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Combinatorics of diagrams of permutations ☆

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A R T I C L E I N F O

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A B S T R A C T

There are numerous combinatorial objects associated to a Grassmannian permutation w_λ that index cells of the totally nonnegative Grassmannian. We study several of these objects and their q-analogues in the case of permutations w that are not necessarily Grassmannian. We give two main results: first, we show that certain acyclic orientations, rook placements avoiding a diagram of w, and fillings of a diagram of w are equinumerous for all permutations w. Second, we give a q-analogue of a result of Hultman–Linusson–Shareshian–Sjöstrand by showing that under a certain pattern condition the Poincaré polynomial for the Bruhat interval of w essentially counts invertible matrices over a finite field avoiding a diagram of w. In addition to our main results, we include at the end a number of open questions.

1. Introduction

In his study [22] of the totally nonnegative Grassmannian $Gr^{\geq 0}_{k,n}(\mathbb{R})$, Postnikov introduced a "zoo" of combinatorial objects that parametrize cells of the matroidal

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decomposition of $Gr_{k,n}^{\geq 0}(\mathbb{R})$. This decomposition refines the Schubert decomposition $Gr_{k,n} = \bigcup_{\lambda \subset \langle (n-k)^k \rangle} \Omega_\lambda$, and the members of the zoo are most easily identified with the *Grassmannian permutations* $w = w_\lambda \in \mathfrak{S}_n$. Among the objects that appear in the zoo are the following four (whose precise definitions will be given later):

(i) The set of *acyclic orientations* of the *inversion graph* of w.
(ii) The set of *placements of* n *non-attacking rooks* on a board associated to w.
(iii) The set of certain *restricted fillings* of a diagram associated to w.
(iv) The set of permutations below w in the *strong Bruhat order*.

Work of Postnikov establishes the following result.

Theorem 1.1. *(See [22, Thm. 24.1].) For a Grassmannian permutation w_λ in $\mathfrak{S}_n$, the sets above are equinumerous.*

This theorem naturally raises the following question:

Problem 1.2. *Characterize the relation among these sets when w is not Grassmannian.*

In the rest of this introduction we give background on previous work and a summary of our own results towards answering this problem and its refinements.

1.1. Definitions

We begin by giving the definitions of the terms in the preceding paragraphs, which will be used throughout this paper. Several definitions are illustrated in Fig. 1.

The pair (i, j) is said to be an **inversion** of the permutation $w \in \mathfrak{S}_n$ if $1 \leq i < j \leq n$ and $w_i > w_j$. The **inversion graph** G_w of w is the graph with vertex set $[n] := \{1, 2, \ldots, n\}$ and with edges given by the inversions of w. We consider the vertices to be ordered with smaller vertices to the *left* or *earlier* and larger vertices to the *right* or *later*. An **acyclic orientation** of a graph G is an orientation of the edges of G so that the oriented graph has no directed cycles. The number of acyclic orientations of G is denoted $AO(G)$.

A **diagram** (or **board**) is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. The **south–east (SE) diagram** E_w (respectively, **south–west (SW) diagram** O_w) of the permutation w is the subset of $[n] \times [n]$ consisting of those elements not directly to the south or east (respectively, south or west) of a nonzero entry in the permutation matrix of w. (In [20, §2.1], the diagram E_w is called the *Rothe diagram* of w.) The size of E_w is the number $\ell(w)$ of inversions of w, while the size of O_w is the number of **anti-inversions**, i.e., the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $w_i < w_j$. (This is also $\binom{n}{2} - \ell(w)$.) Equivalently, E_w is the subset of $[n] \times [n]$ consisting of all pairs (i, w_j) such that $i < j$ and $w_j < w_i$ and O_w is the subset of $[n] \times [n]$ consisting of all pairs (i, w_j) such that $i < j$ and $w_j > w_i$.

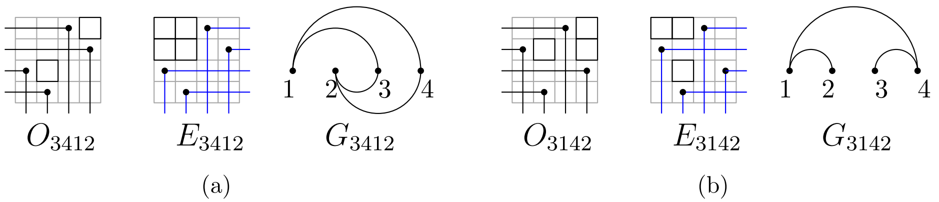


Fig. 1. The SW diagram, the SE diagram, and the inversion graph of the permutations (a) 3412 and (b) 3142. For the Grassmannian permutation 3412 (associated to the shape $\lambda = \langle 2,2\rangle$) we have $AO(G_{3412}) = RP(O_{3412}) = \#[1234, 3412] = 14$.

A **rook placement** on a board B is a set of cells ("rooks") of B such that no two lie in the same row or column. For $B \subseteq [n] \times [n]$, we denote by $RP(B)$ the number of rook placements of n rooks *avoiding* B, i.e., the number of placements of n rooks on $([n] \times [n]) \smallsetminus B$.

The **(strong) Bruhat order** $\preceq$ of $\mathfrak{S}_n$ is the partial order on the symmetric group defined by the cover relations $w \prec w \cdot t_{ij}$ if $\ell(w \cdot t_{ij}) = \ell(w) + 1$ and t_{ij} is the transposition that switches i and j.

We say that a permutation w_λ in the symmetric group $\mathfrak{S}_n$ on n letters is a **Grassmannian permutation** if it has at most one descent; say the position of the descent is k. Each such permutation is associated to a partition λ inside the $k \times (n-k)$ box $\langle (n-k)^k \rangle$ (i.e., a partition with at most k parts and largest part at most $n-k$). This correspondence can be seen from the south–east diagram E_{w_λ}, which is the Ferrers diagram of λ in French notation with possibly some columns in between, see Example 1.3. (Equivalently, this correspondence comes from a certain *wiring diagram* of w_λ, see [22, Sec. 19] and Section 4.7.)

A **filling** of a diagram D is an assignment of 0s and 1s to the elements of D. A filling of D is said to be **percentage-avoiding**[3] [24] if there are no four entries in D at the vertices of a (axis-aligned) rectangle with either of the following fillings: $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$.

Example 1.3. For the Grassmannian permutation $w = 3412$, the SW diagram O_{3412} consists of the two elements $(1,4), (3,2)$ and the SE diagram E_{3412} consists of the four elements $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$. This w has four inversions, at positions $(1,3), (1,4), (2,3), (2,4)$, and two anti-inversions, at positions $(1,2), (3,4)$. The inversion graph G_{3412} has four edges. See Fig. 1(a).

For $w = 3142$, the SW diagram O_{3142} and the SE diagram E_{3142} have three elements each and the inversion graph G_{3142} has three edges. This w is not Grassmannian, and its SE diagram is not the diagram of a partition in French notation. See Fig. 1(b).

[3] Postnikov [22] worked with a related family of fillings he called Ⅎ-diagrams, though percentage-avoidance is also implicit in some parts of his work. The enumerative relationship between Ⅎ-diagrams and percentage-avoiding fillings is made explicit in Spiridonov [26, §4] and is treated bijectively by Josuat-Vergès [14, §4]. For more on Ⅎ-diagrams, see Sections 4.7 and 4.8.

1.2. Previous work

The first result in the direction of Problem 1.2 was the paper [12] of Hultman–Linusson–Shareshian–Sjöstrand settling a conjecture of Postnikov [22, Rem. 24.4]. Their result explains the relation between the number $AO(G_w)$ of acyclic orientations of the inversion graph of w and the size of the Bruhat interval $[\iota, w]$.

Theorem 1.4. *(See [12, Thm. 4.1, Cor. 5.7].) For every permutation w in $\mathfrak{S}_n$, we have $AO(G_w) \leq \#[\iota, w]$. Furthermore, equality holds if and only if w avoids the permutation patterns* 4231, 35142, 42513, *and* 351624.

The permutations on which equality is achieved are very special, and will appear in the sequel. We call them **Gasharov–Reiner permutations** after their first appearance [10] in the literature. (These permutations were recently enumerated by Albert and Brignall [2].)

Various authors have also explored q-analogues of the objects defined above. For example, [12, Thm. 8.1] gives a q-analogue of Theorem 1.4 involving the chromatic polynomial. Also, Oh–Postnikov–Yoo established the following result linking a q-analogue $A_w(q)$ (whose definition we omit) of $AO(G_w)$ to the **Poincaré polynomial** $P_w(q) = \sum_{u \preceq w} q^{\ell(u)}$ (here the sum is over the permutations u in the interval $[\iota, w]$ of the Bruhat order).

Theorem 1.5. *(Oh–Postnikov–Yoo [21, Thm. 7].) For a permutation w in $\mathfrak{S}_n$, $A_w(q) = P_w(q)$ if and only if w avoids* 3412 *and* 4231.

Note that the classes of permutations of Theorem 1.4 and Theorem 1.5 differ, thus the $q = 1$ version of the latter does not imply the former. The class of permutations that appear in Theorem 1.5 are known as **smooth permutations**. They have many very interesting properties (see [1, §4]), of which we mention two here.

Lemma 1.6. *(Lakshmibai–Sandhya [19], Carrell–Peterson [5].) The following are equivalent for w in $\mathfrak{S}_n$:*

(a) *The permutation w is smooth.*
(b) *The Schubert variety X_w associated to w is smooth.*
(c) *The Poincaré polynomial $P_w(q)$ is palindromic, that is, $P_w(q) = q^{\ell(w)} P_w(q^{-1})$.*

1.3. New results

In Section 2, we continue the study of the relationships between the objects in Theorem 1.1 when w is allowed to be an arbitrary permutation. Our first result is a three-way equality involving a suitable generalization of percentage-avoiding fillings (see Definition 2.3).

Theorem 2.1. *Given any permutation w in $\mathfrak{S}_n$, the following are equal: the number $AO(G_w)$ of acyclic orientations of the inversion graph G_w, the number $RP(O_w)$ of placements of n non-attacking rooks on the complement of the SW diagram O_w, and the number of "pseudo-percentage-avoiding fillings" of the SE diagram E_w of w.*

One equality is proved in Section 2 and the other equality is proved by Axel Hultman in Appendix A. An immediate consequence of Theorem 1.4 and Theorem 2.1 is that the number $RP(O_w)$ of rook placements has the same relation with $\#[\iota, w]$ as the number $AO(G_w)$ of acyclic orientations (see Corollary 2.2).

In Section 3, we study relations among q-analogues of the objects described above. The first is the Poincaré polynomial $P_w(q)$, defined in the previous section, which is the natural q-analogue of the size $\#[\iota, w]$ of the Bruhat interval below w. The other is a natural q-analogue of the rook placements avoiding the SW diagram of w.

Definition 1.7. Let $\mathbf{F}_q$ be the finite field with q elements. Define $\mathrm{mat}_w(q)$ to be the number of $n \times n$ invertible matrices over $\mathbf{F}_q$ whose nonzero entries are in $\overline{O_w}$.

It was shown in [17, Prop. 5.1] that $\mathrm{M}_w(q) := \mathrm{mat}_w(q)/(q-1)^n$ is an enumerative q-analogue of $RP(O_w)$, in the sense that

$$\mathrm{M}_w(q) \equiv RP(O_w) \; (= AO(G_w)) \pmod{q-1}. \tag{1}$$

Remarkably, the equality condition between $\mathrm{M}_w(q)$ and (an appropriately rescaled version of) $P_w(q)$ is precisely the same as between their values at $q = 1$ (as in Theorem 1.4). This settles part of a conjecture [15, Conj. 6.6] of Klein and the present authors.

Theorem 3.1. *Let w be a permutation in $\mathfrak{S}_n$. Then $\mathrm{M}_w(q) = q^{\binom{n}{2}+\ell(w)} P_w(q^{-1})$ if and only if w avoids the patterns* 4231, 35142, 42513, *and* 351624.

