# Spanning forests in regular planar maps ☆

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A R T I C L E   I N F O

A B S T R A C T

We address the enumeration of $p$-valent planar maps equipped with a spanning forest, with a weight $z$ per face and a weight $u$ per connected component of the forest. Equivalently, we count $p$-valent maps equipped with a spanning *tree*, with a weight $z$ per face and a weight $\mu := u + 1$ per *internally active* edge, in the sense of Tutte; or the (dual) $p$-angulations equipped with a *recurrent sandpile configuration*, with a weight $z$ per vertex and a variable $\mu := u + 1$ that keeps track of the *level* of the configuration. This enumeration problem also corresponds to the limit $q \to 0$ of the $q$-state Potts model on $p$-angulations.

Our approach is purely combinatorial. The associated generating function, denoted $F(z, u)$, is expressed in terms of a pair of series defined implicitly by a system involving doubly hypergeometric series. We derive from this system that $F(z, u)$ is *differentially algebraic* in $z$, that is, satisfies a differential equation in $z$ with polynomial coefficients in $z$ and $u$. This has recently been proved to hold for the more general Potts model on 3-valent maps, but via a much more involved and less combinatorial proof.

For $u \geq -1$, we study the singularities of $F(z, u)$ and the corresponding asymptotic behaviour of its $n$th coefficient. For $u > 0$, we find the standard behaviour of planar maps, with a subexponential term in $n^{-5/2}$. At $u = 0$ we witness a phase transition with a term $n^{-3}$. When $u \in [-1, 0)$, we obtain an extremely unusual behaviour in $n^{-3}(\ln n)^{-2}$. To our knowledge, this is a new "universality class" for planar maps. We

analyze the phase transition at $u = 0$ in terms of the sandpile model on large maps, and find it to be of infinite order.

## 1. Introduction

A *planar map* is a proper embedding of a connected graph in the sphere. The enumeration of planar maps has received a continuous attention in the past 60 years, first in combinatorics with the pioneering work of Tutte [54], then in theoretical physics [24], where maps are considered as random surfaces modelling the effect of *quantum gravity*, and more recently in probability theory [42,44]. General planar maps have been studied, as well as sub-families obtained by imposing constraints of higher connectivity, or prescribing the degrees of vertices or faces (*e.g.*, triangulations). Precise definitions are given below.

Several robust enumeration methods have been designed, from Tutte's recursive approach (*e.g.* [53]), which leads to functional equations for the generating functions of maps, to the beautiful bijections initiated by Schaeffer [49], and further developed by physicists and combinatorics alike [11,21], via a powerful approach based on matrix integrals [32]. See for instance [18] for a more complete (though non-exhaustive) bibliography.

Beyond the enumerative and asymptotic properties of planar maps, which are now well understood, the attention has also focused on two more general questions: maps on higher genus surfaces [6,26], and maps equipped with an additional structure. The latter question is particularly relevant in physics, where a surface on which nothing happens ("pure gravity") is of little interest. For instance, one has studied maps equipped with a polymer [34], with an Ising model [21,40,17,20] or more generally a Potts model, with a proper colouring [55,56], with loop models [16,15], with a spanning tree [47], or percolation on planar maps [2,10]. These articles parallel the analogous investigation, started in the 80s, of the same list of models on regular planar lattices (square, triangular...), based at the time on the techniques of conformal field theory (CFT).

In particular, several papers have been devoted in the past 20 years to the study of the Potts model on families of planar maps [4,14,31,35,39,58]. One motivation, mentioned by one of our referees, is that "the $q$-state Potts model (for $0 \leq q \leq 4$) covers the full range of universality classes interpolating the discrete family of $(p, p + 1)$ minimal CFT's". In combinatorial terms, solving this model means counting maps equipped with a vertex-colouring in $q$ colours, according to the size (*e.g.*, the number of edges) and the number of *monochromatic edges* (edges whose endpoints have the same colour). Up to a change of variables, this also means counting maps weighted by their *Tutte polynomial*, a bivariate combinatorial invariant which has numerous interesting specializations. By generalizing Tutte's formidable solution of properly coloured triangulations (1973–1982), it has recently been proved that the Potts generating function is *differentially algebraic*,

that is, satisfies a (non-linear) differential equation[1] with polynomial coefficients [9,8,18]. This holds at least for general planar maps and for triangulations (or dually, for cubic maps).

The *nature* of a generating function (rational/algebraic/D-finite[2]/D-algebraic) is measure of the complexity of the class of objects it counts, and is usually believed to reflect some of its combinatorial properties. In particular, it has been recognized since the sixties that most classes of planar maps (with no additional structure) have an algebraic generating function, and this is now understood combinatorially thanks to beautiful bijections with trees (*e.g.* [49,22]). For the Potts model, it is known since 1967 that at least one specialization (maps with a spanning tree) is transcendental but D-finite [47]. The above references [8,18] show that the generating function is in full generality D-algebraic. However, the method that yields these differential equations is extremely involved, and does not shed much light on the structure of $q$-coloured maps. Moreover, one cannot *a priori* exclude the possibility that the generating function would belong to the smaller class of D-finite series. Let us also mention that the asymptotic behaviour of the number of coloured maps, as well as the location of phase transitions, remain unknown.

The aim of this paper is to remedy these problems — so far for a one-variable specialization of the Tutte polynomial. This specialization is obtained by setting one of the variables to 1, or by taking (in an adequate way) the limit $q \to 0$ in the Potts model. Combinatorially, this corresponds to counting maps (in this paper, $p$-valent maps) equipped with a spanning forest. We call them *forested maps*. This problem has already been studied in [25] via a random matrix approach, but with no explicit solution. We use instead a simple combinatorial approach based on the contraction of trees, which appears already in [21, Appendix A] or [25, Section 4]. The generating function $F(z, u)$ that we obtain keeps track of the *size* of the map (the number of faces; variable $z$) and of the number of trees in the forest (minus one; variable $u$). The specialization $u = 0$ thus counts maps equipped with a spanning *tree* and was determined a long time ago [47].

Here is an outline of the paper. We begin in Section 2 with general definitions on maps, and on the Tutte polynomial. We recall some of its combinatorial descriptions, and underline in particular that the series $F(z, \mu - 1)$, once expanded in powers of $z$ and $\mu$, has non-negative coefficients and admits several combinatorial interpretations. This important observation implies that the natural domain of the parameter $u$ is $[-1, +\infty)$ rather than $[0, +\infty)$. In Section 3, we obtain in a purely combinatorial manner an expression of $F(z, u)$ in terms of the solution of a system of two functional equations. In Section 4 we derive from this system that $F(z, u)$ is differentially algebraic in $z$, and give explicit differential equations for cubic ($p = 3$) and 4-valent ($p = 4$) maps. Section 5 is a combinatorial interlude explaining why all series occurring in our equations, like $F(z, u)$ itself, still have non-negative coefficients when $u \in [-1, 0]$.

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[1]  With respect to the size variable.
[2]  Solution of a linear o.d.e.

The rest of the paper is devoted to asymptotic results, for $p = 3$ and $p = 4$: when $u > 0$, forested maps follow the standard asymptotic behaviour of planar maps ($\mu^n n^{-5/2}$) but then there is a phase transition at $u = 0$ (where one counts maps equipped with a spanning tree), and a very unusual asymptotic behaviour in $\mu^n n^{-3} (\ln n)^{-2}$ when $u \in [-1, 0)$. To our knowledge, this is the first time a class of planar maps exhibits this asymptotic behaviour. This proves in particular that $F(z, u)$ is not D-finite in $z$. This is in contrast with the case $u = 0$, since the generating function of maps equipped with a spanning forest is D-finite.

Our key tool in the asymptotic study is the *singularity analysis* of [36]: it relates the asymptotic behaviour of the coefficients of a series $F(z)$ to the singular behaviour of $F$ near its *dominant singularities* (*i.e.*, singularities of minimal modulus). The first case we study (4-valent maps with $u > 0$) is simple: first, one of the two series involved in our system vanishes; the remaining one, denoted $R$, satisfies an inversion equation $\Omega(R(z)) = z$ for which the (unique) dominant singularity $\rho$ of $R$ is such that $R(\rho)$ lies in the domain of analyticity of $\Omega$. One obtains for $R$ a standard square root singularity. This is well understood and almost routine. Two ingredients make the other cases significantly harder:

- when $u < 0$, $R(\rho)$ is a singularity of $\Omega$,
- when $p = 3$ (cubic maps) we have to deal with a system of two equations; the analysis of systems is delicate, even in the so-called *positive case*, which corresponds in our context to $u > 0$ (see [33,5]).

These difficulties, which culminate when $p = 3$ and $u < 0$, are addressed in Sections 6 and 7. Section 6 establishes general results on implicitly defined series. Section 7 focuses on the inversion equation $\Omega(R(z)) = z$ in the case where (up to translation) $\Omega$ has a $z \ln z$ singularity at 0. One then applies these results to the asymptotic analysis of forested maps in Sections 8 (4-valent maps) and 10 (cubic maps). Section 9 exploits the results of Section 8 to study some properties of large random maps equipped with a spanning forest or a spanning tree. In particular, we show that the phase transition at $u = 0$ is rather peculiar, having infinite order (see [13,29,38,50] for other examples in the literature).

We conclude in Section 11 with a few comments.

An extended abstract of this paper has appeared in the proceedings of FPSAC 2013 [19].

## 2. Preliminaries

### 2.1. Planar maps

A *planar map* is a proper embedding of a connected graph (possibly with loops and multiple edges) in the oriented sphere, considered up to continuous deformation. All maps in this paper are planar, and we often omit the term "planar". A *face* is a (topological)
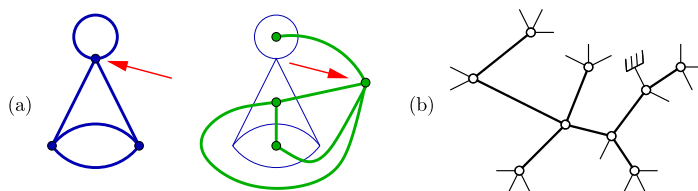
**Fig. 1.** (a) A rooted planar map and its dual (rooted at the dual corner). (b) A 4-valent leaf-rooted tree.

connected component of the complement of the embedded graph. Each edge consists of two *half-edges*, each incident to an endpoint of the edge. A *corner* is an ordered pair $(e_1, e_2)$ of half-edges incident to the same vertex, such that $e_2$ immediately follows $e_1$ in counterclockwise order. The *degree* of a vertex or a face is the number of corners incident to it. A vertex of degree $p$ is called *p-valent*. One-valent vertices are also called *leaves*. A map is *p-valent* if all vertices are $p$-valent. A *rooted map* is a map with a marked corner $(e_1, e_2)$, called the *root* and indicated by an arrow in our figures. The *root vertex* is the vertex incident to the root. The *root half-edge* is $e_2$ and the *root edge* is the edge supporting $e_2$. This way of rooting maps is equivalent to the more standard way where one marks the root edge and orients it from $e_2$ to its other half-edge. All maps of the paper are rooted, and we often omit the term "rooted". The *dual* of a map $M$, denoted $M^*$, is the map obtained by placing a vertex of $M^*$ in each face of $M$ and an edge of $M^*$ across each edge of $M$; we root it in a canonical way (see Fig. 1(a)). The dual of a $p$-valent map is a map with all faces of degree $p$, also called *p-angulation*.

A *(plane) tree* is a planar map with a unique face. A tree is $p$-valent if all non-leaf vertices have degree $p$. We consider the edges leading to the leaves as half-edges, as suggested by Fig. 1(b). A *leaf-rooted* tree (resp. *corner-rooted*) is a tree with a marked leaf (resp. corner). The number of $p$-valent leaf-rooted (resp. corner-rooted) trees with $k$ leaves is denoted by $t_k$ (resp. $t_k^c$) (the notation should be $t_{k,p}$ and $t_{k,p}^c$, but we consider $p$ as a fixed integer, $p \geq 3$). These numbers are well-known [51, Thm. 5.3.10]: they are 0 unless $k = (p-2)\ell + 2$ with $\ell \geq 1$, and in this case,

$$t_k = \frac{((p-1)\ell)!}{\ell!((p-2)\ell+1)!} \qquad \text{and} \qquad t_k^c = p\frac{((p-1)\ell)!}{(\ell-1)!((p-2)\ell+2)!} = t_k \frac{p\ell}{(p-2)\ell+2}. \quad (1)$$

Let $M$ be a rooted planar map with vertex set $V$. A *spanning forest* of $M$ is a graph $F = (V, E)$ where $E$ is a subset of edges of $M$ forming no cycle. Each connected component of $F$ is a tree, and the *root component* is the tree containing the root vertex. We say that the pair $(M, F)$ is a *forested map*. We denote by $F(z, u)$ the generating function of $p$-valent forested maps, counted by faces (variable $z$) and non-root components (variable $u$):

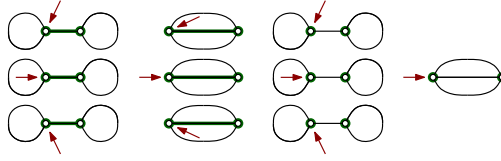$$F(z, u) = \sum_{\substack{M \ p\text{-valent} \\ F \ \text{spanning forest}}} z^{\mathrm{f}(M)} u^{\mathrm{c}(F)-1}, \quad (2)$$

**Fig. 2.** The 10 forested cubic maps with 3 faces.

where f(.) denotes the number of faces and c(.) the number of components. When $p = 3$,

$$F(z, u) = (6 + 4\, u)\, z^3 + \left(140 + 234\, u + 144\, u^2 + 32\, u^3\right) z^4 + O(z^5). \tag{3}$$

The coefficient $(6 + 4u)$ means that there are 10 trivalent (or *cubic*) forested maps with 3 faces: 6 in which the forest is a tree, and 4 in which it has two components (Fig. 2).

### 2.2. Forest counting, the Tutte polynomial, and related models

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The *Tutte polynomial* of $G$ is the following polynomial in two indeterminates (see *e.g.* [12]):

$$\mathrm{T}_G(\mu, \nu) := \sum_{S \subseteq E} (\mu - 1)^{\mathrm{c}(S) - \mathrm{c}(G)} (\nu - 1)^{\mathrm{e}(S) + \mathrm{c}(S) - \mathrm{v}(G)}, \tag{4}$$

where the sum is over all spanning subgraphs of $G$ (equivalently, over all subsets $S$ of edges) and v(.), e(.) and c(.) denote respectively the number of vertices, edges and connected components. The quantity $\mathrm{e}(S) + \mathrm{c}(S) - \mathrm{v}(G)$ is the *cyclomatic number of $S$*, that is, the minimal number of edges one has to delete from $S$ to obtain a forest.

When $\nu = 1$, the only subgraphs that contribute to (4) are the forests. Hence the generating function of forested maps defined by (2) can be written as

$$F(z, u) = \sum_{M \ p\text{-valent}} z^{\mathrm{f}(M)}\, \mathrm{T}_M(u + 1, 1). \tag{5}$$

Note that we write $\mathrm{T}_M$ although the value of the Tutte polynomial only depends on the underlying graph of $M$, not on the embedding.

Even though this is not clear from (4), the polynomial $\mathrm{T}_G(\mu, \nu)$ has non-negative coefficients in $\mu$ and $\nu$. This was proved combinatorially by Tutte [52], who showed that $\mathrm{T}_G(\mu, \nu)$ counts spanning *trees* of $G$ according to two parameters, called *internal* and *external* activities. Other combinatorial descriptions of $\mathrm{T}_G(\mu, \nu)$, in terms of other notions of activity, were given later. Let us present the one due to Bernardi, which is nicely related to maps [7]. Following Mullin [47], we call *tree-rooted map*, a map $M$ equipped with a spanning tree $T$. Walking around $T$ in counter-clockwise order, starting from the root, defines a total order on the edges (those in $T$ and not in $T$, altogether): the first edge that is met is the smallest one, and so on (Fig. 3). An edge $e$ is *internally active* if it belongs to $T$ and is minimal in its cocycle; that is, all the edges $e' \neq e$ such
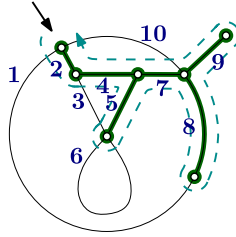
**Fig. 3.** The edges of a tree-rooted map are naturally order by walking around the tree. The active edges are those labelled 1, 3, 6 and 9.

that $(T \setminus \{e\}) \cup \{e'\}$ is a tree are larger than $e$. It is *externally active* if it does not belong to $T$ and is minimal in the cycle created by adding $e$ to $T$. Denoting by $\mathrm{int}(M, T)$ and $\mathrm{ext}(M, T)$ the numbers of internally and externally edges, one has:

$$\mathrm{T}_M(\mu, \nu) = \sum_{T \text{ spanning tree}} \mu^{\mathrm{int}(M,T)} \nu^{\mathrm{ext}(M,T)}.$$

A non-obvious property of this description is that it only depends on the underlying graph of $M$.

Returning to (5), we thus obtain a second description of $F(z, u)$:

$$F(z, u) = \sum_{\substack{M \ p\text{-valent} \\ T \text{ spanning tree}}} z^{\mathrm{f}(M)} (u+1)^{\mathrm{int}(M,T)}. \tag{6}$$

In particular, it makes sense combinatorially to write $u = \mu - 1$ and take $u \in [-1, \infty)$.

We now give four more descriptions of $F(z, u)$ in terms of the dual $p$-angulations. For any planar map $M$, it is known that

$$\mathrm{T}_{M^*}(\mu, \nu) = \mathrm{T}_M(\nu, \mu).$$

Since

$$\mathrm{T}_M(1, \nu) = \sum_{S \subset E, \ S \text{ connected}} (\nu - 1)^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(M)},$$

we first derive from (5) that

$$F(z, u) = \sum_{M \ p\text{-angulation}} z^{\mathrm{v}(M)} \mathrm{T}_M(1, u+1)$$

$$= \sum_{\substack{M \ p\text{-angulation} \\ S \text{ connected subgraph}}} z^{\mathrm{v}(M)} u^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(M)} \tag{7}$$

counts $p$-angulations $M$ equipped with a connected (spanning) subgraph $S$, by the vertex number of $M$ and the cyclomatic number of $S$. Also, the "dual" expression of (6) reads

$$F(z, u) = \sum_{\substack{M \ p\text{-angulation} \\ T \ \text{spanning tree}}} z^{\text{v}(M)}(u+1)^{\text{ext}(M,T)}. \tag{8}$$

Our next interpretation of $F(z, u)$, which we will not entirely detail, relies on the connection between $\text{T}_M(1, \nu)$ and the *recurrent* (or: *critical*) configurations of the sandpile model on $M$. It is known [45,27] that

$$\text{T}_M(1, \nu) = \sum_{C \ \text{recurrent}} \nu^{\ell(C)},$$

where the sum runs over all recurrent configurations $C$, and $\ell(C)$ is the *level* of $C$. Hence

$$F(z, u) = \sum_{\substack{M \ p\text{-angulation} \\ C \ \text{recurrent}}} z^{\text{v}(M)}(u+1)^{\ell(C)} \tag{9}$$

also counts $p$-angulations $M$ equipped with a recurrent configuration $C$ of the sandpile model, by the vertex number of $M$ and the level of $C$.

Our final interpretation is in terms of the *Potts* model. Take $q \in \mathbb{N}$. A *q-colouring* of the vertices of $G = (V, E)$ is a map $c : V \to \{1, \ldots, q\}$. An edge of $G$ is *monochromatic* if its endpoints share the same colour. Every loop is thus monochromatic. The number of monochromatic edges is denoted by $m(c)$. The *partition function of the Potts model* on $G$ counts colourings by the number of monochromatic edges:

$$\text{P}_G(q, \nu) = \sum_{c:V \to \{1, \ldots, q\}} \nu^{m(c)}.$$

The Potts model is a classical magnetism model in statistical physics, which includes (for $q = 2$) the famous Ising model (with no magnetic field) [57]. Of course, $\text{P}_G(q, 0)$ is the chromatic polynomial of $G$. More generally, it is not hard to see that $\text{P}_G(q, \nu)$ is always a *polynomial* in $q$ and $\nu$, and a multiple of $q$. Let us define the *reduced Potts polynomial* $\tilde{\text{P}}_G(q, \nu)$ by

$$\text{P}_G(q, \nu) = q\tilde{\text{P}}_G(q, \nu).$$

Fortuin and Kasteleyn established the equivalence of $\tilde{\text{P}}_G$ with the Tutte polynomial [37]:

$$\tilde{\text{P}}_G(q, \nu) = \sum_{S \subseteq E(G)} q^{\text{c}(S)-1}(\nu-1)^{\text{e}(S)} = (\mu-1)^{\text{c}(G)-1}(\nu-1)^{\text{v}(G)-1} \, \text{T}_G(\mu, \nu),$$

for $q = (\mu-1)(\nu-1)$. Setting $\mu = 1$, we obtain, for a *connected* graph $G$

$$\tilde{\text{P}}_G(0, \nu) = (\nu-1)^{\text{v}(G)-1} \, \text{T}_G(1, \nu).$$

Returning to (7) finally gives

$$F(z, u) = u \sum_{M \ p\text{-angulation}} (z/u)^{\mathrm{v}(M)} \, \tilde{\mathrm{P}}_M(0, u + 1). \tag{10}$$

### 2.3. Formal power series

Let $A = A(z) \in \mathbb{K}[[z]]$ be a power series in one variable with coefficients in a field $\mathbb{K}$. We say that $A$ is *D-finite* if it satisfies a (non-trivial) linear differential equation with coefficients in $\mathbb{K}[z]$ (the ring of polynomials in $z$). More generally, it is *D-algebraic* if there exist a positive integer $k$ and a non-trivial polynomial $P \in \mathbb{K}[z, x_0, \ldots, x_k]$ such that $P\left(z, A, \frac{\partial A}{\partial z}, \ldots, \frac{\partial^k A}{\partial z^k}\right) = 0$.

