Journal of Combinatorial Theory, Series A 175 (2020) 105278

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Licci binomial edge ideals ☆

Viviana Ene [a,*], Giancarlo Rinaldo [b], Naoki Terai [c]

[a] *Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, 900527 Constanta, Romania*
[b] *Department of Mathematics, University of Trento, Via Sommarive, 14, 38123 Povo (Trento), Italy*
[c] *Department of Mathematics, Okayama University, 3-1-1, Tsushima-naka, Kita-ku, Okayama, 700-8530, Japan*

A R T I C L E I N F O

Article history:
Received 8 October 2019
Received in revised form 25 April 2020
Accepted 13 May 2020
Available online xxxx

Keywords:
Licci ideals
Binomial edge ideals
Regularity

A B S T R A C T

We give a complete characterization of graphs whose binomial edge ideal is licci. An important tool is a new general upper bound for the regularity of binomial edge ideals.

0. Introduction

Binomial edge ideals associated to simple graphs have been intensively studied in the last decade. Their algebraic and homological properties are intimately related to the combinatorics of the underlying graph. A lot of effort has been dedicated to study

☆ The second author was supported by GNSAGA of INdAM (Italy). The third author was supported by the JSPS Grant-in Aid for Scientific Research (C) 18K03244.

* Corresponding author.
E-mail addresses: vivian@univ-ovidius.ro (V. Ene), giancarlo.rinaldo@unitn.it (G. Rinaldo), terai@okayama-u.ac.jp (N. Terai).

https://doi.org/10.1016/j.jcta.2020.105278
0097-3165/

the Cohen-Macaulay property of these ideals. As in the case of classical edge ideals, an exhaustive classification of graphs whose binomial edge ideals are Cohen-Macaulay seems to be a hopeless task. There are successful attempts to characterize graphs with specific properties which have Cohen-Macaulay binomial edge ideals. For example, the Cohen-Macaulay property of binomial edge ideals is known for block graphs which include the trees [3] and for bipartite graphs [1]. We refer also to the papers [12,20–22] for other classes of Cohen-Macaulay binomial edge ideals.

Let G be a simple graph (that is, undirected, with no loops, and no multiple edges) on the vertex set $[n] := \{1, 2, \ldots, n\}$ and $S = K[x_1, \ldots, x_n, y_1, \ldots y_n]$ the polynomial ring in $2n$ variables. The binomial edge ideal $J_G \subset S$ of G is generated by all the binomials of the form $f_{ij} = x_i y_j - x_j y_i$ where $\{i, j\}$ is an edge of G. In other words, J_G is generated by the 2-minors of the generic matrix $X = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}$ which correspond to the edges of G.

In this paper, we study binomial edge ideals which are in the linkage class of a complete intersection. We call such ideals licci, in brief. Besides the Cohen-Macaulay property, they satisfy some extra conditions which make possible a full characterization of graphs whose binomial edge ideals are licci. Linkage theory has a rich history in commutative algebra and algebraic geometry. Peskine and Szpiro [19] in 1974 reduced general linkage to questions on ideals over commutative algebras and after then, a lot of work has been done to develop this theory in commutative algebra and algebraic geometry. If I, J are proper ideals in a local regular ring R, they are called *directly linked* and we write $I \sim J$ if there exists a regular sequence $\mathbf{z} = z_1, \ldots, z_g$ in $I \cap J$ such that $J = (\mathbf{z}) : I$ and $I = (\mathbf{z}) : J$. One says that I and J belong to the same *linkage class* if there exists a sequence of direct links

$$I = I_0 \sim I_1 \sim \cdots \sim I_m = J.$$

If J is a complete intersection ideal, then I is said to be licci. The ideals in the same linkage class share several properties. For example, if I and J are linked, then I is Cohen-Macaulay if and only if J is Cohen-Macaulay. In particular, it follows that a licci ideal is Cohen-Macaulay.

The following natural question arises. May we give a full characterization of the graphs G with the property that the associated binomial edge ideal is licci?

In this paper, we give a complete answer to this question. In [9] a necessary condition for a Cohen-Macaulay homogeneous ideal in a polynomial ring to be licci is given. In the case of binomial edge ideals, this condition implies that if $(J_G)_{\mathfrak{m}} \subset S_{\mathfrak{m}}$ (here $\mathfrak{m}$ is the maximal graded ideal of the ring S) is licci, then $\operatorname{reg}(S/J_G) \geq n - 2$. This condition turns to be also sufficient for Cohen-Macaulay binomial edge ideals as we are going to show in this paper.

The regularity of binomial edge ideals have been intensively studied in the last years. In [15] it was proved that the regularity of S/J_G is upper bounded by $n-1$ and it was conjectured that this upper bound is attained if and only if G is a path graph. This conjecture was later proved in [13]. Inspired by the paper [13], we prove a new upper bound for $\operatorname{reg}(S/J_G)$ which is stronger than $n-1$ and it plays an essential role in the characterization of the graphs G whose binomial edge ideal is licci.

The structure of the paper is as follows. In Section 1, we recall the basic results on licci and binomial edge ideals needed in the next sections. In Section 2, we prove that if G is a connected graph, then $\operatorname{reg}(S/J_G) \leq n - \dim \Delta(G)$, where $\Delta(G)$ is the clique complex of G (Theorem 2.1). We believe that this new general upper bound for the regularity of binomial edge ideals will inspire new results on their resolution. In brief, in Theorem 2.1, we prove that for every clique $W \subset [n]$ of the connected graph G, we have $\operatorname{reg}(S/J_G) \leq n - |W| + 1$. The proof is based on a double induction. First we make induction on $n - |W|$ and, secondly, on a combinatorial invariant of G.

The characterization of graphs whose binomial edge ideal is licci is given in Section 3. In Theorem 3.5 we show that, for a connected graph G on n vertices, the following statements are equivalent:

(i) $(J_G)_{\mathfrak{m}} \subset S_{\mathfrak{m}}$ is licci.
(ii) J_G is Cohen-Macaulay and $n-2 \leq \operatorname{reg}(S/J_G) \leq n-1$.
(iii) G is a path graph or it is a triangle with possibly some paths attached to some of its vertices.

The most technical part in the proof is to show that there is no indecomposable graph G with $n \geq 4$ vertices with $\operatorname{reg}(S/J_G) = n-2$ and J_G Cohen-Macaulay. In order to make this part easier to understand, we proved some preparatory lemmas. We can reformulate the above statement by saying that the only indecomposable graphs G with J_G a Cohen-Macaulay ideal and $\operatorname{reg}(S/J_G) = n-2$ are the path with one edge and the triangle. Next we combine this fact with Lemma 3.2 which shows that for any decomposable graph G with $\operatorname{reg}(S/J_G) = n-2$, one of the components must be a path. In this way we derive the combinatorial characterization from Theorem 3.5 (iii).

A straightforward consequence of Theorem 3.5 is Corollary 3.7 which says that for a connected bipartite graph G, the ideal $(J_G)_{\mathfrak{m}} \subset S_{\mathfrak{m}}$ is licci if and only if G is a path graph. The case when G is a disconnected graph is treated in Proposition 3.8.

In the last section of the paper, we show that for chordal graphs, in the equivalent statements of Theorem 3.5, we may replace the Cohen-Macaulay property with the unmixedness of the ideal J_G (Theorem 4.2). For the proof we use a theorem of Dirac which characterizes the chordal graphs in terms of their clique complex.

1. Preliminaries

We recall some notions and fundamental results needed in the later sections.

1.1. Licci ideals

Let R be a regular local ring and I, J proper ideals of R. Then I and J are called *directly linked* and we write $I \sim J$ if there exists a regular sequence $\mathbf{z} = z_1, \ldots, z_g$ in $I \cap J$ such that $J = (\mathbf{z}) : I$ and $I = (\mathbf{z}) : J$. One says that I is *linked* to J or that I and J belong to the same *linkage class* if there exists a sequence of direct links

$$I = I_0 \sim I_1 \sim \cdots \sim I_m = J.$$

If J is a complete intersection ideal, that is, it is generated by a regular sequence, then I is said to be in the **li**nkage **c**lass of a **c**omplete **i**ntersection (*licci* in brief).

Several properties are preserved in the same linkage class. For example, if I is linked to J, then R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay [19]. In particular, any licci ideal is Cohen-Macaulay. A necessary condition for a homogeneous ideal in a polynomial ring to be licci is given in [9].