In Section 4, we give a large number of open questions and other remarks. Notably, in Sections 4.7 and 4.8, we study additional relatives of Postnikov's "Ⅎ-diagrams" and their relations with Bruhat intervals (see Conjecture 4.11) and acyclic orientations. Our results include the following.

Corollary 4.6. *If w avoids* 321 *then the number of Γ-diagrams on E_w is equal to the number of acyclic orientations of the inversion graph of w.*

Supplementary data and code for `Sage` and `Maple` are available at the website http://sites.google.com/site/matrixfinitefields/.

2. Acyclic orientations, rook placements, and fillings

The main result of this section is the following:

Theorem 2.1. *Given a permutation w in $\mathfrak{S}_n$, the following are equal:*

(i) *the number $AO(G_w)$ of acyclic orientations of the inversion graph of w,*
(ii) *the number $RP(O_w)$ of placements of n non-attacking rooks on the complement of the SW diagram O_w of w,*
(iii) *the number of "pseudo-percentage-avoiding fillings" of the SE diagram E_w of w.*

As a corollary of Theorem 2.1 and Theorem 1.4, we have the following result.

Corollary 2.2. *The number of placements of n non-attacking rooks on $\overline{O_w}$ equals the number of permutations in the Bruhat interval $[\iota, w]$ if and only if w avoids* 4231*,* 35142*,* 42513*, and* 351624*.*

The proof of the equality of $AO(G_w)$ and $RP(O_w)$ is deferred to Appendix A, where Axel Hultman gives an elegant proof using some classic results from rook theory. (An alternative, longer proof may be found in the extended abstract [18]; see also Section 4.1.) Then in Section 2.1 we define the pseudo-percentage-avoiding fillings and complete the proof of Theorem 2.1.

2.1. Bijection between acyclic orientations and pseudo-percentage-avoiding fillings

Recall that E_w is the subset of $[n]\times[n]$ consisting of all pairs (i, w_j) such that $i < j$ and $w_j < w_i$ (see Fig. 1, center panels). In this section, we complete the proof of Theorem 2.1 by establishing the equality of the number $AO(G_w)$ of acyclic orientations of the inversion graph of w with the number of *pseudo-percentage-avoiding fillings* of the SE diagram E_w, which we define now.

Definition 2.3. Given a permutation w, we say that a filling A of E_w with 0s and 1s is a **pseudo-percentage-avoiding filling** if it satisfies the following conditions:

(i) A is percentage-avoiding, i.e., if squares (i, j), (i', j), (i, j') and (i', j') are elements of E_w then we do not have $A_{i,j} = A_{i',j'} = 1$ and $A_{i',j} = A_{i,j'} = 0$, nor do we have $A_{i,j} = A_{i',j'} = 0$ and $A_{i',j} = A_{i,j'} = 1$;
(ii) if squares (i, j), (i', j) and (i, j') are elements of E_w and square (i', j') is an entry of w (that is, $j' = w_{i'}$) then we do not have $A_{i,j} = 1$ and $A_{i',j} = A_{i,j'} = 0$, nor do we have $A_{i,j} = 0$ and $A_{i',j} = A_{i,j'} = 1$.

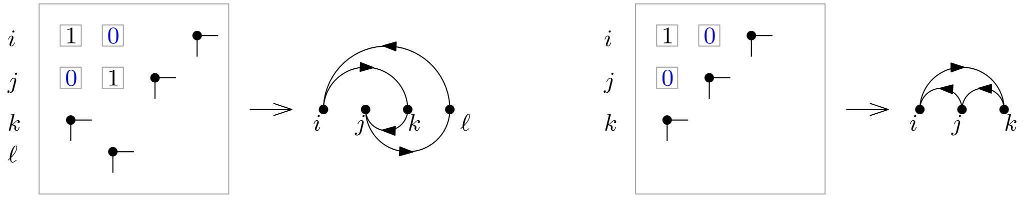
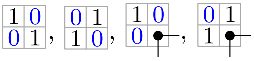


Fig. 2. Given a filling f of E_w and a cell $(i, w_j) \in E_w$, the edge of $\{i, j\}$ of G_w is oriented to the right if $f(i, w_j) = 1$ and to the left if $f(i, w_j) = 0$. Left: a percentage pattern corresponds to an alternating 4-cycle. Right: a pseudo-percentage pattern corresponds to a 3-cycle.

These forbidden patterns can be represented by the images  where the solid dot indicates an entry of the permutation.

The main result of this section is the following:

Proposition 2.4. *Given any permutation w, the number of pseudo-percentage-avoiding fillings of E_w is equal to the number of acyclic orientations of G_w.*

We will need the following property of inversion graphs, whose (easy) proof is left to the reader.

Remark 2.5. Given a permutation w in $\mathfrak{S}_n$ with inversion graph G_w and vertices $1 \leq i < j < k \leq n$, we have that

(i) if $\{i, j\}$ and $\{j, k\}$ are edges of G_w then $\{i, k\}$ is an edge of G_w, and
(ii) if $\{i, k\}$ is an edge of G_w then at least one of $\{i, j\}$ and $\{j, k\}$ is an edge of G_w.

Proof of Proposition 2.4. Call a cycle in an orientation of an inversion graph *alternating* if its edges alternate between being directed to the right and to the left. (In particular, only cycles of even length may be alternating.)

Consider any filling f of E_w. Recall that the elements (i, w_j) of E_w are in correspondence with the inversions (i, j) of w and in turn with the edges $\{i, j\}$ of G_w. In the inversion graph G_w, direct edges corresponding to entries filled with 1 to the right and edges corresponding to entries filled with 0 to the left. One has immediately that f contains a percentage pattern if and only if the corresponding orientation of G_w contains an alternating 4-cycle, and f contains a pseudo-percentage pattern (extended using an entry of w) if and only if the orientation contains a (directed) 3-cycle; see Fig. 2. Thus, it suffices to show that an orientation of an inversion graph is acyclic if and only if it contains no 3-cycles and no alternating 4-cycles. One implication is obvious.

For the other direction, we wish to show that in every orientation of G_w with a directed cycle, there is a 3-cycle or alternating 4-cycle. Choose an orientation of G_w that contains a directed cycle C. We show that if C is not a 3-cycle or an alternating 4-cycle then there is a cycle whose length is strictly less than that of C; this finishes the proof.

Suppose that C contains a chord, i.e., there is an edge of G_w joining two vertices in C that is not an edge of C. In this case, no matter which way one orients the chord, one produces a directed cycle of strictly shorter length than C, as desired.

Observe that if C is not alternating then it necessarily contains a chord: if there are edges $a \to b$ and $b \to c$ of C with $a < b < c$ or $a > b > c$ then by Remark 2.5(i) the inversion graph contains the edge $\{a, c\}$, a chord of C. So we may suppose that C is alternating and of length at least 6.

Let i_0 be the leftmost vertex of C, and write $C = i_0 \to i_1 \to \ldots \to i_{2m-1} \to i_{2m} = i_0$. Choose $k > 0$ minimal so that $i_{2k+2} < i_{2k}$. From the choice of k and the fact that C is alternating it follows that $i_{2k-2} < i_{2k} < i_{2k-1}$ and $i_{2k+2} < i_{2k} < i_{2k+1}$. We have two possibilities: first, if i_{2k+2} lies between i_{2k-2} and i_{2k} then it lies between the endpoints of the edge $i_{2k-2} \to i_{2k-1}$ and so by Remark 2.5(ii) there must be a chord joining i_{2k+2} to one of i_{2k-2}, i_{2k-1}. Alternatively, if $i_{2k+2} < i_{2k-2}$ then i_{2k-2} lies between the endpoints of the edge $i_{2k+1} \to i_{2k+2}$ and so there must be a chord joining i_{2k-2} to one of i_{2k+1}, i_{2k+2}. $\square$

3. q-Analogues of rook placements and Bruhat intervals

Theorems 2.1 and 1.4 show that the size of the Bruhat interval $[\iota, w]$ is equal to the number $RP(O_w)$ of rook placements avoiding the south–west diagram O_w if and only if w avoids the permutation patterns 4231, 35142, 42513, 351624. In this section, we study a natural q-analogue of this result, using a recursive analysis based on that of [12].

The analogue of $\#[\iota, w]$ that we consider is the **Poincaré polynomial**

$$P_w(t) := \sum_{u \preceq w} t^{\ell(u)},$$

where the order relation in the sum is the strong Bruhat order. The analogues $\mathrm{M}_w(q)$ of the number $RP(O_w)$ of rook placements that we consider are the **matrix counting function** and the **normalized matrix-counting function** defined by

$$\mathrm{mat}_w(q) := \#\mathcal{M}(n, O_w) \quad \text{and} \quad \mathrm{M}_w(q) := \frac{1}{(q-1)^n} \cdot \mathrm{mat}_w(q),$$

where

$$\mathcal{M}(n, O_w) := \{n \times n \text{ invertible matrices over } \mathbf{F}_q \text{ with nonzero entries restricted to } \overline{O_w}\}$$

and $\mathbf{F}_q$ is the finite field with q elements.[4] Equation (1) shows that these are indeed q-analogues of $RP(O_w)$.

[4] One could alternatively view $\mathrm{M}_w(q)$ as counting orbits $T\backslash\mathcal{M}(n, O_w)$ of matrices under the action of the (split maximal) torus T of diagonal matrices in $\mathrm{GL}_n(\mathbf{F}_q)$, and indeed all of our proofs could be rephrased in this context.

The main result of this section, answering part of the conjecture [15, Conj. 6.6], is the following.

Theorem 3.1. *Let w be a permutation in $\mathfrak{S}_n$. Then*

$$\mathrm{M}_w(q) = q^{\binom{n}{2}+\ell(w)} P_w(q^{-1}) \tag{2}$$

if and only if w avoids the patterns 4231, 35142, 42513, *and* 351624.

The proof of the "only if" part of Theorem 3.1 is as follows: by Equation (1) we have that $\mathrm{M}_w(1) \equiv RP(O_w) \pmod{q-1}$, while by the definition of $P_w(q)$ we have that $q^{\binom{n}{2}+\ell(w)} P_w(q^{-1}) \mid_{q=1} = \#[\iota, w]$. If w contains one of the patterns 4231, 35142, 42513, or 351624 then $RP(O_w) \neq \#[\iota, w]$ by Corollary 2.2. Therefore for such w the expressions $\mathrm{M}_w(q)$ and $q^{\binom{n}{2}+\ell(w)} P_w(q^{-1})$ cannot be equal for sufficiently large q.

The "if" part of the proof of Theorem 3.1 is shown by induction, and the rest of this section is devoted to its proof. Let $\mathfrak{S}_n(4231, 35142, 42513, 351624)$ be the set of permutations w in $\mathfrak{S}_n$ avoiding the patterns 4231, 35142, 42513, and 351624, i.e., the Gasharov–Reiner permutations. In [12, §5], the authors define two special kinds of descents called *heavy* and *light reduction pairs*. We recall their definition here.

Definition 3.2. Suppose w is a permutation with a descent formed by the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$. We call this descent a **light reduction pair** if

- there is no entry (j, w_j) with $j < i$ and $w_j > w_i$, and
- there is no entry (j, w_j) with $j > i+1$ and $w_{i+1} < w_j < w_i$.

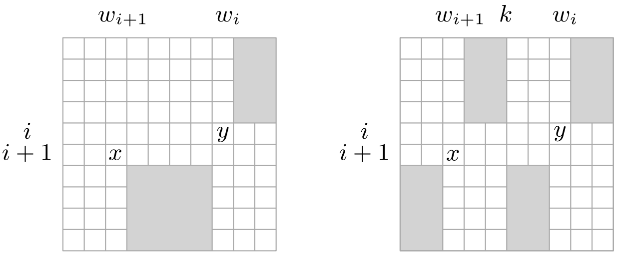
This is illustrated in Fig. 3(a). We call this descent a **heavy reduction pair** if

- there is no entry (j, w_j) with $j > i+1$ and $w_j < w_{i+1}$,
- there is no entry (j, w_j) with $j < i$ and $w_j > w_i$, and
- there is an index k with $w_i \leq k \leq w_{i+1}$ such that there is no entry (j, w_j) with $j < i$ and $w_{i+1} < w_j \leq k$ or with $j > i+1$ and $k < w_j < w_i$.