A $k$-variate power series $A = A(z_1, \ldots, z_k)$ with coefficients in $\mathbb{K}$ is *D-finite* if its partial derivatives (of all orders) span a finite dimensional vector space over $\mathbb{K}(z_1, \ldots, z_k)$.

## 3. Generating functions for forested maps

Fix $p \geq 3$. We give here a system of equations that defines the generating function $F(z, u)$ of $p$-valent forested maps, or, more precisely, the series $zF'_z(z, u)$ that counts forested maps with a marked face. We also give simpler systems for two variants of $F(z, u)$, involving no derivative.

### 3.1. p-Valent maps

**Theorem 3.1.** *Let $\theta$, $\Phi_1$ and $\Phi_2$ be the following doubly hypergeometric series:*

$$\theta(x, y) = \sum_{i \geq 0} \sum_{j \geq 0} t^c_{2i+j} \binom{2i + j}{i, i, j} x^i y^j,$$

$$\Phi_1(x, y) = \sum_{i \geq 1} \sum_{j \geq 0} t_{2i+j} \binom{2i + j - 1}{i - 1, i, j} x^i y^j,$$

$$\Phi_2(x, y) = \sum_{i \geq 0} \sum_{j \geq 0} t_{2i+j+1} \binom{2i + j}{i, i, j} x^i y^j, \tag{11}$$

*where $t_k$ and $t^c_k$ are given by (1) and $\binom{a+b+c}{a,b,c}$ denotes the trinomial coefficient $(a + b + c)!/(a!b!c!)$.*

*There exists a unique pair $(R, S)$ of power series in $z$ with constant term 0 and coefficients in $\mathbb{Q}[u]$ that satisfies*

$$R = z + u \, \Phi_1(R, S), \tag{12}$$

$$S = u \, \Phi_2(R, S). \tag{13}$$

*The generating function $F(z, u)$ of p-valent forested maps is characterized by $F(0, u) = 0$ and*

$$F'_z(z, u) = \theta(R, S). \tag{14}$$

**Remarks. 1.** These equations allow us to compute the first terms in the expansion of $F(z, u)$, for any fixed $p \geq 3$. This is how we obtained (3).

**2.** When $p$ is even, then $t_{2i+1} = 0$ for all $i$. In particular, all terms occurring in the definition (11) of $\Phi_2$ are multiples of $y$, so that $S = 0$. The simplified system reads:

$$F'_z(z, u) = \theta(R) \qquad \text{and} \qquad R = z + u\,\Phi(R), \tag{15}$$

with

$$\theta(x) = \sum_{i \geq 0} t^c_{2i} \binom{2i}{i} x^i \qquad \text{and} \qquad \Phi(x) = \sum_{i \geq 1} t_{2i} \binom{2i-1}{i} x^i.$$

**3.** When $u = 0$, an even more drastic simplification follows from (12)–(13): not only $S = 0$, but also $R = z$, so that (14) becomes an explicit expression of $F'_z$:

$$F'_z(z, 0) = \sum_{i \geq 0} t^c_{2i} \binom{2i}{i} z^i,$$

or equivalently,

$$F(z, 0) = \sum_{i \geq 0} t^c_{2i} \binom{2i}{i} \frac{z^{i+1}}{i+1} = \sum_{\ell \geq 1} \frac{p((p-1)\ell)!}{(\ell-1)!(1+(p-2)\ell/2)!(2+(p-2)\ell/2)!} z^{2+(p-2)\ell/2},$$

$$\tag{16}$$

where we require $\ell$ to be even if $p$ is odd. This series counts $p$-valent maps equipped with a spanning tree, and this expression was already proved by Mullin [47].

**4.** The series $\theta$ and $\Phi_i$ are explicited when $p = 4$ and $p = 3$ in Sections 4.2 and 4.3, respectively.

In order to prove Theorem 3.1, we first relate $F(z, u)$ to the generating function of planar maps counted by the distribution of their vertex degrees. More precisely, let $\bar{M} \equiv \bar{M}(z, u; g_1, g_2, \ldots; h_1, h_2, \ldots)$ be the generating function of rooted planar maps, with a weight $z$ per face, $ug_k$ per *non-root* vertex of degree $k$ and $h_k$ if the root vertex has degree $k$.

**Lemma 3.2.** *The series $F(z, u)$ is related to $M$ through:*

$$F(z, u) = \bar{M}(z, u; t_1, t_2, \ldots; t^c_1, t^c_2, \ldots).$$
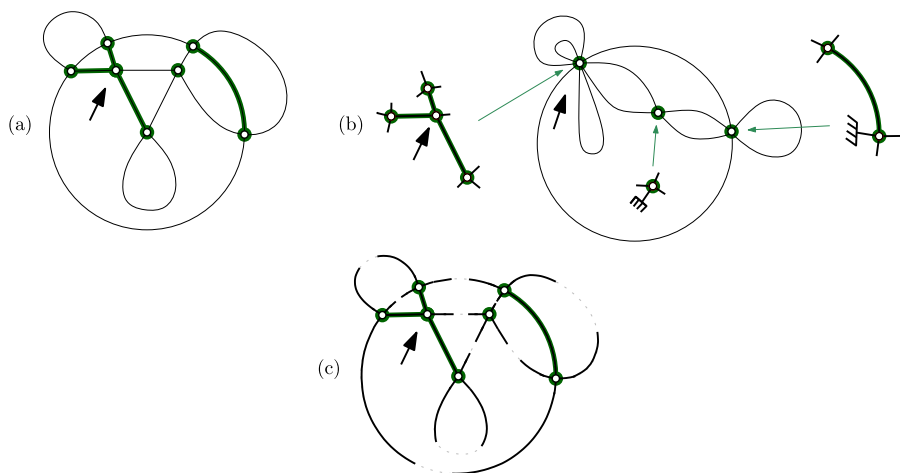
**Fig. 4.** (a) A 4-valent forested map with 9 faces and 2 non-root components. (b) The same map, after contraction of the forest. (c) Assembling the 3 trees gives the original forested map.

**Proof.** The idea is to contract each tree of a spanning forest, incident to $k$ half-edges, into a $k$-valent vertex. It is adapted from [21, Appendix A], where the authors study 4-valent forested maps for which the root edge is not in the forest. It can also be seen as an extension of Mullin's construction for maps equipped with a spanning tree [47]. Finally, it also appears in [25].

Let us now get into the details. First, let us recall that rooted maps have no symmetries: all vertices, edges and half-edges are distinguishable. In particular, one can fix, for every rooted planar map $M'$ (with arbitrary valences) a total order on its half-edges. This order may have a combinatorial significance — a good choice is the order in which half-edges are visited when applying the construction of [22] — but can also be arbitrary.

We now describe a bijection $\Phi$, illustrated in Fig. 4, between forested $p$-valent maps $(M, F)$ and pairs formed of a map $M'$ and a collection $(T_v, v \in V(M'))$ of $p$-valent trees associated with the vertices of $M'$, such that the tree associated with the root vertex of $M'$ is corner-rooted, the others are leaf-rooted, and the number of leaves of $T_v$ is the degree of $v$ in $M'$.

The map $M'$ is obtained by contracting all edges of the forest $F$ (Fig. 4(b)). The arrow that marks the root corner remains at the same place. Now split into two half-edges each edge of $M$ that is not in $F$: this gives a collection of $p$-valent trees, each of them being naturally associated with a vertex $v$ of $M'$. The half-edges of these trees form together the edges of $M'$ (Fig. 4(c)). If $v$ is the root vertex of $M'$, then $T_v$ inherits the corner-rooting of $M$. Otherwise, we root $T_v$ at the smallest of its half-edges, for the total order on half-edges of $M'$.

The following properties are readily checked:

• $T_v$ has $k$ leaves if $v$ has degree $k$ in $M'$,

- $M$ and $M'$ have the same number of faces,
- the number of vertices of $M'$ is the number of components of $F$.

Let us now prove that $\Phi$ is bijective. To recover the forested map $(M, F)$ from the contracted map $M'$ and the associated collection of trees, we inflate each vertex $v$ of $M'$ into the corresponding tree $T_v$. If $v$ is the root vertex of $M'$, the root corner of $T_v$ must coincide with the root corner of $M'$. Otherwise, the root half-edge of $T_v$ is put on the smallest of the half-edges incident to $v$ in $M'$. This proves the injectivity of $\Phi$. Since this reverse construction can be applied to any map $M'$ with a corresponding collection of trees, $\Phi$ is also surjective.  □

**Proof of Theorem 3.1.** In a recent paper, Bouttier and Guitter [23] have expressed the series $\bar{M}$ via a system of equations, established bijectively.[3] Their expression is actually fairly complicated [23, Eq. (1.4)], but the series $z\bar{M}'_z$, which counts maps with a marked face, has a much simpler expression [23, Eq. (2.6)]:

$$\bar{M}'_z = \sum_{i \geq 0} \sum_{j \geq 0} h_{2i+j} \binom{2i+j}{i, i, j} R^i S^j, \tag{17}$$

where $h_0 = 0$ and, by [23, Eq. (2.5)],

$$R = z + u \sum_{i \geq 1} \sum_{j \geq 0} g_{2i+j} \binom{2i+j-1}{i-1, i, j} R^i S^j, \qquad S = u \sum_{i \geq 0} \sum_{j \geq 0} g_{2i+j+1} \binom{2i+j}{i, i, j} R^i S^j. \tag{18}$$

Theorem 3.1 follows by specialization, using Lemma 3.2.

It remains to check that (12)–(13) define a unique pair of series $R$ and $S$ in $z$ with constant terms 0. This is readily proved by observing that (12) determines $R$ up to order $n$ if we know $R$ and $S$ up to order $n-1$; and that (13) determines $S$ up to order $n$ if we know $R$ up to order $n$ and $S$ up to order $n-1$.  □

**Remarks. 1.** The expression of $\bar{M}$ given in [23, Eq. (1.4)] leads to an explicit expression of $F(z, u)$ in terms of $R$ and $S$. However, this expression involves a triple sum (a double sum when $p$ is even, see for instance (100)). This is why we prefer handling the above expression of $F'$. We discuss this further in the final section.

**2.** The inflation of vertices into trees used in the proof of Lemma 3.2 is robust, and allows us to take more parameters into account in the enumeration of forested maps. In particular, we study in Section 8 the size (number of vertices) of the root component of

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[3] Strictly speaking, they do not take the vertex or face number into account, but both are prescribed by the distribution of vertex degrees.
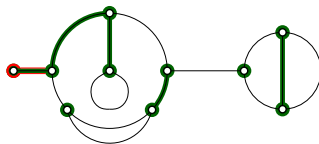
**Fig. 5.** A quasi-cubic forested map with 6 faces and 4 non-root components.

the forest in a large random forested map. Taking this parameter into account with a new variable $s$, Lemma 3.2 becomes

$$F(z, u, s) = \bar{M}(z, u; t_1, t_2, \ldots; t_1^c, \ldots, s^{\frac{k-2}{p-2}} t_k^c, \ldots)$$

since a $p$-valent tree with $k = (p-2)\ell + 2$ leaves has $\ell$ vertices (recall that $t_1^c$ is in fact 0). Thus Theorem 3.1 extends as follows:

$$s^{\frac{2}{p-2}} F_z'(z, u, s) = \theta\left(s^{\frac{2}{p-2}} R, s^{\frac{1}{p-2}} S\right), \tag{19}$$

where $R$, $S$ and $\theta$ are unchanged.

### 3.2. Quasi-p-valent maps (p odd)

A map is said to be *quasi-p-valent* if all its vertices have degree $p$, apart from one vertex which is a leaf. Such maps exist only when $p$ is odd. They are naturally rooted at their leaf: the root corner is the unique corner incident to the leaf and the root edge is the unique edge incident to the leaf. Let $G(z, u)$ denote the generating function of quasi-$p$-valent forested maps counted by faces $(z)$ and non-root components $(u)$ (see Fig. 5). From this section on, the term $1/u$ often occurs, and we adopt the shortcut

$$\bar{u} := 1/u.$$

**Proposition 3.3.** *The generating function of quasi-p-valent forested maps is*

$$G(z, u) = (1 + \bar{u})\left(zS - u\sum_{i \geq 2}\sum_{j \geq 0} t_{2i+j-1}\binom{2i+j-2}{i-2, i, j} R^i S^j\right), \tag{20}$$

*where the series $R$ and $S$ are defined by (12)–(13) and the numbers $t_k$ by (1). Also,*

$$G_z'(z, u) = (1 + \bar{u})S.$$

As in the previous subsection, the key of this result is to relate $G(z, u)$ to a well-understood generating function of maps — here, the generating function $\Gamma_1 \equiv \Gamma_1(z, u; g_1, g_2, \ldots)$ that counts planar maps rooted at leaf, with a weight $z$ per face and $ug_k$ per $k$-valent *non-root* vertex.

**Lemma 3.4.** *The following analogue of Lemma* 3.2 *holds for quasi-p-valent forested maps:*

$$G(z, u) = (1 + \bar{u})\,\Gamma_1(z, u; t_1, t_2, \ldots).$$

**Proof.** The bijection used in the proof of Lemma 3.2 shows that the series $\Gamma_1(z, u; t_1, t_2, \ldots)$ counts quasi-$p$-valent forested maps such that the root edge *is not* in the forest. (With the notation used in that proof, the root vertex of $M'$, of degree 1, remains a trivial tree during the inflation step.) To each such forested map, we can add the root edge to the forest. The resulting forested map has one less component, hence the factor $\bar{u} = 1/u$.  □

**Proof of Proposition 3.3.** The series $\Gamma_1$ has also been expressed by Bouttier et al. in terms of the series $R$ and $S$ of (18) (see [22, Eq. (2.6)]):

$$\Gamma_1 = zS - u \sum_{i \geq 2} \sum_{j \geq 0} g_{2i+j-1} \binom{2i+j-2}{i-2, i, j} R^i S^j. \tag{21}$$

This gives the first part of Proposition 3.3. For the second part, we observe that $\Gamma_1$ is by definition the coefficient of $h_1$ in the series $\bar{M}(z, u; g_1, \ldots; h_1, \ldots)$ defined above Lemma 3.2. Hence it follows from (17) that $\Gamma_1' = S$ (this can also be derived combinatorially from [22]).  □

### 3.3. When the root edge is outside the forest

We now focus on forested maps such that the root edge is outside the forest. Let $H(z, u)$ denote the associated generating function.

**Proposition 3.5.** *The generating function of p-valent forested maps where the root edge is outside the forest is*

$$H(z, u) = \bar{u} z R + \bar{u} z S^2 - \bar{u} z^2$$
$$- 2S \sum_{i \geq 2} \sum_{j \geq 0} t_{2i+j-1} \binom{2i+j-2}{i-2, i, j} R^i S^j - \sum_{i \geq 3} \sum_{j \geq 0} t_{2i+j-2} \binom{2i+j-3}{i-3, i, j} R^i S^j, \tag{22}$$

*where the series $R$ and $S$ are defined by* (12)–(13) *and the numbers $t_k$ by* (1).

   *When $p$ is even, then $S = 0$ and the first double sum disappears. In this case, we also have a very simple expression of $H_z'(z, u)$:*

$$H_z'(z, u) = 2\bar{u}(R - z). \tag{23}$$

Again, the key of this result is to relate $H(z, u)$ to a well-understood generating function of maps — here, the generating function $M \equiv M(z, u; g_1, g_2, \ldots)$ that counts rooted planar maps with a weight $z$ per face and $ug_k$ per vertex of degree $k$.

**Lemma 3.6.** *The following analogue of Lemma 3.2 holds:*

$$H(z, u) = \bar{u} \, M(z, u; t_1, t_2, \ldots).$$

**Proof.** Let us consider again the bijection used in the proof of Lemma 3.2: the fact that the root edge of $M$ is not in the forest $F$ means that, in the corner-rooted tree associated with the root vertex of $M'$, the root half-edge is a leaf. It is then equivalent to root this tree at this leaf. $\square$

**Proof of Proposition 3.5.** The first part of the proposition follows from the known characterization of $M$ (see [22, Eq. (2.1)]):

$$M = \frac{\Gamma_1^2 + \Gamma_2}{z} - z^2,$$

where $\Gamma_1$ is given by (21) and

$$\Gamma_2 = z^2 R - uz \sum_{i \geq 3} \sum_{j \geq 0} \binom{2i + j - 3}{i - 3, i, j} R^i S^j - u^2 \left( \sum_{i \geq 2} \sum_{j \geq 0} g_{2i + j - 1} \binom{2i + j - 2}{i - 2, i, j} R^i S^j \right)^2,$$

with $R$ and $S$ satisfying (18). This gives the first part of the proposition.

Observe that $M(z, u; g_1, g_2, \ldots) = u\bar{M}(z, u; g_1, g_2, \ldots; g_1, g_2, \ldots)$ where $\bar{M}$ is defined just above Lemma 3.2. When $p$ is even, the maps obtained by contracting forests have even degrees ($g_{2k+1} = 0$ for all $k$), the series $S$ given by (18) vanishes, and the combination of (17) and (18) gives $\bar{M}'_z(z, u; g_1, g_2, \ldots; g_1, g_2, \ldots) = 2\bar{u}(R - z)$. Thus $H'_z = \bar{u}M'_z = \bar{M}'_z = 2\bar{u}(R - z)$, as stated in (23). $\square$

## 4. Differential equations

The equations established in the previous section imply that series counting regular forested maps are D-algebraic. We compute explicitly a few differential equations.[4] Recall that the existence of a differential equation is already known in the case of cubic maps (and in fact for the more general Potts model on these maps); but the derivation of this equation is much more difficult [8,18].

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[4] Our asymptotic results will not rely on these equations, but on the functional equations of the previous section. We refer to [48] for an example of an asymptotic study based on a differential equation, also related to the Potts model (properly $q$-coloured triangulations).

## 4.1. The general case

**Theorem 4.1.** *The generating function $F(z, u)$ of p-valent forested maps is D-algebraic (with respect to z). The same holds for the series $G(z, u)$ and $H(z, u)$ of Propositions 3.3 and 3.5.*

**Proof.** We start from the expression (14) of $F'(z, u)$ (as we always differentiate with respect to $z$, we simply denote $F'(z, u)$ for $F'_z(z, u)$). We first observe that the doubly hypergeometric series $\theta$, $\Phi_1$, $\Phi_2$ are D-finite (this follows from the closure properties of D-finite power series [43]).

Then, by differentiating (12) and (13) with respect to $z$, we obtain rational expressions of $R'$ and $S'$ in terms of $u$ and the partial derivatives $\partial\Phi_\ell/\partial x$ and $\partial\Phi_\ell/\partial y$, evaluated at $(R, S)$, for $\ell = 1, 2$. (Indeed, differentiating (12) and (13) gives a linear system in $R'$ and $S'$. Its determinant is a power series in $z$ with coefficients in $\mathbb{Q}[u]$. It is non-zero, since it equals 1 at $u = 0$.)

Let $\mathbb{K}$ be the field $\mathbb{Q}(u)$. Using (14) and the previous point, it is now easy to prove by induction that for all $k \geq 1$, there exists a rational expression of $F^{(k)}(z, u)$ in terms of

$$\left\{ \frac{\partial^{i+j}\Phi_\ell}{\partial x^i \partial y^j}(R, S), \frac{\partial^{i+j}\theta}{\partial x^i \partial y^j}(R, S) \right\}_{i \geq 0, j \geq 0, \ell \in \{1, 2\}}$$

with coefficients in $\mathbb{K}$. But since $\theta$, $\Phi_1$ and $\Phi_2$ are D-finite, the above set of series spans a vector space of finite dimension $d$ over $\mathbb{K}(R, S)$. Therefore there exist $d$ elements $\varphi_1, \ldots, \varphi_d$ in this space, and rational functions $A_k \in \mathbb{K}(x, y, x_1, \ldots, x_d)$, such that $F^{(k)}(z, u) = A_k(R, S, \varphi_1, \ldots, \varphi_d)$ for all $k \geq 1$.

Since the transcendence degree [41, p. 254] of $\mathbb{K}(R, S, \varphi_1, \ldots, \varphi_d)$ over $\mathbb{K}$ is (at most) $d + 2$, the $d + 3$ series $F^{(k)}(z, u)$, for $1 \leq k \leq d + 3$, are algebraically dependent, so that $F'$ (and thus $F$) is D-algebraic.