Theorem 1.1. *[9, Corollary 5.13] Let I be a Cohen-Macaulay homogeneous ideal in a standard graded polynomial ring $S = K[x_1, \ldots, x_n]$ with the graded maximal ideal $\mathfrak{m}$. If $I_{\mathfrak{m}} \subset R = S_{\mathfrak{m}}$ is licci, then*

$$\operatorname{reg}(S/I) \geq (\operatorname{height} I - 1)(\operatorname{indeg} I - 1) \tag{1}$$

where $\operatorname{indeg} I$ *is the initial degree of the ideal I, that is,* $\operatorname{indeg} I = \min\{i : I_i \neq 0\}$.

Although, in general, inequality (1) is not a sufficient condition, if I is the edge ideal of a graph, then $I_{\mathfrak{m}} \subset R = S_{\mathfrak{m}}$ is licci if and only if inequality (1) holds [14]. We will see a similar behavior in Section 3 for binomial edge ideals.

1.2. Graphs and binomial edge ideals

Let G be a simple graph on the vertex set $V(G) := [n]$ with the edge set $E(G)$ and $S = K[x_1, \ldots, x_n, y_1, \ldots y_n]$ the polynomial ring in $2n$ variables over a field K. The binomial edge ideal of the graph G is generated by the binomials $f_e := x_i y_j - x_j y_i$ with $e = \{i, j\} \in E(G)$. In other words, J_G is generated by the 2-minors of the matrix $X = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}$ which correspond to the edges of G. For example, if G is the complete graph K_n on n vertices, then J_G is the ideal $I_2(X)$ generated by all the 2-minors of X. Note that J_{K_n} has a linear resolution by [7, Theorem 7.27]. On the other hand, if G is the path graph P_n on n vertices with edge set $\{\{i, i+1\} : 1 \leq i \leq n-1\}$, then J_G is the ideal of all adjacent maximal minors of X. By [21, Theorem 2.2], if G is a connected graph, J_G is a complete intersection, that is, it is generated by a regular sequence if and only if G is a path graph.

The binomial edge ideals were introduced independently in the papers [6] and [17]. In the last decade, these ideals have been studied by many authors. The interested reader may find a thorough introduction to this topic in the monograph [7]. Fundamental results regarding the minimal free resolutions of binomial edge ideals are surveyed in [24].

In this paper, we need to recall the primary decomposition of binomial edge ideals and some fundamental results on their regularity.

The minimal primary decomposition of a binomial edge ideal is strongly related to the combinatorics of the underlying graph; see [6] or [7, Chapter 7]. Let $\mathcal{S}$ be a (possibly empty) subset of $[n]$ and let $G_{\mathcal{S}}$ be the restriction of G to the vertex subset $[n] \setminus \mathcal{S}$. Let $G_1, \ldots, G_{c(\mathcal{S})}$ be the connected components of this restriction and, for every $1 \leq i \leq c(\mathcal{S})$, let $\tilde{G}_i$ be the complete graph on $V(G_i)$. Then, the ideal

$$P_{\mathcal{S}}(G) = (\{x_i, y_i : i \in \mathcal{S}\}) + J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(\mathcal{S})}}$$

is a prime ideal in S which contains J_G, and by [6, Lemma 3.1] we have

$$\operatorname{height}(P_{\mathcal{S}}(G)) = n - c(\mathcal{S}) + |\mathcal{S}|. \tag{2}$$

Theorem 1.2. *[6] In the above notation, we have*

$$J_G = \bigcap_{\mathcal{S} \subset [n]} P_{\mathcal{S}}(G).$$

In particular, J_G is a radical ideal and its minimal prime ideals are among $P_{\mathcal{S}}(G)$ with $\mathcal{S} \subset [n]$. The following proposition characterizes the sets $\mathcal{S}$ for which the prime ideal $P_{\mathcal{S}}(G)$ is minimal.

Proposition 1.3. *[6, Corollary 3.9] $P_{\mathcal{S}}(G)$ is a minimal prime ideal of J_G if and only if either $\mathcal{S} = \emptyset$ or $\mathcal{S}$ is non-empty and for each $i \in \mathcal{S}$, $c(\mathcal{S} \setminus \{i\}) < c(\mathcal{S})$.*

In graph theoretical terminology, for a connected graph G, $P_{\mathcal{S}}(G)$ is a minimal prime ideal of J_G if and only if $\mathcal{S}$ is empty or $\mathcal{S}$ is non-empty and is a *cut set* of G, that is, i is a cut vertex of the restriction $G_{([n]\setminus\mathcal{S})\cup\{i\}}$ for every $i \in \mathcal{S}$. We recall that a vertex v of the graph H is a *cut vertex* of H if its removing breaks H into more connected components than H has. Let $\mathcal{C}(G)$ be the set of all sets $\mathcal{S} \subset [n]$ such that $P_{\mathcal{S}}(G)$ is a minimal prime ideal of J_G. Equality (2) implies then the following.

Corollary 1.4. *Let G be a connected graph on the vertex set $[n]$. Then J_G is unmixed if and only if for every $\mathcal{S} \in \mathcal{C}(G)$, $c(\mathcal{S}) = |\mathcal{S}| + 1$. In this case, we have* $\operatorname{height} J_G = \operatorname{height} P_{\emptyset}(G) = |V(G)| - 1$.

Proof. The ideal J_G is unmixed if and only if all its minimal prime ideals have the same height. This is the case if and only if, for every $\mathcal{S} \in \mathcal{C}(G)$, $\operatorname{height}(P_{\mathcal{S}}(G)) = \operatorname{height}(P_{\emptyset}(G)) = n - 1$. By (2), this is equivalent to $c(\mathcal{S}) = |\mathcal{S}| + 1$. $\square$

A general upper bound for the regularity of binomial edge ideals was first given in [15], namely, $\operatorname{reg}(S/J_G) \leq n-1$, and in the same paper it was conjectured that $\operatorname{reg}(S/J_G) = n-1$ if and only if G is a path graph. This conjecture was proved in [13].

Theorem 1.5. *[13] Let G be a graph on n vertices which is not a path. Then $\operatorname{reg}(S/J_G) \leq n-2$.*

For a chordal graph G, in [23, Theorem 3.5] it was shown that the number $c(G)$ of maximal cliques of G is an upper bound for $\operatorname{reg}(S/J_G)$.

Recall that a subset $C \subset [n]$ is a *clique* of G if the induced subgraph of G on the vertex set C is a complete graph. The set of cliques of G forms a simplicial complex $\Delta(G)$ called the *clique complex* of G. Its facets are the maximal cliques of G. By a famous theorem of Dirac ([2] or [5, Section 9.2]), a connected graph G is chordal if and only if either G is a complete graph or the facets of $\Delta(G)$ can be ordered as $F_1, \ldots, F_c$ such that, for all $i > 1$, F_i is a leaf of the simplicial complex generated by $F_1, \ldots, F_i$. A *leaf* of a simplicial complex Δ is a facet of Δ which has a *branch*, that is, a facet G such that for all facets F' of Δ with $F' \neq F$, we have $F' \cap F \subseteq G \cap F$.

2. A new upper bound for the regularity of binomial edge ideals

In this section, we give a new general upper bound for the regularity of S/J_G.

Theorem 2.1. *Let G be a connected graph on $[n]$. Then $\operatorname{reg}(S/J_G) \leq n - \dim \Delta(G)$.*

When G is not connected, we derive the following upper bound for the regularity of S/J_G.

Corollary 2.2. *Let G be a graph on n vertices with the connected components $G_1, \ldots, G_c$. Then*

$$\operatorname{reg}(S/J_G) \leq n - (\dim \Delta(G_1) + \cdots + \dim \Delta(G_c)).$$

Let us make some short remarks before proving the above theorem. This new bound will be an essential tool in proving Theorem 3.5. Moreover, it is a substantial improvement of the upper bound given by Matsuda and Murai [15].

In what follows, we will need some notation and known results. If H is a graph and $e \in E(H)$, we denote by $H \setminus e$ the subgraph of H obtained by removing the edge e from $E(H)$ and if $e_1, \ldots, e_m \in E(H)$, we write $H \setminus \{e_1, \ldots, e_m\}$ for the subgraph of H which is obtained by removing the edges $e_1, \ldots, e_m$. If $e = \{i, j\}$ where i, j are vertices of H and $e \notin E(G)$, then $H \cup e$ is the graph with the same vertex set as H and edge set $E(H) \cup \{e\}$, and H_e is the graph with $V(H_e) = V(H)$ and $E(H_e) = E(H) \cup \{\{k, \ell\} : k, \ell \in N(i) \text{ or } k, \ell \in N(j)\}$ where $N(i)$ denotes the set of all neighbors of i in H.

