat{0}\}) \cdot \frac{x^n}{M(n) \cdot n!} = -\log \left(\sum_{n \geq 0} \frac{x^n}{M(n) \cdot n!} \right)$$

$$\sum_{n \geq 0} \mu(S_n \cup \{\hat{0}\}) \cdot \frac{x^n}{N(n) \cdot n!} = - \left(\sum_{n \geq 0} \frac{x^n}{N(n) \cdot n!} \right) \left(\sum_{n \geq 0} \frac{(s \cdot x)^n}{M(n) \cdot n!} \right)^{-1/s},$$

where $s = |G|$.

Let \mathbf{R} be an exponential structure and d a positive integer. Stanley defined the exponential structure $\mathbf{R}^{(d)}$ by letting $R_n^{(d)}$ be the subposet of R_{dn} of all elements x of type (a_1, a_2, \dots) where $a_i = 0$ unless d divides i .

In [14, Example 3.4], Ehrenborg and Readdy define an analogous exponential Dowling structure. Let \mathbf{S} be an exponential Dowling structure associated with the exponential structure \mathbf{R} . Let $S_n^{(d,e)}$ be the subposet of S_{dn+e} consisting of all elements x of type $(b; a_1, a_2, \dots)$ such that $b \geq e$, $b \equiv e \pmod{d}$ and $a_i = 0$ unless d divides i . Then $\mathbf{S}^{(d,e)} = (S_0^{(d,e)}, S_1^{(d,e)}, \dots)$ is an exponential Dowling structure associated with the exponential structure $\mathbf{R}^{(d)}$. The minimal elements of $S_n^{(d,e)}$ are the elements of S_{dn+e} having types given by $b = e$, $a_d = n$ and $a_i = 0$ for $i \neq d$.

If S_n is the Dowling lattice $Q_n(r)$ and $e < d$, then $S_n^{(d,e)}$ is the poset $Q_{dn+e}(r, d) \setminus \{\hat{0}\}$. We show that the posets $Q_{dn+e}(r, d, k) \setminus \{\hat{0}\}$ form an exponential Dowling structure, though the underlying exponential structure is different. In the following proposition, the group G' of Definition 4.2 is the cyclic group μ_r .

Definition 4.5. Suppose $k, d \in \mathbb{Z}$ with $k \mid d$. Let $c_n = \overline{(I_n, \pi_n, \gamma_n)}$ be the element of $Q_{dn}(r, d, k)$ defined as follows:

$$\begin{aligned} I_n &= \{0\}, \\ \pi_n &= \{1, 2, \dots, dn\}, \\ \gamma(i) &= \left\lfloor \frac{(i-1)k}{dn} \right\rfloor + 1 \quad \text{for } i = 1, 2, \dots, dn. \end{aligned}$$

Define $R_n'^{(d,k)}$ to be the lower order ideal generated by c_n in $Q_{dn}(r, d, k)$.

Define $\mathbf{R}'^{(d,k)} = (R_1'^{(d,k)}, R_2'^{(d,k)}, \dots)$.

Define $\mathbf{S}^{(d,e,k)} = (S_0^{(d,e,k)}, S_1^{(d,e,k)}, \dots)$, where $S_n^{(d,e,k)}$ is the poset $Q_{dn+e}(r, d, k) \setminus \{\hat{0}\}$.

Lemma 4.6. Let $\mathbf{R}'^{(d,k)} = (R_1'^{(d,k)}, R_2'^{(d,k)}, \dots)$ be defined as in Definition 4.5. Then $\mathbf{R}'^{(d,k)}$ is an exponential structure.

Proof. It is routine to check axioms (E1)–(E4). The minimal elements of $R_n'^{(d,k)}$ are the elements lying below $c_n \in Q_{dn}(r, d, k)$ having type $(b; a_1, a_2, \dots)$, where $b = 0$, $a_d = n$, and $a_i = 0$ for $i \neq d$. \square

Lemma 4.7. Let $\mathbf{S}^{(d,e,k)}$ be defined as above. Then $\mathbf{S}^{(d,e,k)}$ is an exponential Dowling structure associated with the exponential structure $\mathbf{R}'^{(d,k)}$.

Proof. It is routine to check axioms (D1)–(D4) for $\mathbf{S}^{(d,e,k)}$. The minimal elements of $S_n^{(d,e,k)}$ are the elements of $Q_{dn+e}(r, d, k) \setminus \{\hat{0}\}$ having types given by $b = e$, $a_d = n$, and $a_i = 0$ for $i \neq d$. \square

Lemma 4.8. With $\mathbf{S}^{(d,e,k)}$, $\mathbf{R}'^{(d,k)}$ defined as above,

$$M(n) = \frac{[(\frac{dn}{k})!]^k}{n![(\frac{d}{k})!]^{kn}} \quad \text{and} \quad N(n) = \frac{(dn+e)!r^{(d-1)n}}{n!e![(\frac{d}{k})!]^{kn}k^{dn}}.$$

Proof. To compute $M(n)$, recall, as in the proof of Proposition 4.6, that the minimal elements of $R_j'^{(d)}$ are the elements lying below $c_n \in Q_{dn}(r, d, k)$ having type $(b; a_1, a_2, \dots)$, where $b = 0$, $a_d = n$, and $a_i = 0$ for $i \neq d$.