The proof is similar for the series $G(z, u)$ and $H(z, u)$, with $\theta$ replaced by the adequate D-finite series derived from (20) and (22). Moreover, since these two expressions involve $z$ explicitly, the field $\mathbb{Q}(u)$ used in the above argument must be replaced by $\mathbb{Q}(z, u)$.  □

## 4.2. The 4-valent case

We specialize the above argument to the case $p = 4$. As explained in the second remark following Theorem 3.1, the series $S$ vanishes and $F'(z, u)$ is given by the system (15), with

$$\theta(x) = 4 \sum_{i \geq 2} \frac{(3i - 3)!}{(i - 2)! i!^2} x^i \quad \text{and} \quad \Phi(x) = \sum_{i \geq 2} \frac{(3i - 3)!}{(i - 1)!^2 i!} x^i. \tag{24}$$

The series $\theta(x)$, $\Phi(x)$ and their derivatives lie in a 3-dimensional vector space over $\mathbb{Q}(x)$ spanned (for instance) by 1, $\Phi(x)$ and $\Phi'(x)$. This follows from the following equations, which are easily checked:

$$x(27x - 1)\Phi''(x) + 6\Phi(x) + 6x = 0, \tag{25}$$

$$3\theta(x) = 2(27x - 1)\Phi'(x) - 42\Phi(x) + 12x. \tag{26}$$

By the argument described above, we can now express first $R'$, and then $F'$ and all its derivatives as rational functions of $u$, $R$, $\Phi(R)$ and $\Phi'(R)$. But since $R = z + u\Phi(R)$, this means a rational function of $u$, $z$, $R$ and $\Phi'(R)$. We compute the explicit expressions of $F'$, $F''$ and $F'''$, eliminate $R$ and $\Phi'(R)$ from these three equations, and this gives a differential equation of order 2 and degree 7 satisfied by $F'$, the details of which are not particularly illuminating:

$$9\,F'^{\,2}F''^{\,5}u^6 + 36\,F'^{\,2}F''^{\,3}F'''\,u^5z + 144\,F'^{\,2}F''^{\,4}u^5 - 12\,(21\,z - 1)\,F'\,F''^{\,5}u^5$$

$$+\,432\,F'^{\,2}F''^{\,2}F'''\,u^4z - 48\,(24\,z - 1)\,F'\,F''^{\,3}F'''\,u^4z + 864\,F'^{\,2}F''^{\,3}u^4$$

$$-\,96\,(27\,z - 2)\,F'\,F''^{\,4}u^4 + 4\,(27\,z - 1)\,(15\,z - 1)\,F''^{\,5}u^4 + 1728\,F'^{\,2}F''\,F'''\,u^3z$$

$$-\,288\,(21\,z - 2)\,F'\,F''^{\,2}F'''\,u^3z + 10\,368\,F'\,F'''^{\,2}u^2z^3$$

$$+\,16\,(27\,z - 1)\,(21\,z - 1)\,F''^{\,3}F'''\,u^3z + 2304\,F'^{\,2}F''^{\,2}u^3 - 288\,(31\,z - 4)\,F'\,F''^{\,3}u^3$$

$$-\,64\,\left(6\,uz - 162\,z^2 + 33\,z - 1\right)F''^{\,4}u^3 + 2304\,F'^{\,2}F'''\,u^2z$$

$$-\,2304\,(6\,z - 1)\,F'\,F''\,F'''\,u^2z - 192\,\left(8\,uz - 54\,z^2 + 29\,z - 1\right)F''^{\,2}F'''\,u^2z$$

$$-\,768\,(2\,u + 189\,z - 7)\,F'''^{\,2}uz^3 + 2304\,F'^{\,2}F''\,u^2 - 3072\,(3\,z - 1)\,F'\,F''^{\,2}u^2$$

$$-\,192\,\left(24\,uz - 27\,z^2 + 55\,z - 2\right)F''^{\,3}u^2 - 1536\,(21\,z - 2)\,F'\,F'''\,uz$$

$$-\,768\,\left(12\,uz + 81\,z^2 + 24\,z - 1\right)F''\,F'''\,uz + 1536\,(9\,z + 2)\,F'\,F''\,u$$

$$-\,512\,\left(39\,uz + 81\,z^2 + 51\,z - 2\right)F''^{\,2}u + 36\,864\,F'\,z$$

$$-\,1024\,\left(12\,uz - 162\,z^2 + 33\,z - 1\right)F'''\,z - 1024\,(36\,uz + 27\,z - 1)\,F'' - 24\,576\,z = 0.$$

We argue in Section 11.2 that $F$ does not satisfy a differential equation of order 2.

We have applied the same method to the series $H$ of Proposition 3.5:

$$H(z, u) = \bar{u}zR - \bar{u}z^2 - \Lambda(R)$$

where

$$\Lambda(x) = \sum_{i \geq 3} \frac{(3i - 6)!}{(i - 3)!(i - 2)!i!}x^i$$

satisfies

$$30\Lambda(x) = x(27x - 1)\Phi'(x) + (1 - 24x)\Phi(x) + 3x^2.$$

This gives for $H$ an equation of order 2 and degree 3:

$$3\,(u+1)u^2H'^{\,2}H'' + 12\,u^2zH'H'' + 6\,(u-8)uH'^{\,2} + 240\,H$$
$$+ 4\,(6\,uz - 54\,z + 1)H' + 4\,(3\,uz^2 + 30\,uH + 27\,z^2 - z)H'' + 24\,z^2 = 0.$$

One reason explaining this more modest size is the simplicity of the expression (23) of $H'$.

### 4.3. The cubic case

We start from the expression of $F'$ given in Theorem 3.1. We now have to deal with series $\theta$, $\Phi_1$ and $\Phi_2$ in two variables:

$$\theta(x,y) = 3\sum_{i\geq 0}\sum_{\substack{j\geq 0 \\ 2i+j\geq 3}} \frac{(4\,i+2\,j-4)!}{(2i+j-3)!\,i!^2j!}x^iy^j,$$

$$\Phi_1(x,y) = \sum_{i\geq 1}\sum_{\substack{j\geq 0 \\ 2i+j\geq 3}} \frac{(4i+2j-4)!}{(2i+j-2)!\,(i-1)!\,i!\,j!}x^iy^j, \tag{27}$$

$$\Phi_2(x,y) = \sum_{i\geq 0}\sum_{\substack{j\geq 0 \\ 2i+j\geq 2}} \frac{(4i+2j-2)!}{(2i+j-1)!i!^2j!}x^iy^j. \tag{28}$$

Let us first observe that

$$\theta(x,y) = -2\Phi_1(x,y) + (1-y)\Phi_2(x,y) - 2x - y^2.$$

Consequently, Theorem 3.1 gives:

$$F' = 2z\bar{u} + \bar{u}S - (1+\bar{u})(2R + S^2). \tag{29}$$

Then, the summations over the variable $j$ that occur in $\Phi_1$ and $\Phi_2$ can be performed explicitly, which gives to the cubic case a one-variable flavour. Indeed,

$$\Phi_1(x,y) = (1-4y)^{3/2}\,\Psi_1(t) - x, \tag{30}$$

$$\Phi_2(x,y) = \sqrt{1-4y}\,\Psi_2(t) + \frac{1}{4}\left(1 - \sqrt{1-4y}\right)^2, \tag{31}$$

where $t = x/(1-4y)^2$ and

$$\Psi_1(z) = \sum_{i\geq 1}\frac{(4\,i-4)!}{(2\,i-2)!\,i!\,(i-1)!}\,z^i, \quad \Psi_2(z) = \sum_{i\geq 1}\frac{(4\,i-2)!}{(2\,i-1)!\,i!^2}\,z^i.$$

Our system thus reads:

$$R = z + u \left(1 - 4\,S\right)^{3/2} \Psi_1\left(T\right) - uR, \tag{32}$$

$$S = u\sqrt{1 - 4S}\,\Psi_2\left(T\right) + \frac{u}{4}\left(1 - \sqrt{1 - 4S}\right)^2, \tag{33}$$

with $T = R/(1 - 4S)^2$.

The series $\Psi_1(z)$, $\Psi_2(z)$ and their derivatives lie in a 3-dimensional vector space over $\mathbb{Q}(z)$ spanned (for instance) by 1, $\Psi_1(z)$ and $\Psi_2(z)$. This follows from the following identities, which are easily checked:

$$(1 - 64z)\Psi_1'(z) + 48\Psi_1(z) + 2\Psi_2(z) = 1,$$
$$z(1 - 64z)\Psi_2'(z) + 6\Psi_1(z) + 16z\Psi_2(z) = 8z. \tag{34}$$

By the argument of Section 4.1, we can now express $R'$ and $S'$ as rational functions of $u$, $R$, $S$, $\Psi_1(T)$ and $\Psi_2(T)$. But $\Psi_1(T)$ and $\Psi_2(T)$ can be expressed rationally in terms of $z$, $u$, $R$ and $\sqrt{1 - 4S}$ using (32) and (33). Hence we obtain rational expressions of $R'$ and $S'$ in $u$, $z$, $R$ and $\sqrt{1 - 4S}$. In fact no square root occurs:

$$R' = \frac{R(48z - 1 + 16(u + 1)R + 2(3 + u)S - 8(u + 1)S^2)}{D}, \tag{35}$$

$$S' = \frac{2(3z + (u - 3)R - 12zS + 4(u + 1)RS)}{D}, \tag{36}$$

with

$$D = 36z^2 + (24z - 1 + 24uz)R + 4(u + 1)RS - 4(u + 1)^2RS^2 + 4(u + 1)^2R^2.$$

Combining these two equations with (29), one can now express $F'$, $F''$ and $F'''$ in terms of $u$, $z$, $R$ and $S$, and then eliminate $R$ and $S$ to obtain a differential equation of order 2 satisfied by $F'$ (of degree 17), which we do not report here. For the generating function $G(z, u)$ of quasi-cubic forested maps (Proposition 3.3), we replace (29) by

$$10G = (1 + \bar{u})\left(z - R + 6zS + 2(u + 1)RS\right),$$

and obtain a differential equation of order 2 and degree 5. It becomes a bit simpler when we rewrite $G = (W + z\bar{u})/2$:

$$0 = \left(3\,u^4zW'^{\,4} - u^3(5\,Wu - uz + z)W'^{\,3} + 4\,(u + 1)(5\,Wu - uz + z)^2\right)W''$$
$$- 48\,u^2z(u + 1)W'^{\,3} + 8\,u(u + 1)(5\,Wu - uz + z)W'^{\,2}$$
$$+ 4\,(u^2 - 1)(5\,Wu - uz + z)W'.$$

Introducing the series $W$ is natural in the solution of the Potts model presented in [8], where the above equation was first obtained.

## 5. Combinatorics of forested trees

Eq. (5), and the positivity of the Tutte coefficients, show that the series $F(z, u)$ that counts $p$-valent forested maps has non-negative coefficients when expanded in $(1 + u)$. We say that it is $(u + 1)$-*positive*. Section 2.2 presents several combinatorial descriptions of $F(z, \mu - 1)$ (see (6), (8), (9), (10)). This will lead us to study the asymptotic behaviour of the coefficient of $z^n$ in $F(z, u)$ not only for $u \geq 0$, but for $u \geq -1$. In this study, we will use the fact that several other series are also $(u + 1)$-positive. We prove this in this section thanks to a combinatorial argument that applies to certain classes of *forested trees*.

### 5.1. Positivity in $(1 + u)$

Let $T$ be a tree having at least one edge, and $\mathcal{F}$ a set of spanning forests of $T$. We define a property of $\mathcal{F}$ that guarantees that the generating function $A_{\mathcal{F}}(u)$ that counts forests of $\mathcal{F}$ by the number of components is $(u + 1)$-positive (after division by $u$).

Let $F \in \mathcal{F}$, and let $e$ be an edge of $T$. By *flipping* $e$ in the forest $F$, we mean adding $e$ to $F$ if it is not in $F$, and removing it from $F$ otherwise. This gives a new forest $F'$ of $T$. We say that $e$ is *flippable* for $F$ if $F'$ still belongs to $\mathcal{F}$. We say that $\mathcal{F}$ is *stable* if for each $F \in \mathcal{F}$,

(i) every edge of $T$ not belonging to $F$ is flippable,
(ii) flipping a flippable edge gives a new forest with the same set of flippable edges.

We say that a set $\mathcal{E}$ of forested trees is *stable* if, for each tree $T$, the set of forests $F$ such that $(T, F) \in \mathcal{E}$ is stable. We consider below a generating function $E(z, u)$ of $\mathcal{E}$, where each forested tree $(T, F)$ is weighted by $z^n u^k$, where $n$ is the size of $T$ (number of edges, of leaves...) and $k$ the number of components of $F$, minus 1.

**Lemma 5.1.** *With the above notation, assume $\mathcal{F}$ is stable. Then all elements of $\mathcal{F}$ have the same number, denoted by $f$, of flippable edges. The generating function of forests of $\mathcal{F}$, counted by components, is $A_{\mathcal{F}}(u) = u(1 + u)^f$. Consequently, if $\mathcal{E}$ is a stable set of forested trees, then $E(z, u)$ is $(u + 1)$-positive.*

**Proof.** Condition (i) implies that the forest $F_{\max}$ consisting of all edges of $T$ belongs to $\mathcal{F}$. Moreover, we can obtain $F_{\max}$ from any forest $F$ of $\mathcal{F}$ by adding iteratively flippable edges. By Condition (ii), this implies that any forest $F$ of $\mathcal{F}$ has the same set of flippable edges as $F_{\max}$. It also implies that, to construct a forest $F$ of $\mathcal{F}$, it suffices to choose, for each flippable edge of $F_{\max}$, whether it belongs to $F$ or not. Since $F_{\max}$ has a unique component, and since deleting an edge from a forest increases by 1 the number of components, the expression of $A_{\mathcal{F}}(u)$ follows. $\quad\square$
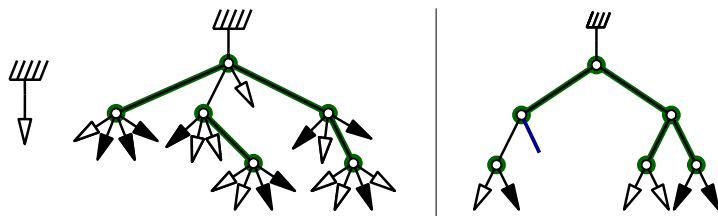
**Fig. 6.** *Left:* The smallest enriched R-tree, and an enriched 5-valent R-tree having 10 leaves (white; charge +1) and 9 buds (black; charge −1). *Right:* A cubic enriched S̃-tree with a dangling half-edge incident to the root component.

### 5.2. Enriched blossoming trees

We now apply the above principle to establish $(u + 1)$-positivity properties for the series $R$ and $S$ given by (12)–(13), and for the series $\tilde{S} \equiv \tilde{S}(z, u)$ defined by

$$\tilde{S}(0, u) = 0, \quad \tilde{S} = u\, \Phi_2(z, \tilde{S}), \tag{37}$$

where $\Phi_2$ is given by (11).

We consider plane trees rooted at a half-edge, which we draw hanging from their root as in Fig. 6. A vertex of degree $d$ is thus seen as the parent of $d - 1$ children. A *subtree* consists of a vertex $v$ and all its descendants. It is naturally rooted at the half-edge incident to $v$ and located just above it. A *blossoming tree* is a plane tree with two kinds of childless vertices: *leaves*, represented by white arrows, and *buds*, represented by black arrows. The edges that carry leaves and buds are considered as half-edges. Also, leaves and buds are not considered as nodes of the tree, and a spanning forest of a blossoming tree does not contain any of its half-edges. The root half-edge does not carry any leaf or bud. Each leaf is assigned a *charge* +1 while each bud is assigned a charge −1. The *charge*, of a blossoming tree is the difference between the number of leaves and buds that it contains. This definition is extended to subtrees.

**Definition 5.2.** Let $p \geq 3$. A $p$-valent blossoming tree $T$ equipped with a spanning forest $F$ is an *enriched R- (resp. S-) tree* if

 (i)  its charge is 1 (resp. 0),
 (ii) any subtree rooted at an edge *not in* $F$ has charge 0 or 1.

We also consider as an enriched R-tree a single root half-edge carrying a leaf.

The pair $(T, F)$ is an *enriched S̃-tree* if each component of $F$ is incident to as many leaves as buds. In this case it is also an enriched S-tree.

An enriched tree $(T, F)$ is *canonical* if the spanning forest $F$ is in fact a tree (and hence consists of all edges of $T$).

Two enriched R-trees are shown in Fig. 6 (left). The readers who are familiar with the R- and S-trees of [22] will recognize that our enriched R- and S-trees are obtained

from them by inflating each vertex of degree $k$ into a (leaf rooted) $p$-valent tree with $k$ leaves.

**Proposition 5.3.** *Let $p \geq 3$. The series $R$, $S$ and $\tilde{S}$ defined by* (12), (13) *and* (37) *count respectively enriched R-, S- and $\tilde{S}$-trees, by the number of leaves (z) and the number of components in the forest (u).*

**Proof.** The equations follow from a recursive decomposition of enriched trees. For instance, an enriched R-tree is either reduced to a single leaf, with no forest at all (and contribution $z$), or contains a root node. This node belongs to a component of the forest. This component is incident to several half-edges (not belonging to the forest), one of them being the root half-edge. Each of the other incident half-edges can be a leaf, a bud, or the root of a non-trivial subtree. In this case, the definition of enriched R-trees implies that this subtree is itself an enriched R-tree (of charge 1) or an enriched S-tree (of charge 0). Since a single leaf is considered as an R-tree, we can say that every half-edge incident to the root component of the forest carries either a bud, or an R- or S-tree. If there are $i$ attached R-trees, we must have $i - 1$ buds for the tree charge to be 1, and an arbitrary number $j$ of S-trees. The root component of the forest is then a leaf-rooted tree with $k = 2i + j$ leaves. This gives (12), where the multinomial coefficient occurring in $\Phi_1$ describes the order in which the $i$ R-trees, the $i - 1$ buds and the $j$ S-trees are attached from left to right.

The proof of (13) is similar, but now as many buds as R-trees must be attached to the root component of the forest in order to make the charge 0.

Finally, an $\tilde{S}$-tree is an S-tree in which all attached R-trees are actually leaves. This explains that (37) is obtained from (13) by replacing each occurrence of $R$ by $z$.  □

We now come back to $(u + 1)$-positivity.

**Proposition 5.4.** *The set of p-valent enriched R- (resp. S-, $\tilde{S}$-) trees having at least one edge is stable, in the sense of Section* 5.1.

**Proof.** For enriched R- and S-trees, an edge is flippable if and only if the attached subtree has charge 0 or 1, and this condition is independent of the forest.

For enriched $\tilde{S}$-trees, an edge is flippable if and only if the attached subtree is incident to as many leaves as buds, and this condition is again independent of the forest.  □

By combining this proposition with Lemma 5.1 and Proposition 5.3, we obtain:

**Corollary 5.5.** *The series $\bar{u}(R - z)$, $\bar{u}S$ and $\bar{u}\tilde{S}$ are $(u + 1)$-positive. Setting $u = \mu - 1$, they count respectively (non-empty) canonical enriched R-, S- and $\tilde{S}$-trees, by the number of leaves (z) and the number of flippable edges ($\mu$).*

When $p = 3$ for instance, we have, writing $\mu = u + 1$,

$$\bar{u}(R - z) = 2(2\mu + 1)z^2 + 4(10\mu^3 + 12\mu^2 + 9\mu + 4)z^3 + O(z^4),$$
$$\bar{u}S = 2z + 6(2\mu^2 + 2\mu + 1)z^2 + 8(16\mu^4 + 28\mu^3 + 30\mu^2 + 22\mu + 9)z^3 + O(z^4),$$
$$\bar{u}\tilde{S} = 2z + 2(2\mu^2 + 8\mu + 5)z^2 + 8(2\mu^4 + 12\mu^3 + 33\mu^2 + 40\mu + 18)z^3 + O(z^4).$$

We will also need the following variant of these results.

**Lemma 5.6.** *Define* $\Phi_2$ *by* (11) *and* $\tilde{S}$ *by* (37). *The series* $\frac{\partial \Phi_2}{\partial y}(z, \tilde{S})$ *is* $(u + 1)$*-positive.*

**Proof.** Let us extend the definition of $\tilde{S}$-enriched trees to $p$-valent blossoming trees that, in addition to leaves and buds, contain also one dangling half-edge, having no charge (Fig. 6, right). Using the arguments of Proposition 5.3, one can prove that the series $u\frac{\partial \Phi_2}{\partial y}(z, \tilde{S})$ counts such $\tilde{S}$-enriched trees for which the half-edge is incident to the root component (as before, $z$ counts leaves and $u$ components).

The set of such trees is again stable: indeed, an edge is flippable if it is flippable in the $\tilde{S}$-sense, and is not on the path from the root to the dangling half-edge.  □

## 6. Implicit functions: some general results

The singular behaviour of a series $Y(z)$ defined by an implicit equation $H(z, Y(z)) = 0$ is well-understood when the singularities of $Y$ occur at a point $z$ such that $H$ is analytic at $(z, Y(z))$, but $H'_y(z, Y(z)) = 0$. A typical situation is the so-called *smooth implicit schema* of [36, Sec. VII.4], which leads to square root singularities in $Y$.

However, in our asymptotic analysis of 4-valent and cubic forested maps, we will have to deal with implicit functions $Y$ that become singular at a point $z$ such that $H$ ceases to be analytic at $(z, Y(z))$. Our series $Y$ have non-negative real coefficients, which implies that their radius is also a dominant singularity, and leads us to pay a special attention to the behaviour of $Y$ along the positive real axis.

