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An upper bound for nonnegative rank

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ABSTRACT

We provide a nontrivial upper bound for the nonnegative rank of rank-three matrices which allows us to prove that $\lceil 6n/7 \rceil$ linear inequalities suffice to describe a convex n-gon up to a linear projection.

1. Preliminaries

Consider a convex polytope $P \subset \mathbb{R}^n$. An *extension* [5,6] of P is a polytope $Q \subset \mathbb{R}^d$ such that P can be obtained from Q as an image under a linear projection from $\mathbb{R}^d$ to $\mathbb{R}^n$. An *extended formulation* [6,10] of P is a description of Q by linear equations and linear inequalities (together with the projection). The *size* [6,10] of the extended formulation is the number of facets of Q. The *extension complexity* [6,10] of a polytope P is the smallest size of any extended formulation of P, that is, the minimal possible number of inequalities in the description of Q. The number of facets of Q can sometimes be significantly smaller [5] than that of P, and this phenomenon can be used to reduce the complexity of linear programming problems useful for numerous applications [3,5,10].

An important result providing the linear algebraic characterization of extended formulations has been obtained in 1991 by Yannakakis [13]. Let a polytope P (with v vertices and f facets) be defined as the set of all points $x \in \mathbb{R}^n$ satisfying the conditions $c_i(x) \geqslant \beta_i$ and $c_j(x) = \beta_j$, for $i \in \{1, \ldots, f\}$ and $j \in \{f+1, \ldots, q\}$, where $c_1, \ldots, c_q$ are linear functionals on $\mathbb{R}^n$. A slack matrix $S = S(P)$ of P is an f-by-v matrix satisfying $S_{it} = c_i(p_t) - \beta_i$, where $p_1, \ldots, p_v$ denote the vertices of P, and we note that S is nonnegative. The following well-known result (see [6, Corollary 5] and also [8, Lemma 3.1]) characterizes the rank of $S(P)$ in terms of the dimension of P.

Proposition 1.1. *A slack matrix of a polytope P has classical rank one greater than the dimension of P.*

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The result by Yannakakis points out the connection between extension complexity and nonnegative factorizations and can now be formulated as follows [6,10,13].

Theorem 1.2. *The extension complexity of a polytope P is equal to the minimal k for which $S(P)$ can be written as a product of f-by-k and k-by-v nonnegative matrices.*

The smallest integer k for which there exists a factorization $A = BC$ with $B \in \mathbb{R}_+^{n\times k}$ and $C \in \mathbb{R}_+^{k\times m}$ is called the *nonnegative rank* of a nonnegative matrix $A \in \mathbb{R}_+^{n\times m}$. Nonnegative factorizations are widely studied and used in data analysis, statistics, computational biology, clustering and numerous other applications [2]. There are still many open questions on nonnegative rank that are interesting for different applications, and a considerable number of them is related to providing bounds on the nonnegative rank in terms of other matrix invariants [4,6,10].

It is easy to show that the nonnegative rank of a matrix equals the classical rank if one of them is less than three [2]. However, even for rank-three m-by-n matrices, the only known upper bound is $\min\{m, n\}$ which is trivial.

Problem 1.3. (See [1].) Assume $n \geqslant 3$. Does there exist a rank-three n-by-n nonnegative matrix with nonnegative rank equal to n?

In view of Proposition 1.1 and Theorem 1.2, one can ask a related question: Does there exist a convex n-gon with extension complexity equal to n, for every n? For $n \leqslant 5$, Problem 1.3 has been solved in the affirmative in [6]. In [7] it was noted that a sufficiently irregular convex hexagon has full extension complexity, providing an affirmative answer for $n = 6$. For $n \geqslant 7$, the problem has been open.

Lin and Chu [11] claimed a positive resolution for Problem 1.3, but their argument has been shown to contain a gap [6,9]. A negative answer for Problem 1.3 has been obtained in [6] for a special case of so-called Euclidean distance matrices. The factorizations of those matrices have been studied subsequently in [9], and the logarithmic upper bounds have been obtained in a number of important special cases. A detailed investigation of extended formulations of convex polygons has been undertaken in [5], but the question about an n-gon with extension complexity equal to n has also been left open.