The next proposition is a direct consequence of the behavior of the regularity with respect to short exact sequences; see [18, Corollary 18.7].

Proposition 2.3. *[13, Proposition 2.1] Let H be a graph on n vertices and $J_H \subset S$ its binomial edge ideal. Let $e = \{i,j\}$ be an edge of H and $f_e = x_iy_j - x_jy_i$. Then, the following inequalities hold:*

(a) $\operatorname{reg}(J_H) \leq \max\{\operatorname{reg}(J_{H\setminus e}), \operatorname{reg}(J_{H\setminus e} : f_e) + 1\}$;
(b) $\operatorname{reg}(J_{H\setminus e}) \leq \max\{\operatorname{reg}(J_H), \operatorname{reg}(J_{H\setminus e} : f_e) + 2\}$;
(c) $\operatorname{reg}(J_{H\setminus e} : f_e) + 2 \leq \max\{\operatorname{reg}(J_{H\setminus e}), \operatorname{reg}(J_H) + 1\}$.

In the settings of the above proposition, we have the following.

Theorem 2.4. *[16, Theorem 3.7]*

$$J_{H\setminus e} : f_e = J_{(H\setminus e)_e} + I_{H,e}$$

where $I_{H,e}$ is the monomial ideal generated by the set

$$\{g_{\pi,t} | \pi : i, i_1, \ldots, i_s, j \text{ is a path between } i \text{ and } j \text{ and } 0 \leq t \leq s\}$$

and

$$g_{\pi,0} = x_{i_1} \cdots x_{i_s}, g_{\pi,t} = y_{i_1} \cdots y_{i_t} x_{i_{t+1}} \cdots x_{i_s} \text{ for } 1 \leq t \leq s.$$

Proof of Theorem 2.1. Clearly, the statement of the theorem follows if we show that for any clique $W \subset [n]$, we have

$$\operatorname{reg}(S/J_G) \leq n - |W| + 1 \text{ or, equivalently, } \operatorname{reg}(J_G) \leq n - |W| + 2. \tag{3}$$

We prove this by induction on $n - |W|$. If $n = |W|$, then G is the complete graph on n vertices and, as we have mentioned in Section 1, we have $\operatorname{reg}(S/J_G) = 1$.

Let $n - |W| > 0$. We proceed with the inductive step. For the remaining part of the proof, we need to define the following. For a vertex $v \in V(G)$, we set $\alpha_G(v) := \binom{\deg v}{2} - |E(G_{N(v)})|$. Here, we used the usual notation G_U for the restriction of G to the subset U of $V(G)$. Obviously, $\alpha_G(v) = 0$ if and only if v is a simplicial vertex in G. Recall that a vertex of a graph is called *simplicial* if it belongs to exactly one maximal clique. In addition, for a subset $W \subset V(G)$, we define $\alpha_G(W) := \min\{\alpha_G(v) : v \in V(G) \setminus W\}$. Further on, we proceed by induction on $\alpha_G(W)$.

Step 1. Let $\alpha_G(W) = 0$. Thus, there exists a simplicial vertex $v \in V(G) \setminus W$. Now we consider two cases, namely $\deg v = 1$ and $\deg v \geq 2$.

Case 1. Let $\deg(v) = 1$ and $e = \{v, w\} \in E(G)$. By Proposition 2.3 (a), we have

$$\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus e}), \operatorname{reg}(J_{G\setminus e} : f_e) + 1\}.$$

Therefore, it is enough to show that

$$\operatorname{reg}(J_{G\setminus e}) \leq n - |W| + 2 \tag{4}$$

and

$$\operatorname{reg}(J_{G\setminus e} : f_e) \leq n - |W| + 1. \tag{5}$$

Since $\deg(v) = 1$, the vertex v becomes isolated in the graph $G \setminus e$, thus $\operatorname{reg}(J_{G\setminus e}) = \operatorname{reg}(J_{(G\setminus e)\setminus v})$. So, for showing inequality (4), we simply apply the inductive hypothesis to the graph $(G \setminus e) \setminus v$. For showing inequality (5), we first apply Theorem 2.4 and get

$$\operatorname{reg}(J_{G\setminus e} : f_e) = \operatorname{reg}(J_{(G\setminus e)_e}),$$

since, $I_{G,e} = (0)$ because the only path connecting v and w in G is the edge $\{v, w\}$. In the graph $(G \setminus e)_e$, v is an isolated vertex, thus,

$$\operatorname{reg}(J_{(G\setminus e)_e}) = \operatorname{reg}(J_{((G\setminus e)_e)\setminus v}).$$

Now we can apply again the inductive hypothesis for $(G \setminus e)_e \setminus v$ and obtain

$$\operatorname{reg}(J_{(G\setminus e)_e\setminus v}) \leq (n - 1) - |W| + 2 = n - |W| + 1.$$

Therefore, Case 1 is completed.

Case 2. Let v be a simplicial vertex of $\deg(v) = t \geq 2$. Before discussing this case, we prove the following.

Claim. *Assume that there exists $v \in V(G) \setminus W$ a simplicial vertex with* $\deg(v) \geq 2$. *Let e be an edge of G which contains v. Then*

$$\operatorname{reg}(J_{G\setminus e} : f_e) \leq n - |W| + 1.$$

Proof of the Claim. Let $\deg(v) = t$, let $N_G(v) = \{v_1, \ldots, v_t\}$ be the set of neighbors of v in G, and set $e_i = \{v, v_i\}$ for $1 \leq i \leq t$. We may assume that $e = e_t$ and let us consider the monomial ideal $I_{G,e}$ from Theorem 2.4. Since v is a simplicial vertex, for any $1 \leq i \leq t - 1$, v_t, v_i, v is a path in G, thus $x_{v_i}, y_{v_i} \in I_{G,e}$ for all $1 \leq i \leq t - 1$. Moreover, every path from v to v_t must pass through some neighbor v_i with $1 \leq i \leq t - 1$. This implies that

$$I_{G,e} = (x_{v_i}, y_{v_i} : 1 \leq i \leq t-1).$$

By Theorem 2.4, we get

$$J_{G\setminus e} : f_e = J_{(G\setminus e)_e} + (x_{v_i}, y_{v_i} : 1 \leq i \leq t-1).$$

Set $H := (G \setminus e)_e$. Then

$$J_{G\setminus e} : f_e = J_{H_{[n]\setminus\{v_1,\ldots,v_{t-1}\}}} + (x_{v_i}, y_{v_i} : 1 \leq i \leq t-1),$$

because the binomial generators of $H = J_{(G\setminus e)_e}$ corresponding to the edges which contain some v_i with $1 \leq i \leq t-1$ are contained in $I_{G,e}$. Since v becomes an isolated vertex in $H_{[n]\setminus\{v_1,\ldots,v_{t-1}\}}$, we get

$$J_{G\setminus e} : f_e = J_{H_{[n]\setminus\{v,v_1,\ldots,v_{t-1}\}}} + (x_{v_i}, y_{v_i} : 1 \leq i \leq t-1),$$

which implies that

$$\operatorname{reg}(J_{G\setminus e} : f_e) = \operatorname{reg}(J_{H_{[n]\setminus\{v,v_1,\ldots,v_{t-1}\}}}).$$

The graph $H_{[n]\setminus\{v,v_1,\ldots,v_{t-1}\}}$ has $n-t$ vertices and the clique $W \setminus \{v, v_1, \ldots, v_{t-1}\}$, thus we may apply the inductive hypothesis because

$$(n-t) - |W \setminus \{v, v_1, \ldots, v_{t-1}\}| \leq n - t - |W| + t - 1 = n - |W| - 1.$$

Therefore, we get

$$\begin{aligned}\operatorname{reg}(J_{G\setminus e} : f_e) = \operatorname{reg}(J_{H_{[n]\setminus\{v,v_1,\ldots,v_{t-1}\}}}) &\leq (n-t) - |W \setminus \{v, v_1, \ldots, v_{t-1}\}| + 2\\ &\leq n - |W| + 1,\end{aligned}$$

and the claim is proved. $\square$

We now go back to the discussion of *Case 2*. Let $N_G(v) = \{v_1, \ldots, v_t\}$ be the set of the neighbors of v in G and $e_i = \{v, v_i\}$ for $1 \leq i \leq t$. By Proposition 2.3 and the Claim, we have

$$\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus e_1}), \operatorname{reg}(J_{G\setminus e_1} : f_{e_1}) + 1\} \leq \max\{\operatorname{reg}(J_{G\setminus e_1}), n - |W| + 2\}.$$

Applying the same argument to $G \setminus e_1$, we obtain

$$\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus\{e_1,e_2\}}), n - |W| + 2\}.$$

After $t-1$ steps, we get

$$\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus\{e_1,e_2,\dots,e_{t-1}\}}), n-|W|+2\}.$$

In the graph $G \setminus \{e_1, e_2, \dots, e_{t-1}\}$, we have $\deg(v) = 1$. Consequently, by Case 1, we derive that $\operatorname{reg}(J_G) \leq n - |W| + 2$ which completes the proof of Step 1.