In this section, we thus examine how far a real series $Y$ defined by an implicit equation can be extended along the positive real axis. We first establish a general result for equations of the form $H(z, Y(z)) = 0$ (Proposition 6.1), which will apply for instance to the series $\tilde{S}$ defined by (37). We then specialize this proposition to an *inversion equation* of the form $\Omega(Y(z)) = z$ (Corollary 6.3). This corollary will apply in particular to the series $R$ defined, in the 4-valent case, by $R = z + u\Phi(R)$ (see (15)).

**Proposition 6.1.** *Let* $H(x, y)$ *be a real bivariate power series, analytic in a neighbourhood of* $(0, 0)$, *satisfying* $H(0, 0) = 0$ *and* $H'_y(0, 0) > 0$. *Let* $Y \equiv Y(z)$ *be the unique power series satisfying* $Y(0) = 0$ *and* $H(z, Y(z)) = 0$. *Then* $Y$ *has a non-zero radius of convergence. Moreover, there exists* $\rho > 0$ *such that:*

(a) $Y$ *has an analytic continuation, still denoted by* $Y$, *in a neighbourhood of* $[0, \rho]$, *which is real valued,*

(b) *H has an analytic continuation, still denoted by H, in a neighbourhood of*
$\{(z, Y(z)), z \in [0, \rho]\}$,
(c) *$H(z, Y(z)) = 0$ for $z \in [0, \rho]$,*
(d) *$H'_y(z, Y(z)) > 0$ for $z \in [0, \rho]$.*

*Moreover, if $\tilde{\rho}$ is the supremum (in $\mathbb{R} \cup \{+\infty\}$) of these values $\rho$, at least one of the following properties holds:*

  (i) *$\tilde{\rho} = +\infty$,*
  (ii) *$\liminf_{z \to \tilde{\rho}^-} H'_y(z, Y(z)) = 0$,*
 (iii) *for each $y \in [\liminf_{z \to \tilde{\rho}^-} Y(z), \limsup_{z \to \tilde{\rho}^-} Y(z)]$, H is singular at $(\tilde{\rho}, y)$,*
 (iv) *$\limsup_{z \to \tilde{\rho}^-} |Y(z)| = +\infty$.*

We begin with a simple lemma.

**Lemma 6.2.** *Let $a < 0 < b$ and let $Y$ be a function analytic in a neighbourhood of $[a, b]$, whose Taylor expansion at 0 has real coefficients. Then $Y$ takes real values on $[a, b]$.*

**Proof.** The functions $z \mapsto Y(z)$ and $z \mapsto \overline{Y(\bar{z})}$ are analytic and coincide in the neighbourhood of 0 where $Y(z)$ is given by its Taylor expansion. Hence they coincide everywhere, and $Y(z)$ is real when $z$ is real. $\square$

**Proof of Proposition 6.1.** The uniqueness of $Y$ comes from the fact that its coefficients can be computed by induction using the equation $H(z, Y(z)) = 0$ and the initial condition $Y(0) = 0$ (the assumption $H'_y(0, 0) \neq 0$ is crucial here). Note that these coefficients are real, so that Lemma 6.2 will apply. But let us first prove that $Y$ has a positive radius of convergence. Since $H'_y(0, 0) > 0$, the analytic implicit function theorem at $z = 0$ implies the existence of a locally analytic solution $\hat{Y}$ to the implicit equation $H(z, \hat{Y}(z)) = 0$ satisfying $\hat{Y}(0) = 0$. The expansion of $\hat{Y}$ around 0 must satisfy this equation as well (in the world of formal power series), and thus coincides with $Y$. Hence $Y$ has a positive radius.

Now consider the set

$$I = \{\rho > 0 \mid \rho \text{ satisfies conditions (a), (b), (c), (d)}\}.$$

This is clearly an open interval of the form $(0, \tilde{\rho})$, and it is non-empty since (a), (b), (c) and (d) hold in the neighbourhood of 0. Assume that none of the properties (i), (ii), (iii) and (iv) hold at $\tilde{\rho}$. In particular, $\tilde{\rho}$ is finite. We will reach a contradiction by proving that $\tilde{\rho} \in I$.

Since (iv) does not hold, $Y$ is bounded on $[0, \tilde{\rho})$. By continuity, the set of accumulation points of $\{Y(z), z \in [0, \tilde{\rho})\}$ is an interval, which coincides with $[\liminf_{z \to \tilde{\rho}^-} Y(z), \limsup_{z \to \tilde{\rho}^-} Y(z)]$. For each $y$ in this interval, the point $(\tilde{\rho}, y)$ is in the closure of the set

$\{(z, Y(z)), z \in [0, \tilde{\rho})\}$ where $H$ is known to be analytic. Since (iii) does not hold, there exists an element $\tilde{y}$ in this interval such that $H$ is analytic at $(\tilde{\rho}, \tilde{y})$. In particular, it is continuous at this point, and (c) implies that $H(\tilde{\rho}, \tilde{y}) = 0$. Finally, since (d) holds, but (ii) does not, $H'_y(\tilde{\rho}, \tilde{y}) > 0$.

These three properties allow us to apply the analytic implicit function theorem: there exists an analytic function $\tilde{Y}$ defined in a neighbourhood of $\tilde{\rho}$ such that $H(z, \tilde{Y}(z)) = 0$ and $\tilde{Y}(\tilde{\rho}) = \tilde{y}$. We want to prove that $\tilde{Y}$ is an analytic continuation of $Y$ at $\tilde{\rho}$, so that, in particular, the interval $[\liminf_{z \to \tilde{\rho}^-} Y(z), \limsup_{z \to \tilde{\rho}^-} Y(z)]$ is reduced to the point $\tilde{y}$.

Since $H'_y(\tilde{\rho}, \tilde{y}) > 0$, there exist $\delta > 0$ and a complex neighbourhood $V$ of $(\tilde{\rho}, \tilde{y})$ such that for $(x, y)$ and $(x, y')$ in $V$,

$$|H(x, y) - H(x, y')| \geq \delta |y - y'|.$$

We can also assume that $\tilde{Y}(x)$ is well-defined for $(x, y) \in V$.

Since $(\tilde{\rho}, \tilde{y})$ is an accumulation point of $\{(z, Y(z)), z \in (0, \tilde{\rho})\}$, and $Y$ is continuous, there exists an interval $[z_0, z_1] \subset (0, \tilde{\rho})$ such that $(z, Y(z)) \in V$ for $z \in [z_0, z_1]$. Then for $z$ in this interval,

$$0 = |H(z, Y(z)) - H(z, \tilde{Y}(z))| \geq \delta |Y(z) - \tilde{Y}(z)|,$$

which shows that the analytic functions $Y$ and $\tilde{Y}$ coincide on $[z_0, z_1]$. So they coincide where they are both defined, and $\tilde{Y}$ is an analytic continuation of $Y$ at $\tilde{\rho}$. This tells us that (a) holds at $\tilde{\rho}$. Now (b) also holds by the choice of $\tilde{y}$, (c) holds by construction of $\tilde{Y}$, and (d) holds as well, as argued above. Thus $\tilde{\rho}$ belongs to $I$, which cannot be true since it is the supremum of the open interval $I$. Hence one of the properties (i), (ii), (iii) and (iv) must hold. $\square$

**Corollary 6.3.** *Let $\Omega(y)$ be a real power series such that $\Omega(0) = 0$ and $\Omega'(0) > 0$. Let $\omega \in (0, +\infty]$ be the first singularity of $\Omega$ on the positive real axis, if it exists, and $+\infty$ otherwise. Let $Y \equiv Y(z)$ be the unique power series satisfying $Y(0) = 0$ and $\Omega(Y(z)) = z$. Then $Y$ has a non-zero radius of convergence. Moreover, there exists $\rho \in (0, \infty]$ such that:*

(1) *$Y$ has an analytic continuation, still denoted by $Y$, in a neighbourhood of $[0, \rho)$, which is real valued,*
(2) *$Y$ is increasing on $[0, \rho)$,*
(3) *$Y(z) \in [0, \omega)$ for $z \in [0, \rho)$,*
(4) *$\Omega(Y(z)) = z$ for $z \in [0, \rho)$,*
(5) *$\lim_{z \to \rho^-} Y(z) = \tau$ and $\lim_{y \to \tau^-} \Omega(y) = \rho$,*

*where*

$$\tau = \min\{y \in [0, \omega) \mid \Omega'(y) = 0\}$$

*if this set is non-empty, and $\tau = \omega$ otherwise.*

This result is stated as an existence result for $\rho$, but (5) actually *determines* the value of $\rho$.

**Proof of Corollary 6.3.** We specialize Proposition 6.1 to $H(x, y) = \Omega(y) - x$. Clearly, $H$ is analytic around $(0, 0)$, $H(0, 0) = 0$ and $H'_y(0, 0) = \Omega'(0) > 0$. We take for $\rho$ the value $\tilde{\rho}$ of Proposition 6.1. Then (1) follows from (a). Conditions (b) and (c) tell us that $\Omega$ has an analytic continuation on $\{Y(z), z \in [0, \rho)\}$, such that $\Omega(Y(z)) = z$ for $z \in [0, \rho)$. By differentiating this identity, we obtain $Y'(z)\Omega'(Y(z)) = 1$, so that (2) now follows from (d). Thus the existence of an analytic continuation of $\Omega$ on $\{Y(z), z \in [0, \rho)\}$ now translates into (3). The monotonicity of $Y$ also allows us to define $Y(\rho) := \lim_{z \to \rho^-} Y(z)$, which is not necessarily finite.

Let us now derive (5) from the second series of properties of Proposition 6.1. We have already seen (this is (3)) that $Y(\rho) \leq \omega$. By Condition (d) of Proposition 6.1, and by definition of $\tau$, the value $Y(\rho)$ is also less than or equal to $\tau$. Assume $Y(\rho) < \tau$. Then $\Omega$ is analytic at $Y(\rho)$, and by continuity of $\Omega$ and $Y$, $\rho = \Omega(Y(\rho)) < +\infty$, so that (i) cannot hold. By definition of $\tau$, we cannot have (ii). It is easy to see that Conditions (iii) and (iv) do not hold either. So we have reached a contradiction, and $Y(\rho) = \tau$. Returning to (4) gives $\rho = \Omega(Y(\rho)) = \Omega(\tau)$.   □

## 7. Inversion of functions with a $z \ln z$ singularity

The inversion of a locally injective analytic function is a well-understood topic: if $\Psi$ is analytic in the disk $C_s$ of radius $s$ centred at 0 and $\Psi(z) \sim z$ as $z \to 0$, then there exist $\rho \in (0, s)$ and $\rho' > 0$, and a function $\Upsilon$ analytic on $C_{\rho'}$, taking its values in $C_\rho$, such that

$$\forall (y, z) \in C_{\rho'} \times C_\rho, \qquad \Psi(z) = y \Longleftrightarrow z = \Upsilon(y).$$

The aim of this section is to see to what extent this can be generalized to a function $\Psi(z)$ having a singularity in $z \ln z$ around 0. Of course we cannot consider disks anymore, and our local domains will be of the following form:

$$D_{\rho, \alpha} := \{z = re^{i\theta} : r \in (0, \rho) \text{ and } |\theta| < \alpha\}.$$

We think that the following theorem could have applications in the future to other combinatorial problems involving a singular inversion.[5]

**Theorem 7.1** *(Log-Inversion). Let $\Psi$ be analytic on $D_{s,\pi}$ for some $s > 0$. Assume that as $z$ tends to $0$ in this domain,*

$$\Psi(z) \sim -cz \ln z$$

*with $c > 0$. Then for each $\alpha \in (0, \pi)$, there exist $\rho \in (0, s)$ and $\rho' > 0$, and a function $\Upsilon$ analytic in $D_{\rho',\alpha}$, taking its values in $D_{\rho,\pi}$, that satisfies*

$$\forall (y, z) \in D_{\rho',\alpha} \times D_{\rho,\pi}, \qquad \Psi(z) = y \Longleftrightarrow z = \Upsilon(y).$$

*Moreover, as $y \to 0$ in $D_{\rho',\alpha}$,*

$$\Upsilon(y) \sim -\frac{y}{c \ln y}.$$

The proof is rather long. The most difficult part is to prove the existence of a unique preimage of $y$ under $\Psi$ in $D_{\rho,\pi}$, for each $y \in D_{\rho',\alpha}$ (Lemma 7.5). This preimage is of course $\Upsilon(y)$. Proving the analyticity of $\Upsilon$ is then a simple application of the analytic implicit function theorem. In order to prove Lemma 7.5, we first study the injectivity and surjectivity of the function $z \mapsto -z \ln z$ around $0$ (Section 7.1), before transferring them to the function $\Psi$ (Section 7.2).

**Note.** As pointed out by one of our referees, the inverse function $\Upsilon$ of Theorem 7.1 is related to the Lambert $W$ function when $\Psi$ is exactly $-z \ln z$. Indeed, the equation $y = -z \ln z$ can be inverted into $z = \exp(W(-y)) = \Upsilon(y)$, where $W$ is the (multivalued) Lambert function [28], defined by $W(u) \exp(W(u)) = u$. We want $W$ to be analytic in a neighbourhood of a real segment of the form $(-\rho', 0)$. By convention, the only branch of the Lambert function that satisfies this property is the principal determination $W_0$. But $W_0(u)$ tends to $0$ when $u$ tends to $0$, hence $\exp(W_0(-y))$ tends to $1$ instead of $0$ (as we want) when $y$ tends to $0$. What we need for $W$ is a mixture of the branches $W_1$ and $W_{-1}$, which is analytic around $(-1/e, 0)$ and has two cuts: for $z \in \mathbb{C} \setminus ((-\infty, -1/e] \cup [0, +\infty))$,

$$W(u) = \begin{cases} W_{-1}(u) & \text{if } \mathrm{Re}(u) \geq 0, \\ W_1(u) & \text{otherwise.} \end{cases}$$

Unfortunately, it seems that this connection with the Lambert function does not spare us to prove the results of the next subsection: first because of this mixture of branches, then because these results deal with very fine properties (see Proposition 7.4 for instance) that cannot be easily found in the literature.

---

[5]  Moreover, as noted by one of the referees, a simple change of variable allows us to study the inversion of a function that behaves like $-z^k \ln z$, with $k \in \mathbb{N}$ — but of course one must then select a branch, as when one inverts $z^k$.

*7.1. The function $z \mapsto -z \ln z$*

Consider the following function

$$
\begin{aligned}
H : \mathbb{C} \setminus \mathbb{R}^- &\to \quad \mathbb{C} \\
z &\mapsto -z \ln z,
\end{aligned}
$$

where ln denotes the principal value of logarithm: if $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$, then $\ln z = \ln r + i\theta$. We also define $\operatorname{Arg} z := \theta$. Let us begin with a few elementary properties of $H$.

**Lemma 7.2.** *The function $H$ satisfies*

$$
H(\bar{z}) = \overline{H(z)}. \tag{38}
$$

*For $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$,*

$$
|H(z)| = r\sqrt{\ln^2 r + \theta^2}. \tag{39}
$$

*The arguments of $z$ and $-\ln z$ have opposite signs. If in addition $r < 1$, then $\operatorname{Arg}(-\ln z) \in (-\pi/2, \pi/2)$. Hence*

$$
\operatorname{Arg} H(z) = \operatorname{Arg} z + \operatorname{Arg}(-\ln z) = \theta + \arctan\left(\frac{\theta}{\ln r}\right). \tag{40}
$$

*If in addition $r \le 1/\sqrt{e}$, then*

$$
|\operatorname{Arg} H(z)| \le |\theta|. \tag{41}
$$

*In particular, $H(z) \notin \mathbb{R}^-$.*

**Proof.** The first two identities are straightforward. The first part of (40) follows from the fact that the arguments of $z$ and $-\ln z$ have opposite signs. The second part follows from $\operatorname{Arg}(-\ln z) \in (-\pi/2, \pi/2)$. Let us now prove (41). Assume $\theta \ge 0$. Then $\operatorname{Arg}(-\ln z) \le 0$ and the first part of (40) gives $\operatorname{Arg} H(z) \le \theta$. Moreover, $\arctan x \ge x$ if $x \le 0$, and thus by the second part of (40),

$$
\operatorname{Arg} H(z) \ge \theta + \frac{\theta}{\ln r} \ge \left(1 - \frac{1}{\ln(1/\sqrt{e})}\right)\theta = -\theta.
$$

The case where $\theta \le 0$ now follows using (38). $\square$

Observe that $H$ is not injective on $\mathbb{C}$: for instance, $H(i) = \pi/2 = H(-i)$. However, $H$ is injective in a (slit) neighbourhood of 0 (Fig. 7).
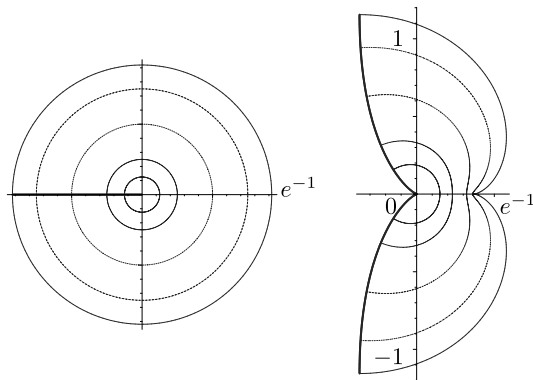
**Fig. 7.** *Left*: The domain $D_{e^{-1},\pi}$, with circles of radius 0.05, 0.1, 0.2 and 0.3. *Right:* The image of the previous picture under the map $H$. The cut along the negative real axis opens into the two thick curves to the left of the picture.

**Proposition 7.3.** *The function $H : z \mapsto -z \ln z$ is injective on $D_{e^{-1},\pi}$.*

**Proof.** Assume there exist $z_1$ and $z_2$ in $D_{e^{-1},\pi}$ such that $H(z_1) = H(z_2)$. By Lemma 7.2, the value $H(z_1)$ is not real and negative, and thus $\ln H(z_1) = \ln H(z_2)$.

This lemma also implies that for $z \in D_{e^{-1},\pi}$, we have

$$\ln H(z) = \ln z + \ln(-\ln z). \tag{42}$$

Hence

$$|\ln z_1 - \ln z_2| = |\ln(-\ln z_1) - \ln(-\ln z_2)|. \tag{43}$$

Let $\kappa = -\max(\ln|z_1|, \ln|z_2|) > 1$. Then $-\ln z_1$ and $-\ln z_2$ lie in $\{z \mid \mathrm{Re}(z) \geq \kappa\}$. This set is convex, so the (vectorial) mean value inequality gives

$$|\ln(-\ln z_1) - \ln(-\ln z_2)| \leq |\ln z_1 - \ln z_2| \sup_{z \in [-\ln z_1, -\ln z_2]} |\ln'(z)| \leq \frac{1}{\kappa}|\ln z_1 - \ln z_2|.$$

Combining this with (43) gives $|\ln z_1 - \ln z_2| = 0$, so that $z_1 = z_2$.  □

We now address the surjectivity of the map $H$.

**Proposition 7.4.** *For $0 < \alpha < \pi$ and $\rho$ small enough (depending on $\alpha$), we have*

$$D_{-\rho \ln \rho, \alpha} \subset H(D_{\rho,\pi}).$$

**Proof.** We are going to prove that the inclusion holds for every $\rho \in (0, 1/e)$ satisfying

$$\arctan\left(\frac{\pi}{|\ln \rho|}\right) \leq \pi - \alpha. \tag{44}$$

Let us fix a complex number $se^{i\gamma}$ with $0 < s < -\rho \ln \rho$ and $|\gamma| < \alpha$. We want to prove the existence of $r < \rho$ and $\theta \in (-\pi, \pi)$ such that $H(re^{i\theta}) = se^{i\gamma}$. We proceed in two steps.

**(1) There exists a continuous function $\theta : (0, \rho) \to (-\pi, \pi)$ such that $\forall r \in (0, \rho)$,**

$$\mathbf{Arg}\, H(r\, e^{i\theta(r)}) = \gamma.$$

Fix $r \in (0, \rho)$. For $\theta \in (-\pi, \pi)$, Lemma 7.2 gives

$$f(r, \theta) := \operatorname{Arg} H(re^{i\theta}) = \theta + \arctan\left(\frac{\theta}{\ln r}\right).$$

Differentiating with respect to $\theta$ gives

$$f'_\theta(r, \theta) = 1 + \frac{1}{\left(1 + \frac{\theta^2}{\ln^2 r}\right) \ln r} \geq 1 + \frac{1}{\ln r} > 0.$$

Hence $f(r, \theta)$ is a continuous increasing function of $\theta$, sending $(-\pi, \pi)$ onto $(-\pi - \arctan(\pi/\ln r), \pi + \arctan(\pi/\ln r))$. Since $r < \rho$ and $\rho$ satisfies (44), this interval contains $(-\alpha, \alpha)$, and thus the value $\gamma$. This proves the existence, and uniqueness (since $f$ increases), of $\theta(r)$.

Now in the neighbourhood of $(r, \theta(r))$, we can apply the implicit function theorem to the equation $f(r, \theta) = \gamma$, and this shows that $\theta$ is continuous on $(0, \rho)$.