In our paper we solve Problem 1.3 and prove that for $n > 6$, the answer is negative. In fact, we provide a nontrivial upper bound for the nonnegative rank and prove that an m-by-n rank-three matrix cannot have nonnegative rank greater than $\lceil 6\min\{m, n\}/7\rceil$. From our results it follows that a convex n-gon has extension complexity at most $\lceil 6n/7\rceil$. That is, we prove that any convex n-gon admits a description with $\lceil 6n/7\rceil$ linear inequalities up to a projection.

The organization of the paper is as follows. In Section 2, we prove the main result in a special case of slack matrices of convex heptagons, thus showing that any convex heptagon admits a description with six linear inequalities. In Section 3, we use those results and prove the main results of our paper, which include the upper bound for the extension complexity of a polygon and for the nonnegative rank of a rank-three matrix.

2. Factoring a slack matrix of a convex heptagon

The problem of constructing nonnegative factorizations is rather hard from the computational point of view. Being NP-hard in general [12], this problem can be also difficult to solve even for explicitly written matrices of relatively small size. In fact, the problem of computing the nonnegative ranks of certain n-by-n rank-three matrices with algebraically independent entries remained open for $n = 7$, see [5].

In this section we present a technique that will allow us to factor the matrices of a certain special form, and we will then be able to prove that slack matrices of convex heptagons have nonnegative ranks less than 7. The considerations of this section deal with matrices having not more than seven rows and seven columns, and we adopt the following convention in order to make the presentation more concise.

Convention 2.1. Throughout this section, the row and column indices of the matrices considered are to be understood as the elements of the ring $\mathbb{Z}/7\mathbb{Z}$. In particular, $A_{3+6,6+1}$ will stand for the $(2,7)$th entry of a matrix A. Also, we will use the letters i and j only for denoting such indices in the present section, and we operate with i and j as with elements from $\mathbb{Z}/7\mathbb{Z}$, throughout the section.

Let us introduce a certain special form of matrices which will be important for the considerations of the present section. By $W[i,j,k]$ we denote the submatrix of W formed by the rows with indices i, j, and k.

Notation 2.2. Given a tuple $\alpha = (a_1, a_2, a_3, b_1, b_2, b_3)$ of six real numbers. By $W(\alpha)$ we will denote the 7-by-3 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & a_1 & a_2 & a_3 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}^{\top},$$

and by $\mathcal{V}(\alpha)$ the 7-by-7 matrix with (i,j)th entry equal to $\det W[i-1, j-2, j-1]$.

The following lemma points out a symmetry in the construction of $\mathcal{V}$.

Lemma 2.3. *Matrices $\mathcal{V}(a_1, a_2, a_3, b_1, b_2, b_3)$ and $\mathcal{V}(b_3, b_2, b_1, a_3, a_2, a_1)$ coincide up to relabeling the rows and columns.*

Proof. Perform the permutation $(16)(25)(34)$ on the row indices and $(17)(26)(35)$ on the column indices of $\mathcal{V}(b_3, b_2, b_1, a_3, a_2, a_1)$. $\square$

Let us present a useful special case when the nonnegative rank of $\mathcal{V}$ is not full.

Lemma 2.4. *Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. If $a_1 + b_1 \geqslant a_2 + b_2$ and $a_3 + b_3 \geqslant a_2 + b_2$, then V has nonnegative rank less than 7.*

Proof. One can check that $V = FG$, where

$$F = \begin{pmatrix} 0 & 0 & 1 & V_{41} + V_{47} & V_{61} & 0 \\ 0 & 0 & 0 & 1 & a_1 - a_2 + b_1 - b_2 & 1 \\ V_{31} & 0 & 0 & 1 & V_{37} & 0 \\ V_{41} & 1 & 0 & 0 & V_{47} & 0 \\ -a_2 + a_3 - b_2 + b_3 & 1 & 0 & 0 & 0 & 1 \\ V_{61} & V_{31} + V_{37} & 1 & 0 & 0 & 0 \\ 0 & V_{31} & 1 & V_{47} & 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & V_{32}/V_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{21}/V_{31} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{13} & 1 & V_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & V_{57}/V_{47} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{65}/V_{47} & 1 \\ V_{72} & 0 & 0 & 1 & 0 & 0 & V_{57} \end{pmatrix}. \quad \square$$

Now we show how can one construct new full-rank matrices from a given vector ψ.

