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Density versions of Schur's theorem for ideals generated by submeasures ☆

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ABSTRACT

We characterize ideals of subsets of natural numbers for which some versions of Schur's theorem hold. These are similar to generalizations shown by Bergelson (1986) in [1] and Frankl, Graham and Rödl (1990) in [7]. Additionally, we prove a generalization of an iterated version of Ramsey's theorem.

1. Introduction

The classical Schur's theorem says that for any finite coloring of the set of natural numbers $\omega = C_1 \cup \dots \cup C_r$, there exist x, y, z having the same color ($x, y, z \in C_i$ for some $i \leqslant r$) such that $x + y = z$. The natural question is "how many x's there are in C_i so that for each of these x there are many y's in C_i so that $x + y$ is also in C_i?" Of course, an answer depends on a definition of the notion "many". If we consider "many" as a set of positive density then this generalization of Schur's theorem remains valid.

Theorem 1.1. *(See [7].) For any partition of $\omega = C_1 \cup C_2 \cup \dots \cup C_r$ there are $\delta = \delta(r) > 0$ and $i \leqslant r$ such that*

$$\bar{d}\big(\big\{x \in \omega\colon\ \bar{d}\big(\{y \in \omega\colon\ x, y, x + y \in C_i\}\big) \geqslant \delta\big\}\big) \geqslant \delta.$$

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Recall that Bergelson earlier pointed out the following density version of Schur's theorem.

Theorem 1.2. *(See [1].) For any partition of $\omega = C_1 \cup C_2 \cup \cdots \cup C_r$, some C_i having $\bar{d}(C_i) > 0$ satisfies for any $\varepsilon > 0$,*

$$\bar{d}\big(\big\{x \in \omega\colon\ \bar{d}\big(\{y \in \omega\colon\ x, y, x+y \in C_i\}\big) \geqslant \bar{d}(C_i)^2 - \varepsilon\big\}\big) > 0.$$

An ideal on ω (by ω we mean a set of all natural numbers) is a family $\mathcal{I} \subset \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the power set of ω) which is closed under taking subsets and finite unions. By Fin we denote the ideal of all finite subsets of ω. If not explicitly said we assume that all considered ideals are proper ($\neq \mathcal{P}(\omega)$) and contain all finite sets.

In this note we show that if the notion "many" means "not in an ideal of subsets of naturals" then the analogous generalization of Schur's theorem holds for a wide class of ideals.

In Section 3 we show that Theorem 1.1 holds for all analytic P-ideals (see definitions below). In order to prove this, we show that an iterated version of Ramsey's theorem holds for every analytic P-ideal (which seems interesting in its own).

In Section 4 we characterize those analytic P-ideals for which the constant δ in Theorem 1.1 does not depend on a number r of cells of the partition.

In Section 5 we show that for another subclass of analytic P-ideals the generalization of Theorem 1.2 holds.

In Section 6 we provide some examples of ideals for which theorems proved in previous sections can be applied. For instance, we consider the class of Erdős–Ulam ideals. This class contains the ideal of statistical density zero sets and the ideal of logarithmic density zero sets.

2. Preliminaries

2.1. Analytic P-ideals

By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (*analytic*) if it is an F_σ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively).

A map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω if

$$\phi(\emptyset) = 0,$$

$$\phi(A) \leqslant \phi(A \cup B) \leqslant \phi(A) + \phi(B),$$

for all $A, B \subset \omega$. It is *lower semicontinuous* if for all $A \subset \omega$ we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\phi : \mathcal{P}(\omega) \to [0, \infty]$ be the submeasure defined by

$$\|A\|_\phi = \limsup_{n\to\infty} \phi(A \setminus n) = \lim_{n\to\infty} \phi(A \setminus n),$$

where the second equality follows by the monotonicity of ϕ. Let

$$\mathrm{Exh}(\phi) = \big\{A \subset \omega\colon\ \|A\|_\phi = 0\big\},$$

$$\mathrm{Fin}(\phi) = \big\{A \subset \omega\colon\ \phi(A) < \infty\big\}.$$

It is clear that $\mathrm{Exh}(\phi)$ and $\mathrm{Fin}(\phi)$ are ideals (not necessarily proper) for an arbitrary submeasure ϕ.

An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for all n, i.e. A_n is almost contained in A for each n.

All analytic P-ideals are characterized by the following theorem of Solecki.

Theorem 2.1. *(See [12].) The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an analytic P-ideal.*
(2) *$\mathcal{I} = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

Moreover, for F_σ ideals the following characterization holds.

Theorem 2.2. *(See [11].) The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an F_σ ideal.*
(2) *$\mathcal{I} = \mathrm{Fin}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

The cardinality of a set X is denoted by $|X|$. We do not distinguish between natural number n and the set $\{0, 1, \ldots, n-1\}$.

The ideal of sets of density 0

$$\mathcal{I}_d = \left\{ A \subset \omega\colon \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}$$

is an analytic P-ideal. If we denote

$$\phi_d(A) = \sup\left\{ \frac{|A \cap n|}{n}\colon n \in \omega \right\},$$

then $\bar{d}(A) = \|A\|_{\phi_d}$ and $\mathcal{I}_d = \mathrm{Exh}(\phi_d)$.

The ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subset \omega\colon \sum_{n\in A} \frac{1}{n} < \infty \right\}$$

is an F_σ P-ideal. If ϕ is a submeasure defined by the formula

$$\phi(A) = \sum_{n\in A} \frac{1}{n},$$

then $\mathcal{I}_{\frac{1}{n}} = \mathrm{Fin}(\phi)$.

The ideal of arithmetic progressions free sets

$$\mathcal{W} = \{W \subset \omega\colon W \text{ does not contain arithmetic progressions of all lengths}\}$$

is an F_σ ideal which is not a P-ideal. The fact that $\mathcal{W}$ is an ideal follows from the non-trivial theorem of van der Waerden. This ideal was firstly considered by Kojman in [9].

We give some examples of ideals in Section 6. A lot more examples can be found in [6], and in Farah's book [4].

2.2. Bolzano–Weierstrass property

Let $\mathcal{I}$ be an ideal on ω, $A \subset \omega$ and $(x_n)_{n\in\omega}$ be a sequence of reals. By $(x_n) \restriction A$ we mean a subsequence $(x_n)_{n\in A}$. We say that $(x_n) \restriction A$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if $\{n \in A\colon |x_n - x| \geqslant \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

An ideal $\mathcal{I}$ on ω is called:

(1) FinBW if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n) \restriction A$ is convergent;
(2) BW if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n) \restriction A$ is $\mathcal{I}$-convergent.