**(2) There exists $r \in (0, \rho)$ such that $|H(r\, e^{i\theta(r)})| = s$.**

The function

$$r \mapsto |H(r\, e^{i\theta(r)})| = r\sqrt{\ln^2 r + \theta(r)^2}$$

is continuous on $(0, \rho)$. It tends to $0$ as $r$ tends to $0$, and to a value at least equal to $-\rho \ln \rho$ as $r$ tends to $\rho$. Since $\alpha < -\rho \ln \rho$, the intermediate value theorem implies that there exists $r \in (0, \rho)$ such that $|H(re^{i\theta(r)})| = s$.

This completes the proof of the proposition. $\square$

### 7.2. The log-inversion theorem

By combining Propositions 7.3 and 7.4, we see that for $\alpha \in (0, \pi)$ and $\rho$ small enough, every point of $D_{-\rho \ln \rho, \alpha}$ has a unique preimage under $H$ in $D_{\rho, \pi}$. We now want to adapt this result to functions $\Psi$ that behave like $H$ in the neighbourhood of the origin.

**Lemma 7.5.** *Let $\Psi$ be analytic on $D_{s, \pi}$ for some $s > 0$. Assume that as $z$ tends to $0$ in this domain,*
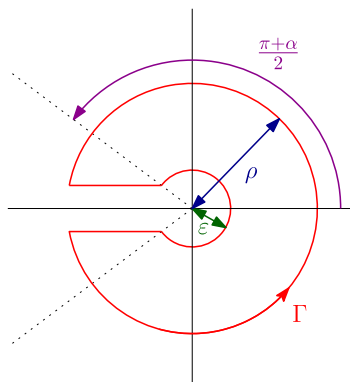
$$\Psi(z) \sim H(z) = -z \ln z.$$

**Fig. 8.** The contour $\Gamma^{(\varepsilon)}$.

*For all $\alpha \in (0, \pi)$, there exist $\rho \in (0, s)$ and $\rho' > 0$ such that every point of $D_{\rho', \alpha}$ has a unique preimage under $\Psi$ in $D_{\rho, \pi}$.*

**Proof.** By assumption, $\Psi(z) - H(z) = o(z \ln z) = o(-|z| \ln |z|)$ as $z$ tends to 0. Let $\rho \in (0, s)$ be small enough for every $z \in D_{\rho, \pi}$ to satisfy

$$|\Psi(z) - H(z)| < -\min\left(\frac{1}{2}, \sin\left(\frac{\pi - \alpha}{4}\right)\right) |z| \ln |z|, \tag{45}$$

$$1 + \frac{1}{\ln |z|} > \frac{1}{2} + \frac{\alpha}{\pi + \alpha}, \tag{46}$$

$$|z \ln z| \leq -2|z| \ln |z|, \tag{47}$$

$$-\ln \frac{|z|}{8} \leq -2 \ln |z|, \tag{48}$$

and assume moreover that $\rho$ is also small enough for the following property to hold:

$$D_{-\frac{\rho}{8} \ln \frac{\rho}{8}, \alpha} \subset H(D_{\frac{\rho}{8}, \pi}). \tag{49}$$

This inclusion is made possible by Proposition 7.4. Several of the above listed conditions can be described by an explicit upper bound on $\rho$ (for instance, (48) just means that $\rho \leq 1/8$), but we will use them in the above form and find convenient to write them so.

Now fix $y_0 \in D_{\rho', \alpha}$ with $\rho' = -\frac{\rho}{8} \ln \frac{\rho}{8}$. We want to prove that $y_0$ has a unique preimage under $\Psi$ in $D_{\rho, \pi}$. By (49) and Proposition 7.3, it has a unique preimage under $H$, denoted by $z_0$, in $D_{\rho, \pi}$ (in fact, $|z_0| < \rho/8$). We thus want to prove that the functions $\Psi - y_0$ and $H - y_0$ have the same number of roots in $D_{\rho, \pi}$, and we will do so using Rouché's theorem.

For $\varepsilon \in (0, |z_0|/8)$, let $\Gamma \equiv \Gamma^{(\varepsilon)}$ be the contour shown in Fig. 8. The interior of $\Gamma$ converges to $D_{\rho, \pi}$ as $\varepsilon \to 0$. Hence for $\varepsilon$ small enough, it contains the point $z_0$, and we just need to prove that $\Psi - y_0$ and $H - y_0$ have the same number of roots inside $\Gamma^{(\varepsilon)}$ for every small enough $\varepsilon$.

By Rouché's theorem, it suffices to show that $|\Psi - H| < |H - y_0|$ on $\Gamma$. Let us decompose $\Gamma$ into three (non-disjoint) parts:

$$\Gamma_1 = \Gamma \cap \{z : |z| = \rho\}, \quad \Gamma_2 = \Gamma \cap \{z : |z| = \varepsilon\} \quad \text{and} \quad \Gamma_3 = \Gamma \cap \left\{z : |\operatorname{Arg} z| > \frac{\pi + \alpha}{2}\right\}.$$

We will use in the study of $\Gamma_1$ and $\Gamma_2$ the following elementary result.

**Lemma 7.6.** *If $\rho \geq |z| \geq 8\,|z'|$ with $z, z' \in \mathbb{C} \setminus (-\infty, 0]$, then*

$$|z \ln z - z' \ln z'| \geq -\frac{1}{2}|z| \ln |z|.$$

**Proof.** We have the following lower bounds:

$$
\begin{aligned}
|z \ln z - z' \ln z'| &\geq |z \ln z| - |z' \ln z'| \\
&\geq -|z| \ln |z| + 2|z'| \ln |z'| && \text{by (39) and (47),} \\
&\geq -|z| \ln |z| + \frac{|z|}{4} \ln \frac{|z|}{8} && \text{because } |z'| \leq |z|/8, \\
&\geq -\frac{1}{2}|z| \ln |z| && \text{by (48).} \qquad \square
\end{aligned}
$$

Since $|z_0| < \rho/8$, we can apply this lemma to $z \in \Gamma_1$ and $z' = z_0$. This gives, using (45):

$$|H(z) - y_0| \geq -\frac{1}{2}|z| \ln |z| > |\Psi(z) - H(z)|.$$

Since $\varepsilon < |z_0|/8$, applying Lemma 7.6 to $z_0$ and $z \in \Gamma_2$ gives:

$$|y_0 - H(z)| \geq -\frac{1}{2}|z_0| \ln |z_0| \geq -\frac{1}{2}|z| \ln |z| > |\Psi(z) - H(z)|.$$

We are left with the contour $\Gamma_3$. If $z \in \Gamma_3$, we claim that

$$|\operatorname{Arg} H(z)| \geq \alpha + \frac{\pi - \alpha}{4}. \tag{50}$$

By (38), it suffices to prove this when $\operatorname{Arg} z \geq 0$. In this case,

$$
\begin{aligned}
\operatorname{Arg} H(z) &= \operatorname{Arg} z + \arctan\left(\frac{\operatorname{Arg} z}{\ln |z|}\right) && \text{by (40),} \\
&\geq \left(1 + \frac{1}{\ln |z|}\right) \operatorname{Arg} z && \text{since } \arctan x \geq x, \\
&> \left(\frac{1}{2} + \frac{\alpha}{\pi + \alpha}\right) \operatorname{Arg} z && \text{by (46),} \\
&\geq \alpha + \frac{\pi - \alpha}{4} && \text{since } \operatorname{Arg} z \geq \frac{\pi + \alpha}{2}.
\end{aligned}
$$

Hence (50) holds on $\Gamma_3$. But since $|\operatorname{Arg} H(z_0)| = |\operatorname{Arg} y_0| < \alpha$, we have

$$| \operatorname{Arg} H(z) - \operatorname{Arg} H(z_0)| > \frac{\pi - \alpha}{4}. \tag{51}$$

We still need one more result to conclude.

**Lemma 7.7.** *For $\beta > 0$ and complex numbers $a$ and $b$ in $\mathbb{C} \setminus (-\infty, 0]$,*

$$| \operatorname{Arg} a - \operatorname{Arg} b| \geq \beta \quad \Longrightarrow \quad |a - b| \geq |a| \sin \beta.$$

**Proof.** (1) If $|\operatorname{Arg} a - \operatorname{Arg} b| \leq \frac{\pi}{2}$, then

$$|a - b| = \left| |a| e^{i(\operatorname{Arg} a - \operatorname{Arg} b)} - |b| \right| \geq \left| \operatorname{Im} \left( |a| e^{i(\operatorname{Arg} a - \operatorname{Arg} b)} \right) \right| \geq |a| \sin \beta.$$

(2) If $|\operatorname{Arg} a - \operatorname{Arg} b| \geq \frac{\pi}{2}$, then

$$|a - b| = \left| |a| - |b| e^{i(\operatorname{Arg} b - \operatorname{Arg} a)} \right| \geq \left| \operatorname{Re} \left( |a| - |b| e^{i(\operatorname{Arg} b - \operatorname{Arg} a)} \right) \right|$$

$$= |a| - |b| \cos \left( \operatorname{Arg} b - \operatorname{Arg} a \right) \geq |a| \geq |a| \sin \beta. \qquad \square$$

By applying this lemma to (51) with $\beta = (\pi - \alpha)/4$, we obtain

$$|H(z) - y_0| \geq |H(z)| \sin \left( \frac{\pi - \alpha}{4} \right)$$

$$\geq - \sin \left( \frac{\pi - \alpha}{4} \right) |z| \ln |z| \qquad \text{by (39)}$$

$$> |\Psi(z) - H(z)| \qquad \text{by (45)}.$$

We have finally proved that $|\Psi(z) - H(z)| < |H(z) - y_0|$ everywhere on the contour $\Gamma \equiv \Gamma^{(\varepsilon)}$, and we can now conclude that $\Psi - y_0$ has, like $H - y_0$, a unique root in $D_{\rho,\pi}$. $\quad \square$

We are finally ready to prove the log-inversion theorem (Theorem 7.1).

**Proof of Theorem 7.1.** Upon writing $\Psi = c\Psi_1$ and $\Upsilon(y) = \Upsilon_1(y/c)$, we can assume without loss of generality that $c = 1$. We then choose $\rho$ and $\rho'$ as in Lemma 7.5. For $y_0 \in D_{\rho',\alpha}$, we define $\Upsilon(y_0)$ as the unique point $z_0$ of $D_{\rho,\pi}$ such that $\Psi(z_0) = y_0$. We now apply the analytic implicit function theorem to the equation $\Psi(\Upsilon(y)) = y$, in the neighbourhood of $(y_0, z_0)$. The function $\Psi$ is analytic at $z_0$ and locally injective by Lemma 7.5. Therefore $\Psi'(z_0) \neq 0$, and there exists an analytic function $\bar{\Upsilon}$ defined in the neighbourhood of $y_0$ such that $\bar{\Upsilon}(y_0) = z_0$ and $\Psi(\bar{\Upsilon}(y)) = y$ in this neighbourhood.

This forces $\Upsilon(y)$ and $\bar{\Upsilon}(y)$ to coincide in a neighbourhood of $y_0$, and implies that $\Upsilon$ is analytic at $y_0$ — and hence in the domain $D_{\rho',\alpha}$.

Let us conclude with the singular behaviour of $\Upsilon$ near 0. The equation $\Psi(\Upsilon(y)) = y$, combined with $\Psi(z) \sim -z \ln z$, implies that $\Upsilon(y) \to 0$ as $y \to 0$. Thus

$$y \sim -\Upsilon(y) \ln(\Upsilon(y))$$

as $y \to 0$. Upon taking logarithms, and using (42), this gives

$$\ln y \sim \ln(\Upsilon(y)) + \ln(-\ln(\Upsilon(y))) \sim \ln(\Upsilon(y)).$$

Combining the last two equations finally gives $\Upsilon(y) \sim -y/\ln y$.  $\square$

## 8. Asymptotics for 4-valent forested maps

Let $F(z,u) = \sum_n f_n(u)z^n$ be the generating function of 4-valent forested maps, given by Theorem 3.1. That is, $f_n(u)$ counts forested 4-valent maps with $n$ faces by the number of non-root components. As recalled in Section 2.2, the polynomial $f_n(\mu - 1)$ has several interesting combinatorial descriptions in terms of maps equipped with an additional structure, and we will study the asymptotic behaviour of $f_n(u)$ for any $u \geq -1$.

Recall that $F(z,u)$ is characterized by (15) where $\theta$ and $\Phi$ are given by (24). As discussed after Theorem 3.1, $F(z,0)$ is explicit and given by (16):

$$F(z,0) = \int \theta(z)dz = 4 \sum_{i \geq 2} \frac{(3i-3)!}{(i-2)!i!(i+1)!}z^{i+1},$$

which makes the case $u = 0$ of the following theorem a simple application of Stirling's formula.

**Theorem 8.1.** *Let $p = 4$, and take $u \geq -1$. The radius of convergence of $F(z,u)$ is*

$$\rho_u = \tau - u\Phi(\tau) \tag{52}$$

*where $\Phi$ is given by (24) and $\tau$ satisfies*

$$\begin{cases} \tau = 1/27 & \text{if } u \leq 0, \\ 1 - u\Phi'(\tau) = 0 & \text{if } u > 0. \end{cases}$$

*The latter condition determines a unique $\tau \equiv \tau_u$ in $(0, 1/27)$.*

*In particular, $\rho_u$ is an affine function of $u$ on $[-1, 0]$:*

$$\rho_u = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right) = \frac{1+u}{27} - u\frac{\sqrt{3}}{12\pi}. \tag{53}$$
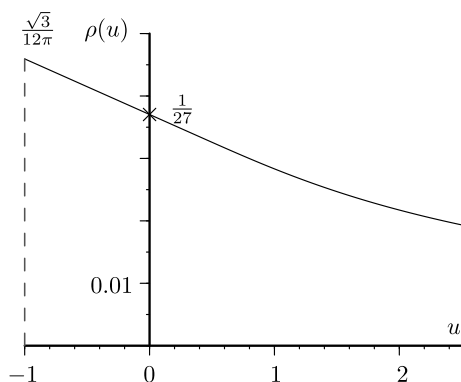
**Fig. 9.** The radius $\rho_u$ of $F(z, u)$, as a function of $u \geq -1$.

The function $\rho_u$ is decreasing, real-analytic everywhere except at $0$, where it is still infinitely differentiable: as $u \to 0^+$,

$$\rho_u = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right) + O\left(\exp\left(-\frac{2\pi}{\sqrt{3}u}\right)\right). \tag{54}$$

Let $f_n(u)$ be the coefficient of $z^n$ in $F(z, u)$. There exists a positive constant $c_u$, depending on $u$ only, such that

$$f_n(u) \sim \begin{cases} c_u\,\rho_u^{-n}n^{-3}(\ln n)^{-2} & \text{if } u \in [-1, 0), \\ c_u\,\rho_u^{-n}n^{-3} & \text{if } u = 0, \\ c_u\,\rho_u^{-n}n^{-5/2} & \text{if } u > 0. \end{cases}$$

The constant $c_u$ is given explicitly in Propositions 8.3 (for $u > 0$) and 8.4 (for $u < 0$), and $c_0 = 2/(3^5\sqrt{3}\pi)$.

The exponent $-5/2$ found for $u > 0$ is standard for planar maps (see for instance Tables 1 and 2 in [3]). The behaviour for $u < 0$ is much more surprising, and, to our knowledge, it is the first time that it is observed in the world of maps.[6] A plot of $\rho_u$ is shown in Fig. 9. Note that $\rho_{-1} = \sqrt{3}/(12\pi)$, a transcendental radius for the series counting 4-valent maps equipped with an internally inactive spanning tree.

The proof of the theorem uses the *singularity analysis* of [36, Ch. VI]. We thus need to locate the *dominant* singularities of the series $F'$ (that is, those of minimal modulus), and to find how $F'$ behave in their vicinity. In order to do this, we begin with the series $R$, defined by $R = z + u\Phi(R)$, and then move to $F' = \theta(R)$. We will find that both series have the same radius $\rho_u$. Moreover, since $F'$ and $\bar{u}(R - z)$ have non-negative

---

[6] Since forested maps correspond to the limit $q \to 0$ of the $q$-state Potts model (see (10)), one should expect a similar behaviour in the phase diagram of the Potts model discussed in [14] for general planar maps. However, the combinatorial relevance of the case $u \in [-1, 0)$ (corresponding in [14] to $J \in [-1, 0)$) is not noted in that paper, so that this phase is not studied.
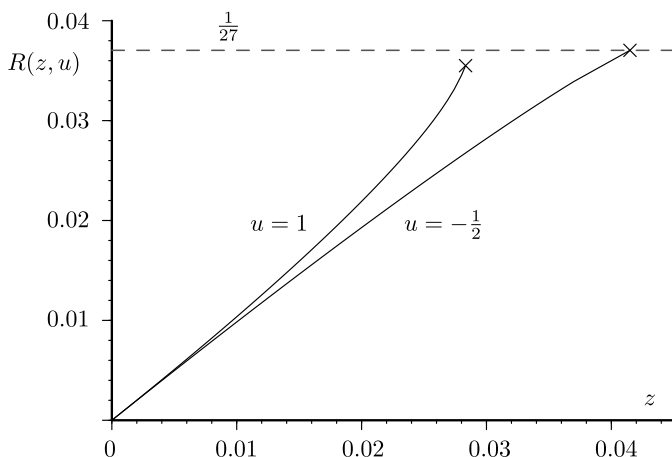
**Fig. 10.** Plot of $R(z, u)$, for $z \in [0, \rho_u]$. *Left:* when $u = 1$, and more generally $u > 0$, $R$ does not reach the dominant singularity of $\Phi$ (which is 1/27) at $\rho_u$. *Right:* When $u = -1/2$, and more generally when $u \in [-1, 0]$, we have $R(\rho_u) = 1/27$.

coefficients in $z$, this radius is a singularity of each (by Pringsheim's theorem). We will prove that neither $F'$ nor $R$ have other dominant singularities, and obtain estimates of these functions near $\rho_u$ (the same estimate, up to a multiplicative factor).

Now the location of $\rho_u$, and its nature as a singularity, depend on whether $u > 0$ or $u < 0$ (Fig. 10). For $u > 0$, the series $R$ will be shown to satisfy the *smooth implicit schema* of [36, Sec. VII.4]. In brief, the dominant singularity $\rho_u$ of $R$ comes from the failure of the assumption $u\Phi'(R(z)) \neq 1$ in the implicit function theorem. The value $R(\rho_u)$ lies in the analyticity domain of $\Phi$ and $\theta$, and the singularities of these series play no role. Both $R$ and $F'$ will be proved to have a square root dominant singularity. If $u < 0$ however, the series $R$ reaches at $\rho_u$ the dominant singularity of $\Phi$ and $\theta$, and the singular behaviours of $R$ and $F'$ at $\rho_u$ depend on the singular behaviours of $\Phi$ and $\theta$. In particular, we find that, around $\rho \equiv \rho_u$, the function $F''(z, u)$ behaves like $1/\ln(1 - z/\rho)$, up to a multiplicative constant. Since this cannot be the singular behaviour of a D-finite series [36, pp. 520 and 582], we have the following corollary.

**Corollary 8.2.** *For $u \in [-1, 0)$, the generating function $F(z, u)$ of 4-valent forested maps is not D-finite. The same holds when $u$ is an indeterminate.*

Recall that $F(z, u)$ is, however, differentially algebraic (Theorem 4.1).

### 8.1. The series $\Phi$ and $\theta$

Recall the definition (24) of these series. The $i$th coefficient of $\theta$ is asymptotic to $27^i/i^2$, up to a multiplicative constant, and the same holds for $\Phi$. Hence both series have radius of convergence 1/27, converge at this point, but their derivatives diverge.

This is as much information as we need to obtain the asymptotic behaviour of $f_n(u)$ when $u > 0$. When $u < 0$, we will need to know singular expansions of $\Phi$ and $\theta$ near $1/27$. Let us first observe that

$$\Phi(x) = x \left( {}_2F_1 \left( \frac{1}{3}, \frac{2}{3}; 2; 27x \right) - 1 \right) \tag{55}$$

where ${}_2F_1(a, b; c; x)$ denotes the standard hypergeometric function with parameters $a$, $b$ and $c$:

$$ {}_2F_1(a, b; c; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, $$

with $(a)_n$ the rising factorial $a(a+1) \cdots (a+n-1)$. The series ${}_2F_1 \left( \frac{1}{3}, \frac{2}{3}; 2; 27x \right)$ can be analytically continued in $\mathbb{C} \setminus [1/27, +\infty)$, and its behaviour as $z$ approaches $1/27$ in this domain is given by [1, Eq. (15.3.11)]. Translated in terms of $\Phi$, this gives, as $\varepsilon \to 0$,

$$\Phi \left( \frac{1}{27} - \varepsilon \right) = \left( \frac{\sqrt{3}}{12\pi} - \frac{1}{27} \right) + \frac{\sqrt{3}}{2\pi} \varepsilon \ln \varepsilon + \left( 1 - \frac{\sqrt{3}}{2\pi} \right) \varepsilon + O(\varepsilon^2 \ln \varepsilon). \tag{56}$$

One also has:

$$\Phi' \left( \frac{1}{27} - \varepsilon \right) = -\frac{\sqrt{3}}{2\pi} \ln \varepsilon - 1 + O(\varepsilon \ln \varepsilon). \tag{57}$$

The series $\theta$ is related to $\Phi$ by (26). It has the same analyticity domain as $\Phi$, with local expansion at $1/27$:

$$\theta \left( \frac{1}{27} - \varepsilon \right) = \left( \frac{2}{3} - \frac{7\sqrt{3}}{6\pi} \right) + \frac{2\sqrt{3}}{\pi} \varepsilon \ln \varepsilon + \frac{7\sqrt{3}}{\pi} \varepsilon + O(\varepsilon^2 \ln \varepsilon). \tag{58}$$

Also,

$$\theta' \left( \frac{1}{27} - \varepsilon \right) = -\frac{2\sqrt{3}}{\pi} \ln \varepsilon - \frac{9\sqrt{3}}{\pi} + O(\varepsilon \ln \varepsilon). \tag{59}$$

### 8.2. When $u > 0$

As in [36, Def. VI.1, p. 389], we call $\Delta$-*domain of radius* $\rho$ any domain of the form

$$\{ z : |z| < r, z \neq \rho \text{ and } | \operatorname{Arg}(z - \rho)| > \phi \}$$

for some $r > \rho$ and $\phi \in (0, \pi/2)$.

