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Bounds on certain classes of Kronecker and q-binomial coefficients

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ABSTRACT

We present a lower bound on the Kronecker coefficients for tensor squares of the symmetric group via the characters of S_n, which we apply to obtain various explicit estimates. Notably, we extend Sylvester's unimodality of q-binomial coefficients $\binom{n}{k}_q$ as polynomials in q to derive sharp bounds on the differences of their consecutive coefficients. We then derive effective asymptotic lower bounds for a wider class of Kronecker coefficients.

1. Introduction

The *Kronecker coefficients* are perhaps the most challenging, deep and mysterious objects in Algebraic Combinatorics. Universally admired, they are beautiful, unapproachable and barely understood. For decades since they were introduced by Murnaghan in 1938, the field lacked tools to study them, so they remained largely out of reach. However, in recent years a flurry of activity led to significant advances, spurred in part by the increased interest and applications to other fields.

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In this paper, we focus on lower bounds for the Kronecker coefficients. We are motivated by applications to the *q-binomial (Gaussian) coefficients*, and by connections to the *Geometric Complexity Theory* (see §7.1). The tools are based on technical advances in combinatorial representation theory obtained in recent years, see [4,6,7,13,28], and our own series of papers [18–21]. In fact, here we give several extensions of our earlier work.

The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are defined by:

$$\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) \chi^\nu , \quad \text{where} \ \ \lambda, \mu \vdash n , \tag{1.1}$$

where χ^α denotes the character of the irreducible representation $\mathbb{S}^\alpha$ of S_n indexed by partition $\alpha \vdash n$. They are integer and nonnegative by definition, have full S_3 symmetry, and satisfy a number of further properties (see §2.2). In contrast with their "cousins", *Littlewood–Richardson (LR) coefficients*, they lack a combinatorial interpretation or any meaningful positive formula, and thus are harder to compute and to estimate.

Our first result is a lower bound of the Kronecker coefficients $g(\lambda, \mu, \mu)$ for multiplicities in tensor squares of self-conjugate partitions:

Theorem 1.1. *Let $\mu = \mu'$ be a self-conjugate partition and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$ be the partition of its principal hooks. Then:*

$$g(\lambda, \mu, \mu) \geq \big| \chi^\lambda[\widehat{\mu}] \big| , \quad \textit{for every} \quad \lambda \vdash n ,$$

where $\chi^\lambda[\widehat{\mu}]$ denotes the value of the character χ^λ at a permutation of cycle type $\widehat{\mu}$.

While it is relatively easy to obtain various upper bounds on the Kronecker coefficients (see e.g. (2.1)), this is the only general lower bound that we know. The theorem strengthens a qualitative result $g(\lambda, \mu, \mu) \geq 1$ given in [21, Lemma 1.3], used there to prove a special case of the Saxl conjecture (see §7.3). We use the bound to give a new proof of Stanley's Theorem 4.1, from [23].

Our second result is motivated by an application of bounds for Kronecker coefficients to the *q-binomial coefficients*, defined as:

$$\binom{m+\ell}{m}_q = \frac{(q^{m+1}-1)\cdots(q^{m+\ell}-1)}{(q-1)\cdots(q^\ell - 1)} = \sum_{n=0}^{\ell\, m} p_n(\ell, m) q^n ,$$

where $p_n(\ell, m)$ is also the number of partitions of n which fit inside an $\ell \times m$ rectangle. In 1878, *Sylvester* proved *unimodality* of the coefficients:

$$p_0(\ell, m) \leq p_1(\ell, m) \leq \ldots \leq p_{\lfloor \ell\, m/2 \rfloor}(\ell, m) \geq \ldots \geq p_{\ell\, m}(\ell, m) ,$$

see [26]. In [18], we used the Kronecker coefficients to prove *strict unimodality*:

$$p_k(\ell,m) - p_{k-1}(\ell,m) \geq 1, \quad \text{for} \quad 2 \leq k \leq \ell m/2, \ \ell, m \geq 8. \tag{1.2}$$

This result was subsequently improved by Zanello [29] and Dhand [9] (see §7.5). Ignoring constraints of ℓ and m, they prove that the l.h.s. of (1.2) is $\Omega(\sqrt{k})$ and $\Omega(k)$, respectively. Here we substantially strengthen these bounds as follows and give an effective bound on certain Kronecker coefficients.

Theorem 1.2. *There is a universal constant $A > 0$, such that for all $m \geq \ell \geq 8$ and $2 \leq k \leq \ell m/2$, we have:*

$$g((\ell m - k, k), m^\ell, m^\ell) = p_k(\ell,m) - p_{k-1}(\ell,m) > A\,\frac{2^{\sqrt{s}}}{s^{9/4}}, \quad \text{where} \quad s = \min\{2k, \ell^2\}.$$

The proof of the theorem gives an effective bound with $A = 0.004$. The lower bound gives the correct exponential behavior of the difference, but perhaps not the base of the exponent. We also discuss an upper bound in §5.3 (see also §7.5).

The proof of the theorem has several ingredients. We use the above mentioned Stanley's theorem, an extension of analytic estimates in the proof of *Almkvist's Theorem* (Theorem 2.2), and the *monotonicity property* of the Kronecker coefficients (Theorem 2.1). Most crucially, we use the following connection between the Kronecker and q-binomial coefficients:

Lemma 1.3 *(Two coefficients lemma). Let $n = \ell m$, $\tau_k = (n-k, k)$, where $1 \leq k \leq n/2$. Then:*

$$g\big(m^\ell, m^\ell, \tau_k\big) = p_k(\ell,m) - p_{k-1}(\ell,m).$$

This simple but very useful lemma was first proved in a special case in [8], and in full generality in [28, §7] and later in [18], but is implicit in [14,19]. Note that it immediately implies Sylvester's unimodality theorem.

Finally, using this result and the semigroup property for Kronecker coefficients we can give an explicit lower bound for a wider classes of partition triples. Here we state an easy corollary of a general (but technical to state) Theorem 6.1 (see below).