Lemma 2.5. *Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. Take $\alpha_1 = (1-a_3-b_3)/(1-b_3)$, $\alpha_2 = (a_1 - a_1b_3 - a_3 + a_3b_1)/(a_1 - a_1b_3)$, $\alpha_3 = (a_2 - a_2b_3 - a_3 + a_3b_2)/(a_2 - a_2b_3)$, $\beta_1 = a_3$, $\beta_2 = a_3/a_1$, $\beta_3 = a_3/a_2$. Then the matrix $U = \mathcal{V}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ satisfies $U_{ij} > 0$ if $i \notin \{j, j+1\}$ and has nonnegative rank equal to that of V.*

Proof. One can check that $V = Q_1 U Q_2$, where

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(1-b_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1/a_3 \\ a_2/a_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_1a_2(1-b_3)}{a_3} \\ a_2(1-b_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1-b_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-b_3}{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a_1(1-b_3)}{a_3} & 0 \end{pmatrix}.$$

Since the numbers $1 - b_3 = V_{42}$, $a_1 = V_{63}$, $a_2 = V_{73}$, and $a_3 = V_{13}$ are positive, the result follows. □

The following six real sequences will be important in our considerations.

Notation 2.6. Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. We will consider the six sequences $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ of reals defined by $\alpha_1(0) = a_1$, $\alpha_2(0) = a_2$, $\alpha_3(0) = a_3$, $\beta_1(0) = b_1$, $\beta_2(0) = b_2$, $\beta_3(0) = b_3$, and also

$$\alpha_1(t+1) = \frac{1 - \alpha_3(t) - \beta_3(t)}{1 - \beta_3(t)},$$

$$\alpha_{\chi+1}(t+1) = \frac{\alpha_\chi(t) - \alpha_\chi(t)\beta_3(t) - \alpha_3(t) + \alpha_3(t)\beta_\chi(t)}{\alpha_\chi(t) - \alpha_\chi(t)\beta_3(t)} \quad \text{for } \chi \in \{1, 2\},$$

$$\beta_1(t+1) = \alpha_3(t), \qquad \beta_2(t+1) = \alpha_3(t)/\alpha_1(t), \qquad \beta_3(t+1) = \alpha_3(t)/\alpha_2(t).$$

Remark 2.7. Lemma 2.5 shows that the sequences $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ are well defined.

It turns out that the sequences introduced are in fact cyclic.

Lemma 2.8. *Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. Then $\alpha_1(7) = a_1$, $\alpha_2(7) = a_2$, $\alpha_3(7) = a_3$, $\beta_1(7) = b_1$, $\beta_2(7) = b_2$, $\beta_3(7) = b_3$.*

Proof. By routine computation. □

The following lemma gives a necessary condition for a matrix to be full-rank.

Lemma 2.9. *Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. If $\alpha_1(2) + \beta_1(2) \leqslant \alpha_2(2) + \beta_2(2)$, then $\alpha_2(6) + \beta_2(6) < \alpha_3(6) + \beta_3(6)$.*

Proof. A routine computation shows that

$$\alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2) = \frac{(-a_3 + a_2(1 - b_3))V_{32}V_{21}}{V_{31}V_{73}V_{42}V_{52}},$$

so the sign of $\alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2)$ equals that of $-a_3 + a_2(1 - b_3)$. Similarly,

$$\alpha_3(6) + \beta_3(6) - \alpha_2(6) - \beta_2(6) = \frac{V_{46}(-a_3 + a_1(1 - b_3))}{V_{15}V_{36}},$$

so the sign of $\alpha_3(6) + \beta_3(6) - \alpha_2(6) - \beta_2(6)$ is that of $-a_3 + a_1(1 - b_3)$. It remains to note that $1 - b_3 = V_{42} > 0$ and $a_1 - a_2 = V_{37} > 0$. □

In fact, we can obtain a stronger condition that holds for full-rank matrices.

Lemma 2.10. *Let $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ be a tuple for which the matrix $V = \mathcal{V}(\psi)$ has full nonnegative rank and $V_{ij} > 0$ if $i \notin \{j-1, j\}$. Then, either $\alpha_1(t) + \beta_1(t) < \alpha_2(t) + \beta_2(t) < \alpha_3(t) + \beta_3(t)$ for every t or $\alpha_1(t) + \beta_1(t) > \alpha_2(t) + \beta_2(t) > \alpha_3(t) + \beta_3(t)$ for every t.*

Proof. Assume that $\alpha_1(t) + \beta_1(t) \leqslant \alpha_2(t) + \beta_2(t)$, for some t. Applying Lemma 2.9 to the tuple $\psi' = (\alpha_1(t+5), \alpha_2(t+5), \alpha_3(t+5), \beta_1(t+5), \beta_2(t+5), \beta_3(t+5))$ and taking into account Lemma 2.8, we obtain that $\alpha_2(t+4) + \beta_2(t+4) < \alpha_3(t+4) + \beta_3(t+4)$. Lemma 2.4 then shows that $\alpha_1(t+4) + \beta_1(t+4) < \alpha_2(t+4) + \beta_2(t+4)$, and we conclude that $\alpha_1(t+4k) + \beta_1(t+4k) < \alpha_2(t+4k) + \beta_2(t+4k) < \alpha_3(t+4k) + \beta_3(t+4k)$, for any positive integer k.