In the first case we say that $\mathcal{I}$ has the finite Bolzano–Weierstrass property, in the second case we say that $\mathcal{I}$ has the Bolzano–Weierstrass property.

By the well-known Bolzano–Weierstrass theorem, the ideal Fin has the FinBW property. For the discussion and applications of these properties see [6], where we examine all BW-like properties. In particular, it is known that the ideal $\mathcal{I}_d$ of sets of density 0 does not have BW, and every F_σ ideal has FinBW.

In the sequel we will use the following characterization of BW-like properties among analytic P-ideals.

Theorem 2.3. *(See [6].) Let ϕ be a lower semicontinuous submeasure. The following conditions are equivalent.*

(1) *The ideal* $\mathrm{Exh}(\phi)$ *is* BW.
(2) *The ideal* $\mathrm{Exh}(\phi)$ *is* FinBW.
(3) *There is $\delta > 0$ such that for any partition $A_1, A_2, \dots, A_n$ of ω there exists $i \leqslant n$ with $\|A_i\|_\phi \geqslant \delta$.*

2.3. Invariant submeasures

We say that an ideal $\mathcal{I}$ is *invariant under translations* if for each $A \in \mathcal{I}$ and $n \in \mathbb{Z}$ (by $\mathbb{Z}$ we denote the set of integers)

$$A + n \in \mathcal{I} \quad \text{where } A + n = \{a + n\colon a \in A\} \cap \omega.$$

We say that a submeasure ϕ is *invariant under translations* if $\phi(A + n) = \phi(A)$ for each $A \subset \omega$ and $n \in \mathbb{Z}$.

Remark. If $\|\cdot\|_\phi$ is invariant under translations then the ideal $\mathrm{Exh}(\phi)$ is invariant under translations.

A submeasure ϕ fulfills a condition Δ if for every $\varepsilon > 0$, $A \subset \omega$ and any $N \in \omega$ there exists a measure $\phi' \leqslant \phi$ such that $\phi' \leqslant 1$ and

$$\phi'(A + n) \geqslant \|A\|_\phi - \varepsilon \quad \text{for each } n \in [-N, N].$$

(Throughout the paper, by a measure we mean a finitely additive measure.)

Proposition 2.4. *Suppose that $\mathcal{I} = \mathrm{Exh}(\phi)$ is an analytic P-ideal, $\phi \in \Delta$ and $\|\cdot\|_\phi$ is invariant under translations. Then $\mathcal{I}$ does not have the* BW *property.*

Proof. For the sake of contradiction, suppose that $\mathcal{I} = \mathrm{Exh}(\phi)$ has the BW property.

For every $n \in \omega$ and $i = 0, 1, \dots, 2^n - 1$ let

$$A_i^n = \bigl\{k \in \omega\colon k \equiv i \pmod{2^n}\bigr\}.$$

Then $\omega = A_0^n \cup \dots \cup A_{2^n-1}^n$ for every $n \in \omega$ and $A_i^n = A_0^n + i$ for every $i < 2^n$. Hence $\|A_0^n\| = \|A_1^n\| = \dots = \|A_{2^n-1}^n\| = \delta_n > 0$ for every $n \in \omega$.

By Theorem 2.3 there is $\delta > 0$ such that $\delta_n \geqslant \delta$ for every $n \in \omega$.

Let $n \in \omega$ be such that $2^n \frac{\delta}{2} > 1$. For $\varepsilon = \frac{\delta}{2}$, $A = A_0^n$ and $N = 2^n$ let $\phi' \leqslant 1$ be a measure required by condition Δ. Then

$$\begin{aligned}
1 &\geqslant \phi'(\omega) = \phi'\bigl(A_0^n\bigr) + \phi'\bigl(A_1^n\bigr) + \dots + \phi'\bigl(A_{2^n-1}^n\bigr) \\
&= \phi'\bigl(A_0^n\bigr) + \phi'\bigl(A_0^n + 1\bigr) + \dots + \phi'\bigl(A_0^n + 2^n - 1\bigr) \geqslant 2^n \bigl\|A_0^n\bigr\| - 2^n\varepsilon \\
&\geqslant 2^n\delta - 2^n\varepsilon = 2^n\frac{\delta}{2} > 1,
\end{aligned}$$

a contradiction. $\square$

2.4. Idempotent ultrafilters

Recall that $\beta\omega$, the Čech–Stone compactification of the set of natural numbers, is the set of all ultrafilters on ω. We consider $\beta\omega$ as the topological space with the basis consisting of all $\{\mathcal{U} \in \beta\omega\colon\ A \in \mathcal{U}\}$ for $A \subset \omega$. One can define an addition operation on $\beta\omega$, which extends the ordinary addition of natural numbers, in the following way. If $\mathcal{U}, \mathcal{V} \in \beta\omega$, then

$$\mathcal{U} + \mathcal{V} = \bigl\{A \subset \omega\colon\ \{n \in \omega\colon\ A - n \in \mathcal{U}\} \in \mathcal{V}\bigr\}.$$

It is known that $\beta\omega$ with this addition is a left-topological semigroup. We say that $\mathcal{U} \in \beta\omega$ is *idempotent* if $\mathcal{U} + \mathcal{U} = \mathcal{U}$. (For more properties of addition on $\beta\omega$ see e.g. [13, Chapter II].)

3. Analytic P-ideals

In [7, Theorem 2.1], the authors proved an iterated version of Ramsey's theorem for the ideal of statistical density zero sets. We will need an analogous result for every analytic P-ideal. By $[\omega]^2$ we mean a family of all two-element subsets of ω, i.e. $[\omega]^2 = \{\{x, y\}\colon\ x, y \in \omega,\ x \neq y\}$.

Theorem 3.1. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal. Then for every coloring $[\omega]^2 = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $\delta = \delta(r)$ and $i \leqslant r$ with*

$$\Bigl\|\bigl\{x \in \omega\colon\ \bigl\|\bigl\{y \in \omega\colon\ \bigl\|\bigl\{z \in \omega\colon\ \{x, y\}, \{x, z\}, \{y, z\} \in C_i\bigr\}\bigr\|_\phi \geqslant \delta\bigr\}\bigr\|_\phi \geqslant \delta\bigr\}\Bigr\|_\phi \geqslant \delta.$$

Proof. Let $S = \omega^{<\omega}$ (i.e. S is the family of all finite sequences of natural numbers) and $S_n = \omega^{\leqslant n}$ (i.e. S_n is the family of all sequences of natural numbers of length less or equal to n). If $s = \langle s_0, s_1, \ldots, s_{n-1}\rangle \in S$ then $|s| = n$ (the length of s). By $\emptyset \in S$ we denote the empty sequence.

