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Positivity of cylindric skew Schur functions



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ABSTRACT

Cylindric skew Schur functions, a generalization of skew Schur functions, are closely related to the well-known problem of finding a combinatorial formula for the 3-point Gromov-Witten invariants of Grassmannians. In this paper, we prove cylindric Schur positivity of cylindric skew Schur functions, as conjectured by McNamara. We also show that all the coefficients appearing in the expansion are the same as the 3-point Gromov-Witten invariants. We start by discussing the properties of affine Stanley symmetric functions for general affine permutations and 321-avoiding affine permutations, and we explain how these functions are related to cylindric skew Schur functions. In addition, we provide an effective algorithm to compute the expansion of cylindric skew Schur functions in terms of cylindric Schur functions, as well as the expansion of affine Stanley symmetric functions in terms of affine Schur functions.

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1. Introduction

Cylindric Schur functions, a generalization of Schur functions, are generating functions for semistandard Young tableaux on a cylindric shape. It is well known that Schur

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functions play an important role in various other subjects, such as representation theory and Schubert calculus. For example, the cohomology ring of the Grassmannian can be described in terms of Schur functions and related combinatorics.

Postnikov [15] observed that combinatorics on cylindric (skew) Schur functions can describe a quantum cohomology of the Grassmannian $Gr(m, n)$. Postnikov showed that the coefficients of a Schur expansion of cylindric skew Schur polynomials in the first m variables are the same as the multiplicative structure constants of the quantum cohomology of the Grassmannian, called (3-point) Gromov-Witten invariants.

When there is no restriction on the number of variables, the cylindric skew Schur functions are often not Schur-positive. In other words, most cylindric skew Schur functions cannot be written as a non-negative linear combination of Schur functions. However, McNamara [11] conjectured that a cylindric skew Schur function is cylindric Schur-positive: it can be written as a non-negative linear combination of cylindric Schur functions. Moreover, these coefficients in the linear combination contain all Gromov-Witten invariants. In this paper, we prove this conjecture and an even stronger statement conjectured by McNamara (see Theorem 1).

For proving the theorem, we first use the result proved by Lam [4] that the cylindric skew Schur functions are special cases of affine Stanley symmetric functions indexed by 321-avoiding affine permutations. Then, we study the affine Nil-Coxeter algebra to derive interesting identities and symmetries related to the affine Stanley symmetric functions for 321-avoiding affine permutations. The cylindric Schur positivity of cylindric skew Schur functions follows from the fact that the affine Stanley symmetric functions can be written as a non-negative linear combination of affine Schur functions and combinatorics of 321-avoiding affine permutations. Note that finding a combinatorial formula for the coefficients of the expansion of the affine Stanley symmetric functions in terms of affine Schur functions is an open problem, and these coefficients contain Littlewood-Richardson coefficients for flag variety and Gromov-Witten invariants for flag variety, which are famous problems in combinatorics. Although the cylindric Schur positivity of cylindric skew Schur functions is proved in this paper, we do not have a manifestly positive formula for the coefficients in the expansion. Instead, we introduce an effective algorithm to compute these coefficients in Section 5, which can also be used to compute the expansion of affine Stanley symmetric functions in terms of affine Schur functions.

The remainder of this paper is organized as follows. In Section 2, we introduce cylindric skew Schur functions and related combinatorics. In Section 3, we review known results of affine Stanley symmetric functions and affine Nil-Coxeter algebras. In Section 4, we study 321-avoiding permutations and relate the properties of affine Stanley symmetric functions to derive Theorem 1. In Section 5, we present an effective algorithm to compute the expansion of cylindric skew Schur functions (resp. affine Stanley symmetric functions) in terms of cylindric Schur functions (resp. affine Schur functions).

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2. Cylindric skew Schur functions

A *partition* $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. Let $\ell(\lambda)$ denote the number of parts in λ . For positive integers $m < n$, let P_{mn} be a set of partitions such that $\lambda_1 \leq n - m$ and $\ell(\lambda) \leq m$. The integers $m < n$ are fixed throughout the paper.

Let $\mathfrak{C}_{m,n}$ be the set $\mathbb{Z}^2/(-n+m, m)\mathbb{Z}$ and let $\langle i, j \rangle$ be $(i, j) + (-n+m, m)\mathbb{Z}$ in $\mathfrak{C}_{m,n}$. $\mathfrak{C}_{m,n}$ inherits a natural partial ordering $<_{\mathfrak{C}}$ from \mathbb{Z}^2 , generated by covering relations $\langle i, j \rangle <_{\mathfrak{C}} \langle i+1, j \rangle$ and $\langle i, j \rangle <_{\mathfrak{C}} \langle i, j+1 \rangle$. A cylindric diagram D is a finite subset of $\mathfrak{C}_{m,n}$ such that for $a, b \in D$, we have $[a, b]_{\mathfrak{C}} \subset D$. For an integer r , the r -th row of a cylindric diagram D is a set of elements $\langle r, j \rangle$ in D , and the r -th column of D is a set of elements $\langle i, r \rangle$ in D . Note that the r -th row of D depends on only r modulo $n - m$ and the r -th column of D depends on only r modulo m . The r -th diagonal is a set of elements $\langle i, j \rangle$ satisfying $j - i = r$ modulo n .

We say that a bi-infinite integer sequence $\alpha = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$ is (m, n) -periodic if $\alpha_i = \alpha_{i+m} - (n - m)$ for any $i \in \mathbb{Z}$, and is increasing if $\alpha_i \leq \alpha_j$ for any $i < j$. For a partition $\lambda \in P_{mn}$ and an integer r , let $\lambda[r]$ be the (m, n) -periodic sequence defined by $\lambda[r]_{i+r} = \lambda_i + r$ for $i = 1, \dots, m$. All increasing (m, n) -periodic sequences are of the form $\lambda[r]$ for some $r \in \mathbb{Z}$ and $\lambda \in P_{mn}$.