Corollary 1.4. *For any partition $\lambda \vdash n$, let $d(\lambda) = m$ be the size of its Durfee square. Then, for any $k \leq m^2/2$, we have:*

$$g(\lambda, \lambda, (n-k,k)) > C\,\frac{2^{\sqrt{2k}}}{(2k)^{9/4}}, \quad \text{where} \quad C = \frac{\sqrt{27/8}}{\pi^{3/2}}.$$

Here the *Durfee square* is the largest square which fits into Young diagram of the partition. In other words, the "thicker" the partition is, the better lower bound we obtain.

The rest of the paper is structured as follows. We begin with a quick recap of definitions, notations and some basic results we are using (Section 2). We prove Theorem 1.1 in Section 3 and give applications of the theorem in Section 4. We then prove Theorems 1.2 in Section 5 and the more general effective bound on the Kronecker coefficients in Section 6 as Theorem 6.1. We conclude with final remarks and open problems (Section 7).

2. Definitions and basic results

2.1. Partitions and Young diagrams

We adopt the standard notation in combinatorics of partitions and representation theory of S_n, as well as the theory of symmetric functions (see e.g. [12,24]).

Let $\mathcal{P}$ denote the set of integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$. We write $|\lambda| = n$ and $\lambda \vdash n$, for $\lambda_1 + \lambda_2 + \ldots = n$. Let $\mathcal{P}_n$ the set of all $\lambda \vdash n$, and let $\mathrm{P}(n) = |\mathcal{P}_n|$ the number of partitions of n. We use $\ell(\lambda)$ to denote the number of parts of λ, and λ' to denote the conjugate partition. Define addition of partitions $\alpha, \beta \in \mathcal{P}$ to be their addition as vectors:

$$\alpha + \beta = (\alpha_1 + \beta_1,\, \alpha_2 + \beta_2, \ldots)\,.$$

We denote by χ^λ the character of the irreducible representation $\mathbb{S}^\lambda$ of S_n corresponding to λ. Denote by $f^\lambda = \chi^\lambda[1^n]$ the dimension of $\mathbb{S}^\lambda$. Finally, *hooks* of a partition μ are defined by $h_{ij} = \mu_i + \mu'_j - i - j + 1$, and the integers $h_{11}, h_{22}, \ldots$ are called *principal hooks*. When $\mu = \mu'$, the sequence of principal hooks is exactly the partition $\widehat{\mu}$ defined in Theorem 1.1.

2.2. Kronecker coefficients

It is well known that

$$g(\lambda,\mu,\nu) \;=\; \frac{1}{n!} \sum_{\omega\in S_n} \chi^\lambda(\omega)\,\chi^\mu(\omega)\,\chi^\nu(\omega)\,.$$

This implies that Kronecker coefficients have full S_3 group of symmetry:

$$g(\lambda,\mu,\nu) \;=\; g(\mu,\lambda,\nu) \;=\; g(\lambda,\nu,\mu) \;=\; \ldots$$

We will use the following *monotonicity property*:

Theorem 2.1 *([13]). Suppose α, β, γ are partitions of n, such that the Kronecker coefficients $g(\alpha,\beta,\gamma) > 0$. Then for any partitions λ, μ, ν with $|\lambda| = |\mu| = |\nu|$ we have*

$$g(\lambda+\alpha, \mu+\beta, \nu+\gamma) \geq g(\lambda,\mu,\nu)\,.$$

This result is an extension of the *semigroup property* for the Kronecker coefficients, which states that the LHS > 0 if $g(\lambda, \mu, \nu) > 0$. This property was originally attributed to Brion and reproved in [7,13].

We also have the following trivial upper bound (see e.g. [24, Exc. 7.83]):

$$g(\lambda, \mu, \nu) \le \min\{f^\lambda, f^\mu, f^\nu\} \quad \text{for all} \quad \lambda, \mu, \nu \vdash n\,. \tag{2.1}$$

2.3. Partition asymptotics

Denote by $\mathrm{P}'(n) = \mathrm{P}(n) - \mathrm{P}(n-1)$ the number of partitions into parts ≥ 2. We have that $\mathrm{P}'(n) \ge 1$ for all $n \ge 2$. Recall the following *Hardy–Ramanujan* and *Roth–Szekeres* formulas, respectively:

$$\mathrm{P}(n) \sim \frac{1}{4\sqrt{3}\,n}\, e^{\pi\sqrt{\frac{2}{3}\,n}}\,, \quad \mathrm{P}'(n) \sim \frac{\pi}{\sqrt{6\,n}}\,\mathrm{P}(n) \quad \text{as} \quad n \to \infty\,, \tag{2.2}$$

see [22] (see also [10, p. 59]).

Denote by $b_k(n)$ the number of partitions of k into distinct odd parts $\le 2n-1$. We have:

$$\prod_{i=1}^{n} \left(1 + q^{2i-1}\right) \;=\; \sum_{k=0}^{n^2} b_k(n) q^k\,.$$

Theorem 2.2 *(Almkvist). The following sequence is symmetric and unimodal for $n > 26$:*

$$(\lozenge) \quad b_2(n)\,,\, b_3(n)\,,\, \ldots\,,\, b_{n^2-2}(n)\,.$$

3. Proof of Theorem 1.1

Denote by $\chi\!\downarrow$ the restriction of the S_n-representation χ to A_n, and by $\psi\!\uparrow$ the induced S_n-representation of the A_n-representation ψ. We refer to [11, §2.5] for basic results in representation theory of A_n. Recall that if $\nu \ne \nu'$, then $\chi^\nu\!\downarrow\, = \chi^{\nu'}\!\downarrow\, = \psi^\nu$ is irreducible in A_n. Similarly, if $\nu = \nu'$, then $\chi^\nu\!\downarrow\, = \psi^\nu_+ \oplus \psi^\nu_-$, where $\psi^\nu_\pm$ are irreducible in A_n, and are related via $\psi^\nu_+[(12)\pi(12)] = \psi^\nu_-[\pi]$.