Let $M = \|\omega\|_\phi$. We claim that there is a family of sets $\{A_n(s) \subset \omega\colon\ n \in \omega \text{ and } s \in S_n\}$ and integers $\{i_n \in \{1, \ldots, r\}\colon\ n \in \omega\}$ such that for all $n \geqslant 0$ and $s \in S_n$:

(1) $\|A_n(s)\|_\phi \geqslant M/r^n$ if $s_i \in A_{n-1}(s \restriction i)$ for every $i < |s|$, and

(2) $\begin{cases} A_n(s) \subset A_{n-1}(s) & \text{if } |s| < n; \\ A_n(s) \subset A_{n-1}(s \restriction n-1) & \text{if } |s| = n, \end{cases}$ and

(3) $\{s_{n-1}, s_n\} \in C_{i_n}$ for each $s_0 \in A_n(\emptyset), s_1 \in A_n(\langle s_0\rangle), \ldots, s_n \in A_n(\langle s_0, s_1, \ldots, s_{n-1}\rangle)$.

Note that from (2) it follows that:

(2′) if $m \leqslant |s| \leqslant n$ then $A_n(s) \subset A_m(s \restriction m)$ for each $0 \leqslant m \leqslant n$.

We will build families $\{A_n(s) \subset \omega\colon\ n \in \omega \text{ and } s \in S_n\}$, and $\{i_n \in \{1, \ldots, r\}\colon\ n \in \omega\}$ by induction on n. Let $A_0(\emptyset) = \omega$. Suppose we have found sets $A_m(s)$ for each $m < n$ and $|s| \leqslant m$ such that conditions (1), (2) and (3) hold.

We define sets $A_n(s)$ in 3 steps. At first we define $A_n(s)$ for $|s| = n - 1$. Then, using backward induction on the length of s, we define $A_n(s)$ for $|s| < n - 1$. In the last step we define i_n and $A_n(s)$ for $|s| = n$.

Step 1. For every $s_0 \in A_{n-1}(\emptyset), s_1 \in A_{n-1}(\langle s_0\rangle), \ldots, s_{n-2} \in A_{n-1}(\langle s_0, \ldots, s_{n-3}\rangle)$ there are $A_n(\langle s_0, \ldots, s_{n-2}\rangle) \subset A_{n-1}(\langle s_0, \ldots, s_{n-2}\rangle)$ and $j_n(\langle s_0, \ldots, s_{n-2}\rangle) \in \{1, \ldots, r\}$ such that

(1) $\|A_n(\langle s_0, \ldots, s_{n-2}\rangle)\|_\phi \geqslant M/r^n$, and
(2) $\|\{w \in A_{n-1}(\langle s_0, \ldots, s_{n-2}\rangle)\colon\ \{s_{n-1}, w\} \in C_{j_n(\langle s_0, \ldots, s_{n-2}\rangle)}\}\|_\phi \geqslant M/r^n$ for every $s_0 \in A_{n-1}(\emptyset), s_1 \in A_{n-1}(\langle s_0\rangle), \ldots, s_{n-2} \in A_{n-1}(\langle s_0, \ldots, s_{n-3}\rangle)$ and $s_{n-1} \in A_n(\langle s_0, \ldots, s_{n-2}\rangle)$.

Indeed, for every $s = \langle s_0, \ldots, s_{n-2}\rangle$, $t \in A_{n-1}(s)$ and $j = 1, \ldots, r$ let $A_n^j(t) = \{w \in A_{n-1}(s)\colon\ \{t, w\} \in C_j\}$. Since $A_{n-1}(s) \setminus \{t\} = \bigcup_{j=1}^r A_n^j(s)$ so there is $j(t) \in \{1, \ldots, r\}$ such that $\|A_n^{j(t)}(t)\|_\phi \geqslant M/r^n$. Let

$A_n^j = \{t \in A_{n-1}(s)\colon\ j(t) = j\}$. Since $A_{n-1}(s) = \bigcup_{j=1}^r A_n^j$ so there is $j \in \{1, \ldots, r\}$ such that $\|A_n^j\|_\phi \geqslant M/r^n$. We put $A_n(s) = A_n^j$ and $j_n(s) = j$.

We put $A_n(s) = \emptyset$ for all $s \in S$ ($|s| = n-1$) such that there is $i < n-1$ with $s_i \notin A_{n-1}(s \restriction i)$.

Step 2 *(inductive).* Suppose that we have defined $A_n(s)$ for $m+1 \leqslant |s| \leqslant n-1$.

Then for every $s_0 \in A_{n-1}(\emptyset), s_1 \in A_{n-1}(\langle s_0\rangle), \ldots, s_{m-1} \in A_{n-1}(\langle s_0, \ldots, s_{m-2}\rangle)$ there are $A_n(\langle s_0, \ldots, s_{m-1}\rangle) \subset A_{n-1}(\langle s_0, \ldots, s_{m-1}\rangle)$ and $j_n(\langle s_0, \ldots, s_{m-1}\rangle) \in \{1, \ldots, r\}$ such that

(1) $\|A_n(\langle s_0, \ldots, s_{m-1}\rangle)\|_\phi \geqslant M/r^n$, and
(2) $\|\{w \in A_{n-1}(\langle s_0, \ldots, s_{n-2}\rangle)\colon\ \{s_{n-1}, w\} \in C_{j_n(\langle s_0,\ldots,s_{m-1}\rangle)}\}\|_\phi \geqslant M/r^n$ for every $s_0 \in A_{n-1}(\emptyset), s_1 \in A_{n-1}(\langle s_0\rangle), \ldots, s_{m-1} \in A_{n-1}(\langle s_0, \ldots, s_{m-2}\rangle)$ and $s_m \in A_n(\langle s_0, \ldots, s_{m-1}\rangle), \ldots, s_{n-1} \in A_n(\langle s_0, \ldots, s_{n-2}\rangle)$.