Recall that a subset of a partially ordered sets is called an ordered ideal if, whenever it contains an element x , it also contains all elements less than x . There is a one-to-one correspondence between a set $\{\lambda[r] \mid r \in \mathbb{Z}, \lambda \in P_{mn}\}$ and the set of ordered ideals $D_{\lambda[r]} = \{\langle i, j \rangle \in \mathfrak{C}_{m,n} \mid (i, j) \in \mathbb{Z}^2, j \leq \lambda[r]_i\}$. The sequence $\lambda[r]$ can be visualized as a boundary of $D_{\lambda[r]}$ (see Fig. 1). For $\lambda, \mu \in P_{mn}$ and $r, s \in \mathbb{Z}$, we call the pair $(\lambda[r], \mu[s])$ a *cylindric shape* of type (m, n) if $\mu[s]_i \leq \lambda[r]_i$ for all i and denote it by $\lambda[r]/\mu[s]$. The inequality condition is equivalent to the inclusion $D_{\mu[s]} \subset D_{\lambda[r]}$. We denote the set of all cylindric shapes of type (m, n) by C_{mn} . Each cylindric diagram in $\mathfrak{C}_{m,n}$ can be written as the difference of two ordered ideals,

$$D_{\lambda[r]/\mu[s]} = D_{\lambda[r]}/D_{\mu[s]} = \{\langle i, j \rangle \in \mathfrak{C}_{m,n} \mid (i, j) \in \mathbb{Z}^2, \mu[s]_i < j \leq \lambda[r]_i\},$$

where $r, s \in \mathbb{Z}$ and $\lambda, \mu \in P_{mn}$. We call $\lambda[r]/\mu[s]$ the *shape* of $D_{\lambda[r]/\mu[s]}$. For partitions $\lambda, \mu \in P_{mn}$ and a non-negative integer d , we denote $\lambda[d]/\mu[0]$ by $\lambda/d/\mu$.

Remark 1. Note that the notation $\lambda[r]$ is from [15], which is different from the notation in [11]. The notation $\lambda[r]$ makes it easier for us to state the second part of Theorem 1.

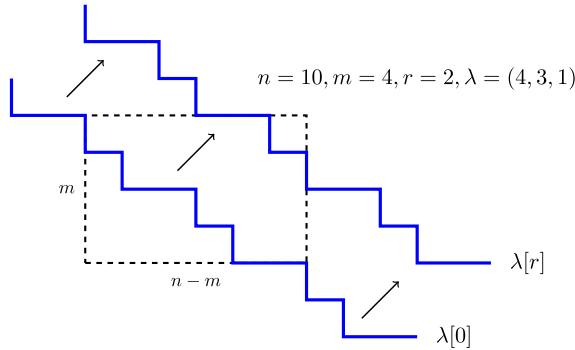


Fig. 1. Boundaries of $\lambda[0]$ and $\lambda[r]$.

A cylindric semistandard Young tableau of shape $\lambda[r]/\mu[s]$ is the map $T : D_{\lambda[r]/\mu[s]} \rightarrow \mathbb{N}$ satisfying $T(\langle i, j \rangle) \leq T(\langle i+1, j \rangle)$ when $\langle i, j \rangle, \langle i+1, j \rangle \in D_{\lambda[r]/\mu[s]}$ and $T(\langle i, j \rangle) < T(\langle i, j+1 \rangle)$ when $\langle i, j \rangle, \langle i, j+1 \rangle \in D_{\lambda[r]/\mu[s]}$. A sequence $(T^{-1}(1), T^{-1}(2), \dots)$ is called the *weight* of T , denoted by $w(T)$. Let \mathbf{x} be the set of variables (x_1, x_2, \dots) . Now, we are ready to define the cylindric skew Schur functions.

Definition 1. A cylindric skew Schur function $s_{\lambda[r]/\mu[s]}(\mathbf{x})$ is defined by

$$s_{\lambda[r]/\mu[s]}(\mathbf{x}) := \sum_T \mathbf{x}^{w(T)},$$

where the sum runs over all cylindric semistandard Young tableaux T of shape $\lambda[r]/\mu[s]$, and $\mathbf{x}^{(a_1, a_2, \dots)} = x_1^{a_1} x_2^{a_2} \dots$.

A cylindric diagram D is called *toric* if each row of D has at most $n-m$ elements and each column of D has at most m elements. If $D_{\lambda/d/\mu}$ is toric, we call the shape $\lambda/d/\mu$ toric as well. Let us define a *toric Schur polynomial* as the specialization

$$s_{\lambda/d/\mu}(x_1, \dots, x_k) = s_{\lambda/d/\mu}(x_1, \dots, x_k, 0, 0, \dots)$$

of the cylindric skew Schur function $s_{\lambda/d/\mu}(\mathbf{x})$ when $\lambda/d/\mu$ is toric.

Recall that P_{mn} is the set of partitions λ with $\lambda_1 \leq n-m$ and $\ell(\lambda) \leq m$. Postnikov [15] showed that for $\lambda, \mu \in P_{mn}$, the toric Schur polynomial $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ is Schur-positive, i.e.,

$$s_{\lambda/d/\mu}(x_1, \dots, x_k) = \sum_{\nu \in P_{mn}} C_{\mu, \nu}^{\lambda, d} s_{\nu}(x_1, \dots, x_k) \quad (1)$$

with non-negative integers $C_{\mu, \nu}^{\lambda, d}$. Moreover, $C_{\mu, \nu}^{\lambda, d}$ is the same as the Gromov-Witten invariants for the Grassmannian $Gr(m, n)$.

Here, we briefly describe the Gromov-Witten invariants $C_{\mu, \nu}^{\lambda, d}$. Let $Gr(m, n)$ be the set of m -dimensional subspaces in \mathbb{C}^n . The set $Gr(m, n)$ is a complex projective variety

called the *Grassmannian*. There is a cellular decomposition of $Gr(m, n)$ into Schubert cells Ω_λ° , where each λ is a partition in P_{mn} . Let Ω_λ be the Zariski closure of Ω_λ° , called the Schubert variety.

For a partition $\lambda \in P_{mn}$, let λ^\vee be the partition in P_{mn} defined by $\lambda_i^\vee = n - m - \lambda_{m-i+1}$ for $i = 1, \dots, m$. For partitions $\lambda, \mu, \nu \in P_{mn}$, the Gromov-Witten invariant $C_{\mu, \nu}^{\lambda, d}$ is the number of rational curves of degree d passing through generic translates of the Schubert varieties $\Omega_{\lambda^\vee}, \Omega_\mu$, and Ω_ν in the Grassmannian $Gr(m, n)$, when the number is finite. This implies that $C_{\mu, \nu}^{\lambda, d}$ is zero unless $|\lambda| + nd = |\mu| + |\nu|$. For $d = 0$, the constants $C_{\mu, \nu}^{\lambda, d}$ are the same as the Littlewood-Richardson coefficients.

In general, cylindric skew Schur functions are not Schur-positive. However, McNamara [11] conjectured the following statement, which is our main theorem.