Now we proceed to prove the inductive step on $\alpha_G(W)$.

Step 2. Let $\alpha_G(W) > 0$. This implies that there exists a non-simplicial vertex $v \in V(G) \setminus W$ such that $\alpha_G(W) = \alpha_G(v)$. Moreover, since v is not simplicial, there exist $v_1, v_2 \in N_G(v)$ such that $e = \{v_1, v_2\} \notin E(G)$. By Proposition 2.3 (b) where $H = G \cup e$, it follows

$$\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\cup e}), \operatorname{reg}(J_G : f_e) + 2\}. \tag{6}$$

By the definition of $\alpha_G(v)$, we have $\alpha_{G\cup e}(v) = \alpha_G(v) - 1$, therefore $\alpha_{G\cup e}(W) \leq \alpha_G(W) - 1$. By induction on $\alpha_G(W)$, we then derive that

$$\operatorname{reg}(J_{G\cup e}) \leq n - |W| + 2.$$

In order to complete this last step, by using (6), it is enough to show that

$$\operatorname{reg}(J_G : f_e) + 2 \leq n - |W| + 2. \tag{7}$$

By Theorem 2.4, we have

$$J_G : f_e = J_{G_e} + I_{G\cup e, e}. \tag{8}$$

Since v_1, v, v_2 is a path, the variables x_v, y_v belong to $I_{G\cup e,e}$. This implies that

$$I_{G\cup e,e} = (x_v, y_v) + I_{(G\setminus v)\cup e,e}.$$

By replacing $I_{G\cup e,e}$ in equality (8), we can rewrite it as

$$J_G : f_e = J_{(G\setminus v)_e} + I_{(G\setminus v)\cup e,e} + (x_v, y_v).$$

This implies that

$$\operatorname{reg}(J_G : f_e) = \operatorname{reg}(J_{(G\setminus v)_e} + I_{(G\setminus v)\cup e,e}).$$

On the other hand, by Theorem 2.4 applied for $G \setminus v$, we get

$$\operatorname{reg}(J_{(G\setminus v)_e} + I_{(G\setminus v)\cup e,e}) = \operatorname{reg}(J_{G\setminus v} : f_e),$$

thus,

$$\operatorname{reg}(J_G : f_e) = \operatorname{reg}(J_{G\setminus v} : f_e).$$

Next, by Proposition 2.3, (c) we have

$$\operatorname{reg}(J_{G\setminus v} : f_e) + 2 \leq \max\{\operatorname{reg}(J_{G\setminus v}), \operatorname{reg}(J_{(G\setminus v)\cup e}) + 1\}.$$

By the inductive hypothesis on $n - |W|$, we have

$$\operatorname{reg}(J_{G\setminus v}) \leq (n-1) - |W| + 2 = n - |W| + 1,$$

and

$$\operatorname{reg}(J_{(G\setminus v)\cup e}) + 1 \leq (n-1) - |W| + 3 = n - |W| + 2.$$

Consequently, we proved inequality (7) and this completes Step 2 and the whole proof of the theorem. □

3. Licci binomial edge ideals

As in the previous section, let G be a simple graph on the vertex set $[n]$ and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring over a field K. Let $\mathfrak{m}$ be the maximal graded ideal of S and set $R = S_{\mathfrak{m}}$.

We recall the notion of decomposable graphs from [8].

Definition 3.1. A connected graph G is called *decomposable* if there exists two subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$ where v is a simplicial vertex in G_1 and G_2. In this case we say that G is decomposable in the vertex v. Otherwise, the graph G is called indecomposable.

As it was proved in [8], if G is decomposable, then $\operatorname{reg}(S/J_G) = \operatorname{reg} S_1/J_{G_1} + \operatorname{reg} S_2/J_{G_2}$ where $S_i = K[\{x_j, y_j : j \in V(G_i)\}]$ for $i = 1, 2$. Moreover, by [20, Theorem 2.7], J_G is Cohen-Macaulay if and only if J_{G_1} and J_{G_2} are Cohen-Macaulay.

Before proving the main theorem of this section, we state some lemmas which are useful in what follows.

Lemma 3.2. *Let G be a decomposable graph as $G = G_1 \cup G_2$ with $|V(G_i)| = n_i$ for $i = 1, 2$ and let $S_i = K[\{x_j, y_j\} : j \in V(G_i)]$ for $i = 1, 2$. If $\operatorname{reg}(S/J_G) = n - 2$, then $\operatorname{reg}(S_1/J_{G_1}) = n_1 - 2$ and G_2 is a path or $\operatorname{reg}(S_2/J_{G_2}) = n_2 - 2$ and G_1 is a path.*

Proof. We have

$$n - 2 = \operatorname{reg}(S/J_G) = \operatorname{reg}(S_1/J_{G_1}) + \operatorname{reg}(S_2/J_{G_2}) \leq (n_1 - 1) + (n_2 - 1) = n - 1.$$

This implies that either $\operatorname{reg}(S_1/J_{G_1}) = n_1-2$ and $\operatorname{reg}(S_2/J_{G_2}) = n_2-1$, or $\operatorname{reg}(S_2/J_{G_2}) = n_2 - 2$ and $\operatorname{reg}(S_1/J_{G_1}) = n_1 - 1$. By Theorem 1.5, in the first case it follows that G_2 is a path, while in the second case, G_1 is a path graph. □

Lemma 3.3. *Let G be a connected graph on the vertex set $[n]$. Suppose that G has a cut vertex v with $\deg_G(v) \geq 4$. Then $\operatorname{reg}(S/J_G) \leq n - 3$.*

Proof. Since v is a cut vertex of G, by [17, Lemma 4.8], we get

$$J_G = J_{G_v} \cap (J_{G\setminus v} + (x_v, y_v))$$

where G_v is the graph on $V(G_v) = V(G)$ with the edge set

$$E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}.$$

Consequently, we have the following exact sequence

$$0 \to \frac{S}{J_G} \to \frac{S}{J_{G_v}} \bigoplus \frac{S}{J_{G\setminus v} + (x_v, y_v)} \to \frac{S}{J_{G_v\setminus v} + (x_v, y_v)} \to 0,$$

since $J_{G_v} + (J_{G\setminus v} + (x_v, y_v)) = J_{G_v\setminus v} + (x_v, y_v)$. From this exact sequence we obtain

$$\operatorname{reg} \frac{S}{J_G} \leq \max\{\operatorname{reg} \frac{S}{J_{G_v}}, \operatorname{reg} \frac{S}{J_{G\setminus v} + (x_v, y_v)}, \operatorname{reg} \frac{S}{J_{G_v\setminus v} + (x_v, y_v)} + 1\}. \tag{9}$$

By our assumption, v has at least 4 neighbors in G. Therefore, in G_v we have a maximal clique with at least 5 vertices. By Theorem 2.1, we have $\operatorname{reg}(S/J_{G_v}) \leq n-4$. The graph $G \setminus v$ has $n - 1$ vertices and at least two connected components, say $G_1, \ldots, G_c$ with $c \geq 2$, because v is a cut vertex of G. Let $S' = K[\{x_j, y_j\} : j \in [n] \setminus \{v\}]$. Then

$$\frac{S'}{J_{G\setminus v}} \cong \frac{S_1}{J_{G_1}} \bigotimes_K \cdots \bigotimes_K \frac{S_c}{J_{G_c}}$$

where $S_i = K[\{x_j, y_j\} : j \in V(G_i)]$ for $i = 1 \ldots, c$. This implies that

$$\operatorname{reg}(S/J_{G\setminus v} + (x_v, y_v)) = \operatorname{reg}(S'/J_{G\setminus v}) = \sum_{i=1}^{c} \operatorname{reg}(S_i/J_{G_i})$$
$$\leq \sum_{i=1}^{c} (|V(G_i)| - 1) = (n-1) - c \leq n-3.$$

If v has at least 4 neighbors in G, then the graph $G_v \setminus v$ has a maximal clique with at least 4 vertices, thus, by Theorem 2.1, we get

$$\operatorname{reg}(S/J_{G_v\setminus v} + (x_v, y_v))) = \operatorname{reg}(S'/J_{G_v\setminus v}) \leq (n-1) - 3 = n-4.$$

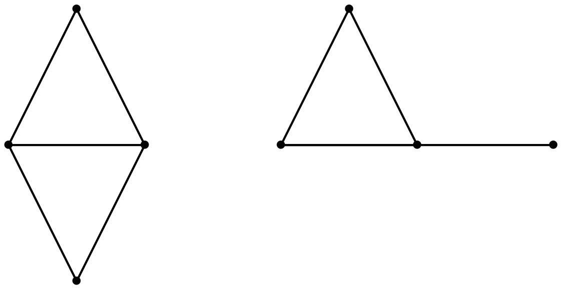

Fig. 1. 4 vertices.