Indeed, let $s = \langle s_0, s_1, \ldots, s_{m-1}\rangle$ be any sequence with $s_0 \in A_{n-1}(\emptyset), s_1 \in A_{n-1}(\langle s_0\rangle), \ldots, s_{m-1} \in A_{n-1}(\langle s_0, s_1, \ldots, s_{m-1}\rangle)$. For each $t \in A_{n-1}(s)$ let $j(t) = j_n(\langle s_0, s_1, \ldots, s_{m-1}, t\rangle)$. From our inductive hypothesis we know that

$$\left\|\left\{w \in A_{n-1}\left(\langle s_0, \ldots, s_{m-1}, t, s_{m+1}, \ldots, s_{n-2}\rangle\right)\colon\ \{s_{n-1}, w\} \in C_{j(t)}\right\}\right\|_\phi \geqslant M/r^n$$

for every $s_{m+1} \in A_n(\langle s_0, \ldots, s_{m-1}, t\rangle), \ldots, s_{n-1} \in A_n(\langle s_0, \ldots, s_{m-1}, t, s_{m+1}, \ldots, s_{n-2}\rangle)$. Put $A_n^j = \{t \in A_{n-1}(s)\colon\ j(t) = j\}$. Since $\|A_{n-1}(s)\|_\phi = M/r^{n-1}$ and $A_{n-1}(s) = \bigcup_{j=1}^r A_n^j$ so there is $j \in \{1, \ldots, r\}$ such that $\|A_n^j\|_\phi \geqslant M/r^n$. We put $A_n(s) = A_n^j$ and $j_n(s) = j$.

We put $A_n(s) = \emptyset$ for all $s \in S$ ($|s| = m$) such that there is $i < m$ with $s_i \notin A_{n-1}(s \restriction i)$.

Step 3. Put $i_n = j(\emptyset)$ and $A_n(\langle s_0, \ldots, s_{n-1}\rangle) = \{t \in A_{n-1}(\langle s_0, \ldots, s_{n-2}\rangle)\colon\ \{s_{n-2}, t\} \in C_{i_n}\}$.

This finishes the construction of families $\{A_n(s) \subset \omega\colon\ n \in \omega \text{ and } s \in S_n\}$ and $\{i_n \in \{1, \ldots, r\}\colon\ n \in \omega\}$.

Now, suppose that $\{A_n(s) \subset \omega\colon\ n \in \omega \text{ and } s \in S_n\}$ and $\{i_n \in \{1, \ldots, r\}\colon\ n \in \omega\}$ fulfill conditions (1), (2′) and (3). Fix $\delta = M/r^{r+1}$. By the pigeonhole principle there are $1 \leqslant a < b \leqslant r+1$ with $i_a = i_b = i$. For every $s_0 \in A_b(\emptyset), s_1 \in A_b(\langle s_0\rangle), \ldots, s_b \in A_b(\langle s_0, s_1, \ldots, s_{b-1}\rangle)$ we have $\{s_{b-1}, s_b\} \in C_i$. Since $A_b(\langle s_0, s_1, \ldots, s_{b-2}\rangle) \subset A_a(\langle s_0, s_1, \ldots, s_{a-1}\rangle)$, $\{s_{a-1}, s_{b-1}\} \in C_i$. Since $A_b(\langle s_0, s_1, \ldots, s_{b-1}\rangle) \subset A_a(\langle s_0, s_1, \ldots, s_{a-1}\rangle)$, $\{s_{a-1}, s_b\} \in C_i$. (The inclusions follow from (2′).) Since $\|A_b(s \restriction m)\|_\phi \geqslant \delta$ for each $m \leqslant b$, we are done. $\square$

As a corollary we get a strengthening of Theorem 1.1.

Theorem 3.2. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal with $\|\cdot\|_\phi$ invariant under translations. Then for every coloring $\omega = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $\delta = \delta(r)$ and $i \leqslant r$ with*

$$\left\|\left\{x \in \omega\colon\ \left\|\{y \in \omega\colon\ x, y, x+y \in C_i\}\right\|_\phi \geqslant \delta\right\}\right\|_\phi \geqslant \delta.$$

Proof. Define a new coloring $[\omega]^2 = D_1 \cup \cdots \cup D_r$ by $\{x, y\} \in D_i \Leftrightarrow |y - x| \in C_i$. By Theorem 3.1 there are $\delta = \delta(r)$ and $i \leqslant r$ such that

$$\left\|\left\{x \in \omega\colon\ \left\|\left\{y \in \omega\colon\ \left\|\left\{z \in \omega\colon\ |x-y|, |x-z|, |y-z| \in C_i\right\}\right\|_\phi \geqslant \delta\right\}\right\|_\phi \geqslant \delta\right\}\right\|_\phi \geqslant \delta.$$

Take any $x \in \omega$ such that

$$\left\|\left\{y \in \omega\colon\ \left\|\left\{z \in \omega\colon\ |x-y|, |x-z|, |y-z| \in C_i\right\}\right\|_\phi \geqslant \delta\right\}\right\|_\phi \geqslant \delta.$$

Then

$$\left\|\left\{y > x\colon\ \left\|\{z > y\colon\ y-x, z-x, z-y \in C_i\}\right\|_\phi \geqslant \delta\right\}\right\|_\phi \geqslant \delta.$$

Since $\|\cdot\|_\phi$ is invariant under translations, so

$$\left\|\left\{y-x\colon\ \left\|\{z-y\colon\ y-x, z-x, z-y\in C_i\}\right\|_\phi\geqslant\delta\right\}\right\|_\phi\geqslant\delta$$

and this finishes the proof. □

Remark. In the classical version of Schur's theorem nothing prevents the case $x=y$. Since for any reasonably defined ideal $\mathcal{I}$ every set $A\notin\mathcal{I}$ has at least two elements, from Corollary 4.3 we get that for every finite coloring of ω there are $x\neq y$ with $\{x,y,x+y\}$ monochromatic, which is Theorem 1 from [3]. (Clearly, it can be also deduced from Theorem 1.2.)

4. Analytic P-ideals with the Bolzano–Weierstrass property

The following lemma was formulated by Bergelson and Hindman for the ideal of statistical density 0 sets (see [2, Le. 1.1]).