This is illustrated in Fig. 3(b).

In [12, Prop. 5.6], it was shown that one can always find a reduction pair in a permutation w in $\mathfrak{S}_n(4231, 35142, 42513, 351624)$.

Proposition 3.3. *(See [12, Prop. 5.6].) Let $w \neq \iota$ be in* $\mathfrak{S}_n(4231, 35142, 42513, 351624)$. *Then either the first descent of w or the first descent of w^{-1} is a reduction pair.*



(a) (b)

Fig. 3. (a) A light reduction pair, and (b) a heavy reduction pair. The gray areas have no entries (j, w_j) of w.

Further, Hultman et al. gave recursions for the size of the Bruhat interval below Gasharov–Reiner permutations using the structure imposed by the reduction pairs. In the following sections, we extend this work by giving recursions for the Poincaré polynomial and matrix counting function of Gasharov–Reiner permutations. Thus, we will establish by induction that the Poincaré polynomials and matrix counts are essentially equal in this case.

3.1. Recursions for permutations with heavy reduction pairs

In this section, we consider the case that the first descent of w is a heavy reduction pair. In order to introduce our result, we must introduce some notation. Given a permutation $w = w_1 \cdots w_n$ whose first descent is a heavy reduction pair in position i, let j be minimal such that $w_j > w_{i+1}$ and define $v = v(w)$ to be the permutation in $\mathfrak{S}_{n-1}$ that satisfies the order-isomorphism

$$v(w) \;\cong\; w_1\, w_2 \,\cdots\, w_{j-1} \quad w_{j+1}\, w_{j+2} \,\cdots\, w_i \quad w_j \quad w_{i+2}\, w_{i+3} \,\cdots\, w_n. \tag{3}$$

(The crucial properties of v for our discussion are proved in Propositions 3.9 and 3.15 below.) In addition, we will make repeated use of the following operation on permutations.

Definition 3.4 *(Deletion in permutations and diagrams).* Suppose that $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is a permutation and $y = (i, w_i)$ is an entry of w. Then the result of **deleting** y from w is the permutation $w - y$ in $\mathfrak{S}_{n-1}$ order-isomorphic to $w_1 \cdots w_{i-1} w_{i+1} \cdots w_n$.

Similarly, for a diagram $D \subset [n] \times [n]$ and a pair $(i, j) \in [n] \times [n]$, **deleting** (i, j) from D yields the diagram $D - (i, j) \subset [n-1] \times [n-1]$ that results from removing the ith row and jth column from D, and reindexing rows and columns as necessary. (The two definitions can be easily seen to agree in the case that D is a diagram with one entry in each row and column, w is the associated permutation, and y is an element of D.)

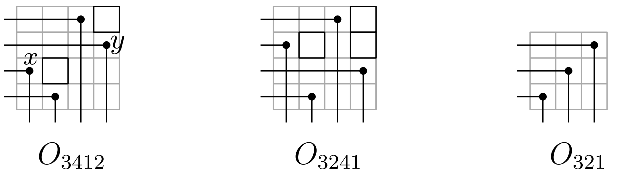


Fig. 4. South–west diagrams of $w = 3412$, $s_2w = 3142$, $v(w) = 321$.

The main result of this section is the following:

Proposition 3.5. *Let w be in $\mathfrak{S}_n(4231, 35142, 42513, 351624)$. If the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i + 1, w_{i+1})$, is a heavy reduction pair then*

$$P_w(t) = P_{s_iw}(t) + t^{\ell(w)-\ell(w-x)} P_{w-x}(t) + t^{\ell(w)-\ell(w-y)} P_{w-y}(t) - t^{\ell(w)-\ell(w-x-y)} P_{w-x-y}(t) \tag{4}$$

and

$$\mathrm{M}_w(q) = \mathrm{M}_{s_iw}(q) + q^n\, \mathrm{M}_v(q), \tag{5}$$

where v is as in (3).

Example 3.6. Let $w = 3412 \in \mathfrak{S}_4$, whose first descent (at position $i = 2$, involving the entries $y = (2, w_2) = (2, 4)$ and $x = (3, w_3) = (3, 1)$) is a heavy reduction pair (Fig. 4). Then $s_2w = 3142 \in \mathfrak{S}_4$ and $w-x = 231 \in \mathfrak{S}_3$, $w-y = 312 \in \mathfrak{S}_3$ and $w-x-y = 21 \in \mathfrak{S}_2$. One can compute the Poincaré polynomials

$$\begin{aligned} P_{3412}(t) &= t^4 + 4t^3 + 5t^2 + 3t + 1, \\ P_{3142}(t) &= t^3 + 3t^2 + 3t + 1, \\ P_{231}(t) = P_{312}(t) &= t^2 + 2t + 1, \\ P_{21}(t) &= t + 1 \end{aligned}$$

and verify that they satisfy the relation

$$P_{3412}(t) = P_{3142}(t) + t^{4-2} \cdot P_{312}(t) + t^{4-2} \cdot P_{231}(t) - t^{4-1} \cdot P_{21}(t).$$

For the matrix counts we have that $v = 321 \in \mathfrak{S}_3$ and one can compute

$$\begin{aligned} \mathrm{M}_{3412}(q) &= q^6(q^4 + 3q^3 + 5q^2 + 4q + 1), \\ \mathrm{M}_{3142}(q) &= q^6(q^3 + 3q^2 + 3q + 1), \\ \mathrm{M}_{321}(q) &= q^3(q^3 + 2q^2 + 2q + 1) \end{aligned}$$

and verify that they satisfy the relation

$$\mathrm{M}_{3412}(q) = \mathrm{M}_{3142}(q) + q^4\, \mathrm{M}_{321}(q).$$

3.1.1. Proof of Equation (4)

Given a Gasharov–Reiner permutation w whose first descent, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a heavy reduction pair, the argument of Hultman et al. leading up to [12, Eq. (3)] establishes that the Bruhat interval $[\iota, w]$ decomposes as the union of the following sets:

- the Bruhat interval $[\iota, s_i w]$,
- the set $S_x = \{u \in [\iota, w] \mid u_{i+1} = i+1\}$ whose elements have an entry at x, and
- the set $S_y = \{u \in [\iota, w] \mid u_i = i\}$ whose elements have an entry at y.

Moreover, we may rephrase several of their observations as follows: they establish that $[\iota, s_i w]$ is disjoint from S_x and S_y; that the maps $u \mapsto u - x$ and $u \mapsto u - y$ are bijections respectively between S_x and the Bruhat interval $[\iota, w - x]$ in $\mathfrak{S}_{n-1}$ [3, Lem. 2.1] and between S_y and $[\iota, w - y]$; and similarly that the map $u \mapsto u - x - y$ is a bijection between $S_x \cap S_y$ and $[\iota, w - x - y]$. Moreover, it follows from Sjöstrand's result [25, Thm. 4] that every permutation $u \in [\iota, w]$ satisfies $u_j < w_i$ for $j < i$ and $u_j > w_{i+1}$ for $j > i+1$. Consequently, among the permutations $u \in S_y$, the ith entry is always involved in exactly the same number of inversions[5], and similarly for those permutations with an entry at position x. Putting everything together, we have

$$\begin{aligned}
P_w(t) &= \sum_{u \preceq w} t^{\ell(u)} \\
&= \sum_{u \preceq s_i w} t^{\ell(u)} + \sum_{u \in S_x} t^{\ell(u)} + \sum_{u \in S_y} t^{\ell(u)} - \sum_{u \in S_x \cap S_y} t^{\ell(u)} \\
&= P_{s_i w}(t) + \sum_{u \preceq w - x} t^{\ell(u) + \ell(w) - \ell(w-x)} + \sum_{u \preceq w - y} t^{\ell(u) + \ell(w) - \ell(w-y)} \\
&\quad - \sum_{u \preceq w - x - y} t^{\ell(u) + \ell(w) - \ell(w-x-y)} \\
&= P_{s_i w}(t) + t^{\ell(w) - \ell(w-x)} P_{w-x}(t) + t^{\ell(w) - \ell(w-y)} P_{w-y}(t) \\
&\quad - t^{\ell(w) - \ell(w-x-y)} P_{w-x-y}(t),
\end{aligned}$$

as desired.

[5] As it happens, this number $\ell(w) - \ell(w-y)$ is equal to $w_i - i$: there are $w_i - 1$ entries of u smaller than $u_i = w_i$, of which $i-1$ occur in u before the ith position, leaving $w_i - i$ to occur after the ith position; and none of the entries in u before the ith position are larger than w_i. One can make a similar computation with y replaced by x.

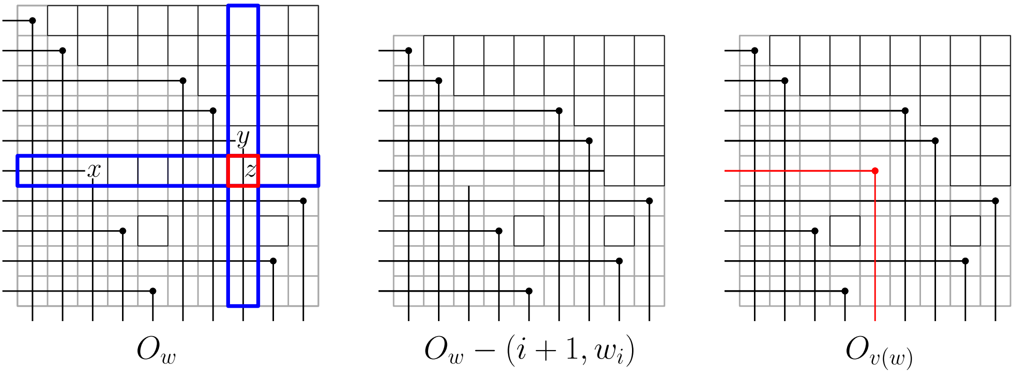


Fig. 5. Example of O_w for the permutation $w = 1\,2\,6\,7\,8\,3\,10\,4\,9\,5$ whose first descent at position $i = 5$ is a heavy reduction pair. Left: we perform Gaussian elimination on matrices in $\mathcal{M}(10, O_w)$ with respect to the entry $z = (i+1, w_i) = (6, 8)$. Center: the diagram after elimination. Right: the diagram $O_{v(w)}$ for the permutation $v(w)$.

3.1.2. Proof of Equation (5)

We get the desired recursion for $\mathrm{M}_w(q)$ by careful applications of Gaussian elimination using the entry $(i+1, w_i)$. Throughout the proof it will be helpful to refer to Fig. 5. We begin by noting some properties of heavy reduction pairs that follow immediately from Definition 3.2.

Remark 3.7. If the first descent of $w = w_1 \cdots w_n$ is a heavy reduction pair in position i, and if j is minimal so that $w_{i+1} < w_j$, then

(i) $(w_1, \ldots, w_{j-1}) = (1, \ldots, j-1)$,
(ii) $(w_j, \ldots, w_i) = (w_i - i + j, w_i - i + j + 1, \ldots, w_i)$, and
(iii) $w_{i+1} = j$.

Proposition 3.8. *Let w be in $\mathfrak{S}_n$. If the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a heavy reduction pair then*

$$\mathrm{M}_w(q) = \mathrm{M}_{s_i w}(q) + q^n \cdot \#\mathcal{M}(n, O_w - (i+1, w_i))/(q-1)^n,$$

where the deleted diagram $O_w - (i+1, w_i)$ is as in Definition 3.4.

Proof. Let $z = (i+1, w_i)$. Since $O_{s_i w}$ equals the diagram $O_w \cup \{z\}$ with rows i and $i+1$ switched, the difference $\mathrm{M}_w(q) - \mathrm{M}_{s_i w}(q)$ is (up to the factor $(q-1)^n$) the number of invertible matrices with support contained in $\overline{O_w}$ having nonzero entry in position z. We now examine the entries of O_w in the row and column of z.