Therefore, from inequality (9), we get $\operatorname{reg}(S/J_G) \leq n-3$. $\square$

Lemma 3.4. *Let G be a connected indecomposable graph on $n \geq 4$ vertices with the following properties:*

(a) *J_G is unmixed;*
(b) *G has a vertex v with exactly two neighbors u_1, u_2 and $\{u_1, u_2\} \in E(G)$.*

Then $\operatorname{reg}(S/J_G) \leq n-3$.

Proof. If $n = 4$, then there are only two graphs which satisfy the condition (b), namely two triangles which share an edge and a triangle with an edge attached to one of its vertices; see Fig. 1.

The first graph does not satisfy the condition (a), while the second graph is decomposable. Thus, we may consider $n \geq 5$.

Let us consider an indecomposable graph G with $n \geq 5$ vertices satisfying the conditions (a) and (b). We claim that $\deg u_1 \geq 4$ or $\deg u_2 \geq 4$. Let us assume that this is not the case, thus $\deg u_1 \leq 3$ and $\deg u_2 \leq 3$. Since G is indecomposable, it follows that $\deg u_1 = 3$, $\deg u_2 = 3$, and there exists a path connecting u_1 and u_2 different from the edge $\{u_1, u_2\}$ and the path u_1, v, u_2. But, in this case, the set $\mathcal{S} = \{u_1, u_2\}$ is a cut set of G with $c(\mathcal{S}) = |\mathcal{S}|$, which is impossible since J_G is an unmixed ideal.

Without loss of generality, we may assume that $\deg u_2 \geq 4$.

We set $e = \{u_1, v\}$. By Proposition 2.3 (a), we have

$$\operatorname{reg} \frac{S}{J_G} \leq \max\left\{\operatorname{reg} \frac{S}{J_{G\setminus e}}, \operatorname{reg} \frac{S}{J_{G\setminus e} : f_e} + 1\right\}. \tag{10}$$

In the graph $G \setminus e$, u_2 is a cut vertex with at least 4 neighbors. Thus, by Lemma 3.3, it follows that

$$\operatorname{reg} \frac{S}{J_{G\setminus e}} \leq n-3.$$

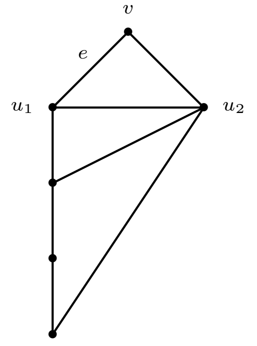

Fig. 2. The graph G when $(G \setminus e)_e \setminus \{u_2, v\}$ is a path.

Now we look at $J_{G\setminus e} : f_e$. By applying Theorem 2.4, we obtain

$$J_{G\setminus e} : f_e = J_{(G\setminus e)_e} + (x_{u_2}, y_{u_2})$$

since all the paths connecting u_1 and v pass trough u_2. Therefore, since v becomes an isolated vertex in the graph $(G \setminus e)_e \setminus u_2$, we get

$$\operatorname{reg} \frac{S}{J_{G\setminus e} : f_e} = \operatorname{reg} \frac{S}{J_{(G\setminus e)_e} + (x_{u_2}, y_{u_2})} = \operatorname{reg} \frac{S'}{J_{(G\setminus e)_e\setminus\{u_2,v\}}}$$

where $S' = K[\{x_j, y_j\} : j \in [n] \setminus \{u_2, v\}]$. If the graph $(G \setminus e)_e \setminus \{u_2, v\}$ is a path, as $\deg u_2 \geq 4$, the graph G looks like in Fig. 2, that is, there are some edges connecting u_2 to some vertices of the path $(G \setminus e)_e \setminus \{u_2, v\}$ different from u_1. But then J_G is not unmixed since $\mathcal{S} = \{u_1, u_2\}$ is a cut set of G with $c(\mathcal{S}) = |\mathcal{S}|$, a contradiction. Therefore, the graph $(G \setminus e)_e \setminus \{u_2, v\}$ is not a path. Thus, by Theorem 1.5, we obtain

$$\operatorname{reg} \frac{S}{J_{G\setminus e} : f_e} = \operatorname{reg} \frac{S'}{J_{(G\setminus e)_e\setminus\{u_2,v\}}} \leq (n-2) - 2 = n - 4,$$

which implies that

$$\operatorname{reg} \frac{S}{J_{G\setminus e} : f_e} + 1 \leq n - 3,$$

and the proof of the lemma is completed. $\square$

We can now state the main result of this section.

Theorem 3.5. *Let G be a connected graph on the vertex set $[n]$. Then the following statements are equivalent:*

(i) $(J_G)_\mathfrak{m} \subset R$ *is licci.*

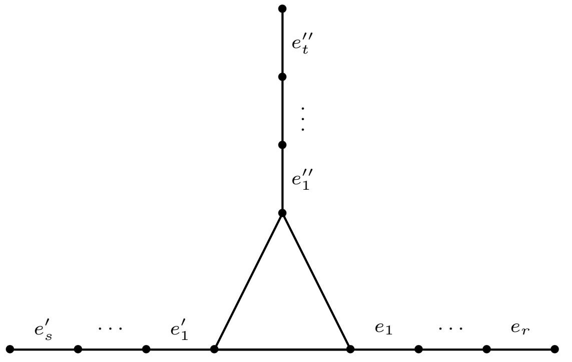

Fig. 3. Licci graphs.

(ii) *J_G is Cohen-Macaulay and $n-2 \leq \operatorname{reg}(S/J_G) \leq n-1$.*

(iii) *G is a path graph or it is isomorphic to one of the graphs depicted in Fig. 3 where r, s, t are non-negative integers. In other words, G is a triangle with possibly some paths connected to some of its vertices.*

Proof. (i) $\Rightarrow$ (ii). Let $(J_G)_{\mathfrak{m}} \subset R$ be licci. By Theorem 1.1, it follows that $\operatorname{reg}(S/J_G) \geq \operatorname{height}(J_G) - 1$. Since J_G is Cohen-Macaulay, thus unmixed, we have $\operatorname{height}(J_G) = \operatorname{height} P_\emptyset(G) = n-1$, by (2). Therefore, if G is connected and $(J_G)_{\mathfrak{m}}$ is licci, then J_G is Cohen-Macaulay and $\operatorname{reg}(S/J_G) \geq n-2$. But we know from [15] that $\operatorname{reg}(S/J_G) \leq n-1$.

Let us prove that (ii) $\Rightarrow$ (iii). Since, by Theorem 1.5, we have $\operatorname{reg}(S/J_G) = n-1$ if and only if G is a path graph, it remains to consider $\operatorname{reg}(S/J_G) = n-2$. By using Lemma 3.2, we may reduce the problem to considering only the case when G is indecomposable. Therefore, in order to get (iii), by taking into account Lemma 3.2, it is enough to show that there is no indecomposable graph G with $|V(G)| \geq 4$ such that J_G is Cohen-Macaulay and $\operatorname{reg}(S/J_G) = n-2$. There is no such graph among those with 4 vertices. Thus, we may consider $n = |V(G)| \geq 5$.