Consider the number of ways of choosing such an atom. First note that labelling of elements in the atom is completely determined by the labelling of x . Therefore the only freedom in choosing an atom comes from the way the elements of the single nonzero block of size dn in x are split up. Within this nonzero block, there are dn/k elements of each of k colours. Pick any colour, and split these dn/k elements into n sets of k/d elements. This can be done in $(n!)^{-1} \binom{dn/k}{d/k, d/k, \dots, d/k}$ (n d/k 's in the denominator) ways, since the order in which these sets are chosen is unimportant. Since there are k colours and we have to split the dn/k elements of each colour up in this way, this gives $((n!)^{-1} \binom{dn/k}{d/k, d/k, \dots, d/k})^k$ ways of splitting the colours. We then have to form nonzero blocks of size d in the atom by combining one set of size d/k of each colour. This can be done in $(n!)^{k-1}$ ways. Hence the total number of such atoms is given by

$$M(n) = \frac{[(\frac{dn}{k})!]^k}{n![(\frac{d}{k})!]^{kn}},$$

as claimed.

To compute $N(n)$, first pick the zero block (of size e) and the n nonzero blocks (of size d) in $(n!)^{-1} \binom{dn+e}{e, d, \dots, d}$ (n d 's in the denominator) ways.

Now count the number of ways of labelling the elements within each nonzero block. It is most convenient to ignore equivalence of labellings at first, and then divide by r at the end to get the required answer. First partition the d elements into k sub-blocks of size d/k . This can be done in $\binom{d}{d/k, d/k, \dots, d/k}$ (k d/k 's in the denominator) ways. Elements in the same sub-block have labels in the same congruence class modulo k . There are now r/k choices for each label. After dividing by r , the number of ways of labelling each nonzero block of size d is $d!r^{d-1}((d/k)!^k k^d)^{-1}$.

There are n such blocks, so

$$N(n) = \frac{1}{n!} \binom{dn+e}{e, d, \dots, d} \left[\frac{d!r^{d-1}}{[(\frac{d}{k})!]^k k^d} \right]^n = \frac{(dn+e)!r^{(d-1)n}}{n!e![(\frac{d}{k})!]^{kn}k^{dn}},$$

as claimed. \square

For the purposes of this work, the main interest lies in the posets $Q_{dn+e}(dr, d, d)$ (and subposets with restricted zero block size).

In the following lemma, the group G' of Definition 4.2 is the cyclic group μ_{dr} .

Lemma 4.9. Let $S_n^{(d,e,d)}$ be the poset $Q_{dn+e}(dr, d, d) \setminus \{\hat{0}\}$. Then $M(n) = (n!)^{d-1}$ and $N(n) = \frac{(dn+e)!r^{(d-1)n}}{n!e!d^n}$.

Proof. This is a special case of the previous lemma, obtained by replacing r with dr , and k with d . \square

Corollary 4.10. With notation as above,

$$\begin{aligned} & \sum_{n \geq 0} \mu(Q_{dn+e}(dr, d, d)) \cdot \frac{x^n e! d^n}{(dn+e)! r^{(d-1)n}} \\ &= - \left(\sum_{n \geq 0} \frac{x^n e! d^n}{(dn+e)! r^{(d-1)n}} \right) \left(\sum_{n \geq 0} \frac{x^n d^n r^n}{(n!)^{d-1}} \right)^{-1/dr}. \end{aligned}$$

Proof. Apply Theorem 4.4 to the exponential Dowling structure $\mathbf{S}^{(d,e,d)} = (S_0^{(d,e,d)}, S_1^{(d,e,d)}, \dots)$, where $S_n^{(d,e,d)} = Q_n(dr, d, d) \setminus \{\hat{0}\}$. The formula for $M(n)$ and $N(n)$ is given in Lemma 4.9. \square

Example 4.11. Consider the case $r = p = 1$, $\gamma = \text{Id}$, $\zeta = -1$ in Theorem 3.18. Case (ii) is the applicable one. Thus $S_{-1}^V(A_{n-1}) \cong Q_n(2, 2, 2) \setminus \{\hat{0}\}$. Suppose further that $n = 2k$ is even.

Corollary 4.10 then gives:

$$\sum_{k \geq 0} \mu(S_{-1}^V(A_{2k-1})) \cdot \frac{2^k x^k}{(2k)!} = - \left(\sum_{k \geq 0} \frac{2^k x^k}{(2k)!} \right) \left(\sum_{k \geq 0} \frac{2^k x^k}{(k!)^2} \right)^{-\frac{1}{2}}.$$

This gives $\mu(S_{-1}^V(A_1)) = 0$, $\mu(S_{-1}^V(A_3)) = -1$, $\mu(S_{-1}^V(A_5)) = 24$, $\mu(S_{-1}^V(A_7)) = -918$ and $\mu(S_{-1}^V(A_9)) = 54\,560$. A separate but similar calculation gives the values of the Möbius functions for n odd.