Let R be the union of the rows indexed by $\{1, 2, \ldots, i-1\}$ and let C be the union of the columns indexed by $\{w_{i+1}+1, \ldots, w_j - 1\} \cup \{w_i + 1, \ldots, n\}$. It follows from Remark 3.7 and the definition of the SW diagram O_w that the entries of O_w in row $i+1$ are exactly those in C and the entries of O_w in column w_i are exactly those in R, and that O_w is contained in $R \cup C$. Consequently, if we superimpose row $i+1$ with any row in R, the entries in O_w in row $i+1$ cover those in the other row; and, similarly, if we superimpose column w_i with any column from C, the entries in O_w in column w_i cover those in the other column.

Now consider the set of matrices in $\mathcal{M}(n, O_w)$ with nonzero entry in position z. Given such a matrix A, perform the following operation: use Gaussian elimination with the nonzero entry in position z to kill the other nonzero entries in its row and column, then delete the row and column of z. The resulting matrix B certainly belongs to $\mathrm{GL}_{n-1}(\mathbf{F}_q)$. Moreover, the analysis of the preceding paragraph guarantees that no step in the elimination procedure disturbs any of the zero entries in positions given by $O_w - z$, so in fact B belongs to $\mathcal{M}(n-1, O_w - z)$. Finally, given a matrix B in $\mathcal{M}(n-1, O_w - z)$, one may reverse this process in precisely $(q-1)q^n$ ways: first, choosing a nonzero entry for position z, then making appropriate row operations to fill in the n entries in row $i+1$ and column w_i that do not belong to R or C. The resulting matrix belongs to $\mathcal{M}(n, O_w)$ by the same analysis. The result follows. $\square$

Proposition 3.9. *Let w be in $\mathfrak{S}_n$. If the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a heavy reduction pair then $O_w - (i+1, w_i)$ and $O_{v(w)}$ are identical up to permutations of rows and columns.*

Proof. Let j be minimal such that $w_{i+1} < w_j$. By construction, the diagram $O_{v-(i,v_i)} = O_v - (i, v_i)$ is identical to the diagram that we get by removing the ith row and w_{i+1}th column from $O_w - (i+1, w_i)$, as both are identical to the diagram that we get by removing the ith and $(i+1)$st rows and w_ith and w_{i+1}th columns from O_w. Thus, it suffices to check that the ith rows of O_v and $O_w - (i+1, w_i)$ are equal and that the $(w_j - 1)$th column of O_v is equal to the w_{i+1}th column of $O_w - (i+1, w_i)$.

First, we consider the columns. By Remark 3.7, the w_{i+1}th column of O_w contains exactly the $j-1$ entries $(1, w_{i+1})$, $(2, w_{i+1})$, $\ldots$, $(j-1, w_{i+1})$, and so the w_{i+1}th column of $O_w - (i+1, w_i)$ consists of these same $j-1$ entries. Similarly, applying Remark 3.7 and the definition of v, we see that the $(w_j - 1)$th column of O_v consists of the entries $(1, w_j - 1)$, $\ldots$, $(j-1, w_j - 1)$, as needed.

Second, we consider the rows. Since i is the position of the first descent, w_i is a left-to-right maximum of w. Thus, the ith row of O_w consists of the $n - w_i$ elements $(i, w_i + 1)$, $(i, w_i + 2)$, $\ldots$, (i, n), and no others. Then the ith row of $O_w - (i+1, w_i)$ also consists of these $n - w_i$ boxes, each shifted one unit to the left. In v, the entries $(j, w_j), \ldots, (i-1, w_i - 1)$ do not form inversions with the entry $(i, w_j - 1)$, while the entries in columns $w_i, \ldots, n-1$ occur in rows with indices larger than i. Thus, the ith row of O_v consists of the same $n - w_i$ entries as the ith row of $O_w - (i+1, w_i)$. $\square$

Finally, we may put these two propositions together to conclude the desired recursion (5) for matrix counts.

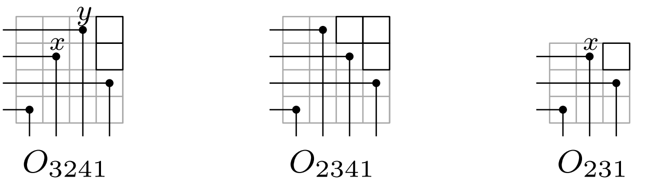


Fig. 6. South–west diagrams of $w = 3241$, $s_1 w = 2341$ and $w - y = 231$.

3.2. Recursions for permutations with light reduction pairs

In this section, we consider the case that the first descent of w is a light reduction pair. The main result of this section is the following one, which gives a pair of related recursions for Poincaré polynomials and matrix counts:

Proposition 3.10. *Let w be in $\mathfrak{S}_n(4231, 35142, 42513, 351624)$. If the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i + 1, w_{i+1})$, is a light reduction pair then*

$$P_w(t) = P_{s_i w}(t) + t^{\ell(w)-\ell(w-y)} P_{w-y}(t) \tag{6}$$

and

$$\mathrm{M}_w(q) = q \cdot \mathrm{M}_{s_i w}(q) + q^{n-1} \cdot \mathrm{M}_{w-y}(q). \tag{7}$$

Example 3.11. With $w = 3241$, the descent of w in position $i = 1$ is a light reduction pair with $y = (1, w_1) = (1, 3)$ and $x = (2, w_2) = (2, 2)$. We have $s_1 w = 2341$ and $w - y = 231$. See Fig. 6 for the south–west diagrams of w, $s_1 w$, and $w - y$. One can compute the Poincaré polynomials

$$\begin{aligned} P_{3241}(t) &= t^4 + 3t^3 + 4t^2 + 3t + 1, \\ P_{2341}(t) &= t^3 + 3t^2 + 3t + 1, \\ P_{231}(t) &= t^2 + 2t + 1, \end{aligned}$$

and verify that they satisfy the relation

$$P_{3241}(t) = P_{2341}(t) + t^{4-2} P_{231}(t).$$

Similarly, one can compute the matrix counts

$$\begin{aligned} \mathrm{M}_{3241}(q) &= q^6(q^4 + 3q^3 + 4q^2 + 3q + 1), \\ \mathrm{M}_{2341}(q) &= q^6(q^3 + 3q^2 + 3q + 1), \\ \mathrm{M}_{231}(q) &= q^3(q^2 + 2q + 1), \end{aligned}$$

and verify that they satisfy the relation

$$\mathrm{M}_{3241}(q) = q\,\mathrm{M}_{2341}(q) + q^3\,\mathrm{M}_{231}(q).$$

3.2.1. Proof of Equation (6)

Given a Gasharov–Reiner permutation w whose first descent, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a light reduction pair, the argument of Hultman et al. leading up to [12, Eq. (1)] establishes that the Bruhat interval $[\iota, w]$ decomposes as the disjoint union of the Bruhat interval $[\iota, s_i w]$ and the set of permutations below w that contain an entry at position y. In addition, the operation $u \mapsto u - y$ is a bijection between the latter set and the interval $[\iota, w - y]$ in the Bruhat order on $\mathfrak{S}_{n-1}$. To finish the proof of (6), it is enough to observe that, as in Section 3.1.1, Sjöstrand's result [25, Thm. 4] can be used to establish that these bijections respect the grading of the Bruhat order.

3.2.2. Proof of Equation (7)

We begin with a useful lemma for how matrix counts avoiding a diagram behave when one adds an entry to the diagram. For $D \subseteq [n] \times [n]$, let $\mathcal{X}(n, D)$ be the set of all $n \times n$ matrices with entries in $\mathbf{F}_q$ and support avoiding D. (Thus $\mathcal{M}(n, D) = \mathcal{X}(n, D) \cap \mathrm{GL}_n(\mathbf{F}_q)$.)

Let $D \subseteq [n] \times [n]$ and $y \in \overline{D}$. Given a matrix $B \in \mathcal{X}(n-1, D-y)$ and $a, b \in \mathbf{F}_q$, define $S_{a\to b}(B)$ to be the number of matrices $A \in \mathcal{M}(n, D)$ such that:

- $A_y = a$,
- when one removes from A the row and column of y, the result is B, and
- the $n \times n$ matrix A' defined by $A'_y = b$ and $A'_z = A_z$ for $z \neq y$ is singular.

We prove a general lemma, showing how to express the difference between two matrix counts in terms of these $S_{a\to b}(B)$.

Lemma 3.12. *Let $D \subseteq [n] \times [n]$ and $y \in \overline{D}$. Then*

$$\#\mathcal{M}(n, D) - q \cdot \#\mathcal{M}(n, D \cup \{y\}) = \sum_{a \in \mathbf{F}_q^\times} \sum_{B \in \mathcal{M}(n-1, D-y)} \left(S_{a\to 0}(B) - S_{0\to a}(B)\right).$$

Proof. First, we give a convenient interpretation of the term $q \cdot \#\mathcal{M}(n, D \cup \{y\})$. This counts pairs (a, A) where $a \in \mathbf{F}_q$ and $A \in \mathcal{M}(n, D \cup \{y\})$. We view this as setting $A_y \mapsto a$ in the invertible matrix A, which might or might not yield an invertible matrix.

Second, given a matrix A in $\mathcal{M}(n, D \cup \{y\})$, the submatrix B that results from removing the row and column of y has support in $\overline{D-y}$ and may or may not be invertible. We show that the difference $\#\mathcal{M}(n, D) - q \cdot \#\mathcal{M}(n, D \cup \{y\})$ cancels all the terms where B is not invertible.

Let $S_{a\to b} = \sum_{B\in\mathcal{X}(n-1,D-y)} S_{a\to b}(B)$ be the number of matrices A in $\mathcal{M}(n,D)$ with $A_y = a$ such that setting $A_y \mapsto b$ (and leaving all other entries of A unchanged) yields a singular matrix. Similarly, let $I_{a\to b}$ be the number of matrices A in $\mathcal{M}(n,D)$ with $A_y = a$ such that setting $A_y \mapsto b$ yields an invertible matrix. We break down $\mathcal{M}(n,D)$ as

$$\#\mathcal{M}(n,D) = \sum_{a\in\mathbf{F}_q} \left(S_{a\to 0} + I_{a\to 0}\right),$$

and we break down $q\cdot\#\mathcal{M}(n,D\cup\{y\})$ as

$$q\cdot\#\mathcal{M}(n,D\cup\{y\}) = \sum_{a\in\mathbf{F}_q} \left(S_{0\to a} + I_{0\to a}\right).$$

Note that $S_{0\to 0} = 0$, and that for all $a\in\mathbf{F}_q$ we have $I_{a\to 0} = I_{0\to a}$. Thus

$$\#\mathcal{M}(n,D) - q\cdot\#\mathcal{M}(n,D\cup\{y\}) = \sum_{a\in\mathbf{F}_q^\times} \left(S_{a\to 0} - S_{0\to a}\right).$$

Next, consider an $n\times n$ matrix A over $\mathbf{F}_q$, thinking of the entry A_y as variable, and let B be the matrix obtained by removing from A the row and column of y. Then $\det(A) = \pm\det(B)\cdot A_y + k$ for some $k\in\mathbf{F}_q$. If $\det(A)$ is nonconstant when viewed as a function of A_y then the linear coefficient $\det(B)$ is nonzero. Therefore if $B\in\mathcal{X}(n-1,D-y)$ is singular and $a\neq 0$ then $S_{a\to 0}(B) = S_{0\to a}(B) = 0$. Thus

$$\begin{aligned}
&\#\mathcal{M}(n,D) - q\cdot\#\mathcal{M}(n,D\cup\{y\})\\
&\quad= \sum_{a\in\mathbf{F}_q^\times}\ \sum_{B\in\mathcal{X}(n-1,D-y)} \left(S_{a\to 0}(B) - S_{0\to a}(B)\right)\\
&\quad= \sum_{a\in\mathbf{F}_q^\times}\ \sum_{B\in\mathcal{M}(n-1,D-y)} \left(S_{a\to 0}(B) - S_{0\to a}(B)\right). \qquad \square
\end{aligned}$$

We note a few points about the diagrams O_w of permutations w whose first descent is a light reduction pair. They follow immediately from Definition 3.2 (see Fig. 7).