Now assume $\alpha_1(t) + \beta_1(t) > \alpha_2(t) + \beta_2(t)$. By Lemma 2.4, we have $\alpha_2(t) + \beta_2(t) > \alpha_3(t) + \beta_3(t)$, and so by Lemma 2.9, $\alpha_1(t+3) + \beta_1(t+3) > \alpha_2(t+3) + \beta_2(t+3)$. Finally, we conclude that $\alpha_1(t+3k) + \beta_1(t+3k) > \alpha_2(t+3k) + \beta_2(t+3k) > \alpha_3(t+3k) + \beta_3(t+3k)$, for any positive k. □

Finally, let us show that a matrix $\mathcal{V}(\psi)$ cannot have full nonnegative rank.

Lemma 2.11. *Given a tuple $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$ for which the matrix $V = \mathcal{V}(\psi)$ satisfies $V_{ij} > 0$ if $i \notin \{j-1, j\}$. Then V has nonnegative rank less than 7.*

Proof. Assume the converse and apply the results of Lemma 2.3 and Lemma 2.10. We can assume without a loss of generality that $\alpha_1(t) + \beta_1(t) < \alpha_2(t) + \beta_2(t) < \alpha_3(t) + \beta_3(t)$, for any nonnegative integer t. Note that $\alpha_3(0) + \beta_3(0) - \alpha_1(0) - \beta_1(0) = a_3 + b_3 - a_1 - b_1$, and routine computations also allow us to check that

$$\alpha_2(1) + \beta_2(1) - \alpha_1(1) - \beta_1(1) = \frac{V_{13}(b_1 + (a_1 - 1)b_3)}{V_{63}V_{42}},$$

$$\alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2) = \frac{V_{13}V_{21}(a_2(1 - b_3) - a_3)}{V_{73}V_{31}V_{42}V_{52}}.$$

Noting that also $1 - b_3 = V_{42} > 0$ and $V_{37} = a_1 - a_2 > 0$, we obtain

$$b_3(1 - a_1) < b_1, \qquad a_3 < a_2(1 - b_3), \qquad a_3 + b_3 > a_1 + b_1, \qquad b_3 < 1, \quad \text{and} \quad a_1 > a_2. \tag{2.1}$$

Now let us check that (2.1) is a contradiction. In fact, the first of these inequalities implies $a_1 + b_1 > a_1 + b_3 - b_3 a_1$, taking into an account the third we obtain $a_3 + b_3 > a_1 + b_3 - b_3 a_1$. Thus we have $a_3 > a_1(1 - b_3)$, which implies $a_3 > a_2(1 - b_3)$ because of the last two inequalities. □

Let us now check that 7-by-7 matrices of a more general form have nonnegative rank at most 6 as well. By $U[r_1, r_2, r_3|c_1, c_2, c_3]$ we denote the submatrix of U formed by the rows with indices r_1, r_2, r_3 and columns with c_1, c_2, c_3.

Lemma 2.12. *Assume that a 7-by-7 matrix U has classical rank 3 and satisfies $U_{ij} = 0$ if $i \in \{j-1, j\}$ and $U_{ij} > 0$ otherwise. Then U has nonnegative rank less than 7.*

Proof. Denote by U' the matrix obtained from U by multiplying the third column by U_{54}/U_{53}, the fifth column by U_{24}/U_{25}, the third row by $\frac{U_{25}}{U_{24}U_{35}}$, the fourth row by $\frac{U_{53}}{U_{43}U_{54}}$, the i'th row by $1/U_{i'4}$ (for i' from $1, 2, 5, 6, 7$). So we have

$$U' = \begin{pmatrix} 0 & 0 & a_3 & 1 & b_3 & U'_{16} & U'_{17} \\ U'_{21} & 0 & 0 & 1 & 1 & U'_{26} & U'_{27} \\ U'_{31} & U'_{32} & 0 & 0 & 1 & U'_{36} & U'_{37} \\ U'_{41} & U'_{42} & 1 & 0 & 0 & U'_{46} & U'_{47} \\ U'_{51} & U'_{52} & 1 & 1 & 0 & 0 & U'_{57} \\ U'_{61} & U'_{62} & a_1 & 1 & b_1 & 0 & 0 \\ 0 & U'_{72} & a_2 & 1 & b_2 & U'_{76} & 0 \end{pmatrix}.$$