Ehrenborg and Readdy also developed a theory of Möbius functions of *restricted structures*. The following material is taken directly from [14].

Definition 4.12. Let I be a subset of \mathbb{N}^+ , and $\mathbf{R} = (R_1, R_2, \dots)$ an exponential structure. Define the *restricted poset* R_n^I to be the poset consisting of all elements $x \in R_n$ whose type (a_1, \dots, a_n) satisfies the condition that $a_i > 0$ implies $i \in I$. For $n \in I$ let $\mu_I(n)$ denote the Möbius function of the poset $R_n^I \cup \{\hat{0}\}$. For $n \notin I$ let $\mu_I(n) = 0$.

Suppose $n \in \mathbb{N}$ and define

$$m_n := \sum_{x \in R_n^I \cup \{\hat{0}\}} \mu_I(\hat{0}, x) = 1 + \sum_{x \in R_n^I} \mu_I(\hat{0}, x).$$

Theorem 4.13. (See [14, Theorem 4.1].) With notation as before,

$$\sum_{i \in I} \mu_I(i) \frac{x^i}{M(i) \cdot i!} = -\log \left(\sum_{n \geq 0} \frac{x^n}{M(n) \cdot n!} - \sum_{n \notin I} m_n \cdot \frac{x^n}{M(n) \cdot n!} \right).$$

There is a corresponding theorem for exponential Dowling structures:

Definition 4.14. Let $I \subseteq \mathbb{N}$, and $J \subseteq \mathbb{Z}_{\geq 0}$. For an exponential Dowling structure $\mathbf{S} = (S_0, S_1, \dots)$ define the *restricted poset* $S_n^{I,J}$ to be all elements $x \in S_n$ whose type $(b; a_1, \dots, a_n)$ satisfies $b \in J$ and $a_i > 0$ implies $i \in I$. For $n \in J$ define $\mu_{I,J}(n)$ to be the Möbius function of the poset $S_n^{I,J} \cup \{\hat{0}\}$. For $n \notin J$ define $\mu_{I,J}(n) = 0$.

For $n \in \mathbb{Z}_{\geq 0}$ define

$$p_n := \sum_{x \in S_n^{I,J} \cup \{\hat{0}\}} \mu_{I,J}(\hat{0}, x) = 1 + \sum_{x \in S_n^{I,J}} \mu_{I,J}(\hat{0}, x).$$

Note that if $n \in J$ then the poset $S_n^{I,J}$ has a maximal element, so $p_n = 0$.

Theorem 4.15. (See [14, Theorem 4.2].) With notation as above,

$$\begin{aligned} & \sum_{b \in J} \mu_{I,J}(b) \cdot \frac{x^b}{N(b) \cdot b!} \\ &= -\sum_{n \geq 0} \frac{x^n}{N(n) \cdot n!} + \sum_{n \notin J} p_n \cdot \frac{x^n}{N(n) \cdot n!} \left(\sum_{n \geq 0} \frac{(r \cdot x)^n}{M(n) \cdot n!} - \sum_{n \notin I} m_n \cdot \frac{(r \cdot x)^n}{M(n) \cdot n!} \right)^{1/r}. \end{aligned}$$

Corollary 4.16. (See [14, Corollary 4.3].) Let $I \subseteq \mathbb{N}$ be a semigroup and $J \subseteq \mathbb{Z}_{\geq 0}$ such that $I + J \subseteq J$. Then the Möbius function of the restricted poset $R_n^I \cup \{\hat{0}\}$ and $S_n^{I,J} \cup \{\hat{0}\}$ respectively has the generating function

$$\begin{aligned} & \sum_{n \in I} \mu_I(n) \cdot \frac{x^n}{M(n) \cdot n!} = -\log \left(\sum_{n \in I \cup \{0\}} \frac{x^n}{M(n) \cdot n!} \right), \\ & \sum_{n \in J} \mu_{I,J}(n) \cdot \frac{x^n}{N(n) \cdot n!} = -\left(\sum_{n \in J} \frac{x^n}{N(n) \cdot n!} \right) \cdot \left(\sum_{n \in I \cup \{0\}} \frac{(r \cdot x)^n}{M(n) \cdot n!} \right)^{-1/r}. \end{aligned}$$

Corollary 4.16 can be used to study the posets $Q_{dn}(dr, d, d, \{0\})$.

Corollary 4.17.

$$\sum_{n \geq 1} \mu(Q_{dn}(dr, d, d, \{0\})) \cdot \frac{x^n d^n}{(dn)! r^{(d-1)n}} = - \left(\sum_{n \geq 1} \frac{x^n d^n}{(dn)! r^{(d-1)n}} \right) \left(\sum_{n \geq 0} \frac{x^n d^n r^n}{(n!)^{d-1}} \right)^{-1/dr}.$$

Proof. In Corollary 4.16 set $\mathbf{R}'^{(d)}, \mathbf{S}^{(d,e,k)} = \mathbf{S}^{(d,0,d)}$ as in Definition 4.5, and let $I = \mathbb{N}$, $J = \mathbb{N} \setminus \{0\}$. \square

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