Lemma 4.1. *Let $\mathcal{I}$ be an ideal which is invariant under translations. There exists an idempotent $\mathcal{U}\in\beta\omega$ with $\mathcal{U}\cap\mathcal{I}=\emptyset$.*

Proof. Let $A=\{\mathcal{U}\in\beta\omega\colon\ \mathcal{I}\cap\mathcal{U}=\emptyset\}$. Then A is non-empty and closed, so compact in $\beta\omega$. Moreover, if $\mathcal{U},\mathcal{V}\in A$ but $\mathcal{U}+\mathcal{V}\notin A$ then there is a $B\in\mathcal{I}$ such that

$$\left\{n\in\omega\colon\ (B-n)\in\mathcal{U}\right\}\in\mathcal{V}.$$

Since $\mathcal{V}$ is non-empty, there exists $n\in\omega$ with $B-n\in\mathcal{U}$. But $B\in\mathcal{I}$ and $\mathcal{I}$ is invariant under translations—a contradiction with $\mathcal{U}\cap\mathcal{I}=\emptyset$. Thus $A+A\subset A$, and consequently A is a compact left-topological semigroup. By Auslander–Ellis theorem (see e.g. [13, Section 15, Lemma 3]) there is an idempotent $\mathcal{U}\in A$. □

Remark. In fact in the proof of Lemma 4.1 (and consequently in Corollary 4.3) we use slightly weaker assumption than invariance of $\mathcal{I}$. We can assume, for example, that for each $A\in\mathcal{I}$

$$\{n\in\omega\colon\ A+n\notin\mathcal{I}\}\in\mathcal{I}.$$

Lemma 4.2. *Suppose that $\mathcal{I}$ is an ideal and there exists an idempotent $\mathcal{U}\in\beta\omega$ with $\mathcal{U}\cap\mathcal{I}=\emptyset$. If $\omega=C_1\cup C_2\cup\cdots\cup C_r$ then there is an $i\leqslant r$ with*

$$\left\{n\in C_i\colon\ C_i\cap(C_i-n)\notin\mathcal{I}\right\}\notin\mathcal{I}.$$

Proof. This lemma follows from the standard argument. We recall it here for a completeness.

Let $\mathcal{U}\in\beta\omega$ be as required. Take $i\leqslant r$ with $C_i\in\mathcal{U}=\mathcal{U}+\mathcal{U}$. Then $\{n\in\omega\colon\ C_i-n\in\mathcal{U}\}\in\mathcal{U}$, hence $\{n\in\omega\colon\ C_i\cap(C_i-n)\in\mathcal{U}\}\in\mathcal{U}$. Finally $\{n\in C_i\colon\ C_i\cap(C_i-n)\in\mathcal{U}\}\in\mathcal{U}$. □

From Lemmas 4.1 and 4.2 follows

Corollary 4.3. *Let $\mathcal{I}$ be an ideal which is invariant under translations. If $\omega=C_1\cup C_2\cup\cdots\cup C_r$ then there is an $i\leqslant r$ with*

$$\left\{n\in C_i\colon\ C_i\cap(C_i-n)\notin\mathcal{I}\right\}\notin\mathcal{I}.$$

The following generalizations of Schur's theorem hold for some subclasses of analytic ideals. Note that in Theorems 4.4 and 4.5 the constant δ does not depend on r.

Theorem 4.4. *Let $\mathcal{I} = \mathrm{Fin}(\phi)$ be an F_σ ideal which is invariant under translations. If $\omega = C_1 \cup C_2 \cup \cdots \cup C_r$ then there is an $i \leqslant r$ with*

$$\phi\big(\big\{x \in \omega\colon\ \phi\big(\{y \in \omega\colon\ x, y, x+y \in C_i\}\big) = \infty\big\}\big) = \infty.$$

Proof. Apply Corollary 4.3 and note that $\phi(A) = \infty$ for every $A \notin \mathcal{I}$. □

Theorem 4.5. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal such that $\|\cdot\|_\phi$ is invariant under translations. Then $\mathcal{I}$ has the* BW *property if and only if there exists $\delta > 0$ such that for every $r \in \omega$ and every coloring $\omega = C_1 \cup C_2 \cup \cdots \cup C_r$ there is $i \leqslant r$ with*

$$\big\|\big\{x \in \omega\colon\ \big\|\{y \in \omega\colon\ x, y, x+y \in C_i\}\big\|_\phi \geqslant \delta\big\}\big\|_\phi \geqslant \delta.$$

Proof. ($\Rightarrow$). By Theorem 2.3(3), there is $\delta > 0$ such that

$$\mathcal{I}_\delta = \big\{A \subset \omega\colon \text{ there exists a partition } A = A_1 \cup \cdots \cup A_n \text{ such that } \|A_i\|_\phi \leqslant \delta \text{ for every } i \leqslant n\big\}$$

is a proper ideal extending $\mathcal{I}$. Now it is enough to apply Corollary 4.3 to the ideal $\mathcal{I}_\delta$ (since $\|A\|_\phi \geqslant \delta$ for every $A \notin \mathcal{I}_\delta$).

($\Leftarrow$). Let $\omega = A_1 \cup \cdots \cup A_n$. Then there is $i \leqslant n$ with

$$\big\|\big\{x \in \omega\colon\ \big\|\{y \in \omega\colon\ x, y, x+y \in A_i\}\big\|_\phi \geqslant \delta\big\}\big\|_\phi \geqslant \delta.$$

Hence $\|A_i\|_\phi \geqslant \delta$, so $\mathcal{I}$ has the BW property by Theorem 2.3. □

Remark. If $\mathcal{U} \in \beta\omega$ is idempotent then $(\mathcal{U} + \mathcal{U}) + \mathcal{U} = \mathcal{U}$. Thus results of this section can be extended to the case of sums of three elements. For instance,

$$\big\{x \in \omega\colon\ \big\{y \in \omega\colon\ \{z \in \omega\colon\ x, y, z, x+y, x+z, y+z, x+y+z \in C_i\} \notin \mathcal{I}\big\} \notin \mathcal{I}\big\} \notin \mathcal{I}.$$

And by induction one can extend it to the case of sums of n-elements for every $n \in \omega$.

Remark. It is possible to prove Theorem 4.5 using a variant of iterated version of Ramsey's theorem (result analogous to Theorem 3.1) with the constant δ independent of the number of colors. In [5], it is proved that this kind of iterated version of Ramsey's theorem holds for every analytic P-ideal with the BW property.

5. Analytic P-ideals generated by submeasures with the Δ property

In this section we are interested in another generalization of a result from [2], which works also for ideals without the Bolzano–Weierstrass property. First, we need a Khintchine recurrence theorem for submeasures. We follow the proof from [14].

Theorem 5.1. *Let ϕ be a submeasure defined on a space S. Let A_i, $i = 0, 1, \ldots,$ be an infinite sequence of sets in S. Suppose also that $m \in [0, 1]$ is such that for every $M \in \omega$ there is a measure $\phi' \leqslant 1$ defined on an algebra containing sets $A_0, \ldots, A_M$ with $\phi' \leqslant \phi$ and $\phi'(A_i) \geqslant m$ for all $i \leqslant M$. Then for any $\lambda < 1$ there exist $i < j$ such that*

$$\phi(A_i \cap A_j) \geqslant \lambda m^2.$$

Moreover, we can assume that $i, j \leqslant N$, where $N = N(m, \lambda)$ depends only on m and λ.