Theorem 1. *For a cylindric shape $\lambda/d/\mu$ in $\mathfrak{C}_{m,n}$, let $s_{\lambda/d/\mu}$ be the cylindric skew Schur functions. Then,*

- (1) $s_{\lambda/d/\mu}$ is cylindric Schur-positive, i.e.,

$$s_{\lambda/d/\mu}(\mathbf{x}) = \sum_{\nu \in P_{mn}, e \geq 0} c_{\nu/e/\emptyset}^{\lambda/d/\mu} s_{\nu/e/\emptyset}(\mathbf{x}),$$

with $c_{\nu/e/\emptyset}^{\lambda/d/\mu} \geq 0$.

- (2) $c_{\nu/e/\emptyset}^{\lambda/d/\mu}$ is the same as $c_{\nu/e-1/\emptyset}^{\lambda/d-1/\mu}$ for all positive integers e , and $c_{\nu/0/\emptyset}^{\lambda/d/\mu} = C_{\mu, \nu}^{\lambda, d}$.

To prove Theorem 1, we first relate the cylindric Schur functions with affine Stanley symmetric functions.

3. Affine Stanley symmetric functions

In this section, we review the theory of affine Stanley symmetric functions and the affine Nil-Coxeter algebra. See [4,5] for further details.

3.1. Affine Stanley symmetric functions

Let \tilde{S}_n denote the affine symmetric group with simple generators s_i for $i \in \mathbb{Z}/n\mathbb{Z}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i && \text{if } i - j \neq 1, -1, \end{aligned}$$

where the indices are taken modulo n . An element of the affine symmetric group may be written as a word in the generators s_i . A *reduced word* of the element is a word of

minimal length. The *length* of w , denoted by $\ell(w)$, is the number of generators in any reduced word of w .

The subgroup of \tilde{S}_n generated by $\{s_1, \dots, s_{n-1}\}$ is naturally isomorphic to the symmetric group S_n . The *0-Grassmannian elements* are minimal length coset representatives of \tilde{S}_n/S_n . In other words, for an element $w \neq \text{id}$ in the affine symmetric group, w is 0-Grassmannian if and only if all reduced words of w end with s_0 . More generally, for $i \in \mathbb{Z}/n\mathbb{Z}$ and $w \in \tilde{S}_n$, w is called *i-Grassmannian* if all reduced words of w end with s_i or w is the identity. We denote the set of *i-Grassmannian elements* by \tilde{S}_n^i . There is a weak order $<$ on \tilde{S}_n defined by the covering relation $w < v$ if $s_i w = v$ with $\ell(w) + 1 = \ell(v)$.

A word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with indices in $\mathbb{Z}/n\mathbb{Z}$ is called *cyclically decreasing* (resp. *cyclically increasing*) if each letter occurs at most once and whenever s_i and s_{i+1} both occur in the word, s_{i+1} precedes s_i (resp. s_i precedes s_{i+1}). For $J \subsetneq \mathbb{Z}/n\mathbb{Z}$, a *cyclically decreasing element* d_J (resp. a *cyclically increasing element* u_J) is the unique cyclically decreasing permutation (resp. cyclically increasing permutation) in \tilde{S}_n that uses exactly the simple generators in $\{s_j \mid j \in J\}$. For example, for $n = 7$ and a subset $J = \{0, 1, 4, 6\}$ of $\mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\}$, we get $d_J = s_4 s_1 s_0 s_6$ and $u_J = s_4 s_6 s_0 s_1$ in \tilde{S}_7 .

For an element w in \tilde{S}_n , we call $w_1 w_2 \cdots w_\ell$ a *cyclically decreasing decomposition* of w if $w_1 w_2 \cdots w_\ell = w$ satisfies $\ell(w_1) + \cdots + \ell(w_\ell) = \ell(w)$ and each w_i is cyclically decreasing. Note that some of w_i can be identity elements. Now, we are ready to define the affine Stanley symmetric functions.

Definition 2. For an element $w \in \tilde{S}_n$, an affine Stanley symmetric function F_w is defined by

$$F_w(\mathbf{x}) = \sum x_1^{\ell(w_1)} x_2^{\ell(w_2)} \cdots x_\ell^{\ell(w_\ell)},$$

where the sum runs over cyclically decreasing decompositions $w_1 \cdots w_\ell$ of w .

If w is 0-Grassmannian, we call F_w an *affine Schur function*. Let $\Lambda^{(n)}$ be the quotient of the ring of symmetric functions Λ by monomial symmetric functions m_λ for $\lambda_1 \geq n$. Let $\Lambda_{(n)}$ be the subalgebra $\mathbb{Z}[h_1, \dots, h_{n-1}]$ of Λ , where h_i is a homogeneous symmetric functions of degree i . Let $\langle \cdot, \cdot \rangle_\Lambda$ be the Hall inner product of Λ . Then, two algebras $\Lambda_{(n)}$ and $\Lambda^{(n)}$ are duals of each other.

Now, we list known theorems about the affine Stanley symmetric functions.

Theorem 2. [5,10] The affine Schur functions $\{F_w \mid w \in \tilde{S}_n^0\}$ form a basis of $\Lambda^{(n)}$.

Theorem 3. [5] The affine Stanley symmetric functions F_w expand positively in terms of the affine Schur functions. Thus, we have

$$F_w = \sum_{v \in \tilde{S}_n^0} c_v^w F_v$$

for a non-negative integer c_v^w .

Let $\{s_u^{(k)} \mid u \in \tilde{S}_n^0\}$ be the dual basis of $\{F_u \mid u \in \tilde{S}_n^0\}$ of $\Lambda_{(n)}$ with respect to the induced Hall inner product $\langle \cdot, \cdot \rangle_\Lambda : \Lambda_{(n)} \times \Lambda^{(n)} \rightarrow \mathbb{Z}$. Here, k is equal to $n - 1$ and will remain so throughout the paper. The symmetric function $s_u^{(k)}$ is called the k -Schur function. It is known that the coefficients c_v^w are the same as the structure constants of the k -Schur functions. We prove this fact by using the affine Nil-Coxeter algebra and noncommutative k -Schur functions in the next section.

3.2. Affine Nil-Coxeter algebra

The *affine Nil-Coxeter algebra* \mathbb{A} is the algebra generated by A_0, A_1, \dots, A_{n-1} over \mathbb{Z} , satisfying

$$\begin{aligned} A_i^2 &= 0 \\ A_i A_{i+1} A_i &= A_{i+1} A_i A_{i+1} \\ A_i A_j &= A_j A_i && \text{if } i - j \neq 1, -1, \end{aligned}$$

where the indices are taken modulo n . A subalgebra of \mathbb{A} generated by A_i for $i \neq 0$ is isomorphic to the Nil-Coxeter algebra studied by Fomin and Stanley [3]. The simple generators A_i can be considered as the *divided difference operators* in Kumar and Kostant's work.