**Proposition 8.3.** *Assume $u > 0$. Then the series $R(z, u)$ is aperiodic and satisfies the smooth implicit schema of [36, Def. VII.4, p. 467]. Its radius is given by (52), and satisfies (54). The series $R$ is analytic in a $\Delta$-domain of radius $\rho \equiv \rho_u$, with a square root singularity at $\rho$:*

$$R(z, u) = \tau - \gamma\sqrt{1 - z/\rho} + O(1 - z/\rho), \tag{60}$$

*where $\tau$ is defined as in Theorem 8.1, and $\gamma = \sqrt{\frac{2\rho}{u\Phi''(\tau)}}$ with $\Phi$ given by (24).*

*The series $F'(z, u)$ is also analytic in a $\Delta$-domain of radius $\rho$, with a square root singularity at $\rho$:*

$$F'(z, u) = \theta(\tau) - \gamma\theta'(\tau)\sqrt{1 - z/\rho} + O(1 - z/\rho), \tag{61}$$

*where $\gamma$ is given above and $\theta$ is defined by (24). Consequently, the $n$th coefficient of $F$ satisfies, as $n \to \infty$,*

$$f_n(u) \sim \theta'(\tau)\sqrt{\frac{\rho^3}{2\pi u\Phi''(\tau)}}\rho^{-n}n^{-5/2}.$$

This proposition establishes the case $u > 0$ of Theorem 8.1.

**Proof.** The results that deal with $R$ are a straightforward application of Definition VII.4 and Theorem VII.3 of [36, pp. 467–468]. Using the notation of this book, $G(z, w) = z + u\Phi(w)$ is analytic for $(z, w) \in \mathbb{C} \times \{|w| < 1/27\}$. The so-called *characteristic system* holds at $(\rho, \tau)$ where $\tau$ is the unique element of $(0, 1/27)$ such that $G_w(\rho, \tau) = u\Phi'(\tau) = 1$, and $\rho := \tau - u\Phi(\tau)$. The existence and uniqueness of $\tau$ is guaranteed by the fact that $\Phi'(w)$ increases (strictly) from 0 to $+\infty$ as $w$ goes from 0 to $1/27$. The aperiodicity of $R$ is obvious from the first terms of its expansion: $R = z + 3z^2u + 6u(3u + 5)z^3 + O(z^4)$.

We now move to $F' = \theta(R)$. Since $R(\rho, u) = \tau < 1/27$, and $R$ has non-negative coefficients, there exists a $\Delta$-domain of radius $\rho$ in which $R$ is analytic and strictly bounded (in modulus) by $1/27$. Since $\theta$ has radius $1/27$, the series $F' = \theta(R)$ is also analytic in this domain, and its singular behaviour around $\rho$ follows from a Taylor expansion. One then applies the Transfer Theorem VI.4 from [36, p. 393] to obtain the behaviour of the $n$th coefficient of $F'$, which is $(n + 1)f_{n+1}(u)$. The estimate of $f_n(u)$ follows.

It remains to find an estimate of $\rho_u$ as $u \to 0^+$. Recall that $u\Phi'(\tau) = 1$. Thus $\tau \equiv \tau_u$ approaches $1/27$ as $u \to 0$, and (57) gives

$$\ln(1/27 - \tau) = -\frac{2\pi(1 + \bar{u})}{\sqrt{3}} + o(1)$$

with $\bar{u} = 1/u$, so that

$$\tau - \frac{1}{27} \sim -\exp\left(-\frac{2\pi(1 + \bar{u})}{\sqrt{3}}\right). \tag{62}$$

Since $\rho = \tau - u\Phi(\tau)$, this gives (54) in view of the expansion (56) of $\Phi$.  $\square$

### 8.3. When $u < 0$

**Proposition 8.4.** *Let $u \in [-1, 0)$. The series $R$ and $F'$ have radius $\rho \equiv \rho_u$ given by* (53). *They are analytic in a $\Delta$-domain of radius $\rho$, and the following estimates hold in this domain, as $z \to \rho$:*

$$R(z) - \frac{1}{27} \sim -\frac{2\pi\rho}{\sqrt{3}u} \frac{1 - z/\rho}{\ln(1 - z/\rho)}, \tag{63}$$

$$F''(z) + 4\bar{u} \sim \frac{72\sqrt{3}\pi\bar{u}^2\rho}{\ln(1 - z/\rho)}. \tag{64}$$

*Consequently, the nth coefficient of $F$ satisfies, as $n \to \infty$,*

$$f_n(u) \sim 72\sqrt{3}\pi\bar{u}^2 \frac{\rho^{-n+3}}{n^3 \ln^2 n}.$$

Since (64) cannot be the singular behaviour of a D-finite series [36, pp. 520 and 582], this proves Corollary 8.2. This proposition also establishes the case $u < 0$ of Theorem 8.1.

**Proof of Proposition 8.4.** We begin as before with the series $R$. The equation $R = z + u\Phi(R)$ reads $\Omega(R) = z$ with $\Omega(y) = y - u\Phi(y)$. Clearly $\Omega(0) = 0$ and $\Omega'(0) = 1 > 0$, so that we can apply Corollary 6.3, in which the role of $Y$ is played by $R$. Let $\omega$, $\tau$ and $\rho$ be defined as in this corollary. It follows from Section 8.1 that $\omega = 1/27$. Since $u < 0$, $\Omega'(y) = 1 - u\Phi'(y)$ does not vanish on $[0, 1/27]$. Hence $\tau = 1/27$ as well. By Property (5) of Corollary 6.3,

$$\rho = \Omega\left(\frac{1}{27}\right) = \frac{1}{27} - u\Phi\left(\frac{1}{27}\right),$$

which, combined with (56), gives (53).

Corollary 6.3 tells us that $R$ has an analytic continuation along $[0, \rho)$. Moreover, $R(z)$ increases from 0 to $1/27$ on $[0, \rho)$, and the equation

$$R = z + u\Phi(R) \tag{65}$$

holds in the whole interval $[0, \rho)$.

By Corollary 5.5, the series $\bar{u}(R - z)$ has non-negative coefficients. As $R$ itself, it is analytic along $[0, \rho)$. By Pringsheim's theorem, its radius is at least $\rho$, and this holds for $R$ as well. We will now study the behaviour of $R$ in the neighbourhood of $\rho$, and prove that it is singular at this point, so that $\rho$ is indeed the radius of $R$.

For $z \in \mathbb{C} \setminus \mathbb{R}^-$, let us define

$$\Psi(z) := \rho + \frac{z - 1}{27} + u\,\Phi\left(\frac{1 - z}{27}\right).$$

As explained above, $1 - 27\,R(\rho - y)$ increases from 0 to 1 as $y$ goes from 0 to $\rho$, and the functional equation (65) satisfied by $R$ reads, for $y \in [0, \rho)$,

$$\Psi(1 - 27\,R(\rho - y)) = y. \tag{66}$$

By (56), we have $\Psi(z) \sim -cz \ln z$ where

$$c = -\frac{\sqrt{3}u}{54\pi} > 0.$$

Let us apply the log-inversion theorem (Theorem 7.1) to $\Psi$, with $\alpha = 3\pi/4$ (we now denote by $r$ and $r'$ the values $\rho$ and $\rho'$ of Theorem 7.1): There exist $r > 0$ and $r' > 0$, and a function $\Upsilon$ analytic on $D_{r', \alpha} = \{|z| < r' \text{ and } |\mathrm{Arg}\,z| < 3\pi/4\}$, such that $\Psi(\Upsilon(y)) = y$. Furthermore, $\Upsilon(y)$ is the only preimage of $y$ under $\Psi$ that can be found in $D_{r, \pi} = \{|z| < r \text{ and } |\mathrm{Arg}\,z| < \pi\}$. Comparing with (66) shows that for $y$ small enough and positive, one has $\Upsilon(y) = 1 - 27\,R(\rho - y)$. Returning to the original variables, this means that, for $z$ real and close to $\rho^-$,

$$R(z) = \frac{1}{27}\left(1 - \Upsilon(\rho - z)\right),$$

so that $R$ can be analytically continued on $\{|z - \rho| < r' \text{ and } |\mathrm{Arg}(z - \rho)| > \pi/4\}$. Moreover, the final statement of Theorem 7.1 gives (63). This shows that $R$ is singular at $\rho$, which is thus the radius of $R$.

In order to prove that $R$ is analytic in a $\Delta$-domain of radius $\rho$, we now have to prove that it has no singularity other than $\rho$ on its circle of convergence. So let $\mu \neq \rho$ have modulus $\rho$. Since $\mathcal{R} := \bar{u}(R - z)$ has positive coefficients and $|\mathcal{R}(\rho)| < +\infty$, the series $\mathcal{R}$ converges at $\mu$, and so does $R$. Recall that $\Phi$ is analytic in $\mathbb{C} \setminus [1/27, +\infty)$. Hence (65), which holds in a neighbourhood of 0, will hold in the closed disk of centre $\rho$ if we can prove the following lemma.

**Lemma 8.5.** *For $|z| \leq \rho$ and $z \neq \rho$, we have $R(z) \notin [1/27, +\infty)$.*

**Proof.** We have already seen that the property holds (since $R$ is increasing) on the interval $[0, \rho)$. On the interval $[-\rho, 0]$, the function $R$ is real (Lemma 6.2) and continuous. Hence, if $R$ exits $(-\infty, 1/27)$ on this interval, there exists $t \in [-\rho, 0]$ such that $R(t) = 1/27$. Let $t$ be maximal for this property. Then $R(z) \in \mathbb{C} \setminus [1/27, +\infty)$ on a complex neighbourhood of $(t, 0]$, and (65) holds there. By differentiating it, we obtain

$$R'(z) = \frac{1}{1 - u\Phi'(R(z))} \neq 0. \tag{67}$$

In particular, $R'(0) = 1$. But since $R(t) = 1/27 > R(0) = 0$, the function $R'(z)$ must vanish in $(t, 0)$, which is impossible in view of its expression above.

Assume now that $z$ is not real, and let us prove that $R(z)$ is not real either. First,

$$|\operatorname{Im} R(z)| = |\operatorname{Im}(z + u\mathcal{R}(z))| \geq |\operatorname{Im} z| + u|\operatorname{Im} \mathcal{R}(z)|. \tag{68}$$

Then:

$$|\operatorname{Im} \mathcal{R}(z)| = |\operatorname{Im}(\mathcal{R}(z) - \mathcal{R}(\operatorname{Re} z))| \leq |\mathcal{R}(z) - \mathcal{R}(\operatorname{Re} z)|$$
$$< |z - \operatorname{Re} z| \max_{y \in [\operatorname{Re} z, z]} |\mathcal{R}'(y)| \leq |\operatorname{Im} z| \max_{|y| \leq \rho} |\mathcal{R}'(y)|. \tag{69}$$

The strict inequality comes from the fact that $\mathcal{R}'$ is not constant over $[\operatorname{Re} z, z]$. But $\mathcal{R}'$ is a power series with positive coefficients, and thus for $|y| \leq \rho$,

$$|\mathcal{R}'(y)| \leq \mathcal{R}'(\rho) = \bar{u}(R'(\rho) - 1) = \bar{u}\left(\lim_{t \to \rho} \frac{1}{1 - u\Phi'(R(t))} - 1\right) = -\bar{u}, \tag{70}$$

because $\Phi'(z)$ tends to $+\infty$ as $z \to 1/27$. Returning to (69) gives $|\operatorname{Im} \mathcal{R}(z)| < -\bar{u}|\operatorname{Im} z|$, and this inequality, combined with (68), gives $|\operatorname{Im} R(z)| > 0$. □

So we now know that (65) holds everywhere in the disk of radius $\rho$, with $R$ only reaching the critical value $1/27$ at $\rho$. By differentiation, (67) holds as well. Let us return to our point $\mu \neq \rho$, of modulus $\rho$. We now want to apply the analytic implicit function theorem to (65) at the point $(\mu, R(\mu))$. We know that $\Phi$ is analytic around $R(\mu)$. Could it be that $u\Phi'(R(\mu)) = 1$? By (67), this would imply that $|R'(z)|$, and thus $|\mathcal{R}'(z)|$, is not bounded as $z$ approaches $\mu$ in the disk. However, $\mathcal{R}'$ has non-negative coefficients and $\mathcal{R}'(\rho)$ has been shown to converge (see (70)). Thus $\mathcal{R}'(z)$ remains bounded in the disk of radius $\rho$, and in particular $u\Phi'(R(\mu)) \neq 1$. The analytic implicit function theorem then implies that $R$ is analytic at $\mu$.

In conclusion, we have proved that there exists a $\Delta$-domain of radius $\rho$ where $R$ is analytic and avoids the half-line $[1/27, +\infty)$.

Let us now turn our attention to $F' = \theta(R)$. Since $\theta$ is analytic in $\mathbb{C} \setminus [1/27, +\infty)$, the series $F'$ is analytic in the same $\Delta$-domain as $R$. The estimate (63) of $R$, combined with the expansion (58) of $\theta$, does not give immediately the singular behaviour of $F'$, and it is more direct to work with $F''$. Indeed,

$$F''(z) = R'(z)\theta'(R(z)) = \frac{\theta'(R(z))}{1 - u\Phi'(R(z))}. \tag{71}$$

By (57) and (59),

$$\frac{\theta'(1/27 - \varepsilon)}{1 - u\Phi'(1/27 - \varepsilon)} = -4\bar{u} - 2\bar{u}\left(9 - \frac{4\pi(1 + \bar{u})}{\sqrt{3}}\right)\frac{1}{\ln \varepsilon} + O(1/\ln^2 \varepsilon)$$
$$= -4\bar{u} + \frac{72\sqrt{3}\pi\bar{u}^2\rho}{\ln \varepsilon} + O(1/\ln^2 \varepsilon),$$

in view of (53). This, combined with (71) and the estimate (63) of $R(z)$, gives (64). One finally applies the Transfer Theorem VI.4 from [36, p. 393] to obtain the behaviour of the $n$th coefficient of $F''$, which is $(n+2)(n+1)f_{n+2}(u)$. The estimate of $f_n(u)$ follows.   □

## 9. Large random maps equipped with a forest or a tree

We still focus in this section on 4-valent maps, equipped either with a spanning forest or with a spanning tree. In each case, we define a Boltzmann probability distribution on maps of size $n$, which involves a parameter $u$ and takes into account the number of components of the spanning forest, or the number of internally active edges of the spanning tree (equivalently, the level of a recurrent sandpile configuration, as explained in Section 2.2). We observe on several statistics of these models the effect of the phase transition found at $u = 0$ in the previous section.

### 9.1. Forested maps: Number and size of components

Fix $n \in \mathbb{N}$ and $u \in [0, +\infty)$. Consider the following probability distribution on 4-valent forested maps $(M, F)$ having $n$ faces:

$$\mathbb{P}_c(M, F) = \frac{u^{c(F)-1}}{f_n(u)},$$

where $c(F)$ is the number of components of $F$, and $f_n(u)$ counts 4-valent forested maps by the number of non-root components. Under this distribution, let $C_n$ be the number of components of $F$, and $S_n$ the size (number of vertices) of the root component. When $u = 0$, only tree-rooted maps have a positive probability, $C_n = 1$ and $S_n = n - 2$, the total number of vertices in the map. Let us examine how this changes when $u > 0$.

**Proposition 9.1.** *Assume $u > 0$. Under the distribution $\mathbb{P}_c$, we have, as $n \to \infty$:*

$$\mathbb{E}_c(C_n) \sim \frac{u\Phi(\tau)}{\tau - u\Phi(\tau)}\, n,$$

*where $\Phi$ is given by (24) and $\tau \equiv \tau_u$ is the unique solution in $(0, 1/27)$ of $u\Phi'(y) = 1$. The size $S_n$ of the root component admits a discrete limit law: for $k \geq 1$,*

$$\lim_{n \to +\infty} \mathbb{P}_c(S_n = k) = \frac{4\,(3\,k)!}{(k-1)!\,k!\,(k+1)!} \frac{\tau^k}{\theta'(\tau)} \tag{72}$$

*with $\theta$ defined by (24).*

**Proof.** We have

$$\mathbb{E}_c(C_n - 1) = \sum_{(M,F)} (c(F) - 1) \frac{u^{c(F)-1}}{f_n(u)} = u \frac{f_n'(u)}{f_n(u)} = u \frac{[z^{n-1}]F_{zu}''(z, u)}{[z^{n-1}]F_z'(z, u)}. \tag{73}$$

It follows from the definition (15) of $R$ and $F$ that

$$F''_{zu}(z, u) = \frac{\Phi(R)\theta'(R)}{1 - u\Phi'(R)}. \tag{74}$$

We now use singularity analysis. The functions $\Phi$ and $\theta$ are analytic at $\tau = R(\rho, u)$, the value $\tau$ satisfies $1 = u\Phi'(\tau)$, and a singular estimate of $R - \tau$ is given by (60). This gives, as $z \to \rho$,

$$F''_{zu}(z, u) \sim \frac{\Phi(\tau)\theta'(\tau)}{u\Phi''(\tau)\gamma\sqrt{1 - z/\rho}}$$

where $\gamma$ is as in Proposition 8.3. An estimate of $F'_z(z, u)$ is given by (61). Our estimate of $\mathbb{E}_c(C_n)$ then follows from a transfer theorem, and the fact that $\rho = \tau - u\Phi(\tau)$.

To study $S_n$, we add to our generating function $F(z, u)$ a weight $s$ for each vertex belonging to the root component. As discussed at the end of Section 3.1, we now have, by specializing (19):

$$sF'_z(z, u, s) = \theta(s R),$$

where $R = R(z, u)$ is as before. We can express $\mathbb{P}_c(S_n = k)$ in terms of $F'_z$:

$$\mathbb{P}_c(S_n = k) = \frac{[z^{n-1}s^k]F'_z(z, u, s)}{[z^{n-1}]F'_z(z, u, 1)} = \frac{[z^{n-1}s^{k+1}]\theta(s R)}{[z^{n-1}]\theta(R)}.$$

We now apply Proposition IX.1 from [36, p. 629]. Proposition 8.3 guarantees that its hypotheses are indeed satisfied, and this gives (72), using the expression (24) of $\theta$.  $\square$

### 9.2. Tree-rooted maps: Number of internally active edges

Fix $n \in \mathbb{N}$ and $u \in [-1, +\infty)$. Consider the following probability distribution on 4-valent tree-rooted maps $(M, T)$ having $n$ faces:

$$\mathbb{P}_i(M, T) = \frac{(u + 1)^{i(M,T)}}{f_n(u)},$$

where $i(M, T)$ is the number of internally active edges in $(M, T)$. Eq. (6) shows that this is indeed a probability distribution. Under this distribution, let $I_n$ denote the number of internally active edges. As shown by (9), $I_n$ can also be described as the level $\ell(C)$ of a recurrent sandpile configuration $C$ of an $n$-vertex quadrangulation $M$, drawn according to the distribution

$$\mathbb{P}_s(M, C) = \frac{(u + 1)^{\ell(C)}}{f_n(u)}.$$

We assume below that $u > -1$, otherwise $I_n = 0$.

**Proposition 9.2.** *The expected number of internally active edges undergoes a phase transition of infinite order at $u = 0$: as $n \to \infty$,*

$$\mathbb{E}_i(I_n) \sim \kappa_u n, \tag{75}$$

*with*

$$\kappa_u = \frac{(1+u)\Phi(\tau)}{\tau - u\Phi(\tau)}$$

*where $\Phi$ is given by* (24) *and $\tau \equiv \tau_u$ is defined in* Proposition 8.1. *The function $\kappa_u$ is rational for $u \le 0$, real-analytic everywhere except at 0, where it is still infinitely differentiable: as $u \to 0^+$,*

$$\kappa_u = \frac{(1+u)\Phi(1/27)}{1/27 - u\Phi(1/27)} + O\left(\exp\left(-\frac{2\pi}{\sqrt{3}u}\right)\right).$$

Of course, this non-analytic behaviour originates in the similar behaviour of the function $\tau_u$ of Proposition 8.1. The convergence of $\mathbb{E}_i(I_n)/n$ was proved by Mullin when $u = 0$ for general planar maps having $n$ edges (but the limit is not given explicitly) [46].