Remark 3.13. Suppose the first descent of $w\in\mathfrak{S}_n$ is in position i and is a light reduction pair. Then:

(i) the ith and $(i+1)$st rows of O_w have entries in exactly the same set of columns, namely those with indices $\{w_i+1,\ldots,n\}$; and
(ii) all the entries in the north-east rectangle $[1,i-1]\times[w_i+1,n]$ are in O_w.

It follows from Remark 3.13(i) that $O_{s_iw} = O_w \cup \{(i,w_i)\}$. Then by Lemma 3.12 applied to $D = O_w$ and $y = (i,w_i)$ we have

$$\mathrm{mat}_w(q) - q \cdot \mathrm{mat}_{s_i w}(q) = \sum_{a \in \mathbf{F}_q^\times} \sum_{B \in \mathcal{M}(n-1, O_{w-y})} \left(S_{a\to 0}(B) - S_{0\to a}(B)\right). \tag{8}$$

Fix a matrix B in $\mathcal{M}(n-1, O_{w-y})$. From B we build matrices $A = \begin{bmatrix} a & \mathbf{u} \\ \mathbf{v} & B \end{bmatrix}$ where $a \in \mathbf{F}_q$, $\mathbf{u}$ is a row vector in $\mathbf{F}_q^{n-1}$ whose first $r = w_i - 1$ entries are free and the rest set to zero; and $\mathbf{v}$ is column vector in $\mathbf{F}_q^{n-1}$ whose last $c = n-1-i$ entries are free and the rest set to zero. The motivation for this construction is that these matrices are simply rearrangements of matrices with support avoiding O_w. In particular, for any choice of $a, \mathbf{u}, \mathbf{v}, B$, the resulting matrix A satisfies $(1, 2, \ldots, i) \cdot A \cdot (1, 2, \ldots, w_i)^{-1} \in \mathcal{X}(n, O_w)$; see Fig. 7. The determinant of such a matrix is

$$\det(A) = a \det(B) - \mathbf{u}B^{-1}\mathbf{v}. \tag{9}$$

There are q^{r+c} of these matrices of the form $\begin{bmatrix} 0 & \mathbf{u} \\ \mathbf{v} & B \end{bmatrix}$. Each of these is invertible or has rank $n-1$. Let $N(B)$ be number of such matrices that have rank $n-1$, so the remaining $q^{r+c} - N(B)$ matrices are invertible.

We proceed to compute the terms $S_{a\to 0}(B)$ and $S_{0\to a}(B)$.

- By (9), a matrix A is counted in $S_{a\to 0}(B)$ if and only if $a\det(B) \neq 0$ and $\mathbf{u}B^{-1}\mathbf{v} = 0$. This in turn is equivalent to $a \neq 0$ and $\begin{bmatrix} 0 & \mathbf{u} \\ \mathbf{v} & B \end{bmatrix}$ having rank $n-1$. Thus for each $a \in \mathbf{F}_q^\times$ the number of such cases is $S_{a\to 0}(B) = N(B)$.
- By (9), a pair (a, A) is counted in $S_{0\to a}(B)$ if and only if $a\det(B) = \mathbf{u}B^{-1}\mathbf{v} \neq 0$. This implies that $\begin{bmatrix} 0 & \mathbf{u} \\ \mathbf{v} & B \end{bmatrix}$ has rank n and thus

$$S_{0\to a}(B) = \begin{cases} q^{r+c} - N(B) & \text{if } a = \mathbf{u}B^{-1}\mathbf{v}/\det(B), \\ 0 & \text{otherwise.} \end{cases}$$

Substituting into Equation (8) gives

$$\begin{aligned} &\mathrm{mat}_w(q) - q \cdot \mathrm{mat}_{s_i w}(q) \\ &\quad = \sum_{B \in \mathcal{M}(n-1, O_{w-y})} (q-1)N(B) - \sum_{B \in \mathcal{M}(n-1, O_{w-y})} \left(q^{r+c} - N(B)\right) \\ &\quad = \sum_{B \in \mathcal{M}(n-1, O_{w-y})} \left(q \cdot N(B) - q^{r+c}\right). \end{aligned} \tag{10}$$

Finally, we compute $N(B)$. This is the number of choices of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}B^{-1}\mathbf{v} = 0$ and $\mathbf{u}$ and $\mathbf{v}$ have support as described in the paragraph preceding (9). Let $\mathbf{u}' = \mathbf{u}B^{-1}$. If $\mathbf{u}'$ has support in the first $n-1-c$ entries then $\mathbf{u}'\mathbf{v} = 0$ for all q^c choices of $\mathbf{v}$. By Remark 3.13(ii), the matrix B has a zero block matrix in its north-east

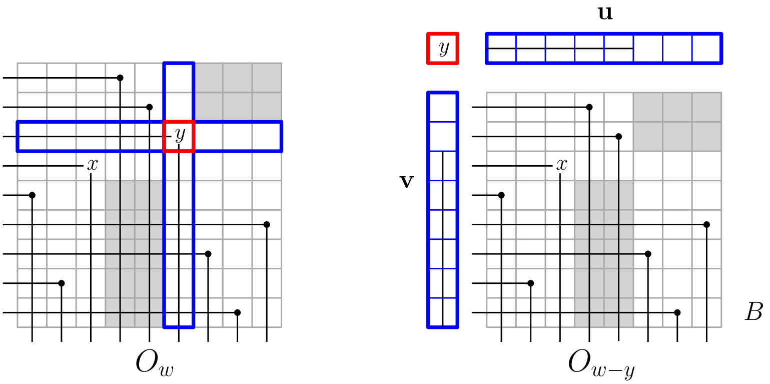


Fig. 7. Left: the diagram O_w for $w = 456319728$, whose first descent is a light reduction pair. The key features of O_w are that the two rows involved in the descent have entries in the same columns and there is north-east rectangle (in gray) in O_w. Right: rearrangement of the rows and columns of O_w yields the subdiagram O_{w-y} where $w - y = 45318627$.

corner of size $(n-1-c) \times (n-1-r)$ (see Fig. 7), and so every vector $\mathbf{u}'$ with support in the first $n-1-c$ entries is sent by B to a vector $\mathbf{u}'B$ with support in the first r entries. Since B is invertible this implies that each of the q^{n-1-c} vectors $\mathbf{u}'$ with support in the first $n-1-c$ entries is the image under B^{-1} of a vector $\mathbf{u}$ with support in the first r entries.

For the remaining $q^r - q^{n-1-c}$ choices of $\mathbf{u}$, the matrix $\mathbf{u}B^{-1}$ has support intersecting the last c entries and so there are q^{c-1} choices of $\mathbf{v}$ such that $\mathbf{u}B^{-1}\mathbf{v} = 0$. From the preceding two paragraphs it follows that

$$N(B) = q^{n-1-c} \cdot q^c + (q^r - q^{n-1-c}) \cdot q^{c-1} = q^{n-1} + q^{r+c-1} - q^{n-2}. \qquad (11)$$

Finally combining this with (10) yields

$$\begin{aligned} \mathrm{mat}_w(q) - q \cdot \mathrm{mat}_{s_i w}(q) &= \sum_{B \in \mathcal{M}(n-1, O_{w-y})} \big(q(q^{n-1} + q^{r+c-1} + q^{n-2}) - q^{r+c}\big) \\ &= q^{n-1}(q-1)\,\mathrm{mat}_{w-y}(q). \end{aligned}$$

Dividing by $(q-1)^n$ gives (7), as desired.

3.3. End of proof of Theorem 3.1

Finally, in this section we put the preceding results together in order to finish the inductive proof of the "if" part of Theorem 3.1. We induct simultaneously on the size (i.e., number of entries) and the length (i.e., number of inversions) of the permutation w, starting with the base case of identity permutations.

Proposition 3.14. *For the identity permutation ι, we have*

$$\mathrm{M}_\iota(q) = q^{\binom{n}{2}+\ell(\iota)} P_\iota(q^{-1}) = q^{\binom{n}{2}}.$$

Proof. The result is trivial: $(q-1)^n \cdot \mathrm{M}_\iota(q) = (q-1)^n q^{\binom{n}{2}}$ is the number of invertible $n \times n$ lower triangular matrices and $P_w(t) = 1$. $\square$

Now suppose that w is a Gasharov–Reiner permutation. We have by Proposition 3.3 that either the first descent of w or the first descent of w^{-1} is a reduction pair. (Note that w^{-1} is also Gasharov–Reiner.) For any permutation w it is well-known that $P_w(t) = P_{w^{-1}}(t)$, and by [15, Prop. 5.2(ii)] the diagrams O_w and $O_{w^{-1}}$ are rearrangements of each other and so $\mathrm{M}_w(q) = \mathrm{M}_{w^{-1}}(q)$. Thus without loss of generality we may assume that the first descent of w is a reduction pair.

3.3.1. The case of a light reduction pair

If the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a light reduction pair then by (7) and induction we have

$$\begin{aligned}\mathrm{M}_w(q) &= q \cdot \mathrm{M}_{s_i w}(q) + q^{n-1}\,\mathrm{M}_{w-y}(q) \\ &= q \cdot q^{\binom{n}{2}+\ell(s_i w)} P_{s_i w}(q^{-1}) + q^{n-1} \cdot q^{\binom{n-1}{2}+\ell(w-y)} P_{w-y}(q^{-1}).\end{aligned}$$

Then from (6) it follows that

$$\begin{aligned}\mathrm{M}_w(q) &= q^{\binom{n}{2}+\ell(w)} \left(P_{s_i w}(q^{-1}) + q^{-(\ell(w)-\ell(w-y))} P_{w-y}(q^{-1})\right) \\ &= q^{\binom{n}{2}+\ell(w)} P_w(q^{-1}),\end{aligned}$$

as desired.

3.3.2. The case of a heavy reduction pair

Suppose the first descent of w is a heavy reduction pair. We first consider the case $i = j$, i.e., that $w_1 < \ldots < w_{i-1} < w_{i+1} < w_i$. In this case, the definition of heavy reduction pairs implies that

$$w = 1 \cdots (i-1)\, w_i\, i\, \tau$$

for some permutation τ of $[n] \smallsetminus \{1, \ldots, i, w_i\}$. Thus, we have that $s_i w = 1 \cdots i\, w_i\, \tau$ and that $w - x$ is order-isomorphic to $1 \cdots (i-1)\, w_i\, \tau$, so $P_{s_i w}(t) = P_{w-x}(t)$. Similarly, we have that $w - y$ is order-isomorphic to $1 \cdots i\, \tau$ and that $w - x - y$ is order-isomorphic to $1 \cdots (i-1)\, \tau$, so $P_{w-x-y}(t) = P_{w-y}(t)$. Then Equation (4) reduces to

$$P_w(t) = (1+t) \cdot P_{w-x}(t).$$

Moreover, when $i = j$ we have $v(w) = w - x$, so Equation (5) reduces to

$$\mathrm{M}_w(q) = \mathrm{M}_{s_i w}(q) + q^n \cdot \mathrm{M}_{w-x}(q).$$

Thus, by induction we have

$$\begin{aligned}
\mathrm{M}_w(q) &= \mathrm{M}_{s_i w}(q) + q^n \cdot \mathrm{M}_{w-x}(q) \\
&= q^{\binom{n}{2}+\ell(s_i w)} P_{s_i w}(q^{-1}) + q^n \cdot q^{\binom{n-1}{2}+\ell(w-x)} P_{w-x}(q^{-1}) \\
&= \left(q^{\binom{n}{2}+\ell(w)-1} + q^{n+\binom{n-1}{2}+\ell(w)-1}\right) P_{w-x}(q^{-1}) \\
&= q^{\binom{n}{2}+\ell(w)}(q^{-1}+1) P_{w-x}(q^{-1}) \\
&= q^{\binom{n}{2}+\ell(w)} P_w(q^{-1}),
\end{aligned}$$

as desired.