Consider now the conjugacy classes of A_n and the corresponding character values. Denote by C^α the conjugacy class of S_n of permutations of cycle type α, and by $\mathcal{D} \subset \mathcal{P}$ the set of partitions into distinct odd parts. We have two cases:

(**1**) For $\alpha \notin \mathcal{D}$, we have C^α is also a conjugacy class of A_n. Then

$$\chi^\nu\!\downarrow[C^\alpha] \;=\; \chi^\nu[C^\alpha] \quad \text{if} \quad \nu \ne \nu'\,,$$

$$\psi^\nu_\pm[C^\alpha] \;=\; \frac{1}{2}\chi^\nu[C^\alpha] \quad \text{if} \quad \nu = \nu'\,.$$

(**2**) For $\alpha \in \mathcal{D}$, we have $C^\alpha = C^\alpha_+ \cup C^\alpha_-$, where $C^\alpha_\pm$ are conjugacy classes of A_n. Then

$$\begin{aligned} \chi^\nu\!\downarrow[C^\alpha_\pm] \,&=\, \chi^\nu[C^\alpha] \quad \text{if} \quad \nu \ne \nu'\,, \\ \psi^\nu_\pm[C^\alpha_\pm] \,&=\, \frac{1}{2}\chi^\nu[C^\alpha] \quad \text{if} \quad \nu = \nu' \quad \text{and} \quad \alpha \ne \widehat{\nu}\,, \\ \psi^\nu_\pm[C^{\widehat{\nu}}_+] - \psi^\nu_\pm[C^{\widehat{\nu}}_-] \,&=\, \pm e_\nu \quad \text{if} \quad \nu = \nu' \quad \text{and} \quad e_\nu = (\widehat{\nu}_1\,\widehat{\nu}_2\,\cdots)^{1/2} > 0\,. \end{aligned}$$

Now, by the Frobenius reciprocity, for every $\mu = \mu'$ we have:

$$\langle \psi^\mu_\pm\!\uparrow, \chi^\alpha\rangle = \langle \psi^\mu_\pm, \chi^\alpha\!\downarrow\rangle\,,$$

which is nonzero exactly when $\alpha = \mu$ and so $\psi^\mu_\pm\!\uparrow = \chi^\mu$. This implies

$$\begin{aligned} g(\lambda,\mu,\mu) \,&=\, \langle \chi^\mu \otimes \chi^\lambda, \chi^\mu\rangle \,=\, \langle \chi^\mu \otimes \chi^\lambda,\, \psi^\mu_\pm\!\uparrow\rangle \,=\, \big\langle (\chi^\mu \otimes \chi^\lambda)\!\downarrow, \psi^\mu_\pm\big\rangle \\ &=\, \big\langle \psi^\mu_+ \otimes \chi^\lambda\!\downarrow,\, \psi^\mu_\pm\big\rangle + \big\langle \psi^\mu_- \otimes \chi^\lambda\!\downarrow,\, \psi^\mu_\pm\big\rangle\,. \end{aligned} \tag{3.1}$$

We can now estimate the Kronecker coefficient in the theorem. First, decompose the following tensor product of the A_n representations:

$$\psi^\mu_+ \otimes \chi^\lambda\!\downarrow \,=\, \oplus_\tau m_\tau\, \psi^\tau\,, \tag{3.2}$$

where ψ^τ are all the irreducible representations of A_n, the coefficients m_τ are their multiplicities in the above tensor product, and τ goes over the appropriate indexing.

Note that for any character χ of S_n and $\pi \in A_n$ we trivially have $\chi\!\downarrow[\pi] = \chi[\pi]$. Evaluating that tensor product on the classes $C^{\widehat{\mu}}_\pm$ gives

$$\big(\psi^\mu_+ \otimes \chi^\lambda\!\downarrow\big)[C^{\widehat{\mu}}_+] - \big(\psi^\mu_+ \otimes \chi^\lambda\!\downarrow\big)[C^{\widehat{\mu}}_-] \,=\, \chi^\lambda\!\downarrow[C^{\widehat{\mu}}_\pm]\Big(\psi^\mu_+[C^{\widehat{\mu}}_+] - \psi^\mu_+[C^{\widehat{\mu}}_-]\Big) \,=\, \chi^\lambda[C^{\widehat{\mu}}]\, e_\nu\,.$$

On the other hand, evaluating the right-hand side of equation (3.2) gives

$$\begin{aligned} \big(\psi^\mu_+ \otimes \chi^\lambda\!\downarrow\big)[C^{\widehat{\mu}}_+] - \big(\psi^\mu_+ \otimes \chi^\lambda\!\downarrow\big)[C^{\widehat{\mu}}_-] \,&=\, \sum_\tau m_\tau\Big(\psi^\tau[C^{\widehat{\mu}}_+] - \psi^\tau[C^{\widehat{\mu}}_-]\Big) \\ = m_{\mu+}\Big(\psi^\nu_+[C^{\widehat{\nu}}_+] - \psi^\nu_+[C^{\widehat{\nu}}_-]\Big) + m_{\mu-}\Big(\psi^\nu_-[C^{\widehat{\nu}}_+] - \psi^\nu_-[C^{\widehat{\nu}}_-]\Big) \,&=\, \big(m_{\mu+} - m_{\mu-}\big)\, e_\nu. \end{aligned}$$

Here we used the fact that all characters are equal at the two classes $C^{\widehat{\mu}}_\pm$, except for the ones corresponding to μ. Equating the evaluations and using $e_\nu > 0$, we obtain

$$m_{\mu+} - m_{\mu-} = \chi^\lambda[C^{\widehat{\mu}}]\,.$$

This immediately implies

$$\max\{m_{\mu+}, m_{\mu-}\} \,\ge\, \Big|\chi^\lambda[C^{\widehat{\mu}}]\Big| \tag{3.3}$$

On the other hand, since all inner products are nonnegative, the equation (3.1) gives

$$g(\lambda,\mu,\mu) \,\geq\, \max\big\{\langle\psi_{+}^{\mu}\otimes\chi^{\lambda}\!\downarrow, \psi_{+}^{\mu}\rangle, \langle\psi_{+}^{\mu}\otimes\chi^{\lambda}\!\downarrow, \psi_{-}^{\mu}\rangle\big\} \,=\, \max\big\{m_{\mu+}, m_{\mu-}\big\}\,,$$

and now equation (3.3) implies the result. □

4. Bounds on Kronecker coefficients via characters

4.1. Stanley's theorem

We give a new proof of the following technical result by Stanley [23, Prop. 11]. Our proof uses Theorem 1.1 and Almkvist's Theorem 2.2. Both results are crucially used in the next section.