**Proof of Proposition 9.2.** We have

$$\mathbb{E}_i(I_n) = \sum_{(M,T)} i(M,T)\frac{(u+1)^{i(M,T)}}{f_n(u)} = (u+1)\frac{f_n'(u)}{f_n(u)} = (u+1)\frac{[z^{n-1}]F_{zu}''(z,u)}{[z^{n-1}]F_z'(z,u)}. \tag{76}$$

Comparing with (73), we see that for $u > 0$, we have $\mathbb{E}_i(I_n) = (1+\bar{u})\mathbb{E}_c(C_n)$. Thus (75) follows from Proposition 9.1 when $u > 0$. The expansion of $\kappa_u$ near $0^+$ follows from the estimate (62) of $\tau$ and the expansion (56) of $\Phi$.

Let us now take $u \in [-1, 0)$. The series $F_{zu}''$ is still given by (74), which can also be written $\Phi(R)F_{zz}''$ (by (71)), or $\bar{u}(R-z)F_{zz}''$. In view of the estimates (63) and (64) of $R$ and $F_{zz}''$, we find

$$[z^{n-1}]F_{zu}''(z,u) \sim \bar{u}(1/27 - \rho)[z^{n-1}]F_{zz}''(z,u).$$

Returning to (76) gives (75) by singularity analysis, since $\rho = 1/27 - u\Phi(1/27)$.

When $u = 0$, we have $R = z$. Hence (74) reads $F_{zu}''(z,0) = \Phi(z)\theta'(z)$, while $F_z'(z,0) = \theta(z)$. As above, (75) follows from (76) by singularity analysis, using (56), (58) and (59). $\square$

**A limit law for $I_n$.** As suggested by one of the referees, we could try to determine a limit law (rather than the limiting first moment) for the random variable $I_n$. Its probability generating function is given by

$$p_n(v) := \sum_{k \ge 0} \mathbb{P}_i(I_n = k)v^k = \frac{f_n((u+1)v - 1)}{f_n(u)}.$$

When $u \in (-1, \infty) \setminus \{0\}$ and $v$ is real and close enough to 1, Theorem 8.1 implies that

$$p_n(v) \sim \frac{c_{(u+1)v-1}}{c_u} \left( \frac{\rho_u}{\rho_{(u+1)v-1}} \right)^n .$$

We expect this behaviour to be sufficiently uniform in $v$ to meet the assumptions of the quasi-power theorem [36, Thm. IX.8, p. 645]. In particular, we expect $(I_n - \mathbb{E}(I_n))/\sqrt{n}$ to converge to a centred Gaussian distribution. We have actually proved this for $u > 0$. Let us sketch our argument: first, we apply [33, Prop. 1] to prove that $R(z, u)$ satisfies an "algebraic singularity schema", in the sense of [36, Thm. IX.12, p. 676]. Using $F'_z = \theta(R)$, we then establish the same property for $F'_z$. We finally check that the assumptions of [36, Thm. IX.12, p. 676] hold and conclude.

The most interesting question may be whether this Gaussian behaviour persists at $u = 0$, even though the behaviour of $p_n(v)$ now depends on the sign of $v - 1$:

$$p_n(v) \sim \frac{c_{v-1}}{c_0} \left( \frac{\rho_0}{\rho_{v-1}} \right)^n \times \begin{cases} (\ln n)^2 & \text{if } v < 1, \\ n^{1/2} & \text{if } v > 1. \end{cases}$$

For any $k \geq 0$, the generating function

$$\sum_{M,T} \mathrm{f}(M) z^{\mathrm{f}(M)-1} i(M, T)^k$$

can be computed explicitly in terms of $\theta$, $\Phi$ and their derivatives, by differentiating the system (15) several times with respect to $u$, and then setting $u = 0$. In this way, we have computed the first 9 limiting moments of $(I_n - n\mu)/(\sigma\sqrt{n})$, with

$$\mu = \frac{9\sqrt{3}}{4\pi} - 1 \quad \text{and} \quad \sigma^2 = \frac{9\sqrt{3}}{4\pi} \left( \frac{9\sqrt{3}}{4\pi} - 1 \right).$$

We have found them to be those of a standard Gaussian law, namely $0, 1, 0, 3, 0, 15, 0, 105, 0$, and we conjecture that the limit distribution of $I_n$ remains Gaussian (after normalization) when $u = 0$.

## 10. Asymptotics for cubic forested maps

We study in this section the singular behaviour of the series $F(z, u)$ that counts cubic forested maps by the number of components, and the asymptotic behaviour of its $n$th coefficient $f_n(u)$. As expected, we observe a "universality" phenomenon: our results are qualitatively the same as for 4-valent maps (Theorem 8.1). However, the cubic case is more difficult since we now have to deal with a pair of equations:

$$R = z + u\Phi_1(R, S), \quad S = u\Phi_2(R, S),$$

where $\Phi_1$ and $\Phi_2$ are given by (27) and (28). Our results are less complete than in the 4-valent case: when $u < 0$, we only determine the singular behaviour of $F'(z, u)$ as $z$ approaches the radius of $F'$ on the real axis. We do not know if $F'$ has dominant singularities other than its radius. Consequently, we have not obtained the asymptotic behaviour of $f_n(u)$ when $u < 0$. Our real analysis still provides a result on the asymptotic behaviour of a sum (related to the Cesàro mean) involving these numbers.

**Theorem 10.1.** *Let $p = 3$, and take $u \geq -1$. The radius of convergence of $F(z, u)$ reads*

$$\rho_u = \tau - u\Phi_1(\tau, \sigma)$$

*where the pair $(\tau, \sigma)$ satisfies*

$$\sigma = u\Phi_2(\tau, \sigma)$$

*and*

$$\begin{cases} 64\tau = (1 - 4\sigma)^2 & \text{if } u \leq 0, \\ \left(1 - u\Phi_1^x(\tau, \sigma)\right)\left(1 - u\Phi_2^y(\tau, \sigma)\right) = u^2\Phi_1^y(\tau, \sigma)\Phi_2^x(\tau, \sigma) & \text{if } u > 0. \end{cases}$$

*The series $\Phi_1$ and $\Phi_2$ are given by (27) and (28), and $\Phi_i^x$ (resp. $\Phi_i^y$) denotes the derivative of $\Phi_i$ with respect to its first (resp. second) variable.*

*In particular, $\rho_u$ is an algebraic function of $u$ on $[-1, 0]$:*

$$\rho_u = \frac{3(1 - u^2)^2\pi^4 + 96u^2\pi^2(1 - u^2) + 512u^4 + 16u\sqrt{2}\left(\pi^2(1 - u^2) + 8u^2\right)^{3/2}}{192\pi^4(1 + u)^3}. \tag{77}$$

*Let $f_n(u)$ be the coefficient in $z^n$ in $F(z, u)$. There exists a positive constant $c_u$, depending on $u$ only, such that*

$$f_n(u) \sim \begin{cases} c_u\,\rho_u^{-n}n^{-3} & \text{if } u = 0, \\ c_u\,\rho_u^{-n}n^{-5/2} & \text{if } u > 0. \end{cases}$$

*For $u \in [-1, 0)$, the series $F'(z) \equiv F'(z, u)$ has the following singular expansion as $z \to \rho_u^-$:*

$$F'(z) = F'(\rho_u) + \alpha(\rho_u - z) + \beta\,\frac{\rho_u - z}{\ln(\rho_u - z)}\left(1 + o(1)\right), \tag{78}$$

*where*

$$\beta = \frac{4u - 3\sqrt{2}\sqrt{\pi^2(1 - u^2) + 8u^2}}{2u^2} < 0.$$

**Remarks. 1.** We expect the singular expansion (78) to transfer to coefficients, so that

$$f_n(u) \sim -\beta\rho_u^{-n+2}n^{-3}(\ln n)^{-2}. \tag{79}$$

Proving this would require more information on $F'$ than we have (behaviour in a complex neighbourhood of the radius, and around the other dominant singularities, if any). However, Tauberian theorems provide estimates of Cesàro means from real estimates of the function [36, Thm. VI.13], and by computing real estimates of $F'$, $F''$ and $F'''$ we can conclude that

$$\sum_{k=0}^{n} k^3 \rho_u^k f_k(u) \sim -\beta \rho_u^2 \, n \ln n^{-2},$$

in accordance with (79).

**2.** As in the 4-valent case, the singular behaviour of $F'$ obtained when $u < 0$ is incompatible with D-finiteness [36, pp. 520 and 582].

**Corollary 10.2.** *For $u \in [-1, 0)$, the generating function $F(z, u)$ of cubic forested maps is not D-finite. The same holds when $u$ is an indeterminate.*

**3.** The series $F(z, 0)$ has a simple explicit expression given by (16):

$$F(z, 0) = 3 \sum_{\ell \geq 1} \frac{(4\ell)!}{(2\ell - 1)!(\ell + 1)!(\ell + 2)!} z^{\ell+2}.$$

The above theorem follows in this case from Stirling's formula. One has $\sigma = 0$ and $\rho_0 = \tau = 1/64$. We will thus focus below on the cases $u > 0$ and $u < 0$.

**4.** At $u = -1$, one finds $\rho_{-1} = \pi^2/384$, a beautiful transcendental radius of convergence for the series counting cubic maps equipped with an internally inactive spanning tree.
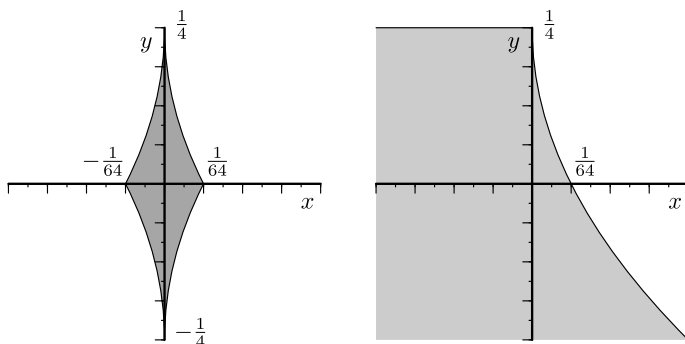
*10.1. The series $\Phi_1$, $\Phi_2$, $\Psi_1$ and $\Psi_2$*

We have performed in Section 4.3 a useful reduction by showing that the bivariate series $\Phi_1(x, y)$ and $\Phi_2(x, y)$ can be expressed in terms of the univariate hypergeometric series $\Psi_1$ and $\Psi_2$ (see (30)–(31)). The $i$th coefficient of $\Psi_1$ is asymptotic to $64^i/i^2$, up to a multiplicative constant, and the same holds for $\Psi_2$. Hence both series have radius of convergence $1/64$, converge at this point, but their derivatives diverge. In fact,

$$\Psi_1(z)/z = {}_2F_1(1/4, 3/4; 2; 64z),$$

so that $\Psi_1$ can be analytically defined on $\mathbb{C} \setminus [1/64, +\infty)$. The same holds for $\Psi_2(z)$ in view of (34). It follows from [1, Eq. (15.3.11)] that, as $\varepsilon \to 0$ in $\mathbb{C} \setminus \mathbb{R}^-$,

$$\Psi_1\left(\frac{1}{64} - \varepsilon\right) = \frac{\sqrt{2}}{24\,\pi} + \frac{\sqrt{2}}{2\,\pi}\,\varepsilon \ln \varepsilon - \frac{\sqrt{2}}{2\,\pi}\varepsilon + O\left(\varepsilon^2 \ln \varepsilon\right). \tag{80}$$

**Fig. 11.** *Left:* The domain of absolute convergence of the series $\Phi_1$ and $\Phi_2$, in the real plane. *Right:* A domain where an analytic continuation exists. No analytic continuation exists at a point of the parabola $64x = (1 - 4y)^2$.

By (34), we also have

$$\Psi_2 \left( \frac{1}{64} - \varepsilon \right) = \left( \frac{1}{2} - \frac{\sqrt{2}}{\pi} \right) + \frac{4\sqrt{2}}{\pi} \, \varepsilon \, \ln \varepsilon + \frac{12\sqrt{2}}{\pi} \varepsilon + O\left( \varepsilon^2 \ln \varepsilon \right). \qquad (81)$$

Let us now return to $\Phi_1$ and $\Phi_2$. The series $\sqrt{1 - 4y}$ has radius $1/4$, the series $\Psi_1$ and $\Psi_2$ have radius $1/64$, and thus $\Phi_1(x, y)$ and $\Phi_2(x, y)$ converge absolutely for $|y| < 1/4$ and $64|x| < (1 - 4|y|)^2$ (Fig. 11, left). The expressions (30) and (31) show that $\Phi_1$ and $\Phi_2$ have an analytic continuation for $y \in \mathbb{C} \setminus [1/4, +\infty)$ and $x/(1-4y)^2 \in \mathbb{C} \setminus [1/64, +\infty)$ (Fig. 11, right). As $\Psi_1'(t)$ and $\Psi_2'(t)$ tend to $+\infty$ when $t \to 1/64$, there is no way to extended analytically $\Phi_1$ or $\Phi_2$ at a point of the *critical parabola* $\{64x = (1 - 4y)^2\}$.

### 10.2. When $u > 0$

**Proposition 10.3.** *Assume $u > 0$. The series $R$, $S$ and $F'$ have the same radius of convergence, denoted $\rho_u$, which satisfies the conditions stated in Theorem 10.1. The three series are analytic in a $\Delta$-domain of radius $\rho_u$, with a square root singularity at $\rho_u$. In particular,*

$$f_n(u) \sim c_u \rho_u^{-n} n^{-5/2}$$

*for some positive constant $c_u$.*

**Proof.** Recall that these three series are defined by the system (29), (32), (33). The analysis of systems of functional equations can be a tricky exercise, even in the *positive case*[7] and with 2 equations only. In particular, the connection between the location of the radius and the solution(s) of the so-called *characteristic system* is subtle (see [33,5]). In

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[7] By this, we mean a system given by equations of the form $R_i = F_i(R_1, \ldots, R_m)$ where the series $F_i$ have non-negative coefficients.

our case however, the equation that defines $S$ does not involve the variable $z$ explicitly, and this allows us to proceed safely in two steps. As in Section 5.2, we first define $\tilde{S} \equiv \tilde{S}(z, u)$ as the unique power series in $z$ satisfying $\tilde{S}(0, u) = 0$ and

$$\tilde{S} = u\,\Phi_2(z, \tilde{S}) \tag{82}$$

$$= u\sqrt{1 - 4\tilde{S}}\ \Psi_2\left(\frac{z}{(1 - 4\tilde{S})^2}\right) + \frac{u}{4}\left(1 - \sqrt{1 - 4\tilde{S}}\right)^2. \tag{83}$$

We will first study $\tilde{S}$, and then move to $R$, which is now defined by the following equation:

$$R = z + u\Phi_1(R, \tilde{S}(R)) \tag{84}$$

$$= z + u(1 - 4\tilde{S}(R))^{3/2}\Psi_1\left(\frac{R}{(1 - 4\tilde{S}(R))^2}\right) - uR, \tag{85}$$

where we have denoted for short $\tilde{S}(z) = \tilde{S}(z, u)$. Of course, $S = \tilde{S}(R)$.

So let us begin with $\tilde{S}$. One can prove that (82) fits in the smooth implicit function schema of [36, Def. VII.4], but we can actually content ourselves with an application of Proposition 6.1, where $\tilde{S}$ plays the role of $Y$. The series $H(x, y) = y - u\Phi_2(x, y)$ satisfies the assumptions of this proposition. Define $\tilde{\rho}$ as in the proposition. Since $\tilde{S}$ has non-negative coefficients, the points $(z, \tilde{S}(z))$ form, as $z$ goes from 0 to $\tilde{\rho}$, an increasing curve starting from $(0, 0)$ in the plane $\mathbb{R}^2$. Condition (b), together with the properties of $\Phi_2$ described in Section 10.1, implies that this curve cannot go beyond the parabola $64x = (1 - 4y)^2$. This rules out the possibilities (i) and (iv). Now $H'_y(x, y) = 1 - u\Phi'_2(x, y)$ approaches $-\infty$ as $(x, y)$ approach the parabola, and thus Condition (d) rules out the possibility (iii). The curve $(z, \tilde{S}(z))$ thus ends (at $z = \tilde{\rho}$) before reaching the parabola. Moreover (ii) holds: $H'_y(\tilde{\rho}, \tilde{S}(\tilde{\rho})) = 0$, or equivalently,

$$1 = u\frac{\partial \Phi_2}{\partial y}(\tilde{\rho}, \tilde{S}(\tilde{\rho})). \tag{86}$$

(The lim inf of (ii) becomes here a true limit because of the positivity of the coefficients of $\Phi_2$ and $\tilde{S}$.) By (a), the radius of $\tilde{S}$ is at least $\tilde{\rho}$. Finally, it follows from (82) that for $z \in [0, \tilde{\rho})$,

$$\tilde{S}'(z) = u\,\frac{\frac{\partial \Phi_2}{\partial x}(z, \tilde{S}(z))}{1 - u\frac{\partial \Phi_2}{\partial y}(z, \tilde{S}(z))}. \tag{87}$$

By (86), this derivative tends to $+\infty$ as $z \to \tilde{\rho}$. Hence $\tilde{S}$ has radius $\tilde{\rho}$. Fig. 12 (left) illustrates the behaviour of $\tilde{S}$ on $[0, \tilde{\rho}]$.

Let us now consider Eq. (84) that defines $R$, and prove that it fits in the smooth implicit function schema of [36, Def. VII.4, pp. 467–468]. With the notation of this definition, $G(z, w) = z + u\Phi_1(w, \tilde{S}(w))$. The properties of $\tilde{S}$ established above show that
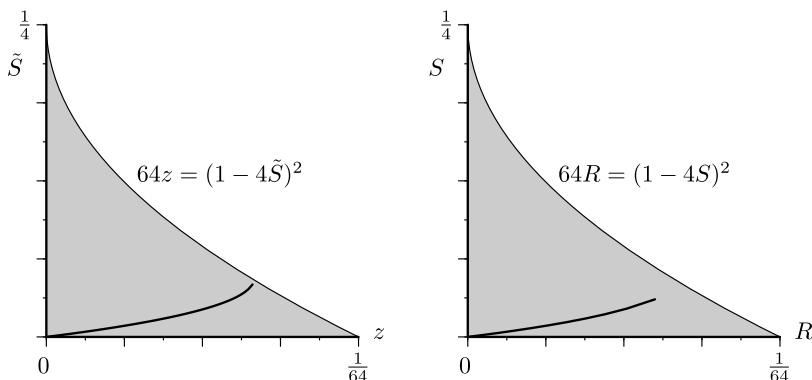
**Fig. 12.** *Left:* Plot of $\tilde{S}(z)$ for $u = 1$ and $z \in [0, \tilde{\rho}]$. The points $(z, \tilde{S}(z))$ remain below the parabola $64z = (1 - 4\tilde{S})^2$. The plot was obtained using the expansion of $\tilde{S}(z)$ at order 80 (this is why the divergence of $\tilde{S}'$ at $\tilde{\rho}$ is not very clear on the picture), and the estimate $\tilde{\rho} \simeq 0.01032$. *Right:* Plot of $(R(z), S(z))$ for $u = 1$ and $z \in [0, \rho]$, with $\rho \simeq 0.00980$. This curve follows the plot of $\tilde{S}$, but stops at the point $(R(\rho), S(\rho))$, for which $R(\rho) < \tilde{\rho}$.

$G$ is analytic in $\mathbb{C} \times \{w : |w| < \tilde{\rho}\}$. The characteristic equation $1 = G'_w(\rho, \tau)$ does not involve $\rho$ and reads

$$1 = u \left( \frac{\partial \Phi_1}{\partial x}(\tau, \tilde{S}(\tau)) + \tilde{S}'(\tau) \frac{\partial \Phi_1}{\partial y}(\tau, \tilde{S}(\tau)) \right). \tag{88}$$

The right-hand side of this equation increases from 0 to $+\infty$ as $\tau$ goes from 0 to $r$ (because, as observed above, $\tilde{S}'(\tilde{\rho}) = +\infty$). Hence (88) determines a unique value of $\tau$ in $(0, \tilde{\rho})$. The equation $\tau = G(\rho, \tau)$ gives the value of $\rho$:

$$\rho = \tau - u\Phi_1(\tau, \tilde{S}(\tau)). \tag{89}$$

Let $\sigma = \tilde{S}(\tau)$. The combination of (89), (82), (88) and (87) proves the properties of $\rho, \tau$ and $\sigma$ stated in Theorem 10.1.

The rest of the argument is analogous to the end of the proof of Proposition 8.3. First, $R$ is irreducible as shown by the first terms of its expansion at 0:

$$R = z + 2u(2u + 3)z^2 + 4u(42u^2 + 63u + 10u^3 + 35)z^3 + O(z^4).$$

By Theorem VII.3 of [36, p. 468], it has radius $\rho$, and is analytic in a $\Delta$-domain of radius $\rho$. It takes the value $\tau$ at $\rho$, with a square root singularity there. By composition with the series $\tilde{S}$, which has radius $\tilde{\rho} > \tau$, the same properties hold for $S = \tilde{S}(R)$, and finally for the series $F'$ given by (14) (since $\theta(x, y)$ is analytic in $\mathbb{R}^2$ for $64x < (1 - 4y)^2$).