Note that A_i satisfy the same braid relations as s_i in \tilde{S}_n , i.e., $A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$. Therefore, it makes sense to define

$$\begin{aligned} A_w &= A_{i_1} \cdots A_{i_l} \text{ where} \\ w &= s_{i_1} \cdots s_{i_l} \text{ is a reduced decomposition.} \end{aligned}$$

One can check that

$$A_v A_w = \begin{cases} A_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that d_J is the cyclically decreasing element corresponding to a proper subset J of $\mathbb{Z}/n\mathbb{Z}$. For $i < n$, let

$$\mathbf{h}_i = \sum_{\substack{J \subset \mathbb{Z}/n\mathbb{Z} \\ |J|=i}} A_{d_J} \in \mathbb{A},$$

where $\mathbf{h}_0 = 1$ and $\mathbf{h}_i = 0$ for $i < 0$ by convention. Lam [4] showed that the elements $\{\mathbf{h}_i\}_{i < n}$ commute and freely generate a subalgebra \mathbb{B} of \mathbb{A} , called the *affine Fomin-Stanley algebra*. It is well known that \mathbb{B} is isomorphic to $\mathbb{Z}[h_1, \dots, h_k]$ via the map sending \mathbf{h}_i to h_i , where h_i is a complete homogeneous symmetric function of degree i . Therefore, the set $\{\mathbf{h}_\lambda = \mathbf{h}_{\lambda_1} \cdots \mathbf{h}_{\lambda_l} \mid \lambda_1 \leq k\}$ forms a basis of \mathbb{B} .

There is another basis of \mathbb{B} , called the *noncommutative k -Schur functions* $\mathbf{s}_\lambda^{(k)}$, indexed by partitions λ with $\lambda_1 \leq k$. We call such a partition a *k -bounded partition*. There is a bijection between the set of k -bounded partitions and \tilde{S}_n^0 [9]. Hereafter, we also index the noncommutative k -Schur functions with 0-Grassmannian elements using the bijection. We will review the bijection between the set of k -bounded partitions and \tilde{S}_n^0 at the end of this section.

For an element $w \in \tilde{S}_n^0$, the noncommutative k -Schur function $\mathbf{s}_w^{(k)}$ is the image of $s_w^{(k)}$ via the isomorphism $\Lambda_{(k)} \cong \mathbb{B}$. Lam [5] showed that the noncommutative k -Schur function $\mathbf{s}_w^{(k)}$ is the unique element in \mathbb{B} that has the unique 0-Grassmannian term A_w . The noncommutative k -Schur functions are non-equivariant versions of the j functions studied by Peterson [14] for affine type A . For details and the original definition of noncommutative k -Schur functions, see [5,6].

Example 1. For a positive integer $i < n$, we have $\mathbf{s}_{s_{i-1}s_{i-2}\dots s_0}^{(k)} = \mathbf{h}_i$ since \mathbf{h}_i is in \mathbb{B} and $s_{i-1}s_{i-2}\dots s_0$ is the unique cyclically decreasing element of length i in \tilde{S}_n^0 . Note that the k -bounded partition corresponding to $s_{i-1}s_{i-2}\dots s_0$ is a partition (i) .

Lam showed the following theorem.

Theorem 4. [5] For $w \in \tilde{S}_n$, assume that we have an expansion

$$F_w = \sum_u c_u^w F_u,$$

where the sum runs over 0-Grassmannian elements u . Then, we have

$$\mathbf{s}_u^{(k)} = \sum_{w \in \tilde{S}_n} c_u^w A_w.$$

The structure constants of k -Schur functions are the same as coefficients c_u^w appearing in Theorem 4. For $w \in \tilde{S}_n^0$, consider the coefficients of A_w on each side of the identity

$$\mathbf{s}_u^{(k)} \mathbf{s}_v^{(k)} = \sum_{\alpha \in \tilde{S}_n^0} d_{u,v}^\alpha \mathbf{s}_\alpha^{(k)}.$$

The coefficient of A_w on the right-hand side is $d_{u,v}^w$. On the other hand, the coefficient of A_w on the left-hand side is $\sum c_u^{u'} c_v^{v'}$, where the sum runs over elements u', v' in \tilde{S}_n such that $u'v' = w$ with $\ell(u') + \ell(v') = \ell(w)$. Since w is 0-Grassmannian, v' has to be 0-Grassmannian as well. By the property of the noncommutative Schur functions, v' must be equal to v and the summation becomes $c_u^{u'}$, where u' satisfies $u'v = w$ with $\ell(u') + \ell(v) = \ell(w)$. Therefore, we have the following.

Theorem 5. For $w, u, u', v \in \tilde{S}_n$ satisfying $w = u'v$ with $\ell(w) = \ell(u') + \ell(v)$, we have $c_u^{u'} = d_{u,v}^w$.

Since the affine Fomin-Stanley algebra is commutative, we have the following symmetry of the coefficients $c_u^{u'}$.

Corollary 1. *For $\alpha = w_1 w_2 = v_1 v_2$ in \tilde{S}_n^0 such that $\ell(w_1) = \ell(v_2)$, $\ell(w_2) = \ell(v_1)$, and $\ell(\alpha) = \ell(w_1) + \ell(w_2) = \ell(v_1) + \ell(v_2)$, we have $c_{v_2}^{w_1} = c_{w_2}^{v_1}$.*

Proof. By Theorem 5, we have $c_{v_2}^{w_1} = d_{v_2, w_2}^\alpha = d_{w_2, v_2}^\alpha = c_{w_2}^{v_1}$. \square

Now, we review Denton's work regarding affine permutations to explain the bijection between the set of k -bounded partitions and \tilde{S}_n^0 . For $w \in \tilde{S}_n$, consider a unique maximal $J \subsetneq \mathbb{Z}/n\mathbb{Z}$ such that $w = vd_J$ with $\ell(w) = \ell(v) + \ell(d_J)$ for some $v \in \tilde{S}_n$. Indeed, Denton showed that such a d_J is unique in the following sense:

Lemma 1. [2, Corollary 18] *For a given affine permutation w , there is a unique subset J of $\mathbb{Z}/n\mathbb{Z}$ such that whenever we have $w = ud_{J'}$ for some u and a cyclically decreasing element $d_{J'}$ with $\ell(w) = \ell(u) + \ell(d_{J'})$, J contains J' .*

We denote such a set J by $D(w)$ and $|J|$ by $\max_r(w)$. Similarly, consider the unique maximal set $J' \subsetneq \mathbb{Z}/n\mathbb{Z}$ such that $w = vu_{J'}$ with $\ell(w) = \ell(v) + \ell(u_{J'})$ for some $v \in \tilde{S}_n$ and denote J' (resp. $|J'|$) by $U(w)$ (resp. $\max_c(w)$). Note that by repeating Lemma 1, we get the following corollary.