Theorem 4.1 *(Stanley). The following polynomial in q is symmetric and unimodal*

$$\binom{2n}{n}_q - \prod_{i=1}^{n}\big(1+q^{2i-1}\big)\,.$$

Proof. Let $\mu=(n^n)$ and $\tau_k=(n^2-k,k)$, where $k\le n^2/2$. By the two coefficients lemma (Lemma 1.3), we have

$$g(\tau_k,\mu,\mu) \,=\, p_k(n,n) - p_{k-1}(n,n)\,.$$

By the Jacobi–Trudi identity and the Murnaghan–Nakayama rule, we have:

$$\chi^{\tau_k}[\widehat{\mu}] \,=\, \chi^{(n^2-k)\circ(k)}[\widehat{\mu}] - \chi^{(n^2-k+1)\circ(k-1)}[\widehat{\mu}] \,=\, b_k(n)-b_{k-1}(n)\,.$$

(cf. [19,21]). Applying Theorem 1.1 with $\lambda=\tau_k$ and μ as above, we have:

$$p_k(n,n)-p_{k-1}(n,n) \,=\, g(\lambda,\mu,\mu) \,\geq\, \chi^{\lambda}[\widehat{\mu}] \,=\, b_k(n)-b_{k-1}(n)\,.$$

Reordering the terms, we conclude

$$p_k(n,n)-b_k(n) \,\geq\, p_{k-1}(n,n)-b_{k-1}(n)\,,$$

which implies unimodality. The symmetry is straightforward. □

4.2. Asymptotic applications

Let $\rho_m=(m,m-1,\ldots,2,1)$ be the *staircase shape*, $n=|\rho_m|=\binom{m+1}{2}$. The coefficient $g(\rho_m,\rho_m,\nu)$ first appeared in connection with the *Saxl conjecture* [21], and was further studied in [28, §8].

For simplicity, let $m = 1 \bmod 2$, so $\widehat{\rho}_m = (2m-1, \ldots, 5, 1)$. Let $\tau_k = (n-k, k)$. Applying Theorem 1.1 and the Murnaghan–Nakayama rule as above, we have

$$g(\rho_m, \rho_m, \tau_k) \, \geq \, \bigl|\chi^{\tau_k}\bigl[\widehat{\rho}_m\bigr]\bigr| \, = \, \mathrm{P}_R(k) - \mathrm{P}_R(k-1)\,,$$

where $\mathrm{P}_R(k)$ is the number of partitions of k into parts from $R = \{1, 5, \ldots, 2m-1\}$.

In the "small case" $k \leq 2m$, by the Roth–Szekeres theorem [22], we have:

$$g(\rho_m, \rho_m, \tau_k) \, \geq \, \mathrm{P}_R(k) - \mathrm{P}_R(k-1) \, \sim \, \frac{\pi\sqrt{2}}{3\,k^{3/2}} e^{\pi\sqrt{k/6}}\,,$$

i.e. independent of n. On the other hand, by equation (2.1), we have

$$g(\rho_m, \rho_m, \tau_k) \, \leq \, f^{\tau_k} \, < \, \frac{n^k}{k!}\,,$$

leaving a substantial gap between the upper and lower bounds. For $k = O(1)$ bounded, Theorem 8.10 in [28], gives

$$g(\rho_m, \rho_m, \tau_k) \, \sim \, m^k \, \sim \, (2\,n)^{k/2} \quad \text{as} \quad n \to \infty\,,$$

suggesting that the upper bound is closer to the truth. In fact, the proof in [28] seems to hold for all $k = o(m)$.

In the "large case" $k = n/2 \sim m^2/4$, the Odlyzko–Richmond result [17, Thm. 3] gives

$$g(\rho_m, \rho_m, \tau_k) \, \geq \, \mathrm{P}_R(k) - \mathrm{P}_R(k-1) \, \sim \, \frac{3^{3/2}}{2^{15/4}\sqrt{\pi}\,m^3} 2^{m/4} \, \sim \, \frac{3^{3/2}}{2^{47/4}\sqrt{\pi}\,k^{3/2}} 2^{\sqrt{k}/2}\,.$$

For the upper bound, equation (2.1) gives

$$g(\rho_m, \rho_m, \tau_k) \, \leq \, f^{\tau_k} \, \lesssim \, \frac{1}{\sqrt{\pi}\,k^{3/2}}\, 4^k\,.$$

4.3. Lower bounds for border equal partitions

Two partitions $\lambda, \mu \vdash n$ are called *s-border equal* if they have equal the first s principal hooks. By $\lambda^{\langle s \rangle}$ denote the partition with the first s rows and s columns removed.