Let us assume that such a graph does exist. By [1, Remark 5.3], since J_G is Cohen-Macaulay, the graph G must have a cut vertex, say v. Since G is indecomposable, v has at least 3 neighbors in G. If v has at least 4 neighbors, by Lemma 3.3, it follows that $\operatorname{reg}(S/J_G) \leq n-3$, a contradiction. Thus, v has exactly 3 neighbors, say w, u_1, u_2. Since G is indecomposable and v is a cut vertex in G, it follows that none of the edges $\{u_1,u_2\}, \{u_1,w\}, \{u_2,w\}$ belongs to $E(G)$. On the other hand, as J_G is unmixed, the graph $G \setminus v$ has exactly two connected components, say G_1 and G_2. We may assume that u_1, u_2 are vertices in G_1 and w is a vertex in G_2. Let $e = \{v, w\}$. By Proposition 2.3 (a), we have

$$n-2=\operatorname{reg}\frac{S}{J_G}\leq \max\left\{\operatorname{reg}\frac{S}{J_{G\setminus e}},\operatorname{reg}\frac{S}{J_{G\setminus e}:f_e}+1\right\}. \tag{11}$$

We observe that $G \setminus e$ has two connected components, namely G' with $V(G') = V(G_1) \cup \{v\}$ and $E(G') = E(G_1) \cup \{\{u_1, v\}, \{u_2, v\}\}$ and $G'' = G_2$. Obviously, G' is not a path graph since G_1 is connected, thus there exists at least one path connecting u_1 and u_2 in G_1 which does not contain v and is not the edge $\{u_1, u_2\}$. On the other hand, if G_2 does not consist only of the isolated vertex w, then G_2 cannot be a path since the graph G is indecomposable. Let $S' = K[\{x_j, y_j\} : j \in V(G')]$ and $S'' = K[\{x_j, y_j\} : j \in V(G'')]$. Then, by Theorem 1.5, we have

$$\operatorname{reg}\frac{S'}{J_{G'}}+\operatorname{reg}\frac{S''}{J_{G''}}\leq (|V(G')|-2)+(|V(G'')|-2)=n-4.$$

Therefore,

$$\operatorname{reg}\frac{S}{J_{G\setminus e}}=\operatorname{reg}\frac{S'}{J_{G'}}+\operatorname{reg}\frac{S''}{J_{G''}}<n-3.$$

If G_2 consist only of the isolated vertex w, then we get

$$\operatorname{reg}\frac{S}{J_{G\setminus e}}=\operatorname{reg}\frac{S'}{J_{G'}}\leq |V(G')|-2=n-3.$$

Thus, in any case we have

$$\operatorname{reg}\frac{S}{J_{G\setminus e}}\leq n-3. \tag{12}$$

Now we look at the term $\operatorname{reg}(S/J_{G\setminus e} : f_e)$ of inequality (11). By Theorem 2.4, it follows that $J_{G\setminus e} : f_e = J_{(G\setminus e)_e}$ since there is no path in G connecting v and w except the edge $e = \{v, w\}$. This is due to the fact that when we remove the cut vertex v from G, we get two connected components by the unmixedness of J_G. The graph $(G \setminus e)_e$ consists as well of two connected components, say H_1 which contains v and H_2 which contains w. If H_2 contains some other vertices together with w, then H_2 cannot be a path since G is indecomposable. The component H_1 is not a path since it contains at least the triangle with vertices u_1, u_2, v. Therefore, if $S_i = K[\{x_j, y_j\} : j \in V(H_i)]$ for $i = 1, 2$, by Theorem 1.5, we obtain

$$\begin{aligned}\operatorname{reg}\frac{S}{J_{G\setminus e}:f_e}&=\operatorname{reg}\frac{S}{J_{(G\setminus e)_e}}=\operatorname{reg}\frac{S_1}{J_{H_1}}+\operatorname{reg}\frac{S_2}{J_{H_2}}\\&\leq (|V(H_1)|-2)+(|V(H_2)|-2)=n-4.\end{aligned}$$

This inequality and (12) contradicts inequality (11).

It remains to analyze the case when H_2 consists of the isolated vertex w. In this case we have

$$\operatorname{reg} \frac{S}{J_{(G\setminus e)_e}} = \operatorname{reg} \frac{S_1}{J_{H_1}}. \tag{13}$$

We claim that H_1 satisfies the conditions of Lemma 3.4. Clearly, H_1 satisfies the condition (b). It remains to prove that J_{H_1} is an unmixed ideal because if H_1 is decomposable in u_1 or u_2, then G is decomposable, and this is impossible by our hypotheses on G. We first observe that any non-empty cut set of H_1 does not contain the vertex v which is a simplicial vertex in H_1. Let us assume that there exists a non-empty cut set $\mathcal{S} \subset V(H_1)$ such that $c_{H_1}(\mathcal{S}) \neq |\mathcal{S}| + 1$. The set $\mathcal{S}$ is obviously a cut set for the graph G as well. Moreover, if $H_1, \ldots, H_{c_{H_1}(\mathcal{S})}$ are the connected components of the restriction of H_1 to the vertex set $V(H_1) \setminus \mathcal{S}$, with $v \in V(H_1)$, then the connected components of the restriction of G to $V(G) \setminus \mathcal{S}$ are $H_1 \cup \{v, w\}, H_2, \ldots, H_{c_{H_1}(\mathcal{S})}$. Hence $c_G(\mathcal{S}) = c_{H_1}(\mathcal{S}) \neq |\mathcal{S}| + 1$, a contradiction to the unmixedness of J_G. Since H_1 is a graph on $n - 1$ vertices which satisfies the conditions of Lemma 3.4, we get $\operatorname{reg}(S_1/J_{H_1}) \leq (n - 1) - 3 = n - 4$. Thus, we have proved that

$$\operatorname{reg} \frac{S}{J_{G\setminus e} : f_e} = \operatorname{reg} \frac{S}{J_{(G\setminus e)_e}} = \operatorname{reg} \frac{S_1}{J_{H_1}} \leq n - 4.$$

This inequality together with (12) contradicts inequality (11) and the proof of (ii) $\Rightarrow$ (iii) is completed.

Finally, we prove the implication (iii) $\Rightarrow$ (i).

As it was observed in the proof of [8, Proposition 3], if $G = G_1 \cup G_2$ is a decomposable graph, then we have $\operatorname{Tor}_i(S/J_{G_1}, S/J_{G_2}) = 0$ for all $i > 0$. In particular, it follows that J_{G_1} and J_{G_2} are transversal ideals in the sense of [10, Section 2]. Now, let G_1 be a triangle with the vertices v_1, v_2, v_3. Then J_{G_1} is a Cohen-Macaulay ideal of height 2, thus it is licci by [19]. If we attach a path G_2 to G_1 in one of its vertices, say v_1, the resulting graph G is decomposable in v_1 and J_{G_2} is a complete intersection ideal. According to [10, Theorem 2.6] or [11, Theorem 4.4], it follows that $(J_G)_\mathfrak{m}$ is a licci ideal. We repeat this argument by attaching a path in the vertex v_2 to G and, next another path in the vertex v_3. In each step, we get a licci ideal. $\square$

Remark 3.6. One may prove the implication (iii)$\Rightarrow$(i) by finding an explicit link of J_G to a complete intersection for a graph G as in Fig. 3. However the proof involves repetitive and technical calculations which we do not include here. Instead, we indicate the main ingredient to derive the constructive proof. Set

$$e_i = \{v_i, v_{i+1}\}, f_i = f_{e_i} = x_i y_{i+1} - y_i x_{i+1} \ (i = 1, 2, \ldots, r),$$
$$e'_i = \{v'_i, v'_{i+1}\}, f'_i = f_{e'_i} = x'_i y'_{i+1} - y'_i x'_{i+1} \ (i = 1, 2, \ldots, s),$$

$$e''_i = \{v''_i, v''_{i+1}\}, f''_i = f_{e''_i} = x''_i y''_{i+1} - y''_i x''_{i+1} \ (i = 1, 2, \ldots, t),$$

where

$$\begin{aligned} x_i &= x_{v_i}, y_i = y_{v_i} \ (i = 1, 2, \ldots, r+1), \\ x'_i &= x_{v'_i}, y'_i = y_{v'_i} \ (i = 1, 2, \ldots, s+1), \\ x''_i &= x_{v''_i}, y''_i = y_{v''_i} \ (i = 1, 2, \ldots, t+1). \end{aligned}$$

We also set

$$\begin{aligned} f &= f_{\{v_1, v'_1\}} = x_1 y'_1 - y_1 x'_1, \\ f' &= f_{\{v'_1, v''_1\}} = x'_1 y''_1 - y'_1 x''_1, \\ f'' &= f_{\{v''_1, v_1\}} = x''_1 y_1 - y''_1 x_1. \end{aligned}$$

We put

$$S = K[x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}, x'_1, \ldots, x'_{s+1}, y'_1, \ldots, y'_{s+1}, x''_1, \ldots, x''_{t+1}, y''_1, \ldots, y''_{t+1}].$$

Then $J_G = (f, f', f'', f_1, \ldots, f_r, f'_1, \ldots, f'_s, f''_1, \ldots, f''_t)$.