Corollary 2. [2, Corollary 20] *Every affine permutation w has a unique maximal decomposition into cyclically decreasing elements, i.e., $w = d_{J_p} \cdots d_{J_1}$, where $\ell(w) = \sum_{i=1}^p |J_i|$ and the sequence $(|J_1|, |J_2|, \dots, |J_p|, 0, 0, \dots)$ is the maximum with respect to the lexicographic order.*

Denton also showed that a sequence $(|J_1|, |J_2|, \dots, |J_p|, 0, 0, \dots)$ forms a partition with $|J_1| \leq k$, which is a k -bounded partition. When w is 0-Grassmannian, the map sending w to a k -bounded partition

$$\lambda = (|J_1|, |J_2|, \dots, |J_p|)$$

becomes a bijection. Each set J_i is determined by the size $|J_i| = \lambda_i$. Thus, we have $J_i = [-j + 1, \lambda_j - j]$ [2, Corollary 39].

4. Combinatorics on 321-avoiding permutations

An element $w \in \tilde{S}_n$ is called *321-avoiding* if any reduced word of w does not contain a subword $s_i s_{i+1} s_i$ for any $i \in I$. Lam [4] showed that a cylindric skew Schur function is an affine Stanley symmetric function for some $w \in \tilde{S}_n$. In this section, we develop combinatorics on 321-avoiding affine permutations and reprove and generalize the above-mentioned result more systematically. Then, we use these results to prove Theorem 1.

One of the main objects in this section is an n -connected ribbon, a ribbon with maximum size n ; understanding n -connected ribbons enables us to prove (2) of Theorem 1.

If $\lambda[r]/\mu[s]$ is a cylindric shape of type (m, n) , let $s_i \cdot (\lambda[r]/\mu[s])$ be a cylindric shape $\lambda'[r']/\mu[s]$ of type (m, n) such that $D_{\lambda'[r']/\mu[s]} = D_{\lambda[r]/\mu[s]} \cup \{\langle p, q \rangle \mid q - p = i\}$ for some $p, q \in \mathbb{Z}$ if such p and q exist, and $s_i \cdot (\lambda[r]/\mu[s])$ is not defined otherwise. In other words, one can obtain $D_{\lambda'[r']}$ from $D_{\lambda[r]}$ by simply adding a box at the i -th diagonal when possible. If $\alpha = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$ is the (m, n) -periodic increasing bi-infinite sequence corresponding to $\lambda[r]$, then $s_i \cdot (\lambda[r]/\mu[s])$ is well defined if and only if there is an integer p such that $\alpha_p + 1 - p = i$ modulo n and $\alpha_p < \alpha_{p+1}$. In this case, $s_i \cdot (\lambda[r]/\mu[s])$ is $\lambda'[r']/\mu[s]$, where the (m, n) -periodic increasing bi-infinite sequence corresponding to $\lambda'[r']$ is β , where $\beta_j = \alpha_j + 1$ if $j = p$ modulo m and α_j otherwise. Hence, we also define an action s_i on the set of (m, n) -periodic increasing sequences similarly. In this paper, bi-infinite sequences are always (m, n) -periodic and increasing.

For an element A_i in \mathbb{A} , one can define an action on the set of cylindric shapes of type (m, n) by

$$A_i \cdot (\lambda[r]/\mu[s]) = \begin{cases} s_i \cdot (\lambda[r]/\mu[s]) & \text{if } s_i \cdot (\lambda[r]/\mu[s]) \text{ is well-defined} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this gives an action of \mathbb{A} on the set C_{mn} of cylindric shapes of type (m, n) . In fact, the following is true.

Lemma 2. *As an action on the set C_{mn} of cylindric shapes of type (m, n) (or the set of (m, n) -periodic bi-infinite sequences), the generators A_i satisfy the following.*

- (1) $A_i^2 = 0$,
- (2) $A_i A_j = A_j A_i$ if $i - j \neq \pm 1$,
- (3) $A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1} = 0$,
- (4) For a cyclically decreasing element w of length $> n - m$, $A_w = 0$,
- (5) For a cyclically increasing element w of length $> m$, $A_w = 0$.

Proof. Note that it is sufficient to show that the A_i 's satisfy the above-mentioned relations as an action on the set of (m, n) -periodic bi-infinite sequences.

(1) is obvious. To prove (2), for distinct elements $i, j \in \mathbb{Z}/n\mathbb{Z}$ satisfying $i - j \neq \pm 1$, note that $A_i A_j \cdot \lambda[r]$ is not 0 if and only if one can add two boxes at diagonals i and j , and in this case, $A_i A_j \cdot \lambda[r]$ is obtained by adding two such boxes in $\mathfrak{C}_{m,n}$. Since $j - i \neq \pm 1$, two such boxes cannot be adjacent and we have $A_i A_j \cdot \lambda[r] = A_j A_i \cdot \lambda[r]$. For (3), assume that $A_i A_{i+1} A_i \cdot \alpha = \beta$ for some (m, n) -periodic sequence β . Then, $D_{\beta/\alpha}$ has cardinality 3 and two distinct boxes $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle$ at the i -th diagonal. Since there is only one i -th diagonal in $\mathfrak{C}_{m,n}$, we have $\langle a_1 + 1, a_2 + 1 \rangle = \langle b_1, b_2 \rangle$. Since $D_{\beta/\alpha}$ is a cylindric diagram, both $\langle a_1 + 1, a_2 \rangle$ and $\langle a_1, a_2 + 1 \rangle$ are in $D_{\beta/\alpha}$. Since each box lies in the $(i - 1)$ -th and

$(i + 1)$ -th diagonals, respectively, $A_i A_{i+1} A_i \cdot \alpha$ cannot be β . Moreover, one can use the same argument to show that $A_i A_{i-1} A_i \cdot \alpha = 0$.

To prove (4), assume that for a cyclically decreasing element w , we have $A_w \cdot \alpha = \beta$ for some (m, n) -periodic sequences α, β . Then, one can show that $D_{\beta/\alpha}$ is a *horizontal strip*, i.e., there are no two boxes in the same column. Indeed, if there are two adjacent boxes in the same column, then the boxes are at the i -th diagonal and $(i - 1)$ -th diagonal, respectively, for some $i \in \mathbb{Z}/n\mathbb{Z}$. In any reduced word of w , s_i precedes s_{i-1} so that w is not a cyclically decreasing element. Note that for any cylindric diagram in $\mathfrak{C}_{m,n}$, there are at most $n - m$ columns so that $\ell(w) \leq n - m$, proving (4). The proof of (5) is similar to the proof of (4). \square

From Lemma 2, we define the quotient algebra $\mathbb{A}_{m,n}$ of \mathbb{A} by the relations (3)–(5) in Lemma 2. One can rephrase the conditions (4)–(5) using the following lemma.