Corollary 4.2. *Let $\lambda, \mu \vdash n$ be s-border equal partitions such that $\mu = \mu'$ is self-conjugate. Denote by $\alpha = \lambda^{\langle s \rangle}$, $\beta = \mu^{\langle s \rangle}$, and let $\widehat{\beta} = (2\beta_1 - 1, 2\beta_2 - 3, \ldots) \vdash n$. Then:*

$$g(\lambda, \mu, \mu) \, \geq \, \bigl|\,\chi^{\alpha}[\widehat{\beta}]\,\bigr|\,.$$

Proof. By the Murnaghan–Nakayama rule, for the s-border equal partitions λ and μ, there is a unique way to fit the first s rim hooks of length $(\widehat{\mu}_1, \ldots, \widehat{\mu}_s)$ into the shape λ. Therefore, we have

$$\left|\, \chi^\lambda[\widehat{\mu}] \,\right| \, = \, \left|\, \chi^\alpha[\widehat{\beta}] \,\right|.$$

Now Theorem 1.1 implies the result. □

Example 4.3. Fix $r \geq 4$ and $s \geq 0$. Consider $\lambda = \big(2r^2 + 2r + s + 1\big)^{s+1}\big(s+1\big)^{2r^2+2r}$, $\mu = \big(2r^2 + 2r + s + 1\big)^{s}(2r + s + 1)^{2r+1}s^{2r^2}$, and observe that $|\lambda| = |\mu|$, $\mu = \mu'$. Furthermore, λ and μ are s-border equal. In notation of the corollary, we have $\alpha = \big(2r^2 + 2r + 1, 1^{2r^2+2r}\big)$, $\beta = \big(2r+1\big)^{2r+1}$, and $\widehat{\beta} = (4r+1, 4r-1, \ldots, 3, 1)$. Using Corollary 4.2, the Murnaghan–Nakayama rule and the Giambelli formula as in [21], we conclude that

$$g(\lambda, \mu, \mu) \, \geq \, \left|\, \chi^\alpha[\widehat{\beta}] \,\right| \, \geq \, b_k(2r+1) \, - \, b_{k-1}(2r+1)\,.$$

We therefore have:

$$g(\lambda, \mu, \mu) \, \geq \, 0.32 \, \frac{2^{\sqrt{2k}}}{(2k)^{9/4}} \, \geq \, \frac{4^r}{3\,(2r)^{9/2}}\,, \qquad \text{where} \quad k \, = \, 2\,r\,(r+1)\,, \tag{4.1}$$

where the inequality (4.1) follows from Theorem 5.3 given in the next section (note that $r \geq 4$ implies $k \geq 26$, assumed in the theorem). We omit the details.

5. Bounds on the q-binomial coefficients

5.1. Analytic estimates

The proof of Almkvist's Theorem 2.2 is based on the following technical results. Denote

$$\vartheta_k(n) \, = \, b_k(n) - b_{k-1}(n)\,,$$

setting $\vartheta_0(n) = b_0(n) = 1$.

Lemma 5.1. *Let $n \geq 28$. For all $k \leq n^2/2$ with $k \neq 2n+1$, we have that*

$$\vartheta_k(n) \geq \vartheta_k(n-1).$$

For $k = 2n+1$ we have that

$$\vartheta_k(n) = \vartheta_k(n-4).$$

Proof. This statement follows directly from the generating function identities. We have that

$$\begin{aligned}\sum_k \vartheta_k(n)q^k &= \sum_k b_k(n)q^k - q\sum_k b_{k-1}(n)q^{k-1} \\ &= \prod_{i=1}^{n}(1+q^{2i-1}) - q\left(\prod_{i=1}^{n}(1+q^{2i-1}) - q^{n^2}\right) \\ &= (1-q)\prod_{i=1}^{n}(1+q^{2i-1}) - q^{n^2+1} \\ &= (1-q)\prod_{i=1}^{n-1}(1+q^{2i-1})\cdot(1+q^{2n-1}) + q^{n^2+1}.\end{aligned}$$

Expanding the last product above as the generating function for $\vartheta_k(n-1)$, we obtain the recursion

$$\vartheta_k(n) = \vartheta_k(n-1) + \vartheta_{k-2n+1}(n-1),$$

where we set $\vartheta_{k'}(n') = 0$ if $k' < 0$. By Almkvist's Theorem, when $n \ge 28$, we have that $\vartheta_{k-2n+1}(n-1) \ge 0$ for $k-2n+1 \ne 2$, so $\vartheta_k(n) \ge \vartheta_k(n-1)$. This proves the first part.

Now, if $k = 2n+1$ we have $\vartheta_2(n-1) = -1$, so we expand the above recursion further as

$$\begin{aligned}\vartheta_{2n+1}(n) &= \vartheta_{2n+1}(n-1) - 1 = \vartheta_{2n+1}(n-2) + \vartheta_4(n-2) - 1 \\ &= \vartheta_{2n+1}(n-3) + \vartheta_6(n-3) - 1 = \vartheta_{2n+1}(n-4) + \vartheta_8(n-4) - 1 = \vartheta_{2n+1}(n-4),\end{aligned}$$

giving the second part of the statement. □

Lemma 5.2. *For $n \ge 86$ and $(n-1)^2/2 \le k \le n^2/2$, we have:*

$$b_k(n) - b_{k-1}(n) \ge C\frac{2^n}{n^{9/2}}, \quad \textit{where} \quad C = \frac{3\sqrt{3}}{2\sqrt{2}\,\pi^{3/2}} \approx 0.329\,.$$

Proof. We invoke details from the computations in [1]. It is shown there that

$$\frac{\partial b_k(n)}{\partial k} \ge \frac{2^{n+2}}{\pi}I \ge \frac{2^{n+3}}{\pi}(I_1 - |I_2| - |I_3| - |I_4|),$$

where I_j are certain explicit integrals. By the bounds in this proof we have:

$$I_1 \ge \frac{3\sqrt{3}}{4\sqrt{2\pi}}\frac{1}{n^{9/2}} = \frac{C\pi}{2}\frac{1}{n^{9/2}}, \qquad |I_2| \le \frac{7\pi^3}{96n^2}\exp\left(-\frac{(5\pi-6)n}{16\pi} + \frac{\pi}{32n}\right), \quad (5.1)$$

$$|I_3| \le \frac{\pi^2}{8}\exp\left(-\frac{(5\pi-3)n}{16\pi} + \frac{\pi}{16n}\right), \qquad |I_4| \le \frac{\pi\sqrt{\pi n}}{4}\left(\frac{3}{\sqrt{2en}}\right)^n \quad (5.2)$$