Set

$$I := (f + f', f'', f_1, \ldots, f_r, f'_1, \ldots, f'_s, f''_1, \ldots, f''_t),$$

and

$$L := (x_1 - x''_1, y_1 - y''_1, f_1, \ldots, f_r, f'_1, \ldots, f'_s, f''_1, \ldots, f''_t).$$

Then one can show that I, L are complete intersections with height $r + s + t + 2$, and, moreover, the equality $L = I : J_G$ holds.

An immediate consequence of Theorem 3.5 is the following.

Corollary 3.7. *Let G be a connected bipartite graph. Then the ideal $(J_G)_\mathfrak{m} \subset R = S_\mathfrak{m}$ is licci if and only if G is a path graph, or equivalently, J_G is a complete intersection.*

We now turn to the disconnected graphs.

Proposition 3.8. *Let G be a graph with the connected components $G_1, \ldots, G_c$ where $c \geq 2$. Then $(J_G)_\mathfrak{m} \subset R = S_\mathfrak{m}$ is licci if and only if either all the connected components of G are paths or one component of G is isomorphic to a graph of Fig. 3 and all the other components are paths.*

Proof. We first remark that, by [10, Theorem 2.6] or [11, Theorem 4.4], if the components of G satisfy the conditions of the proposition, then $(J_G)_\mathfrak{m}$ is licci since the ideals J_{G_i} are pairwise transversal by [4, Lemma 3.1].

For the converse, let $(J_G)_\mathfrak{m}$ be a licci ideal. Then J_G is Cohen-Macaulay which implies that all the ideals J_{G_i} are Cohen-Macaulay and

$$\operatorname{reg}(S/J_G) \geq \operatorname{height}(J_G) - 1 = \operatorname{height}(J_{G_1}) + \cdots + \operatorname{height}(J_{G_c}) - 1 = n - c - 1.$$

On the other hand, we have

$$\operatorname{reg}(S/J_G) = \sum_{i=1}^{c} \operatorname{reg}(S_i/J_{G_i}) \leq \sum_{i=1}^{c} (|V(G_i)| - 1) = n - c.$$

Here $S_i = K[\{x_j, y_j\} : j \in V(G_i)]$ for $1 \leq i \leq c$. The above inequalities imply that $\operatorname{reg}(S/J_G) = n-c$ or $\operatorname{reg}(S/J_G) = n-c-1$. In the first case, it follows that $\operatorname{reg}(S_i/J_{G_i}) = |V(G_i)| - 1$ for all i, which implies that all the connected components of G are path graphs.

Let $\operatorname{reg}(S/J_G) = n - c - 1$. This means that for one of the connected components, say G_1, we have $\operatorname{reg}(S_1/J_{G_1}) = |V(G_1)| - 2$ and all the other components of G are path graphs. Then, by Theorem 3.5, it follows that G_1 is isomorphic to one of the graphs displayed in Fig. 3. $\square$

4. Licci binomial edge ideals of chordal graphs

In this section we show that if we restrict to chordal graphs, we may relax the condition (ii) in Theorem 3.5, namely, we may ask that J_G is only unmixed instead of being Cohen-Macaulay. Before proving the main theorem of this section, we need a preparatory result. We first recall that for a graph G, $c(G)$ denotes the number of maximal cliques of G, that is, the number of facets of the clique complex $\Delta(G)$.

Lemma 4.1. *Let G be a connected chordal graph with n vertices. Then $c(G) = n - 2$ if and only if the following conditions hold:*

(i) *the maximal cliques of G have at most 3 vertices;*
(ii) *G has at least one maximal clique with 3 vertices;*
(iii) *G has exactly one maximal clique with 3 vertices or, for any two triangles F_1, F_2 of $\Delta(G)$, there is a sequence of triangles $F_1 = F_{i_1}, \ldots, F_{i_r} = F_2$ such that for any $1 \leq j \leq r - 1$, F_{i_j} and $F_{i_{j+1}}$ share an edge.*

Proof. Let $c(G) = n - 2$. Then (i) follows by [23, Proposition 3.1]. If G has no maximal clique with 3 vertices, then G is a tree, thus $c(G) = n - 1$, contradiction. Therefore, condition (ii) holds.

We prove (iii) by induction on n. Since G is chordal, by Dirac's theorem, we may order the facets of $\Delta(G)$ as $F_1, \ldots, F_c$ where $c = c(G)$ such that F_i is a leaf of $\langle F_1, \ldots, F_i \rangle$ for all i. If F_c is an edge, say $F_c = \{v, w\}$ with $\deg w = 1$, then the graph $G \setminus w$ has $n-1$ vertices and $n-3$ cliques, thus, by induction, it satisfies (iii), and, consequently, G satisfies (iii) as well.

Let F_c be a triangle with the vertices u, v, w and assume that F_j with $j < c$ is a branch of F_c. If $F_j \cap F_c$ consists of just one vertex, say $F_j \cap F_c = \{v\}$, then the subgraph $G' = G \setminus \{u, w\}$ has $n-2$ vertices and $n-3$ maximal cliques, therefore G' is a tree. This implies that $\Delta(G)$ has exactly one facet with 3 elements, and condition (iii) is automatically fulfilled. Let us now assume that the branch F_j intersects F_c in the edge $\{v, w\}$. We consider the graph $G \setminus u$. This is a graph on $n-1$ vertices with $n-3$ maximal cliques, thus, by the inductive hypothesis, it satisfies (iii). Let us choose two triangles F, F' in $\Delta(G)$. If they are facets in $\Delta(G \setminus u)$, then they satisfy (iii). Otherwise, we may assume that $F' = F_c$. But then, by the inductive hypothesis on $G \setminus u$ there is a sequence of triangles $F = F_{i_1}, \ldots, F_{i_r} = F_j$ such that for any $1 \leq s \leq r-1$, F_{i_s} and $F_{i_{s+1}}$ share an edge. Then the sequence $F = F_{i_1}, \ldots, F_{i_r} = F_j, F_{i_{r+1}} = F_c$ satisfies the required condition for G.

For the converse, let us assume that G is a connected chordal graph with n vertices which satisfies the three conditions of the statement. By condition (ii) and [23, Proposition 3.1], it follows that $c(G) \leq n-2$.

Let us assume that there exists a connected chordal graph G satisfying conditions (i)–(iii) and such that $c(G) < n-2$ and choose one with the minimal number of vertices. We consider again the leaf order $F_1, \ldots, F_c$ on the facets of $\Delta(G)$ and take F_j with $j < c$ a branch of F_c. If F_c is an edge, $F_c = \{v, w\}$ with $\deg w = 1$, then the graph $G \setminus w$ has $n-1$ vertices and satisfies conditions (i)–(iii), thus, by our assumption on G we have $c(G \setminus w) = n-3$, which implies that $c(G) = n-2$, contradiction.

If F_c is a triangle, $F_c = \{u, v, w\}$, and F_j intersects F_c in just one vertex, say v, then we have the following cases.

Case 1. The facet F_c is the only triangle in $\Delta(G)$. Then, the subgraph $G \setminus \{u, w\}$ is a tree on $n-2$ vertices, thus $\Delta(G \setminus \{u, w\})$ has $n-3$ maximal cliques, which implies that $c(G) = n-2$, contradiction.

Case 2. There exists a triangle $F \in \Delta(G \setminus \{u, w\})$. Then, as G satisfies condition (iii), there exists a triangle $F' \neq F_c$ which intersects F_c along an edge. But this is impossible since the branch F_j intersects F_c in one vertex.

Finally, we have to consider that F_j shares an edge with F_c, say $F_j \cap F_j = \{v, w\}$. Since F_j is a branch of F_c, there is no other facet F of $\Delta(G)$ with $F \cap F_c = \{u, w\}$ or $F \cap F_c = \{u, v\}$. Then the graph $G \setminus u$ obviously satisfies conditions (i)–(iii) and has $n-1$ vertices. By the choice of G, we have $c(G \setminus u) = n-3$, thus $c(G) = n-2$, contradiction. $\square$

Theorem 4.2. *Let G be a connected chordal graph on the vertex set $[n]$. Then the following statements are equivalent:*

(i) $(J_G)_\mathfrak{m} \subset R$ *is licci.*
(ii) J_G *is Cohen-Macaulay and* $n-2 \leq \operatorname{reg}(S/J_G) \leq n-1$.
(iii) J_G *is unmixed and* $n-2 \leq \operatorname{reg}(S/J_G) \leq n-1$.
(iv) G *is a path graph or it is isomorphic to a graph depicted in Fig. 3.*

Proof. We have to prove only the implication (iii) $\Rightarrow$ (iv). Let J_G be unmixed and let $\operatorname{reg}(S/J_G) = n-1$. Then, by Theorem 1.5, G is a path graph. Let us now discuss the case when $\operatorname{reg}(S/J_G) = n-2$. By [23, Theorem 3.5], we have $\operatorname{reg}(S/J_G) \leq c(G)$. Thus, we get $c(G) \geq n-2$. If $c(G) = n-1$, then G is a tree, but since J_G is unmixed, by [3, Corollary 1.2], it follows that G is a path graph.