Finally, we are left with the case that $i > j$. In this case, the three permutations w, $s_i w$ and $w - y$ have first descents in positions i, $i-1$ and $i-1$, respectively, and satisfy the hypotheses of Proposition 3.5. Thus, applying Proposition 3.5 to each of these permutations and rearranging gives

$$\begin{aligned}
q^n\, \mathrm{M}_{v(w)}(q) &= \mathrm{M}_w(q) - \mathrm{M}_{s_i w}(q), \\
q^n\, \mathrm{M}_{v(s_i w)}(q) &= \mathrm{M}_{s_i w}(q) - \mathrm{M}_{s_{i-1} s_i w}(q), \qquad \text{and} \\
q^{n-1}\, \mathrm{M}_{v(w-y)}(q) &= \mathrm{M}_{w-y}(q) - \mathrm{M}_{s_{i-1}(w-y)}(q).
\end{aligned} \tag{12}$$

To make use of these equations, we need some basic properties of the permutation $v(w)$.

Proposition 3.15. *Suppose that $w \in \mathfrak{S}_n$ is a Gasharov–Reiner permutation whose first descent, involving the entries (i, w_i) and $(i+1, w_{i+1})$, is a heavy reduction pair. If $i > j$ then, with $v = v(w)$ as in (3), we have that $v \in \mathfrak{S}_{n-1}(4231, 35142, 42513, 351624)$ and the first descent of v is in position $i-1$ and is a light reduction pair.*

Proof. Certainly $v \in \mathfrak{S}_{n-1}$. To show that v avoids the four bad patterns, it suffices to show that the sequence

$$w' = w_1\, w_2\, \cdots\, w_{j-1} \quad w_{j+1}\, w_{j+2}\, \cdots\, w_i \quad w_j \quad w_{i+2}\, w_{i+3}\, \cdots\, w_n$$

(to which v is order-isomorphic) avoids them.

Suppose that w' contains one of the four forbidden patterns, and let μ be a subsequence of w' order-isomorphic to one of these patterns. We will derive a contradiction.

By Remark 3.7(i), the entries $w_1, \ldots, w_{j-1}$ are all smaller than all other entries of w' and occur in increasing order, but none of the four forbidden patterns begins with its smallest element. Thus, μ does not contain any of these entries.

Removing the entry w_j from w' leaves a sequence order-isomorphic to a subsequence of w. Since w avoids the four patterns in question, it follows that μ must contain the entry w_j. The same is true if one removes (simultaneously) the entries $w_{j+1}, w_{j+2}, \ldots, w_i$ from w', so μ must contain at least one of these entries.

Thus, the first descent of μ occurs between one of the values $w_{j+1}, \ldots, w_i$ and w_j. Therefore, by Remark 3.7(ii), in the permutation order-isomorphic to μ, the entries of the shortest prefix including the bottom of the first descent form an interval. However, none of the four forbidden patterns have this property. This is a contradiction. Thus v is Gasharov–Reiner.

Finally, it is easy to see that the first descent of v is in position $i-1$ and is a light reduction pair. $\square$

Continuing with the notation of the preceding proof, we have that the first descent of $v(w)$ is between the entries $y' = (i-1, w_i - 1)$ and $x' = (i, w_j - 1)$. Since this descent is a light reduction pair, we may apply (7) to conclude that

$$\mathrm{M}_{v(w)}(q) = q\,\mathrm{M}_{s_{i-1}v(w)}(q) + q^{n-2}\,\mathrm{M}_{v(w)-y'}(q). \tag{13}$$

It is easy to check that

$$v(s_i w) = s_{i-1} \cdot v(w), \qquad v(w-y) = v(w) - y', \qquad \text{and} \qquad s_{i-1}(w-y) = s_i w - y'.$$

Thus, we may multiply (13) through by q^n and substitute from (12) to conclude that

$$\mathrm{M}_w(q) - \mathrm{M}_{s_i w}(q) = q(\mathrm{M}_{s_i w}(q) - \mathrm{M}_{s_{i-1}s_i w}(q)) + q^{n-1}(\mathrm{M}_{w-y}(q) - \mathrm{M}_{s_i w - y'}(q)). \tag{14}$$

Now we derive the same recursion for Poincaré polynomials. The first descent of $s_i w$ is a heavy reduction pair involving the entries $y' = (i-1, w_i - 1)$ and $x'' = (i, w_{i+1})$, so by (4) we have

$$\begin{aligned} P_{s_i w}(t) = {} & P_{s_{i-1}s_i w}(t) + t^{\ell(s_i w) - \ell(s_i w - x'')} P_{s_i w - x''}(t) \\ & + t^{\ell(s_i w) - \ell(s_i w - y')} P_{s_i w - y'}(t) - t^{\ell(s_i w) - \ell(s_i w - x'' - y')} P_{s_i w - x'' - y'}(t). \end{aligned} \tag{15}$$

It is easy to check that

$$s_i w - x'' = w - x \qquad \text{and} \qquad s_i w - x'' - y' = w - x - y,$$

and so subtracting t times (15) from (4) (keeping in mind that $\ell(s_i w) = \ell(w) - 1$) yields

$$\begin{aligned} & P_w(t) - tP_{s_i w}(t) \\ & \quad = \big(P_{s_i w}(t) - tP_{s_{i-1}s_i w}(t)\big) + t^{\ell(w)} \cdot (t^{-\ell(w-y)} P_{w-y}(t) - t^{-\ell(s_i w - y')} P_{s_i w - y'}(t)). \end{aligned}$$

Finally, we make the substitution $t = q^{-1}$ and multiply through by $q^{\binom{n}{2} + \ell(w)}$ to conclude

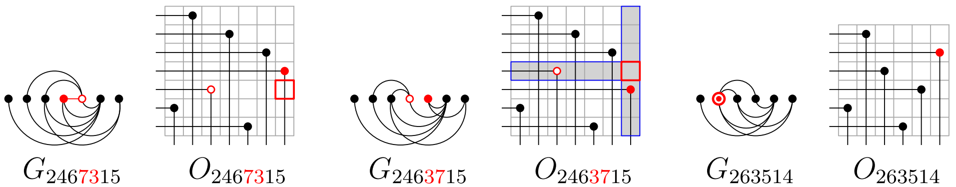


Fig. 8. The inversion graph and SW diagram for (a) the permutation 2467315, (b) the result of deletion, and (c) the result of contraction.

$$\begin{aligned}
&q^{\binom{n}{2}+\ell(w)}P_w(q^{-1}) - q^{\binom{n}{2}+\ell(s_iw)}P_{s_iw}(q^{-1})\\
&\quad = q\left(q^{\binom{n}{2}+\ell(s_iw)}P_{s_iw}(q^{-1}) - q^{\binom{n}{2}+\ell(s_{i-1}s_iw)}P_{s_{i-1}s_iw}(q^{-1})\right)\\
&\qquad + q^{n-1}\left(q^{\binom{n-1}{2}+\ell(w-y)}P_{w-y}(q^{-1}) - q^{\binom{n-1}{2}+\ell(s_iw-y')}P_{s_iw-y'}(q^{-1})\right).
\end{aligned}$$

Comparing with (14) and applying the inductive hypothesis gives the desired result.

4. Further remarks and questions

4.1. Bijective proofs

One can give an alternate proof of the first part of Theorem 2.1 via a recursive argument: given $w \in \mathfrak{S}_n$, one produces permutations $w' \in \mathfrak{S}_n$ and $w'' \in \mathfrak{S}_{n-1}$ such that the inversion graphs $G_{w'}$ and $G_{w''}$ are isomorphic respectively to the graphs that we get by deleting or contracting a particular edge in G_w, and also such that rook placements on the south–west diagrams $O_{w'}$ and $O_{w''}$ correspond naturally to rook placements on O_w where a particular cell respectively does not or does contain a rook. (See Fig. 8, and [18] for more details.) Then the result follows from the deletion-contraction recursion for acyclic orientations. In principle, this can be unravelled (noncanonically) to give a bijection. Is it possible instead to give a single, explicit (i.e., nonrecursive) bijection between rook placements avoiding O_w and acyclic orientations of G_w? Cf. Appendix A.

4.2. Other types

The acyclic orientations of the inversion graph of a permutation w are in correspondence with the regions of the hyperplane arrangement $\mathcal{A}_w$ consisting of the hyperplanes $x_i - x_j = 0$ in $\mathbb{R}^n$ for each inversion (i, j) of w. This arrangement has a natural analogue when the symmetric group $\mathfrak{S}_n$ is replaced by any Weyl group W. In this setting, Hultman [13] has proved an analogue of Theorem 1.4 of Hultman–Linusson–Shareshian–Sjöstrand. Is there an analogue of rook placements avoiding a diagram associated to an element w in W that allows one to extend Theorem 2.1 or Theorem 3.1 to this context? Barrese and Sagan (personal communication) have made some initial progress on this direction.

4.3. A nicer recurrence for Poincaré polynomials

As a consequence of Theorem 3.1 and Equation (5), the Poincaré polynomial for a Gasharov–Reiner permutation w with a heavy reduction pair in its first descent satisfies

$$P_w(t) = t \cdot P_{s_i w}(t) + P_{v(w)}(t).$$

This latter recursion is arguably simpler than (4). Is it possible to prove such a result directly, without going through the painful contortions following the proof of Proposition 3.15? For example, can one exhibit a bijection between the relevant Bruhat intervals that shifts lengths appropriately?

4.4. Connection with Schubert varieties

Theorem 3.1 gives a relationship between a function counting invertible matrices and a Poincaré polynomial. The Poincaré polynomial $P_w(t)$ has a geometric, as well as combinatorial, meaning: it gives the decomposition of the Schubert variety X_w over $\mathbb{C}$ into Schubert cells, or equivalently it counts $\mathbf{F}_q$ points in X_w. In addition, the Gasharov–Reiner permutations w characterize the Schubert varieties X_w *defined by inclusions* [10]. (For an overview of connections between Schubert varieties and combinatorics, see [1].) Thus, it seems natural to suppose that there should be an explanation for Theorem 3.1 involving the associated Schubert varieties. At present, we have no such explanation for Gasharov–Reiner permutations.

However, it is possible to give a simple proof in the special case of permutations avoiding the pattern 312. In this case, the diagram O_w is a (reflection of a) Young diagram. Thus its complement $\overline{O_w}$ is also a partition shape. The Schubert variety X_w is one of Ding's *partition varieties*, and Ding showed [6, Thm. 33] that the Poincaré polynomial of this variety is equal to the Garsia–Remmel *rook polynomial* [9]. Next, work of Haglund [11, Thm. 1] shows that for a partition shape, the rook polynomial and matrix count $\mathrm{M}_w(q)$ are equal up to powers of q. Finally, 312-avoiding permutations avoid the patterns 3412 and 4231, so by the work of Lakshmibai–Sandhya [19] and Carrell–Peterson [5] (Lemma 1.6) we may replace $P_w(q)$ with $q^{\ell(w)}P_w(q^{-1})$ to complete the proof.

The following result for smooth permutations follows easily from Theorem 3.1; it was independently proven by Linusson–Shareshian (personal communication). Recall that $w \in \mathfrak{S}_n$ is smooth if w avoids the patterns 3412 and 4231.

Corollary 4.1. *Let w be a permutation in $\mathfrak{S}_n$. We have*

$$\mathrm{M}_w(q) = q^{\binom{n}{2}} P_w(q)$$

if and only if w is smooth.