A calculation shows that for $n \geq 86$ we have

$$\frac{C\pi}{4}\frac{1}{n^{9/2}} \geq \frac{7\pi^3}{96n^2}\exp\left(-\frac{(5\pi-6)n}{16\pi} - \frac{\pi}{32n}\right) + \frac{\pi^2}{8}\exp\left(-\frac{(5\pi-3)n}{16\pi} + \frac{\pi}{16n}\right) + \frac{\pi\sqrt{\pi n}}{4}\left(\frac{3}{\sqrt{2en}}\right)^n.$$

Applying this to the inequality (5.1) we get

$$b_k(n) - b_{k-1}(n) \geq \frac{\partial b_k(n)}{\partial k} \geq \frac{2^{n+2}}{\pi}(I_1 - |I_2| - |I_3| - |I_4|) \geq \frac{2^{n+1}}{\pi}\frac{C\pi}{2}\frac{1}{n^{9/2}}. \quad \square$$

Based on this setup, we refine Almkvist's Theorem 2.2 as follows

Theorem 5.3. *For any $n \geq 31$, and $27 \leq k \leq n^2/2$ we have:*

$$b_k(n) - b_{k-1}(n) \geq C2^{\sqrt{2\,k}}\frac{1}{(2\,k)^{9/4}}, \quad \textit{where } C \textit{ is as above.}$$

Proof. First, a direct computer calculation shows that the inequality holds for all $n \in \{31, \ldots, 86\}$ and $27 \leq k \leq n^2/2$.

First, suppose that $k \geq 31^2/2$. Let $m = \lfloor\sqrt{2k}\rfloor + 1$, so that $\frac{(m-1)^2}{2} \leq k < \frac{m^2}{2}$ and $m \geq 31$. If $m \geq 86$, we have by Lemma 5.2 that

$$\vartheta_k(m) \geq C2^m\frac{1}{m^{9/2}} \geq C2^{\sqrt{2\,k}}\frac{1}{(2k)^{9/4}}$$

where the last inequality follows since the function $f(x) = \log 2\sqrt{x} - 9/4\log(x)$ is increasing. By the direct calculation for values of $m \in \{31, \ldots, 86\}$, we conclude that the inequality

$$\vartheta_k(m) \geq C2^{\sqrt{2\,k}}\frac{1}{(2k)^{9/4}}$$

holds for all $m \geq 31$ with $m^2/2 \geq k \geq (m-1)^2/2$. If $2k+1 > n$, then by induction on n applying Lemma 5.1 we get $\vartheta_k(n) \geq \vartheta_k(m)$, since $2k+1 > m+4$ under the given constraints. If $2k+1 \leq n$, then if $n' = 2k+1$ we have again by Lemma 5.1 that $\vartheta_k(n) \geq \vartheta_k(n-1) \geq \vartheta_k(n') \geq \vartheta_k(n'-4) \geq \cdots \geq \vartheta_k(m)$. In both cases we get the desired inequality.

Next, if $27 \leq k \leq 31^2/2$ and again $2k+1 = n' \geq 55$, by induction on n applying Lemma 5.1 we have $\vartheta_k(n) \geq \vartheta_k(n-1) \geq \cdots \geq \vartheta_k(n') \geq \vartheta_k(n'-4) \geq \cdots \vartheta_k(31)$, and the direct calculation gives the bound. $\square$

Corollary 5.4. *Let $n \geq 8$, $1 \leq k \leq n^2/2$, $\mu = (n^n)$ and $\tau_k = (n^2 - k, k)$. Then*

$$g(\mu, \mu, \tau_k) \geq C \frac{2^{\sqrt{2k}}}{(2k)^{9/4}}, \quad \text{where} \quad C = \frac{\sqrt{27/8}}{\pi^{3/2}}.$$

Proof. Following the proof of Stanley's Theorem 4.1, for all $27 \leq k \leq n^2/2$ and $n \geq 31$ Theorem 5.3 gives:

$$g(\mu, \mu, \tau_k) = p_k(n, n) - p_{k-1}(n, n) \geq b_k(n) - b_{k-1}(n) \geq C \frac{2^{\sqrt{2k}}}{(2k)^{9/4}}.$$

For the remaining values of n and k we check the inequality by computer, noticing that $p_k(n, n) = p_k(26, 26)$ when $k \leq 26$. □

5.2. Partitions in rectangles

Let $\tau_k = (m\ell - k, k)$, by Lemma 1.3, we have

$$\delta_k(\ell, m) := p_k(\ell, m) - p_{k-1}(\ell, m) = g(m^\ell, m^\ell, \tau_k).$$

Theorem 5.5. *Let $8 \leq \ell \leq m$ and $1 \leq k \leq m\ell/2$. Define n as*

$$n = \begin{cases} 2\lfloor \frac{\ell-8}{2} \rfloor, & \text{when } \ell m \text{ is even,} \\ 2\lfloor \frac{\ell-8}{2} \rfloor - 1, & \text{when } \ell m \text{ is odd,} \end{cases}$$

and let $v = \min(k, n^2/2)$. Then:

$$\delta_k(\ell, m) \geq C \frac{2^{\sqrt{2v}}}{(2v)^{9/4}} \qquad \text{where} \quad C = \frac{3\sqrt{3}}{2\sqrt{2}\pi^{3/2}}.$$

Proof. We apply Theorem 2.1 to bound the Kronecker coefficient for rectangles with an appropriate Kronecker coefficient for a square and then apply Corollary 5.4.

By strict unimodality (1.2), we have that $g(m^\ell, m^\ell, (m\ell - k, k)) > 0$ for all $\ell, m \geq 8$. By Corollary 5.4, we can assume $\ell < m$. Assume first that $\ell > 16$.