As in the proof of Theorem 3.5, it is enough to show that there is no indecomposable chordal graph with $n \geq 4$ vertices which satisfies the conditions J_G unmixed and $\operatorname{reg}(S/J_G) = c(G) = n-2$. Let us assume that such a graph G does exists.

By Theorem 2.1, it follows that the maximal cliques of G have at most three vertices. As G is a chordal graph, by Dirac's theorem, it follows that the facets of the clique complex $\Delta(G)$ of G have a leaf order, say $F_1, \ldots, F_{n-2}$. In particular, this means that F_{n-2} has a branch. Let F_j with $j \leq n-3$ be a branch of F_{n-2}.

Case 1. Assume that the intersection $F_j \cap F_{n-2}$ consists of only one vertex of G, say $F_j \cap F_{n-2} = \{v\}$. If F_{n-2} has only the branch F_j, then G is decomposable which contradicts our assumption on G. Thus F_{n-2} has $q \geq 2$ branches, say $F_{j_1}, \ldots, F_{j_q}$. Then, as J_G is unmixed, it follows that the induced subgraph of $G \setminus v$ on the vertex set $\bigcup_{i=1}^q F_{j_i} \setminus v$ is connected. This implies that all the facets $F_{j_1}, \ldots, F_{j_q}$ are triangles. If F_{n-2} is also a triangle, we get a contradiction to Lemma 4.1. Thus, F_{n-2} must be an edge and then v is a cut vertex of G with $\deg_G(v) \geq 4$. By Lemma 3.3, it follows that $\operatorname{reg}(S/J_G) \leq n-3$, a contradiction.

Case 2. Assume that the intersection $F_j \cap F_{n-2}$ consists of two vertices of G, say $F_j \cap F_{n-2} = \{v, w\}$. In this case, F_{n-2} is a triangle with the vertices u, v, w. Since J_G is unmixed, there must be other facets of $\Delta(G)$ whose intersection with F_{n-2} is contained in $\{v, w\}$ or equal to $\{v, w\}$. Let $F_{j_1}, \ldots, F_{j_q}$ with $q \geq 2$ and $j_q = j$ be the facets of $\Delta(G)$ with $F_{j_s} \cap F_{n-2} \subseteq \{v, w\}$ for $1 \leq s \leq q$. As v is not a simplicial vertex in G, we may apply again [17, Lemma 4.8] and get

$$J_G = J_{G_v} \cap (J_{G\setminus v} + (x_v, y_v)).$$

We use the following exact sequence of S–modules:

$$0 \to \frac{S}{J_G} \to \frac{S}{J_{G_v}} \bigoplus \frac{S}{J_{G\setminus v} + (x_v, y_v)} \to \frac{S}{J_{G_v \setminus v} + (x_v, y_v)} \to 0,$$

to derive that

$$\operatorname{reg}\frac{S}{J_G}\leq \max\{\operatorname{reg}\frac{S}{J_{G_v}},\operatorname{reg}\frac{S}{J_{G\setminus v}+(x_v,y_v)},\operatorname{reg}\frac{S}{J_{G_v\setminus v}+(x_v,y_v)}+1\}. \tag{14}$$

By [23, Lemma 3.4], it follows that $c(G_v)\leq c(G)-q$, hence, by our assumption on q, we get $c(G_v)\leq n-4$. On the other hand, by [23, Lemma 3.3], we have $c(G_v\setminus v)\leq c(G_v)$, thus $c(G_v\setminus v)\leq n-4$. In particular, it follows that

$$\operatorname{reg}(S/J_{G_v})\leq n-4 \text{ and } \operatorname{reg}(S/J_{G_v\setminus v}+(x_v,y_v))\leq n-4. \tag{15}$$

Therefore, by (14), we must have

$$\operatorname{reg}\frac{S}{J_{G\setminus v}+(x_v,y_v)}=\operatorname{reg}\frac{S'}{J_{G\setminus v}}\geq n-2,$$

where $S'=K[\{x_j,y_j\}:j\in[n]\setminus\{v\}]$. As $G\setminus v$ has $n-1$ vertices, it follows by Theorem 1.5 that $G\setminus v$ is a path graph. But in this case, $\mathcal{S}=\{v,w\}$ is a cut set of G because G is indecomposable. In addition, the restriction of G to the vertex set $[n]\setminus\{v,w\}$ has two connected components, which is a contradiction to the unmixedness of J_G. $\square$

References

[1] D. Bolognini, A. Macchia, F. Strazzanti, Binomial edge ideals of bipartite graphs, Eur. J. Comb. 70 (2018) 1–25.
[2] G.A. Dirac, On rigid circuit graphs, Abh. Math. Semin. Univ. Hamb. 25 (1961) 71–76.
[3] V. Ene, J. Herzog, T. Hibi, Cohen-Macaulay binomial edge ideals, Nagoya Math. J. 204 (2011) 57–68.
[4] H.T. Hà, H.D. Nguyen, N.V. Trung, T.N. Trung, Symbolic powers of sums of ideals, Math. Z. 294 (2020) 1499–1520.
[5] J. Herzog, T. Hibi, Monomial Ideals, Grad. Texts in Math., vol. 260, Springer, London, 2010.
[6] J. Herzog, T. Hibi, F. Hreinsdotir, T. Kahle, J. Rauh, Binomial edge ideals and conditional independence statements, Adv. Appl. Math. 45 (2010) 317–333.
[7] J. Herzog, T. Hibi, H. Ohsugi, Binomial Ideals, Grad. Texts in Math., vol. 279, Springer, London, 2018.
[8] J. Herzog, G. Rinaldo, On the extremal Betti numbers of binomial edge ideals of block graphs, Electron. J. Comb. 25 (1) (2018) #P1.63.
[9] C. Huneke, B. Ulrich, The structure of linkage, Ann. Math. 126 (1987) 277–334.
[10] M.R. Johnson, Linkage and sums of ideals, Trans. Am. Math. Soc. 350 (5) (1998) 1913–1930.
[11] M.R. Johnson, P. Mantero, Ubiquity of complete intersection liaison classes, Ill. J. Math. 58 (3) (2014) 647–663.
[12] D. Kiani, S. Saeedi Madani, Some Cohen-Macaulay and unmixed binomial edge ideals, Commun. Algebra 43 (2015) 5434–5453.
[13] D. Kiani, S. Saeedi Madani, The Castelnuovo–Mumford regularity of binomial edge ideals, J. Comb. Theory, Ser. A 139 (2016) 80–86.
[14] K. Kimura, N. Terai, K. Yoshida, Licci squarefree monomial ideals generated in degree two or with deviation two, J. Algebra 390 (2013) 264–289.
[15] K. Matsuda, S. Murai, Regularity bounds for binomial edge ideals, J. Commut. Algebra 5 (1) (2013) 141–149.
[16] F. Mohammadi, L. Sharifan, Hilbert function of binomial edge ideals, Commun. Algebra 42 (2014) 688–703.
[17] M. Ohtani, Graphs and ideals generated by some 2–minors, Commun. Algebra 39 (2011) 905–917.

[18] I. Peeva, Graded Syzygies, Algebr. Appl., vol. 14, Springer, 2010.
[19] C. Peskine, L. Szpiro, Liaison des variétés algébriques, I, Invent. Math. 26 (1974) 271–302.
[20] A. Rauf, G. Rinaldo, Construction of Cohen-Macaulay binomial edge ideals, Commun. Algebra 42 (2014) 238–252.
[21] G. Rinaldo, Cohen-Macaulay binomial edge ideals of small deviation, Bull. Math. Soc. Sci. Math. Roum. 56(104) (4) (2013) 497–503.
[22] G. Rinaldo, Cohen-Macaulay binomial edge ideals of cactus graphs, J. Algebra Appl. 18 (4) (2019) 1950072.
[23] M. Rouzbahani Malayeri, S. Saeedi Madani, D. Kiani, Regularity of binomial edge ideals of chordal graphs, arXiv:1810.03119.
[24] S. Saeedi Madani, Binomial edge ideals: a survey, in: V. Ene, E. Miller (Eds.), Multigraded Algebras and Applications, in: Springer Proceedings in Mathematics & Statistics, 2018, pp. 83–94.