Lemma 3. For $w \in \tilde{S}_n$ and a positive integer $q < n$, $\max_r(w) > q$ if and only if there exist $v_1, v_2 \in \tilde{S}_n$ and $S \subsetneq \mathbb{Z}/n\mathbb{Z}$ satisfying $w = v_1 d_S v_2$ and $\ell(w) = \ell(v_1) + \ell(v_2) + |S|$ and $|S| > q$. Similarly, $\max_c(w) > q$ if and only if $w = v_1 u_S v_2$ for some $S \subsetneq \mathbb{Z}/n\mathbb{Z}$ with $\ell(w) = \ell(v_1) + \ell(v_2) + |S|$ and $|S| > q$.

Proof. The ‘only if’ statement is obvious, since there exist an affine permutation u and $S \subset \mathbb{Z}/n\mathbb{Z}$ satisfying $w = u d_S$ with $\ell(w) = \ell(u) + |S|$ and $|S| = \max_r(w)$ by the definition of $\max_r(w)$. To prove the ‘if’ statement, assume that we have $w = v_1 d_S v_2$ satisfying the conditions in the above-mentioned lemma. First, we prove that there exist an element $u \in \tilde{S}_n$ and $S' \subsetneq \mathbb{Z}/n\mathbb{Z}$ satisfying $d_S v_2 = u d_{S'}$ with $\ell(v_2) = \ell(u)$ and $|S'| = |S|$. Let ℓ be $\ell(v_2)$ and p be $|S|$, and consider the identity $(\mathbf{s}_{s_0}^{(k)})^\ell \mathbf{h}_p = \mathbf{h}_p (\mathbf{s}_{s_0}^{(k)})^\ell$. The coefficient of $A_{d_S v_2} = A_{d_S} A_{v_2}$ in $\mathbf{h}_p (\mathbf{s}_{s_0}^{(k)})^\ell$ is at least 1, since the coefficient of A_{v_2} at $(\mathbf{s}_{s_0}^{(k)})^\ell$ is positive (it is equal to the number of reduced words of v_2) and the coefficient of A_{d_S} at \mathbf{h}_p is one. Therefore, the coefficient of $A_{d_S v_2}$ in $(\mathbf{s}_{s_0}^{(k)})^\ell \mathbf{h}_p$ is positive. Since every term in $(\mathbf{s}_{s_0}^{(k)})^\ell \mathbf{h}_p$ is of the form $A_u d_{S'}$ for some affine permutation $u \in \tilde{S}_n$ with $\ell(u) = \ell$ and $S' \subsetneq \mathbb{Z}/n\mathbb{Z}$ with $|S'| = p$, there are such u and S' satisfying $d_S v_2 = u d_{S'}$, and we have proved the claim. By the claim, we have $w = v_1 u d_{S'}$ with $\ell(w) = \ell(v_1) + \ell(u) + |S'|$ and $\max_r(w) \geq |S'| > q$ by the definition of \max_r . \square

By Lemma 3, $\mathbb{A}_{m,n}$ can also be defined by the quotient of \mathbb{A} by the relation (1)–(3) and the relation

$$A_w = 0 \quad \text{if } \max_r(w) > n - m \text{ or } \max_c(w) > m.$$

We abuse a notation that the image of A_i under the projection $\mathbb{A} \rightarrow \mathbb{A}_{m,n}$ is denoted by the same A_i . Then, the set $\{A_w \mid w \in \tilde{S}_n \text{ is 321-avoiding, } \max_c(w) \leq m, \max_r(w) \leq n - m\}$ forms a basis of $\mathbb{A}_{m,n}$. We denote the set $\{w \mid w \in \tilde{S}_n \text{ is 321-avoiding, } \max_c(w) \leq m, \max_r(w) \leq n - m\}$ by $A_{(n-m,m)}$. We denote the set $\tilde{S}_n^0 \cap A_{(n-m,m)}$ by $A_{(n-m,m)}^0$. We

will show that there is a bijection between the set $A_{(n-m,m)}^0$ and the set of all cylindric shapes $\lambda/d/\emptyset$ (Theorem 7) by analyzing the elements of $A_{(n-m,m)}$ and $A_{(n-m,m)}^0$.

Let $\mathbb{B}_{m,n}$ be the subalgebra of $\mathbb{A}_{m,n}$ generated by $\bar{\mathbf{h}}_1, \dots, \bar{\mathbf{h}}_{n-m}$, where $\bar{\mathbf{h}}_i$ is the image of \mathbf{h}_i via the projection $\mathbb{A} \rightarrow \mathbb{A}_{m,n}$. For $i < n$, let \mathbf{e}_i be an element in \mathbb{A} defined by $\sum_w A_w$, where the sum runs over cyclically increasing elements w of length i . Let $\bar{\mathbf{e}}_i$ be the image of \mathbf{e}_i via the projection $\mathbb{A} \rightarrow \mathbb{A}_{m,n}$. For $w \in \tilde{S}_n^0$, let $\mathbf{s}_w^{(m,n)}$ be the image of $\mathbf{s}_w^{(k)}$ via the projection $\mathbb{A} \rightarrow \mathbb{A}_{m,n}$. We show the following lemmas for later purposes.

Lemma 4. For $J \subsetneq \mathbb{Z}/n\mathbb{Z}$, $A_{d_J}A_i$ is zero in $\mathbb{A}_{m,n}$ if $i \in J$. Similarly, $A_iA_{d_J}$, $A_{u_J}A_i$, and $A_iA_{u_J}$ are zero in $\mathbb{A}_{m,n}$ if $i \in J$.

Proof. Let $[p, q]$ be the maximal interval containing i and contained in J . Here, $[p, q] = \{p, p+1, \dots, q-1, q\}$ modulo n . It is sufficient to show that $A_qA_{q-1} \dots A_{p+1}A_pA_i$ is zero in $\mathbb{A}_{m,n}$. If $p = i$, then it is clearly zero; hence, assume that $p \neq i$. Since i is contained in $[p, q] = [p, i-2] \cup [i-1, q]$, we have

$$A_qA_{q-1} \dots A_{p+1}A_pA_i = (A_q \dots A_{i+1}A_iA_{i-1})A_i(A_{i-2}A_{i-3} \dots A_{p+1}A_p);$$

hence, this element contains a word $A_iA_{i-1}A_i$ and it is therefore zero in $\mathbb{A}_{m,n}$. The proof of the second statement is similar and is hence omitted. \square

Remark 2. Note that the proof of Lemma 4 uses only the relations (1)–(3) and the 321-avoiding condition.