First, suppose that ℓm is even and let $n = 2\lfloor \frac{\ell-8}{2} \rfloor$. Then for any $1 < k \leq \frac{\ell m}{2}$ we can find $1 \neq r \leq \frac{(m-n)\ell}{2}$ and $1 \neq s \leq \frac{n\ell}{2}$, such that $k = r + s$. Take $s = \min(k, n\ell/2)$. Let $\tau_k = (m\ell - k, k)$ and $\tau_r = ((m - n)\ell - r, r)$, $\tau_s = (n\ell - s, s)$. Apply Theorem 2.1 to the triples $\big((m - n)^\ell, (m - n)^\ell, \tau_r\big)$ and $\big(n^\ell, n^\ell, \tau_s\big)$ to obtain

$$\delta_k(\ell, m) = g(m^\ell, m^\ell, \tau_k) \geq \max\Big(g\big((m - n)^\ell, (m - n)^\ell, \tau_r\big), g\big(n^\ell, n^\ell, \tau_s\big)\Big) \geq \delta_s(\ell, n).$$

Similarly, dividing the $n \times \ell$ rectangle into $n \times n$ square and $n \times (n - \ell)$ rectangle, where $n\ell$ is even, we have

$$\delta_s(\ell, n) \geq \delta_v(n, n),$$

where $v = \min(s, n^2/2) = \min(k, n^2/2)$.

In the case that both ℓ and m are odd, the only case where the above reasoning fails is when $k = \lfloor m\ell/2 \rfloor$ and r, s don't exist. In this case we take $n = 2\lfloor \frac{\ell-8}{2} \rfloor - 1$ and we can always find r, s. In summary, we have that

$$\delta_k(\ell, m) \geq \delta_v(n, n),$$

where

$$n = \begin{cases} 2\lfloor \frac{\ell-8}{2} \rfloor, & \text{when } \ell m \text{ is even} \\ 2\lfloor \frac{\ell-8}{2} \rfloor - 1, & \text{when } \ell m \text{ is odd} \end{cases}$$

and $v = \min(k, n^2/2)$. Now apply Corollary 5.4 to bound $\delta_v(n, n)$ and obtain the result for $\ell > 16$.

When $\ell \leq 16$, and $m \geq 24$, we can apply the same reasoning as above to show $\delta_k(\ell, m) \geq \delta_v(\ell, 16)$. Finally, for $\ell, m \leq 16$ the statement is easily verified by direct calculation. □

Proof of Theorem 1.2. For $n \geq \ell - 9$, the desired inequality then follows from Theorem 5.5 and the observation that

$$\frac{2^{n/\sqrt{2}}}{n^{9/2}} \geq 2^{-9/\sqrt{2}} \frac{2^{\ell/\sqrt{2}}}{\ell^{9/2}}.$$

Taking $A = 2^{-9/\sqrt{2}} C \approx 0.004$ gives the desired inequality for all values. □

5.3. Other bounds

Let $k \leq \ell \leq m$ and $n = \ell m$. We have:

$$\delta_k(\ell, m) = p_k(\ell, m) - p_{k-1}(\ell, m) = \mathrm{P}(k) - \mathrm{P}(k-1) \geq \mathrm{P}'(k) \sim \frac{\pi}{12\sqrt{2}\, k^{3/2}} \; e^{\pi \sqrt{\frac{2}{3} k}}$$

Compare this with the lower bound in Theorem 1.2:

$$\delta_k(\ell, m) > A \frac{2^{\sqrt{2 k}}}{(2 k)^{9/4}}.$$

There is only room to improve the base of exponent here:

$$\text{from } \; 2^{\sqrt{2}} \approx 2.26 \; \text{ to } \; e^{\pi \sqrt{\frac{2}{3}}} \approx 13.00\,.$$

In fact, using our methods, the best lower bound we can hope to obtain is

$$e^{\pi\sqrt{\frac{1}{6}}} \approx 3.61\,,$$

which is the base of exponent in the Roth–Szekeres formula for the number $b_k(n)$ of unrestricted partitions into distinct odd parts, where $n \ge k$.

For a different extreme, let $m = \ell$ be even, and $k = m^2/2$. We have the following sharp upper bound:

$$\delta_k(m,m) \le p_k(m,m) \sim \sqrt{\frac{3}{\pi\, m}}\binom{2\,m}{m} \sim \frac{\sqrt{3}\,4^m}{\pi\, m}\,.$$

On the other hand, the lower bound in Theorem 1.2 gives:

$$\delta_k(m,m) \,>\, A\,\frac{2^m}{m^{9/2}}\,.$$

Again, we cannot improve the base of the exponent 2 with our method, simply because the total number of partitions into distinct odd parts $\le 2m-1$, is equal to 2^m.

6. Effective bounds on Kronecker coefficients for more general triples

Here we prove a lower bound for a wide class of partition triples.

Let μ be a partition with Durfee square of size $\ge n$. We say that μ is *decomposed* as $(n^n+\alpha,\beta)$ if α, β are the partitions which remain after an $n\times n$ square is removed from λ: α is the remaining partition occupying the first n rows and β is the partition below, i.e. $\beta=(\lambda_{n+1},\lambda_{n+2},\ldots)$.

Theorem 6.1. *Let λ, μ, ν be partitions of the same size, such that the Durfee squares of μ and ν have sizes at least n, and such that the Young diagrams of λ, μ, ν can be decomposed as $\mu=(n^n+\alpha^1,\alpha^2)$, $\nu=(n^n+\beta^1,\beta^2)$ and $\lambda=\gamma^1+\gamma^2+\tau$, where $\tau=(n^2-k,k)$ or $\tau=(n^2-k,k)'$ for some $k>1$. Suppose that $g(\alpha^1,\beta^1,\gamma^1)>0$ and $g(\alpha^2,\beta^2,\gamma^2)>0$. Then*

$$g(\lambda,\mu,\nu) \,\ge\, C\,\frac{2^{\sqrt{2k}}}{(2k)^{9/4}}\,, \quad \textit{where} \quad C=\frac{\sqrt{27/8}}{\pi^2}\,.$$