Proof. First we claim that there are $N = N(m, 1/3)$ and two sets A_i and A_j ($i < j \leqslant N$) such that

$$\phi(A_i \cap A_j) \geqslant \frac{1}{3} m^2.$$

Suppose on the contrary that $\phi(A_i \cap A_j) < \frac{1}{3}m^2$ for every $i < j$. Let

$$F_0 = A_0 \quad \text{and} \quad F_i = A_i \setminus \bigcup_{k=0}^{i-1} (A_i \cap F_k) \quad \text{for } i > 0.$$

Clearly all sets F_i are pairwise disjoint and $F_i \subset A_i$ for each i.

Fix $M \in \omega$. There is a measure ϕ' with

$$\begin{aligned}\phi'(F_i) &= \phi'\left(A_i \setminus \bigcup_{k=0}^{i-1} A_i \cap F_k\right) = \phi'(A_i) - \phi'\left(\bigcup_{k=0}^{i-1} A_i \cap F_k\right)\\ &\geqslant m - \phi\left(\bigcup_{k=0}^{i-1} A_i \cap F_k\right) > m - i\frac{1}{3}m^2\end{aligned}$$

for each $i \leqslant M$. Thus

$$1 \geqslant \phi'(S) \geqslant \sum_{i=0}^{M} \phi'(F_i) > (M+1)m - \frac{M(M+1)}{2}\frac{1}{3}m^2 = \frac{(M+1)m}{6}(6 - Mm).$$

On the other hand, if $3/m - 1 \leqslant M \leqslant 3/m$ then

$$\frac{(M+1)m}{6}(6 - Mm) \geqslant \frac{(M+1)m}{2} \geqslant \frac{3}{m}\frac{m}{2} > 1,$$

a contradiction. This finishes the proof of our first claim (for $N = N(m, 1/3) = 3/m$).

Consider the product space S^r ($r \in \omega$) with submeasure ϕ_r defined by a formula

$$\phi_r(U) = \inf\left\{ \sum_{i=0}^{n} \prod_{t=0}^{r-1} \phi\big(X_i^t\big) \colon U \subset \bigcup_{i=0}^{n} X_i^0 \times X_i^1 \times \cdots \times X_i^{r-1} \right\}$$

for each $U \subset S^r$.

It is not difficult to check that the submeasure ϕ_r and a sequence of sets $A_0^r, A_1^r, \ldots$ satisfy the hypotheses of the theorem (indeed, for $m^r \in [0,1]$ and $M \in \omega$ take ϕ_r' to be a product measure of r measures ϕ').

Applying the previous claim to the submeasure ϕ_r and the sequence A_k^r there are $i < j < N(m^r, 1/3)$ such that

$$\frac{1}{3}\big(m^r\big)^2 < \phi_r\big(A_i^r \cap A_j^r\big) = \phi_r\big((A_i \cap A_j)^r\big) = \big(\phi(A_i \cap A_j)\big)^r.$$

Thus $\phi(A_i \cap A_j) \geqslant (\frac{1}{3})^{1/r} m^2$. To finish the proof it is enough to fix any r such that $(1/3)^{1/r} \geqslant \lambda$. Then $N = N(m, \lambda) = N(m^r, 1/3)$. $\square$

Corollary 5.2. *Let $\phi \in \Delta$, $\varepsilon > 0$, $A \subset \omega$ and $n_k \in \mathbb{Z}$ ($k \in \omega$). Then*

$$\phi\big((A + n_i) \cap (A + n_j)\big) \geqslant \|A\|_\phi^2 - \varepsilon$$

for some $i, j \leqslant K$ (where $K = K(\|A\|_\phi, \varepsilon)$ depends only on the norm of a set A and ε).

Proof. We will apply Theorem 5.1 to the submeasure ϕ and a sequence $A_k = A + n_k$ ($k \in \omega$).

Let $m = \sqrt{\|A\|_\phi^2 - \varepsilon/2}$ and $M \in \omega$. Since $\phi \in \Delta$ so there is a measure $\phi' \leqslant \phi$ and $\phi' \leqslant 1$ such that

$$\phi'(A + i) \geqslant \|A\|_\phi - \varepsilon'$$

for every $|i| \leqslant N = \max\{|n_0|, \ldots, |n_M|\}$ and $\varepsilon' = \|A\|_\phi - \sqrt{\|A\|_\phi^2 - \varepsilon/2}$. Hence $\phi'(A_k) \geqslant m$ for every $k \leqslant M$. Thus we can apply Theorem 5.1.

Let $\lambda = \frac{\|A\|_\phi^2 - \varepsilon}{\|A\|_\phi^2 - \varepsilon/2}$. Then there are $i < j < N(m, \lambda)$ with

$$\phi(A_i \cap A_j) \geqslant \lambda m^2 = \|A\|_\phi^2 - \varepsilon.$$

Since m and λ depend only on $\|A\|_\phi$ and ε, so we put $K(\|A\|_\phi, \varepsilon) = N(m, \lambda)$. $\square$

Given a subset $S \subset \omega$ denote by $\mathcal{D}(S)$ the difference set

$$\mathcal{D}(S) = \{s_1 - s_2\colon\ s_1 > s_2 \text{ and } s_1, s_2 \in S\}.$$

The following lemma was formulated by Bergelson for the ideal of sets of statistical density 0 (see [1, Prop. 2.2]).

Lemma 5.3. *Let S be an infinite subset of ω. Let $\mathcal{I} = \mathrm{Exh}(\phi)$ for a submeasure $\phi \in \Delta$ with $\|\cdot\|_\phi$ invariant under translations. Then for any $A \subset \omega$ with $A \notin \mathcal{I}$ and $\varepsilon > 0$ there exists $n \in \mathcal{D}(S)$ such that*

$$\left\|A \cap (A - n)\right\|_\phi \geqslant \|A\|_\phi^2 - \varepsilon.$$

Proof. Arrange elements of S in an increasing sequence $\{s_k\colon\ k \in \omega\}$. Let $A_k^l = (A \setminus l) - s_k$ for every $k, l \in \omega$.

Let $K = K(\|A\|_\phi, \varepsilon)$ be as in Corollary 5.2. Then by Corollary 5.2

$$\phi\left(A_{i_l}^l \cap A_{j_l}^l\right) \geqslant \|A \setminus l\|_\phi^2 - \varepsilon = \|A\|_\phi^2 - \varepsilon$$

for some $i_l, j_l \leqslant K(\|A \setminus l\|_\phi, \varepsilon) = K$ and every $l \in \omega$.