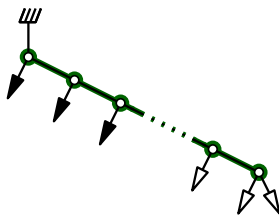The behaviour of $R$ and $S$ is illustrated in Fig. 12 (right). $\qquad \square$

**Fig. 13.** A cubic enriched $\tilde{S}$-tree with no flippable edge.

*10.3. When $u < 0$*

**Proposition 10.4.** *Let $u \in [-1, 0)$. The series $R$, $S$ and $F'$ have radius $\rho \equiv \rho_u$ given by* (77). *As $z \to \rho_u^-$, these three series admit an expansion of the form* (78), *with $\beta > 0$.*

**Proof.** As a preliminary remark, recall that $F(z, u)$ is $(u + 1)$-positive, with several combinatorial interpretations described in Section 2.2. By Pringsheim's theorem, the radius of $F$ is also its smallest real positive singularity. By Corollary 5.5, the same holds for $R$, $S$ and $\tilde{S}$. This will be used frequently in the proof, without further reference to Pringsheim's theorem.

As in the case $u > 0$, we proceed in two steps, and study first the series $\tilde{S}$ defined by (82), and then the series $R$ defined by (84). Let us begin with $\tilde{S}$, and apply Proposition 6.1 with $H(x, y) = y - u\Phi_2(x, y)$. Let us rule out the possibilities (i), (ii) and (iv).

**(i)** Could $\tilde{S} \equiv \tilde{S}(z, u)$ have an analytic continuation on $(0, +\infty)$? That is, an infinite radius of convergence? Corollary 5.5 implies that the radius of $\tilde{S}$ is at most the radius of $\tilde{S}(z, -1)$, which counts (by the number of leaves) enriched $\tilde{S}$-trees with no flippable edge. Since these trees can have arbitrary large size (Fig. 13), $\tilde{S}(z, -1)$ is not a polynomial. Its coefficients are non-negative integers, and hence its radius is at most 1. The same thus holds for $\tilde{S}(z, u)$.

**(ii)** By Lemma 5.6, the series $\frac{\partial \Phi_2}{\partial y}(z, \tilde{S}(z))$ has non-negative coefficients. Since its constant term is 0, the function $1 - u\frac{\partial \Phi_2}{\partial y}(z, \tilde{S}(z))$ is increasing on $[0, \tilde{\rho})$, with initial value 1: this rules out (ii).
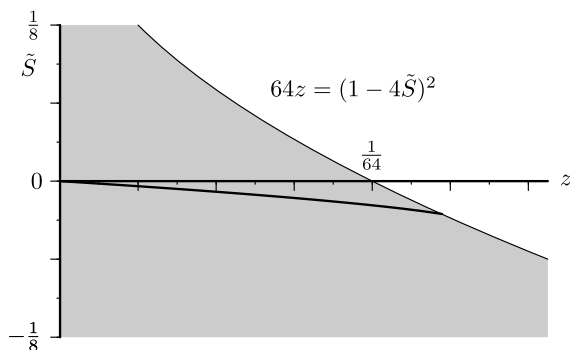
**(iv)** By Corollary 5.5, $\tilde{S}$ is negative and decreases on $[0, \tilde{\rho})$. Assume that it tends to $-\infty$. Since $\tilde{\rho}$ is finite, this implies that

$$\lim_{z \to \tilde{\rho}^-} \Psi_2\left(\frac{z}{(1 - 4\tilde{S}(z))^2}\right) = \Psi_2(0) = 0.$$

But then (83) gives

$$(1 + \bar{u})\tilde{S} = -u\sqrt{1 - 4\tilde{S}}/2 + o\left(\sqrt{1 - 4\tilde{S}}\right),$$

which is impossible if $\tilde{S} \to -\infty$.

**Fig. 14.** A plot of $\tilde{S}(z)$ for $u = -1/2$ and $z \in [0, \tilde{\rho}]$. The curve reaches the parabola $64z = (1 - 4\tilde{S})^2$ at $\tilde{\rho}$. The plot was obtained using the expansion of $\tilde{S}(z)$ up to order 25. Plotting the pairs $(R(z), S(z))$ for $z \in [0, \rho)$ gives the same curve.

We conclude that (iii) holds, so that $\Phi_2$ has no analytic continuation at $(\tilde{\rho}, \tilde{S}(\tilde{\rho}))$. Given the properties of $\Phi_2$ described in Section 10.1, this means that

$$64\tilde{\rho} = (1 - 4\tilde{S}(\tilde{\rho}))^2.$$

The radius of $\tilde{S}$ is at least $\tilde{\rho}$, the value of which we will determine explicitly later. Fig. 14 shows a plot of $\tilde{S}$ for $u = -1/2$. One can in fact prove that $\tilde{\rho}$ *is* the radius of $\tilde{S}$, but we will not use that.

Let us now consider Eq. (84) that defines $R$, and apply Corollary 6.3 with $\Omega(y) = y - u\Phi_1(y, \tilde{S}(y))$. We have just seen that, as $y$ goes from 0 to $\tilde{\rho}$, the pair $(y, \tilde{S}(y))$ reaches for the first time the critical parabola at $\tilde{\rho}$. Hence, with the notation of Corollary 6.3, the first singularity of $\Omega$ on the positive real axis satisfies $\omega \geq \tilde{\rho}$. Let us define $\tau$ and $\rho$ as in Corollary 6.3.

Could it be that $\Omega'(\tau) = 0$? By Corollary 6.3, $R(z)$ increases on $(0, \rho)$ and $\Omega'(R(z)) = 1/R'(z) \geq 0$. So could it be that $R'(z)$ tends to $+\infty$ as $z$ tends to $\rho$? No: by Corollary 5.5, $\bar{u}(R'(z) - 1)$ has non-negative coefficients, and thus it is always larger than its value at $z = 0$, which is 0. Since $u < 0$, this gives $R'(z) \leq 1$ on $(0, \rho)$, and we conclude that $\Omega'(\tau) > 0$. Hence $\tau = \omega \geq \tilde{\rho}$.

Since $R(z)$ increases from 0 to $\omega$ on $[0, \rho]$, there exists a unique $\hat{\rho}$ such that $R(\hat{\rho}) = \tilde{\rho}$. Since $\tilde{S}$ has radius at least $\tilde{\rho}$, the series $S = \tilde{S}(R)$ has also radius at least $\hat{\rho}$. The plot of the pairs $(R(z), S(z))$ for $z \in [0, \hat{\rho}]$ coincides with the plot of $(z, \tilde{S}(z))$ for $z \in [0, \tilde{\rho}]$ shown in Fig. 14.

We will now use the system (32)–(33) defining $R$ and $S$ to obtain expansions of $R$ and $S$ near $\hat{\rho}$. These expansions will be found to be singular at $\hat{\rho}$: this implies that $\hat{\rho} = \rho$ is the radius of $R$ and $S$.

We adopt the following notation: $z = \hat{\rho} - x$, $R(z) = \tilde{\rho} - r$, $S(z) = S(\hat{\rho}) - s$ and

$$\frac{R(z)}{(1 - 4S(z))^2} = \frac{1}{64} - \varepsilon. \tag{90}$$

The quantities $x$, $r$, $s$ and $\varepsilon$ tend to 0 as $z$ tends to $\hat{\rho}$. Let us begin by expanding (33) for $z$ close to $\hat{\rho}$. Using the expansion (81) of $\Psi_2$ near $1/64$, we obtain

$$a_1 + b_1 s + c_1 \varepsilon \ln \varepsilon + d_1 \varepsilon = O(\varepsilon^2 \ln \varepsilon) + O(s^2) + O(s \varepsilon \ln \varepsilon), \tag{91}$$

with

$$a_1 = \frac{1+u}{4}\delta^2 - \frac{u\sqrt{2}}{\pi}\delta + \frac{u-1}{4},$$

$$b_1 = -\frac{2u\sqrt{2}}{\pi\delta} + 1 + u, \quad c_1 = \frac{4\sqrt{2}}{\pi}u\delta, \quad d_1 = 3c_1,$$

and $\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})}$. In particular, $a_1$ must vanish, which gives the value of $\delta$:

$$\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})} = \frac{2\sqrt{2}u + \sqrt{\pi^2(1-u^2) + 8u^2}}{\pi(1+u)}. \tag{92}$$

(The choice of a minus sign before a square root would give a negative value, which is impossible for $\delta = \sqrt{1 - 4\tilde{S}(\tilde{\rho})}$.) Note that $u$ has a rational expression in terms of $\delta$:

$$u = -\frac{\pi(\delta^2 - 1)}{\pi\delta^2 - 4\sqrt{2}\delta + \pi}.$$

We will replace all occurrences of $u$ by this expression, to avoid handling algebraic coefficients.

Let us now return to the expansion (91). Given that $\delta > 0$, we have $b_1 > 0$ for $u \in [-1, 0)$. Hence $s = O(\varepsilon \ln \varepsilon)$, and (91) can be rewritten as

$$b_1 s + c_1 \varepsilon \ln \varepsilon + d_1 \varepsilon = O(\varepsilon^2 \ln^2 \varepsilon). \tag{93}$$

Let us now move to (90). Using $\tilde{\rho} = \delta^4/64$, it gives

$$b_2 s + d_2 \varepsilon + e_2 r = O(\varepsilon^2 \ln^2 \varepsilon) \tag{94}$$

with $b_2 = -8\delta^2, d_2 = 64\delta^4, e_2 = -64$. Finally, Eq. (33) that defines $R$ gives

$$a_3 + b_3 s + c_3 \varepsilon \ln \varepsilon + d_3 \varepsilon + e_3 r + f_3 x = O(\varepsilon^2 \ln^2 \varepsilon), \tag{95}$$

where, in particular,

$$a_3 = 96(\pi\delta^2 - 4\sqrt{2}\delta + \pi)\hat{\rho} + \delta^3(2\sqrt{2}\delta^2 - 3\pi\delta + 4\sqrt{2}).$$

Since $a_3$ must vanish, we obtain a rational expression of $\hat{\rho}$ in terms of $\delta$, and then, using (92), an explicit expression which coincides with (77). We do not give here the

expressions of $b_3, c_3, d_3, e_3$ and $f_3$, which are rational in $\delta$. They are easy to compute. Let us just mention that $f_3 \neq 0$.

Now, using (93), (94) and (95) in this order, we obtain for $s$, $r$ and finally $x$ expansions in $\varepsilon$ of the form

$$s = c_4 \, \varepsilon \ln \varepsilon + d_4 \, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon), \tag{96}$$

$$r = c_5 \, \varepsilon \ln \varepsilon + d_5 \, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon), \tag{97}$$

$$x = c_6 \, \varepsilon \ln \varepsilon + d_6 \, \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon). \tag{98}$$

In particular, $c_6 \neq 0$ for $u \in [-1, 0)$ and the latter equation gives $x \sim c_6 \, \varepsilon \ln \varepsilon$, so that $\ln x \sim \ln \varepsilon$ and thus

$$\varepsilon = \frac{x}{c_6 \ln x} \, (1 + o(1)) \, . \tag{99}$$

To conclude, we use (98) to express $\varepsilon \ln \varepsilon$ as a linear combination of $x$ and $\varepsilon$ (plus $O()$ terms), and (99) to express $\varepsilon$ in terms of $x$. This replaces (96) and (97) by

$$s = \frac{c_4}{c_6} x + \frac{d_4 c_6 - c_4 d_6}{c_6^2} \frac{x}{\ln x} (1 + o(1)),$$

$$r = \frac{c_5}{c_6} x + \frac{d_5 c_6 - c_5 d_6}{c_6^2} \frac{x}{\ln x} (1 + o(1)).$$

These equations, written explicitly, read

$$S(z) = \frac{1 - \delta^2}{4} + \frac{4\pi}{\delta \sqrt{\pi^2 (1 - u^2) + 8u^2}} (\hat{\rho} - z) - \frac{2\sqrt{2}\pi}{u\delta} \frac{\hat{\rho} - z}{\ln(\hat{\rho} - z)} (1 + o(1)) \, ,$$

$$R(z) = \tilde{\rho} - \frac{\pi\delta}{2\sqrt{\pi^2 (1 - u^2) + 8u^2}} (\hat{\rho} - z) - \frac{\sqrt{2}\pi\delta}{4u} \frac{\hat{\rho} - z}{\ln(\hat{\rho} - z)} (1 + o(1)) \, ,$$

as $z \to \hat{\rho}$. In particular, $R$ and $S$ are singular at $\hat{\rho}$, so that $\hat{\rho} = \rho$ is their common radius.

Using (29), we finally compute an expansion of $F'(z)$ near $\rho$, which gives (78). The coefficient $\beta$ of $(\rho - z)/\ln(\rho - z)$ does not vanish on $[-1, 0)$, and $F'$ has radius $\rho$ as well. Note that we can also compute expansions of $F''$ and $F'''$ using (29), (35) and (36). This is how we obtain the result stated in the first remark below Theorem 10.1. $\quad \square$

**The series $R$, $S$ and $F'$ in the complex plane.** We have contented ourselves with a real study of $R$, $S$ and $F'$ near their common radius $\rho$. Going further with complex analysis would require, among other results, a better knowledge of the series $\tilde{S}$ beyond its disk of convergence (we expect $R(z)$ to exit this disk for some $z$ with $|z| < \rho$, in the same way it exits the disk of convergence of $\Phi$ in the 4-valent case for some $z$ with $|z| < \rho$). We expect for $\tilde{S}$ at least a second singularity on the negative real axis, at which $\tilde{S}$ approaches $1/4$ (this prediction comes from Eq. (83) which defines $\tilde{S}$, and from the fact

that $\Psi_2(y)$ diverges as $y$ tends to $-\infty$). These observations indicate that the complex analysis of $F'$ is likely to be much more difficult (and longer) than the real analysis we have performed.

## 11. Final comments

### 11.1. Universality of the asymptotic behaviour

Our asymptotic results remain incomplete when $p = 3$, as we have not been able to obtain the asymptotic behaviour of $f_n(u)$ for negative values of $u$ (but only the real singular behaviour of $F'(z,u)$). We still predict $f_n(u)$ to behave like $c_u \rho_u^{-n} n^{-3} (\ln n)^{-2}$, as in the 4-valent case.

The second author has obtained complete results — and proved universality of the asymptotic behaviour — for all *even* values of $p$. As explained below Theorem 3.1, the series $S$ vanishes in this case, so that we only deal with one equation (in $R$). When $p = 6$ for instance, it reads:

$$R = z + u\Phi(R) = z + u \sum_{\ell \geq 1} \frac{(5\ell)!}{\ell!(4\ell+1)!} R^{2\ell+1}.$$

New difficulties arise from the periodicity of $\Phi$ and $R$. In particular, $R$ and $F$ have several singularities on their circle of convergence. The proof involves a difficult generalization of Section 5, proving that if $p = 2q$, the series $(R^n - z^n)/u$ are $(u+1)$-positive for $1 \leq n \leq q$. These results will appear in a forthcoming paper; see also [30]. The same asymptotic results hold for general (non-regular) Eulerian forested maps [30].

### 11.2. A differential equation involving $F$, rather than $F'$?

The two differential equations (DEs) obtained for the series $F$ in Section 4, for the 4-valent, and then for the cubic case, are in fact equations of order 2 satisfied by $F'$. It is natural to ask if $F$ itself satisfies a DE of order 2. Let us examine in detail the case $p = 4$.

Returning to Lemma 3.2, we first need an expression of $\bar{M}$. Since $t_{2i+1} = t_{2i+1}^c = 0$ when $p = 4$, we can content ourselves with an expression of $\bar{M}$ valid when $g_{2i+1} = 0$ for all $i$. Such an expression is easily obtained from the expression (17) of $\bar{M}_z'$. Indeed, $S = 0$ in the even case, and Eqs. (17) and (18), written as

$$\bar{M}_z' = \bar{\theta}(R), \quad R = z + u\bar{\Phi}(R),$$

imply at once

$$\bar{M} = \bar{\Psi}(R)$$

where

$$\bar{\Psi}(x) = \int \bar{\theta}(x) \left(1 - u\bar{\Phi}'(x)\right) \, \mathrm{d}x$$

$$= \sum_{i \geq 1} h_{2i} \binom{2i}{i} \frac{x^{i+1}}{i+1} - u \sum_{i \geq 1, j \geq 0} h_{2i} g_{2j+2} (2j+1) \binom{2i}{i} \binom{2j}{j} \frac{x^{i+j+1}}{i+j+1}.$$

This should be compared to Eq. (1.4) in [23], which reads, in the even case,

$$\bar{M} = \sum_{n \geq 1} h_{2n} \binom{2n}{n} \frac{R^{n+1}}{n+1} - u \sum_{n \geq 1, q \geq 0, k > q} h_{2n} g_{2k} \binom{2n+2q}{n+q} \binom{2k-2q-2}{k-q-1} \frac{R^{n+k}}{n+q+1}.$$

Our (simpler) expression is obtained by summing over $q$.

Hence for $p = 4$, Lemma 3.2 gives

$$F(z, u) = 4 \sum_{i \geq 2} \frac{(3i-3)!}{(i-2)!i!^2} \frac{R^{i+1}}{i+1} - u \sum_{i \geq 2, j \geq 1} \frac{(3i-3)!}{(i-2)!i!^2} \frac{(3j)!}{j!^3} \frac{R^{i+j+1}}{i+j+1} = \Psi(R), \quad (100)$$

where $\Psi(x) = \Psi_1(x) - u\Psi_2(x)$,

$$\Psi_1(x) = \int \theta(x) \, \mathrm{d}x, \qquad \Psi_2(x) = \int \theta(x) \Phi'(x) \, \mathrm{d}x,$$

and now $R$ is defined by $R = z + u\Phi(R)$, where $\Phi$ is given by (24).

Now assume that $F$ is differentially algebraic of order 2: there exists a non-zero polynomial $P$ such that

$$P(F, F', F'', z, u) = 0.$$

Equivalently,

$$P(\Psi(R), \theta(R), R'\theta'(R), z, u) = 0.$$

Using $z = R - u\Phi(R)$, $R' = (1 - u\Phi'(R))^{-1}$ and Eqs. (25) and (26) that relate $\theta$, $\Phi$, and their derivatives, we conclude that either $\Psi(x)$ is algebraic over $\mathbb{Q}(x, u, \Phi(x), \Phi'(x))$, or $\Phi(x)$ and $\Phi'(x)$ are algebraically related over $\mathbb{Q}(x)$. Let us examine these two possibilities.

1. Can $\Psi(x)$ be algebraic over $\mathbb{Q}(x, u, \Phi(x), \Phi'(x))$? Given that

$$15\Psi_1(x) = 15 \int \theta(x) \, \mathrm{d}x = 54x^2 - 2(1 + 81x)\Phi(x) + 8x(27x - 1)\Phi'(x)$$

and

$$3\Psi_2(x) = 3 \int \theta(x)\Phi'(x) \, \mathrm{d}x$$

$$= 12x\Phi(x) - 2(1 - 27x)\Phi(x)\Phi'(x) - 48\Phi^2(x) + 12 \int \frac{\Phi^2(x)}{x} \, \mathrm{d}x,$$

this is equivalent to saying that $\int \Phi(x)^2/x \, \mathrm{d}x$ is algebraic over $\mathbb{Q}(x, \Phi(x), \Phi'(x))$. Or, using (55), that the hypergeometric function

$$f(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; x\right)$$

is such that $g(x) := \int x f^2(x) \, \mathrm{d}x$ is algebraic over $\mathbb{Q}(x, f(x), f'(x))$ (here, we use the fact that

$$20 \int x f(x) \, \mathrm{d}x = 9x^2 f(x) + 9x^2(1 - x)f'(x)\bigg).$$

A related question (apparently more restrictive) is whether $g$ is a linear combination of $f^2, ff', (f')^2$. Given that

$$2f(x) + 18(x - 1)f'(x) + 9(x - 1)f''(x) = 0,$$

the vector space spanned over $\mathbb{Q}(x)$ by these 3 series contains all products $f^{(i)}f^{(j)}$ and is closed by differentiation. This would imply that $g$ satisfies a linear DE of order 3 with coefficients in $\mathbb{Q}(x)$. Starting from the order 4 DE satisfied by $g'$,

$$-4g'(x) + 8x(x - 1)g''(x) + 27x(x - 1)^2 g^{(3)}(x) + 9x^2(x - 1)^2 g^{(4)}(x) = 0,$$

the Maple command `ode_int_y` tells us that $g$ satisfies no linear DE of order 3. We have learnt from discussions with Alin Bostan, Bruno Salvy and Michael Singer that this implies that $g$ is not algebraic over $\mathbb{Q}(x, f, f')$.

2. Now could it be that $\Phi$ and $\Phi'$ are algebraically linked over $\mathbb{Q}(x)$, or, equivalently, that $f$ and $f'$ are algebraically linked over $\mathbb{Q}(x)$? One can prove that this is not the case, by combining the fact that $f'(x)$ diverges at 1 like $\ln(1 - x)$, while $f(1) = \sqrt{3}/(12\pi)$ is finite and transcendental.

These considerations imply that we have found in Section 4.2 the equation of minimal order satisfied by $F$.

### Acknowledgments

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