Proof. Since transposing two partitions in a triple leaves the Kronecker unchanged, by Corollary 5.4 we have:

$$g(n^n,n^n,\tau)=g(n^n,(n^n)',(n^2-k,k)')=g(n^n,n^n,(n^2-k,k)) \,\ge\, C\frac{2^{\sqrt{2k}}}{(2k)^{9/4}}\,, \qquad (6.1)$$

so in particular $g(n^n,n^n,\tau)>0$ with the above effective lower bound. Now we apply the transposition property and the monotonicity property several times as follows:

$$\begin{aligned} g(\lambda,\mu,\nu) &= g(\lambda,\mu',\nu') = g\big(\tau+\gamma^1+\gamma^2,(n^n+\alpha^1)'+(\alpha^2)',(n^n+\beta^1)'+(\beta^2)'\big) \\ &\ge \max\big\{g\big(\tau+\gamma^1,(n^n+\alpha^1)',(n^n+\beta^1)'\big),g\big(\gamma^2,(\alpha^2)',(\beta^2)'\big)\big\} \\ &\ge \max\big\{g(\tau+\gamma^1,n^n+\alpha^1,n^n+\beta^1),g(\gamma^2,\alpha^2,\beta^2)\big\} \\ &\ge g(\tau+\gamma^1,n^n+\alpha^1,n^n+\beta^1) \\ &\ge \max\{g(\tau,n^n,n^n),g(\gamma^1,\alpha^1,\beta^1)\} \ge g(\tau,n^n,n^n)\,. \end{aligned}$$

The lower bound now follows from equation (6.1). □

Proof of Corollary 1.4. Let λ be decomposed as $\lambda = (m^m+\alpha,\beta)$, where $\alpha \vdash a$, $\beta \vdash b$. Since

$$g(\alpha,\alpha,(a)) = g(\beta,\beta,(b)) = 1 > 0\,,$$

we can apply Theorem 6.1 for the triple (λ,λ,ν), where $\nu = (m^2-k,k)+(a)+(b) = (n-k,k)$. □

7. Final remarks

7.1. It is rather easy to justify the importance of the Kronecker coefficients in Combinatorics and Representation Theory. Stanley writes: "One of the main problems in the combinatorial representation theory of the symmetric group is to obtain a combinatorial interpretation for the Kronecker coefficients" [24].

Geometric Complexity Theory (GCT) is a more recent interdisciplinary area, where computing the Kronecker coefficients is crucial (see [15]). Bürgisser voices a common complaint of the experts: "frustratingly little is known about them" [5].

Part of this work is motivated by questions in GCT. Specifically, the interest is in estimating the coefficients

$$g\big(m^\ell,m^\ell,\lambda\big)\,, \quad \text{where} \quad \lambda \vdash \ell m\,.$$

Both Theorems 1.1 and 1.2 are directly applicable to this case, when $m=\ell$ and $\lambda=\lambda'$, and when $\ell(\lambda)=2$, respectively. We plan to return to this problem in the future.

7.2. The notion of s-border equal partitions given in §4.3 is perhaps new but rather natural. It would be interesting to see if a stronger bound

$$g(\lambda,\mu,\mu) \ge g\big(\lambda^{\langle s\rangle},\mu^{\langle s\rangle},\mu^{\langle s\rangle}\big)$$

holds under the assumptions of Corollary 4.2. Note that when one of the partitions of decomposition is empty, $\lambda = (m^m+\alpha,\varnothing)$, $m=d(\lambda)$, the bound follows immediately from Theorem 6.1. Note also that one cannot drop the $\mu=\mu'$ assumption here. For example, when $\lambda=\mu=(2,2,1)$ and $s=1$, we have $\lambda^{\langle 1\rangle}=\mu^{\langle 1\rangle}=(1)$, and $g(\lambda,\mu,\mu)=0$ while $g(\lambda^{\langle 1\rangle},\mu^{\langle 1\rangle},\mu^{\langle 1\rangle})=1$.

7.3. In a special case $\lambda = \mu = \mu'$, Theorem 1.1 and the Murnaghan–Nakayama rule gives a weak bound $g(\mu, \mu, \mu) \geq 1$ proved earlier in [3]. In [21], we apply a qualitative version of Theorem 1.1 to a variety of partitions generalizing hooks and two-row partitions. Unfortunately, computing the characters of S_n is #P-hard [20]. It is thus unlikely that this approach can give good bounds for general *tensor squares* $g(\lambda, \mu, \mu)$ (cf. [28]).

7.4. As we showed in § 5.3, there is a gap in the base of the exponent between lower and upper bounds even for $\delta_k(m, m)$. Finding sharper lower bounds in this case would be very interesting.

On the other hand, there is perhaps also room for improvement for the rectangular case $\delta_k(\ell, m)$, where $\ell = o(m)$ and k is in the "middle range" $\ell^2/2 \leq k \leq \ell m/2$. Since our proof of a lower bound in this case relies crucially on Theorem 2.1 and no other tools in this case are available, extending the inequality in the theorem would be very useful.

7.5. As we mentioned in the introduction, the first lower bound $\delta_k(\ell, m) \geq 1$ was obtained by the authors in [18]. This was quickly extended in followup papers [9] and [29], both of which employed O'Hara's combinatorial proof [16]. First, Zanello's proof gives:

$$\delta_k(\ell, m) > d\,, \quad \text{for} \;\; \ell \geq d^2 + 5d + 12\,, \;\; m \geq 2d + 4\,, \;\; 4d^2 + 10d + 7 \leq k \leq \ell m/2\,,$$

see the proof of Prop. 4 in [29]. Similarly, Dhand proves somewhat stronger bounds

$$\delta_k(\ell, m) > d\,, \quad \text{for} \;\; \ell, m \geq 8\,d, \;\; 2d \leq k \leq \ell m/2\,,$$

see Theorem 1.1 in [9].

In conclusion, let us mention that the sequence $p_k(\ell, m)$ has remarkably sharp asymptotics bounds, including the *central limit theorem* (CLT) with the error bound [27], and the Hardy–Ramanujan type formula [2]. When ℓ is fixed, sharp asymptotic bounds are given in [25].

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