Proof. To prove the "if" direction, first compare the definitions to see that if w is smooth then w is also Gasharov–Reiner. Thus it follows by Theorem 3.1 that $\mathrm{M}_w(q) = q^{\binom{n}{2}+\ell(w)}P_w(q^{-1})$. Then by Lemma 1.6 we have that $q^{\binom{n}{2}+\ell(w)}P_w(q^{-1}) = q^{\binom{n}{2}}P_w(q)$.

Next we prove the "only if" direction. The argument in Section 3 following the statement of Theorem 3.1 establishes that w must be Gasharov–Reiner. So, again using Theorem 3.1, we have that $\mathrm{M}_w(q) = q^{\binom{n}{2}+\ell(w)}P_w(q^{-1})$. This fact and the hypothesis imply that $P_w(q)$ is palindromic, and by Lemma 1.6 it follows that w is smooth. □

4.5. Positivity in matrix-counting for other permutations

Computational evidence suggests [15, Conj. 5.1] that $\mathrm{M}_w(q)$ is a polynomial in $\mathbb{N}[q]$ for all permutations w, not just for Gasharov–Reiner permutations. (In general, the number of invertible matrices over $\mathbf{F}_q$ with restricted support is not necessarily a polynomial in q [28, §8.1].) It would be very interesting if one could explain this fact geometrically, e.g., via some sort of cellular decomposition of the set of matrices counted by $\mathrm{M}_w(q)$. A more naive approach is to look for a recursion along the lines of Equations (5) and (7) that is valid for all permutations. The next example gives some discouraging evidence for the latter approach.

Example 4.2. For $w = 4312$, we have that

$$\mathrm{M}_{4312}(q) - \mathrm{M}_{3412}(q) = q^{11} + 2q^{10} + 2q^9 - q^7,$$

which has a negative coefficient. Curiously, for all w in $\mathfrak{S}_n$ for $n \leq 7$ we have that $\mathrm{M}_w(q) - q \cdot \mathrm{M}_{s_i w}(q) \in \mathbb{N}[q]$. However, for $w = 3412$, the difference

$$\mathrm{M}_{3412}(q) - q \cdot \mathrm{M}_{3142}(q) = 2q^8 + 3q^7 + q^6$$

is not of the form $q^a \cdot \mathrm{M}_u(q)$ for any integer a and permutation u.

Remark 4.3. Note that if w is *not* Gasharov–Reiner then $\#[\iota, w] > AO(G_w)$ by Theorem 1.4. The q-analogue of this fact is the following conjecture in [15]: for all w the difference $q^{\binom{n}{2}+\ell(w)}P_w(q^{-1}) - \mathrm{M}_w(q)$ belongs to $\mathbb{N}[q]$. On the other hand, there is no inequality of coefficients between $\mathrm{M}_w(q)$ and $q^{\binom{n}{2}}P_w(q)$ for the non-smooth permutations. For example, $\mathrm{M}_{3412}(q) = q^{10}+3q^9+5q^8+4q^7+q^6$ and $q^6P_{3412}(q) = q^{10}+4q^9+5q^8+3q^7+q^6$ are not comparable coefficientwise.

Remark 4.4. For $n \leq 7$, computations show that the polynomials $\mathrm{M}_w(q)$ are *unimodal* for all $w \in \mathfrak{S}_n$, i.e., their coefficients first increase, then decrease. However, they are not generally *log-concave*: when $w = 5673412$ we have that the sequence of coefficients of M_w is $(1, 4, 17, 52, 116, 203, 289, 346, 355, 316, 246, 167, 98, 49, 20, 6, 1)$, and $4^2 < 1 \cdot 17$ is a violation of log-concavity.

4.6. Matrices of lower rank

By [17, Prop. 5.1], the counting function for matrices of rank r with support avoiding a permutation diagram O_w is a q-analogue of placements of r non-attacking rooks avoiding O_w. We conjecture [15, Conj. 5.1] that this function belongs to $(q-1)^r\mathbb{N}[q]$; what are its coefficients counting? Are there corresponding “lower rank” analogues of any other members of Postnikov’s “zoo”, either for all permutations or for some nice subclass (e.g., Grassmannian permutations, smooth permutations, Gasharov–Reiner permutations)?

4.7. Counting and q-counting fillings of permutation diagrams

Above, we have studied percentage-avoiding fillings of the SE diagram E_w for a permutation w. When $w = w_\lambda$ is Grassmannian, E_w is the Young diagram of λ in French notation. For such shapes, percentage-avoiding fillings are in bijection with a large family of similarly restricted fillings (see [26,14]), including the *⅂-fillings*[6] studied by Postnikov. Here, we mention some additional results and conjectures relating to these other restricted fillings of the diagram E_w when w is not necessarily Grassmannian.

Given a binary filling f of a diagram D, we say that f is a ⅃-filling if it avoids the patterns $\begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array}$. Similarly, we say that f is

- a Γ-filling if it avoids the patterns $\begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array}$,
- a L-filling if it avoids the patterns $\begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array}$, and
- a ⅂-filling if it avoids the patterns $\begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array}$.

In this section, we focus on the case that the diagram E_w has a particular nice structure. We say that a diagram D has the **south–east (SE) property** if whenever (i, j), (i', j) and (i, j') are in D with $i' > i$ and $j' > j$ then (i', j') is also in D. For such diagrams, condition (ii) in Definition 2.3 is never relevant, and so pseudo-percentage avoidance reduces to percentage avoidance in this case.

It is easy to see that the SE diagram E_{321} fails to have the SE property, and consequently that E_w also fails to have the SE property for any permutation w containing 321 as a pattern. The converse of this statement is also true:

Proposition 4.5. *(See [20, Prop. 2.2.13].) If w avoids 321 then E_w is, up to removing rows and columns that do not intersect E_w, a skew Young shape in French notation. In this case E_w has the SE property.*

We start by giving a corollary of Proposition 2.4.

[6] In [22], Postnikov used English notation for partitions, while we use French notation; thus, his “⅃ diagrams” are equivalent to our ⅂-fillings and what would be his “L diagrams” correspond to our Γ-fillings.

Corollary 4.6. *If w avoids 321 then the number of Γ-fillings of E_w is equal to the number of acyclic orientations of the inversion graph of w.*

Proof. By Proposition 2.4, for all w the number of acyclic orientations of the inversion graph of w equals the number of pseudo-percentage-avoiding fillings of E_w. By Proposition 4.5 and the paragraphs that precede it, if w avoids 321 then a filling of E_w is pseudo-percentage-avoiding if and only if it is percentage-avoiding. Moreover, since E_w is a skew Young shape, we have by work of Spiridonov [26] (see also Josuat-Vergès [14, §4]) that the number of percentage-avoiding fillings and the number of Γ-fillings of E_w are equal. $\square$

Remark 4.7. Although the number of L-fillings and Γ-fillings of E_w coincide when w is Grassmannian, these numbers can differ for other 321-avoiding permutations. For example, the permutation 351624 avoids 321, and one can check that there are 98 Γ-fillings and 100 L-fillings of E_{351624}. By Theorem 4.9 below, the latter are in bijection with the elements in the interval $[\iota, 351624]$.

We now focus on L-fillings and ⅂-fillings. Given a binary filling f of a diagram, the size $|f|$ is the number of 1s in the filling. For w in $\mathfrak{S}_n$ and $\pi \in \{\mathrm{L}, \daleth\}$, let $F_w^{\pi}(q)$ be the generating function $\sum_f q^{|f|}$ where the sum is over π-fillings of E_w. By abuse of notation, $F_\lambda^{\pi}(q) = F_{w_\lambda}^{\pi}(q)$ is the generating function of π-fillings of the Young diagram of the partition λ. The following is a corollary of [22, Thm. 19.1].

Theorem 4.8 *(Postnikov). For a Grassmannian permutation w_λ in $\mathfrak{S}_n$ associated to a partition $\lambda \subseteq k \times (n-k)$, we have that $F_\lambda^{\mathrm{L}}(q) = F_\lambda^{\daleth}(q) = q^{\ell(w_\lambda)} P_{w_\lambda}(q^{-1})$.*

Proof sketch. Given w in $\mathfrak{S}_n$, fix a reduced decomposition of w and the corresponding *wiring diagram* of the decomposition. Then each u in $[\iota, w]$ is obtained as a subword of the reduced decomposition [20, Prop. 2.1.3]. To rule out repetitions one can choose the *lexicographically maximal* (*minimal*) subword that is a reduced expression for u. Postnikov then characterized these subwords as certain *pipe dreams* of the wiring diagram, obtained by changing crossings of wires to uncrossings, with two restrictions: if two wires cross at a point P then they cannot cross or uncross before (after) P. Call these *lexicographically maximal (minimal) pipe dreams*; see Fig. 9(b). It follows that they are in bijection with the elements u in $[\iota, w]$.

Next, for a Grassmannian permutation w_λ in $\mathfrak{S}_n$, Postnikov describes a wiring diagram with crossings exactly on the cells of the Young diagram of λ. Then pipe dreams of this wiring diagram correspond to fillings of λ. Moreover, the lexicographically maximal (minimal) pipe dreams are exactly the ⅂-fillings (L-fillings) of λ. This yields a correspondence $u \leftrightarrow f$ between u in $[\iota, w_\lambda]$ and ⅂-fillings (L-fillings) f of λ such that $\ell(u) = \ell(w) - |f|$, as desired. $\square$

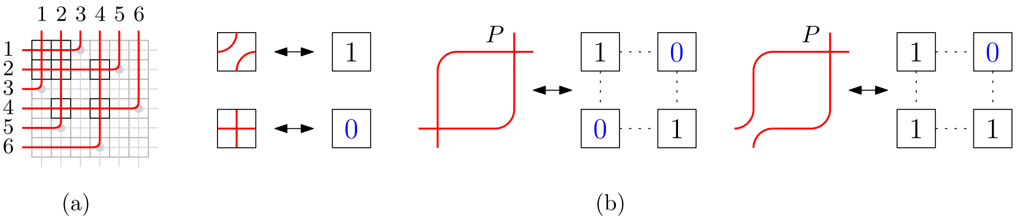


Fig. 9. (a) The hook wiring diagram of $w = 351624$ which has crossings in exactly the elements of E_{351624}, (b) correspondence between (un)crossings in a pipe dream and binary fillings; the forbidden wires for lexicographic-maximal pipe dreams and the corresponding Ꞁ-patterns.

Theorem 4.8 can be extended to 321-avoiding permutations. (This extension is due to Postnikov–Spiridonov (personal communication), but it seems that the statement has not been written down anywhere before.)

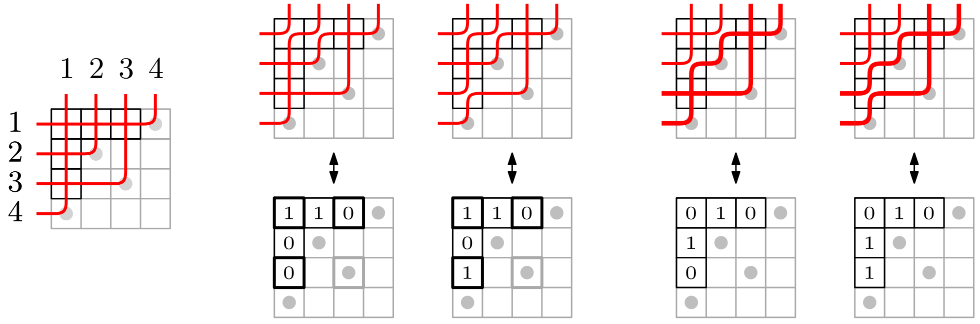
Theorem 4.9. *For w in $\mathfrak{S}_n$ avoiding 321 we have that $F^{\mathrm{L}}_w(q) = F^{\daleth}_w(q) = q^{\ell(w)} P_w(q^{-1})$.*

Proof sketch. The argument is essentially the same as that of Theorem 4.8 sketched above. It is necessary to give a wiring diagram for w analogous to the one for a Grassmannian permutation with crossings exactly on the cells of the diagram of w. Given w, for each $i = 1, \ldots, n$ we draw a wire starting from the first entry of the ith row that goes right until it reaches the entry (i, w_i) where it turns 90° and continues up to end in the first entry of the w_ith column. This collection of n wires is a wiring diagram of w with crossings in exactly the elements $(i, j) \in E_w$. See Fig. 9(a) for an example and [20, Rem. 2.1.9] for a similar construction. We call this wiring diagram the **hook wiring diagram** of w. If w is a Grassmannian permutation w_λ or w avoids 321 then E_w is, up to removing rows and columns that do not intersect E_w, the Young diagram of λ or of a skew Young shape respectively. The rest of the argument in the proof of Theorem 4.8 follows for this wiring diagram on the skew shape. However, the argument can fail for w containing 321 (see Fig. 10). $\square$

Remark 4.10. A consequence of this result and Lemma 1.6 is that when w avoids 321 and 3412 then $F^{\mathrm{L}}_w(q) = F^{\daleth}_w(q) = P_w(q)$. From computations, both statements appear to be if-and-only-ifs. This class of permutations has also appeared several times in the literature [23,29].