Lemma 5. For $J, J' \in \mathbb{Z}/n\mathbb{Z}$ such that $J \cap J'$ is nonempty, $A_{u_J}A_{d_{J'}}$ is zero in $\mathbb{A}_{m,n}$. Similarly, $A_{d_J}A_{u_{J'}}$ is also zero in $\mathbb{A}_{m,n}$.

Proof. Assume that $\ell(u_J) + \ell(d_{J'}) = \ell(u_Jd_{J'})$ and $u_Jd_{J'}$ is 321-avoiding; otherwise, $A_{u_J}A_{d_{J'}}$ is zero in $\mathbb{A}_{m,n}$. Choose an element i in $J \cap J'$. Since s_i appears twice in any reduced word of $u_Jd_{J'}$ and $u_Jd_{J'}$ is 321-avoiding, both s_{i-1} and s_{i+1} appear between two s_i 's in the reduced word. However, $i-1$ is not in J since s_{i-1} cannot appear to the right of s_i because u_J is cyclically increasing. Similarly, $i-1$ is not in J' since s_{i-1} cannot appear to the left of s_i because $d_{J'}$ is cyclically decreasing. Therefore, it creates a contradiction and the proof is completed. One can prove the second statement similarly. \square

An element $w \in \tilde{S}_n$ is called an n -connected ribbon if w is of the form $u_Jc d_J$ for some $J \subsetneq \mathbb{Z}/n\mathbb{Z}$. For such a w , if A_w is nonzero in $\mathbb{A}_{m,n}$, then $|J^c| \leq m$ and $|J| \leq n-m$ by Lemma 3; hence, the inequalities become equalities.

In the rest of the section, we prove lemmas about n -connected ribbons to prove Proposition 10. Then, we prove Theorem 1.

Corollary 3. *If w_1, w_2 are n -connected ribbons such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ and $A_{w_1 w_2}$ is nonzero in $\mathbb{A}_{m,n}$, then $w_1 = w_2$.*

Proof. Since w_i is an n -connected ribbon, $w_i = u_{J_i^c} d_{J_i}$ for some $J_i \subsetneq \mathbb{Z}/n\mathbb{Z}$. Since $u_{J_1^c} d_{J_1} u_{J_2^c} d_{J_2}$ is reduced, we have $\ell(d_{J_1} u_{J_2^c}) = \ell(d_{J_1}) + \ell(u_{J_2^c})$. Note that $|J_1| = |J_2| = n - m$ and $|J_2^c| = m$ because $A_{w_1} A_{w_2}$ is nonzero in $\mathbb{A}_{m,n}$. Therefore, by Lemma 5, J_1 and J_2^c do not intersect and we get $J_1 = J_2$ and $w_1 = w_2$. \square

Let r_m be $u_{[-m,-1]} d_{[0,n-m-1]}$, which is

$$s_{-m} s_{-m+1} \cdots s_{-1} s_{n-m-1} s_{n-m-2} \cdots s_1 s_0.$$

We show that r_m is the unique 0-Grassmannian n -connected ribbon such that A_{r_m} is nonzero in $\mathbb{A}_{m,n}$. In fact, we show a stronger statement.

Theorem 6. *As an element in $\mathbb{A}_{m,n}$, we have*

$$s_{r_m}^{(m,n)} = \sum_v A_v$$

where v runs over all n -connected ribbons $u_{J^c} d_J$ for some $J \subsetneq \mathbb{Z}/n\mathbb{Z}$ with $|J| = n - m$. Therefore, r_m is the unique 0-Grassmannian n -connected ribbon such that A_{r_m} is nonzero in $\mathbb{A}_{m,n}$.

Proof. First, we show that $s_{r_m}^{(m,n)} = \bar{\mathbf{e}}_m \bar{\mathbf{h}}_{n-m}$. It is sufficient to show that for a 0-Grassmannian element v , A_v appears in $\bar{\mathbf{e}}_m \bar{\mathbf{h}}_{n-m}$ if and only if $v = r_m$. Let v be a 0-Grassmannian element $u_{J'} d_{J'}$ for some $J, J' \subsetneq \mathbb{Z}/n\mathbb{Z}$ with $|J| = n - m, |J'| = m$, appearing in $\bar{\mathbf{e}}_m \bar{\mathbf{h}}_{n-m}$. Since v is 0-Grassmannian, $d_{J'}$ has to be $s_{n-m-1} \cdots s_1 s_0$. By Lemma 5, J' has to be J^c .

Every (nonzero) element A_x appearing in $\bar{\mathbf{e}}_m \bar{\mathbf{h}}_{n-m}$ is of the form $u_{K_1} d_{K_2}$ for some $K_1, K_2 \in \mathbb{Z}/n\mathbb{Z}$ with $|K_1| = m$ and $|K_2| = n - m$. Since A_x is not zero in $\mathbb{A}_{m,n}$, $K_1 = K_2^c$ by Lemma 5 and the proof is completed. \square

Lemma 6. *For $v \in \tilde{S}_n$ such that vr_m is 321-avoiding and $\ell(vr_m) = \ell(v) + \ell(r_m)$, v is 0-Grassmannian.*

Proof. If $v = v' s_i$ for some nonzero i , then it suffices to show that $\ell(s_i r_m) < \ell(r_m)$. Assume that we have $\ell(s_i r_m) > \ell(r_m)$ for some nonzero i . Since $r_m = u_{[-m,-1]} d_{[0,n-m-1]}$, i is not $-m$. If i is in $[-m+1, -1]$, then $s_i u_{[-m,-1]}$ is not 321-avoiding by Lemma 4. The last case is when i is in $[1, n-m-1]$. If i is $n-m-1$, then $s_i u_{[-m,-1]} s_{n-m-1}$ is $s_{n-m-1} s_{n-m} s_{n-m-1} u_{[-m+1,-1]}$; hence, it is not 321-avoiding. If i is in $[1, n-m-2]$, then $s_i u_{[-m,-1]} d_{[0,n-m-1]}$ is $u_{[-m,-1]} s_i d_{[0,n-m-1]}$ and $s_i d_{[0,n-m-1]}$ is not 321-avoiding by Lemma 4, which is a contradiction. \square

Lemma 7. Assume that for $w \in A_{(n-m,m)}$, s_i occurs in a reduced word of w exactly once for all $i \in \mathbb{Z}/n\mathbb{Z}$. Then, w is a n -connected ribbon.