Since there are only finitely many pairs $i_l < j_l \leqslant K$ and l can be arbitrarily large, there are $i' < j' \leqslant K$ with $i' = i_l$ and $j' = j_l$ for infinitely many l. Then

$$\begin{aligned}\phi\left(\left((A - s_{i'}) \cap (A - s_{j'})\right) \setminus l\right) &\geqslant \phi\left(\left(\left(A \setminus (l + s_K)\right) - s_{i'}\right) \cap \left(\left(A \setminus (l + s_K)\right) - s_{j'}\right)\right) \\ &= \phi\left(A_{i'}^{l+s_K} \cap A_{j'}^{l+s_K}\right) \geqslant \|A\|_\phi^2 - \varepsilon\end{aligned}$$

for infinitely many l. Thus

$$\left\|(A - s_{i'}) \cap (A - s_{j'})\right\|_\phi \geqslant \|A\|_\phi^2 - \varepsilon.$$

Since $\|\cdot\|_\phi$ is invariant under translations,

$$\|A\|_\phi^2 - \varepsilon \leqslant \left\|(A - s_{i'}) \cap (A - s_{j'})\right\|_\phi = \left\|A \cap \left(A - (s_{j'} - s_{i'})\right)\right\|_\phi,$$

which finishes the proof. $\square$

Lemma 5.4 is essentially the same as Bergelson's and Hindman's Lemma 2.1 from [2]. In the original paper it was formulated for the ideal of sets of statistical density 0. We provide a proof of our slightly modified version for the completeness.

Lemma 5.4. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ for a submeasure $\phi \in \Delta$ with $\|\cdot\|_\phi$ invariant under translations. Let $\mathcal{U} \in \beta\omega$ be idempotent with $\mathcal{U} \cap \mathcal{I} = \emptyset$. Let $A \in \mathcal{U}$ and $\varepsilon > 0$. Then*

$$\left\{x \in A\colon\ A - x \in \mathcal{U} \text{ and } \left\|A \cap (A - x)\right\|_\phi \geqslant \|A\|_\phi^2 - \varepsilon\right\} \in \mathcal{U}.$$

Proof. For any $S \subset \omega$ define

$$\mathrm{FS}(S) = \left\{ \sum F\colon\ F \text{ is a finite non-empty subset of } S \right\}.$$

It is well known that for any $\mathcal{U} \in \beta\omega$ with $\mathcal{U} + \mathcal{U} = \mathcal{U}$ and $B \in \mathcal{U}$ there exists an infinite $C \subset B$ such that $\mathrm{FS}(C) \subset B$ (see e.g. [13, Section 15, Lemma 4].)

Let

$$B = \left\{ x \in \omega\colon\ \left\| A \cap (A - x) \right\|_\phi \geqslant \|A\|_\phi^2 - \varepsilon \right\}.$$

Since $A \in \mathcal{U}$ and $\mathcal{U} + \mathcal{U} = \mathcal{U}$, $\{x \in \omega\colon\ A - x \in \mathcal{U}\} \in \mathcal{U}$. So, it is enough to show that $B \in \mathcal{U}$ (then $A \cap B \cap \{x \in \omega\colon\ A - x \in \mathcal{U}\} \in \mathcal{U}$ is as required).

Suppose that $B \notin \mathcal{U}$. Then $\omega \setminus B \in \mathcal{U}$, so there exists $C \subset \omega \setminus B$ with $\mathrm{FS}(C) \subset \omega \setminus B$. Fix such $C = \{x_n\colon\ n \in \omega\}$, where x_n is increasing. Let $S = \{\sum_{k=0}^n x_k\colon\ n \in \omega\}$.

By Lemma 5.3 there are $y < z$ in S such that $z - y \in B$. On the other hand, $y = \sum_{k=0}^n x_k$ and $z = \sum_{k=0}^m x_k$ for some $n < m$, so

$$z - y = \sum_{k=n+1}^{m} x_k \in \mathrm{FS}(C) \subset \omega \setminus B,$$

a contradiction. $\square$

By Lemmas 4.1 and 5.4 the following generalization of Theorem 1.2 holds.

Theorem 5.5. *Suppose that $\mathcal{I} = \mathrm{Exh}(\phi)$ is an analytic P-ideal, $\phi \in \Delta$ and $\|\cdot\|_\phi$ is invariant under translations. For any partition of $\omega = C_1 \cup C_2 \cup \cdots \cup C_r$, some C_i having $\|C_i\|_\phi > 0$ satisfies for any $\varepsilon > 0$,*

$$\left\| \left\{ x \in \omega\colon\ \left\| \{ y \in \omega\colon\ x, y, x + y \in C_i \} \right\|_\phi \geqslant \|C_i\|_\phi^2 - \varepsilon \right\} \right\|_\phi > 0.$$

6. Examples

6.1. *Erdős–Ulam ideals*

Let $f : \omega \to [0, +\infty)$ be such that

$$\sum_{i=0}^{\infty} f(i) = +\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{f(n)}{\sum_{i\in n} f(i)} = 0,$$

then

$$\mathcal{EU}_f = \left\{ A \subset \omega\colon\ \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i\in n} f(i)} = 0 \right\}$$

is called an *Erdős–Ulam ideal* [8]. The ideal $\mathcal{I}_d$, of statistical density zero sets, is an Erdős–Ulam ideal (generated by any constant positive function f). The ideal $\mathcal{I}_{\log}$ of logarithmic density zero sets also is an Erdős–Ulam ideal, where

$$\mathcal{I}_{\log} = \left\{ A \subset \omega\colon\ \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} \frac{1}{i}}{\sum_{i\in n} \frac{1}{i}} = 0 \right\} = \left\{ A \subset \omega\colon\ \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} \frac{1}{i}}{\log n} = 0 \right\}.$$

Every Erdős–Ulam ideal $\mathcal{EU}_f$ is an analytic P-ideal of the form $\mathrm{Exh}(\phi_f)$, where

$$\phi_f(A) = \sup_{n\in\omega} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i\in n} f(i)} \quad \text{and} \quad \|A\|_{\phi_f} = \limsup_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i\in n} f(i)}.$$

There are Erdős–Ulam ideals which are not invariant under translations (hence their norms are not invariant under translations), and do not have the Δ property, e.g. $\mathcal{EU}_f$ generated by $f(n) = (-1)^n + 1$. The next proposition gives a sufficient condition for a function f to define a norm $\|\cdot\|_{\phi_f}$ which is invariant under translations and has the Δ property.