4.8. q-Counting pseudo-fillings of permutation diagrams

In this section we look briefly at fillings of E_w where the diagram might not have the SE property. Because of this defect, we put extra restrictions on the fillings just as we did with the percentage avoiding fillings in Section 2.1. We say that a filling f of E_w is



(a) (b) (c)

Fig. 10. (a) The hook wiring diagram of $w = 4231$ associated to the reduced word $s_3s_2s_1s_2s_3$. (b) Two fillings of E_{4231} containing the pseudo-⅂ pattern that correspond to lexicographically maximal pipe dreams. (c) Two lexicographically non-maximal pipe dreams of the hook wiring diagram of 4231 that do correspond to pseudo-⅂ fillings of E_{4231}.

- a **pseudo-L-filling** if it avoids the patterns $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&1\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&0\\0&\bullet\end{smallmatrix}$, $\begin{smallmatrix}1&1\\0&\bullet\end{smallmatrix}$, where the solid dot indicates an entry of the permutation, and
- a **pseudo-⅂-filling** if it avoids the patterns $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&0\\1&\bullet\end{smallmatrix}$, $\begin{smallmatrix}1&0\\0&\bullet\end{smallmatrix}$, where the solid dot indicates an entry of the permutation.

For w in $\mathfrak{S}_n$ and $\pi \in \{\mathrm{L}, ⅂\}$, let $PF^{\pi}_w(q)$ be the generating function $\sum_f q^{|f|}$ where the sum is over fillings of E_w avoiding the appropriate pseudo-π pattern. Note that if E_w has the SE property then the last two patterns to avoid in pseudo-π-fillings will never be relevant, and so these fillings reduce to the usual π-fillings. For such w we have that $PF^{\pi}_w(q) = F^{\pi}_w(q)$.

The next conjecture suggests an extension of Theorem 4.9 for Gasharov–Reiner permutations and pseudo-fillings.

Conjecture 4.11. *For w in $\mathfrak{S}_n(4231, 35142, 42513, 351624)$ we have that*

$$PF^{\mathrm{L}}_w(q) = PF^{⅂}_w(q) = q^{\ell(w)}P_w(q^{-1}).$$

This conjecture has been verified by brute force for $n \leq 7$. A proof of this conjecture, combined with Theorems 1.4 and 2.1, would extend the equivalence of Theorem 1.1 from Grassmannian to Gasharov–Reiner permutations. Recall that the combinatorial objects in Theorem 1.1 identified with Grassmannian permutations w_λ also count and parametrize *positroid cells* inside a *Schubert cell* Ω_λ. Do some of the objects described in this paper linked to other permutations w count cells in a decomposition of a generalization of $Gr^{\geq 0}_{k,n}(\mathbb{R})$?

Remark 4.12. Note that the number of pseudo-L-fillings and the number of pseudo-⅂-fillings of E_w can differ for certain permutations w. For example for $w = 35241$ the number of pseudo-L fillings of E_{35241} is 56 and the number of pseudo-Ꞁ fillings of E_{35241} is 60. The numbers differ also for the inverse 53142 of w. These are the only permutations in $\mathfrak{S}_5$ where the number of these two fillings differ.

Similarly, the number of pseudo-L-fillings of E_w and the size $\#[\iota, w]$ of the Bruhat interval can differ for certain permutations w. For example, for $w = 52341$, we have that $PF^{\mathrm{L}}_{52341}(1) = 72$ and $\#[\iota, 52341] = 68$.

Remark 4.13. One approach to prove Conjecture 4.11 would be to extend Postnikov's correspondence from Theorem 4.8 to lexicographically maximal (minimal) pipe dreams encoding u in $[\iota, w]$ and pseudo-Ꞁ- (pseudo-L-) fillings of E_w. Brute force calculations suggest there is such a correspondence for all w in $\mathfrak{S}_n(4231, 35142, 42513, 351624)$ up to $n \leq 6$ but, it may fail for other permutations; see Fig. 10.

Another approach to prove the conjecture is the reduction pairs used in the proof of Theorem 3.1. One can show, using an analysis similar to the one by Williams in [30], that if the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a light reduction pair then

$$PF^{\text{Ꞁ}}_{w}(q) = q \cdot PF^{\text{Ꞁ}}_{s_i w}(q) + PF^{\text{Ꞁ}}_{w-y}(q), \tag{16}$$

and it is also not difficult to show that if the first descent of w, involving the entries $y = (i, w_i)$ and $x = (i+1, w_{i+1})$, is a heavy reduction pair then

$$PF^{\mathrm{L}}_{w}(q) = q \cdot PF^{\mathrm{L}}_{s_i w}(q) + PF^{\mathrm{L}}_{w-y}(q) + PF^{\mathrm{L}}_{w-x}(q) - PF^{\mathrm{L}}_{w-y-x}(q). \tag{17}$$

These recursions match those for $P_w(q)$ in Propositions 3.10 and 3.5; however, we have been unable to prove the two corresponding recursions necessary to complete the induction. We have also been unable to prove $PF^{\pi}_{w}(q) = PF^{\pi}_{s_i w}(q) + q \cdot PF^{\pi}_{v}(q)$ for π in $\{\mathrm{L}, \text{Ꞁ}\}$, which would be analogous to (5).

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Appendix A. Acyclic orientations, rook placements, inversion graphs and the chromatic polynomial (by Axel Hultman)

In this appendix we provide an independent proof of the part of Theorem 2.1 which asserts $AO(G_w) = RP(O_w)$. More precisely, we prove the following statement:

Theorem A.1. *For any $w \in \mathfrak{S}_n$, the chromatic polynomial of the inversion graph G_w satisfies*

$$\chi_{G_w}(t) = \sum_{i=0}^{n} r_{n-i}(O_w)t(t-1)\cdots(t-i+1), \tag{18}$$

where the rook number *$r_k(O_w)$ is the number of placements of k non-attacking rooks on O_w.*

From (18), the desired assertion follows if one sets $t = -1$ and invokes the standard results

$$AO(G_w) = (-1)^n \chi_{G_w}(-1) \tag{19}$$

and

$$RP(O_w) = \sum_{i=0}^{n} (-1)^i r_i(O_w)(n-i)!. \tag{20}$$

The identity (19) was originally obtained by Stanley [27] whereas (20) is due to Kaplansky and Riordan [16].

The idea behind the proof of Theorem A.1 is essentially that employed by Goldman, Joichi and White for proving [8, Theorem 2]. Some care is required, though, since O_w is not in general *proper* in the sense of [8]. It is, however, possible to make it proper by a suitable rearrangement of its columns. Then one could apply [8, Theorem 2] directly. After observing that the associated graph $\Gamma_n(B)$ of the rearranged board is isomorphic to G_w, Theorem A.1 would follow. Instead of taking this route, let us state a direct proof.

Proof of Theorem A.1. In a graph G, a subset of the vertices is called *independent* if it induces an edgeless subgraph of G. For a positive integer k, denote by $P(w, k)$ the set of partitions of the vertex set of the inversion graph G_w into k independent subsets. Equivalently, we may think of $P(w, k)$ as the set of transitively closed subgraphs of the complement graph $\overline{G_w}$ with k connected components and all n vertices.

Let us say that an *n-spine* is a graph on vertex set $[n]$ in which every connected component is a path whose vertices can be traversed in increasing (or, going the other way, decreasing) order. Equivalently, a graph on $[n]$ is an n-spine if every vertex has at most one smaller neighbour and at most one larger neighbour.

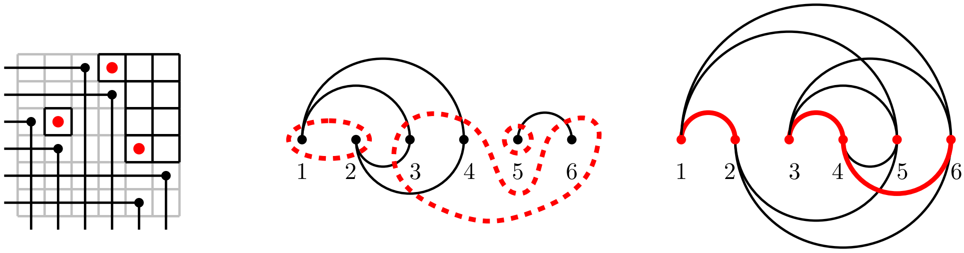

Fig. 11. A non-attacking 3-rook placement on the SW diagram of $w = 341265 \in \mathfrak{S}_6$ (left), the corresponding partition of G_w into $6 - 3 = 3$ independent sets (center) and the associated 3-edge subgraph of $\overline{G_w}$ which forms a 6-spine (right).

A rook on O_w corresponds to a noninversion of w, i.e. an edge of $\overline{G_w}$. In this way, the non-attacking rook placements on O_w are in bijective correspondence with the sets of edges of $\overline{G_w}$ that contain no two edges with a common smallest vertex and no two edges with a common largest vertex. That is, the non-attacking k-rook placements on O_w correspond bijectively to the k-edge subgraphs of $\overline{G_w}$ that are n-spines.

If $1 \leq i_1 < i_2 < i_3 \leq n$ and (i_1, i_2) and (i_2, i_3) are noninversions of w, then so is (i_1, i_3). Hence, the transitive closure of any k-edge n-spine subgraph of $\overline{G_w}$ is an element of $P(w, n-k)$. Conversely, every element of $P(w, n-k)$ is clearly the closure of a unique n-spine. This shows that $P(w, n-k)$, too, is in bijection with the k-edge subgraphs of $\overline{G_w}$ that are n-spines. Hence, $r_{n-i}(O_w) = \#P(w, i)$. Now observe that

$$\chi_{G_w}(t) = \sum_{i=0}^{n} \#P(w, i) \cdot t(t-1) \cdots (t-i+1),$$

since (for a positive integer t) the term indexed by i in the sum counts the proper vertex colourings of G_w that use exactly i out of t given colours. This concludes the proof. $\square$

An illustration of the constructions occurring in the proof is found in Fig. 11.

Remark A.2 *(by AHM and JBL)*. The equality (18) is particularly nice when the reverse of w is *vexillary*, i.e., when w avoids 3412. In this case O_w is, up to permuting rows and columns, a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. Then calculating the right side of (18) is straightforward: by [7] we have that $\sum_{i=0}^{n} r_{n-i}(O_w) t(t-1) \cdots (t-i+1) = \prod_{i=1}^{n} (t + \lambda_i - i + 1)$. On the other hand, we say that a graph is *chordal* if every cycle of four or more edges in the graph has a chord, i.e., an edge joining two non-consecutive vertices in the cycle. It is well known that the chromatic polynomial of a chordal graph G may be written as $\prod_{i=1}^{n}(t - e_i)$ for certain nonnegative integers e_i depending on G (see e.g. [21, Prop. 12]). One can show that the inversion graph G_w is chordal if and only if w avoids 3412 and that in this case the multisets $\{e_i\}_{i=1}^{n}$ and $\{i - \lambda_i - 1\}_{i=1}^{n}$ are equal.

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