Since U' has classical rank 3, there are certain real constants $c_1, \ldots, c_7$ such that $U'_{ij} = c_j \det U'[i, j-1, j|3, 4, 5]$, for any i and j. Therefore, we obtain $U'_{ij} = c_j V_{ij}$ for any i and j, where V is the matrix $\mathcal{V}(a_1, a_2, a_3, b_1, b_2, b_3)$ from Notation 2.2. Since $V_{13} = V_{32}$ and $V_{72} = V_{21}$, the numbers c_1, c_2, and c_3 are of the same sign. Similarly, $V_{65} = V_{46}$ and $V_{76} = V_{57}$, so that the numbers c_5, c_6, and c_7 are of the same sign as well. Further, since $V_{24} = V_{25} = V_{43} = 1$, we obtain $c_3 = c_4 = c_5 = 1$, and the numbers $c_1, \ldots, c_7$ are thus all positive. So we can conclude that U and V coincide up to multiplying the rows and columns by positive numbers, and the result then follows from Lemma 2.11. □

Now we can prove the main result of the present section.

Theorem 2.13. *A slack matrix of a convex heptagon has nonnegative rank at most* 6.

Proof. Proposition 1.1 shows that the slack matrix S of a convex heptagon has classical rank equal to 3. Therefore, S satisfies the assumptions of Lemma 2.12 up to renumbering the rows and columns. □

3. Main results

In this section we prove the main results of our paper. Let us start with a corollary of Theorem 2.13 which gives a solution for Problem 1.3 in the case $n = 7$.

Theorem 3.1. *Let A be a nonnegative 7-by-n matrix with classical rank equal to 3. Then the nonnegative rank of A does not exceed* 6.

Proof. Consider the standard simplex Δ consisting of points $(x_1, \ldots, x_7)$ with nonnegative coordinates satisfying $\sum_{i=1}^{7} x_i = 1$. Since Δ contains 7 facets, the intersection of Δ with the column space of A is a polygon I with k vertices, and $k \leqslant 7$. Form a matrix S of column coordinate vectors of vertices of I, then every column of A belongs to the convex hull of the columns of S up to scaling. Thus we have $A = SB$ with B nonnegative, so the result of the theorem is immediate if $k < 7$. If $k = 7$, then by Theorem 2.13, S has nonnegative rank less than 7 being a slack matrix for I. □

Now we can provide a nontrivial upper bound for the nonnegative rank of matrices with classical rank equal to 3, thus providing a solution for Problem 1.3 in the case $n \geqslant 7$.

Theorem 3.2. *The nonnegative rank of a rank-three matrix* $A \in \mathbb{R}_{+}^{m\times n}$ *does not exceed* $\lceil 6\min\{m,n\}/7\rceil$.

Proof. By Theorem 3.1, any seven rows of A can be expressed as linear combinations with nonnegative coefficients of certain six nonnegative rows, so the nonnegative rank of A does not exceed $\lceil 6m/7\rceil$. The nonnegative rank is invariant under transpositions, so the result follows. □

Together with the result from [7], where it was noted that a sufficiently irregular convex hexagon has full extension complexity, Theorem 3.2 provides a full answer for Problem 1.3. Namely, the following result is true.

Corollary 3.3. *If* $n \geqslant 7$, *then the nonnegative rank of any rank-three* m*-by-*n *nonnegative matrix is less than* n. *For* $k \in \{3, 4, 5, 6\}$, *there are* k*-by-*k *rank-three matrices with nonnegative rank equal to* k.

Finally, we can prove an upper bound for the extension complexity of convex polygons.

Corollary 3.4. *The extension complexity of any convex* n*-gon does not exceed* $\lceil 6n/7\rceil$.

Proof. By Proposition 1.1 and Theorem 3.2, the nonnegative rank of a slack matrix does not exceed $\lceil 6n/7\rceil$, so the result follows from Theorem 1.2. □

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