Proposition 6.1. *Let $\mathcal{I} = \mathrm{Exh}(\phi_f)$ be an Erdős–Ulam ideal. If*

$$\lim_{n\to\infty} \frac{\sum_{i\in n} |f(i+k) - f(i)|}{\sum_{i\in n} f(i)} = 0 \quad \text{for every } k \in \mathbb{Z} \tag{$\star$}$$

then $\|\cdot\|_{\phi_f}$ is invariant under translations, and $\phi_f \in \Delta$. (We assume that $f(i) = 0$ for any $i < 0$.)

Proof. The fact that $\|\cdot\|_{\phi_f}$ is invariant under translations follows from an easy calculation. We show below that $\phi_f \in \Delta$.

Fix $\varepsilon > 0$, $A \subset \omega$ and $M \in \omega$. By ($\star$) it is possible to find an $N \in \omega$ such that

$$\frac{\sum_{i\in n} |f(i+k) - f(i)|}{\sum_{i\in n} f(i)} < \frac{\varepsilon}{3} \quad \text{and} \quad \frac{\sum_{i=n-k}^{n-1} f(i)}{\sum_{i=0}^{n-1} f(i)} < \frac{\varepsilon}{3}$$

for each $k \in [-M, M]$ and $n \geqslant N$.

There exists $n' \geqslant N$ with

$$\frac{\sum_{i\in A\cap n'} f(i)}{\sum_{i\in n'} f(i)} \geqslant \|A\|_{\phi_f} - \frac{\varepsilon}{3}.$$

Then the measure ϕ'_f given by a formula

$$\phi'_f(B) = \frac{\sum_{i\in B\cap n'} f(i)}{\sum_{i\in n'} f(i)}$$

is such that $\phi' \leqslant \phi$ and for each $k \in [0, M]$

$$\begin{aligned}
\phi'_f(A+k) &= \frac{\sum_{i\in A\cap(n'-k)} f(i+k)}{\sum_{i\in n'} f(i)} \\
&= \frac{\sum_{i\in A\cap(n'-k)} (f(i+k) - f(i)) - \sum_{i\in A\cap[n'-k,\dots,n'-1]} f(i) + \sum_{i\in A\cap n'} f(i)}{\sum_{i\in n'} f(i)} \\
&\geqslant -\frac{\varepsilon}{3} - \frac{\varepsilon}{3} + \frac{\sum_{i\in A\cap n'} f(i)}{\sum_{i\in n'} f(i)} \geqslant \|A\|_{\phi_f} - \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\phi'_f(A-k) &= \frac{\sum_{i\in A\cap[k,\dots,n'+k-1]} f(i-k)}{\sum_{i\in n'} f(i)} = \frac{\sum_{i\in A\cap(n'+k)} f(i-k)}{\sum_{i\in n'} f(i)} \\
&= \frac{\sum_{i\in A\cap n'} (f(i-k) - f(i)) + \sum_{i\in A\cap[n',\dots,n'+k-1]} f(i-k) + \sum_{A\cap n'} f(i)}{\sum_{i\in n'} f(i)} \\
&\geqslant -\frac{\varepsilon}{3} + 0 + \frac{\sum_{i\in A\cap n'} f(i)}{\sum_{i\in n'} f(i)} \geqslant \|A\|_{\phi_f} - \varepsilon.
\end{aligned}$$

Thus $\phi_f \in \Delta$. $\square$

Remark. If $\mathcal{I} = \mathrm{Exh}(\phi_f)$ is an Erdős–Ulam ideal and f is monotone then f fulfills the condition ($\star$).

By the above proposition, every Erdős–Ulam ideal with the property ($\star$) satisfies the hypotheses of Theorems 5.1 and 5.5. In particular, Theorems 5.1 and 5.5 hold for the ideal $\mathcal{I}_d$ of statistical density zero sets, and for the ideal $\mathcal{I}_{\log}$ of logarithmic density zero sets.

6.2. Louveau–Veličković ideals

Let $\{n_i\}_{i\in\omega}$ be an increasing sequence of natural numbers. Let I_i be pairwise disjoint intervals on ω such that $|I_i| = 2^{n_i}$. Let ϕ_i be a submeasure on I_i given by

$$\phi_i(A) = \frac{\log_2(|A \cap I_i| + 1)}{n_i}.$$

Then $\phi = \sup_i \phi_i$ is a lower semicontinuous submeasure and $\mathcal{LV}_{\{n_i\}} = \mathrm{Exh}(\phi)$ is called the *Louveau–Veličković ideal* [10].

Proposition 6.2. *Let $\mathcal{LV}_{\{n_i\}} = \mathrm{Exh}(\phi)$ be a Louveau–Veličković ideal. Then $\|\cdot\|_\phi$ is invariant under translations.*

Proof. Let $A \subset \omega$. It is not difficult to see that $\|A\|_\phi = \lim_{n\to\infty} \sup_{i>n} \phi_i(A)$. Hence there is a subsequence $(i_m)_{m\in\omega}$ such that $\|A\|_\phi = \lim_{m\to\infty} \phi_{i_m}(A)$.

Let $k \in \mathbb{Z}$. Then

$$\frac{\log_2(|A \cap I_{i_m}| + 1 - k)}{n_{i_m}} \leqslant \phi_{i_m}(A+k) \leqslant \frac{\log_2(|A \cap I_{i_m}| + 1 + k)}{n_{i_m}}$$

for every $m \in \omega$. Since

$$\lim_{m\to\infty} \frac{\log_2(|A \cap I_{i_m}| + 1 \pm k)}{n_{i_m}} = \lim_{m\to\infty} \phi_{i_m}(A),$$

so $\|A+k\|_\phi \geqslant \|A\|_\phi$. Moreover, $\|A\|_\phi = \|(A+k)-k\|_\phi \geqslant \|A+k\|_\phi$. Thus $\|A+k\|_\phi = \|A\|_\phi$. $\square$

In [6], the authors showed that Louveau–Veličković ideals have the BW property. Hence these ideals satisfy the hypotheses of Theorem 4.5.

6.3. Summable ideals

For $f : \omega \to \mathbb{R}^+$ such that $\sum_{n\in\omega} f(n) = +\infty$ we define the *summable ideal* [11] by

$$\mathcal{I}_f = \left\{ A \subset \omega\colon \sum_{n\in A} f(n) < \infty \right\}.$$

It is not difficult to see that every summable ideal is F_σ. Moreover, an easy calculation shows that if f fulfills the condition $(\star\star)$:

$$\sum_{i\in\omega} \bigl| f(i+k) - f(i) \bigr| < +\infty \quad \text{for every } k \in \mathbb{Z}, \tag{$\star\star$}$$

then $\mathcal{I}_f$ is invariant under translations. Thus, for example, every summable ideal $\mathcal{I}_f$ with f monotone satisfies the hypotheses of Theorem 4.4.

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