Proof. First, we write $w = vd_J$ for $v \in \tilde{S}_n$ and $J \subset \mathbb{Z}/n\mathbb{Z}$ with maximal J . Assume that we have an interval $[p, q] \subset \mathbb{Z}/n\mathbb{Z}$ such that the interval $[p, q]$ and J do not intersect. Choose a maximal interval $[p, q]$ such that $p-1, q+1 \in J$. Since s_i does not appear in d_J for $i \in [p, q]$, it occurs in a reduced word of v . Since s_{p-1} and s_{q+1} does not appear in a reduced word of v , one can write $v = v_1 v_2$ such that s_i occurs in a reduced word of v_2 if and only if $i \in [p, q]$. If v_2 is not a cyclically increasing element, there exists $i \in [p, q-1]$ such that s_{i+1} precedes s_i . Choose a minimal possible i ; then, both s_{i-1} and s_{i+1} precede s_i or $i = p$, such that one can write $v_2 = v_3 s_i$ with $\ell(v_2) = \ell(v_3) + 1$. Since i is in $[p, q-1]$, $s_i d_J$ is cyclically decreasing and it contradicts the definition of J and the assumption $i+1 \notin J$. By applying this technique to all possible $[p, q]$ such that $[p, q]$ does not intersect with J , we prove that v is a cyclically increasing element. By Lemma 5, v is equal to u_{J^c} and $w = u_{J^c} d_J$ since $\ell(w) = n$. Therefore, w is an n -connected ribbon. \square

Lemma 8. For $w \in \tilde{S}_n$, assume that for all $i \in \mathbb{Z}/n\mathbb{Z}$, the generator s_i occurs in some reduced word of w . Further, assume that A_w is nonzero in $\mathbb{A}_{m,n}$. Then, there exist unique $v, u \in \tilde{S}_n$ such that $w = vu$, $\ell(w) = \ell(v) + \ell(u)$, and for $i \in \mathbb{Z}/n\mathbb{Z}$, s_i occurs in a reduced word of u exactly once. Similarly, assuming the same condition on w , there exist unique $u', v' \in \tilde{S}_n$ such that $w = u'v'$, $\ell(w) = \ell(u') + \ell(v')$, and for all $i \in \mathbb{Z}/n\mathbb{Z}$, s_i occurs in a reduced word of u' exactly once.

Proof. It is sufficient to prove the first statement by the symmetry. First, we show the existence of u and v . Since A_w is nonzero in $\mathbb{A}_{m,n}$, then $\max_r(w) \leq n-m$ and $\max_c(w) \leq m$ by Lemma 3. Assume that $w = v_1 v_2$ for some $v_1, v_2 \in \tilde{S}_n$ satisfying $\ell(w) = \ell(v_1) + \ell(v_2)$ and the generator s_i occurs in a reduced word of v_2 at most once. Choose v_1, v_2 such that the length of v_2 is maximal.

If $\ell(v_2)$ is equal to n , then the lemma is proved; hence, assume that $\ell(v_2) < n$. Choose an interval $[p, q] \subset \mathbb{Z}/n\mathbb{Z}$ such that for all $i \in [p, q]$, s_i does not occur in all reduced words of v_2 . Choose a maximal $[p, q]$ such that s_{p-1} and s_{q+1} does appear in a reduced word of v_2 .

Consider a reduced word $s_{i_1} \cdots s_{i_\ell}$ of v_1 such that the maximum of the set $\{j \mid i_j \in [p, q]\}$ is the largest and denote the maximum by j . If i_a is equal to $p-1$ for some $a > j$, since $v_1 v_2$ is 321-avoiding and a reduced word of v_2 contains s_{p-1} , s_p should appear in $s_{i_a} \cdots s_{i_\ell}$. However, since $a > j$, it is a contradiction and i_a cannot be equal to $p-1$ for any $a > j$. By a similar argument, i_a cannot be equal to $q+1$ for any $a > j$.

Further, note that by the definition of j , i_a is not in $[p, q]$ for all $a > j$. Hence, i_a is in $[q+2, p-2]$ and s_{i_a} commutes with s_j . If j is not $\ell = \ell(v_1)$, then, since $s_{i_1} \cdots s_{i_\ell} = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} s_{i_j} s_{i_{j+2}} \cdots s_{i_\ell}$, it contradicts the definition of j . Therefore, we have $j = \ell$. Then, by setting $w_1 = v_1 s_{i_\ell}$ and $w_2 = s_{i_\ell} v_2$, we have $w = w_1 w_2$ with

$\ell(w) = \ell(w_1) + \ell(w_2)$ and the generator s_i occurs in a reduced word of w_2 at most once for all $i \in \mathbb{Z}/n\mathbb{Z}$. Since $\ell(w_2) = \ell(v_2) + 1$, it contradicts the assumption on v_2 .

To prove the uniqueness, we first write $w = vu = vu_Jcd_J$ for some $J \subset \mathbb{Z}/n\mathbb{Z}$. Since $w \in A_{(n-m,m)}$, $|J| = n - m$ and by Corollary 2, the set J is uniquely determined. Therefore, $u = u_Jcd_J$ is also uniquely determined. \square

Corollary 4. For $w \in A_{(n-m,m)}$, let d be the minimum of the number of times s_i occurs in a reduced word of w for $i \in \mathbb{Z}/n\mathbb{Z}$. Then, we have $w = w^{(0)}v^d$ for $w^{(0)} \in A_{(n-m,m)}$ with $\ell(w) = \ell(w^{(0)}) + nd$ such that s_i does not occur in a reduced word of $w^{(0)}$ for some i and v is a connected n -ribbon. We call the decomposition $w = w^{(0)}v^d$ a ribbon decomposition. Moreover, if w is 0-Grassmannian, then v is equal to r_m and $w^{(0)}$ is also 0-Grassmannian.

Proof. First, note that the number of s_i that appears in a reduced word of w for given i is independent of a reduced word, since w is 321-avoiding. By recursively applying Lemma 7 and 8, we have a decomposition $w = w^{(0)}v^d$ for some connected n -ribbon by Corollary 3. If w is 0-Grassmannian, then v is also 0-Grassmannian and hence equal to r_m . By Lemma 6, $w^{(0)}$ is also 0-Grassmannian. \square

Lemma 9. For an element w in $A_{(n-m,m)}^0$ and a ribbon decomposition $w = w^{(0)}v^d$, s_{n-m} does not appear in a reduced word of $w^{(0)}$. Therefore, d is the number of times s_{n-m} appears in a reduced word of w .

Proof. Assume that s_{n-m} appears in a reduced word of $w^{(0)}$. We first claim that either $\max_c(w^{(0)}) < m$ or $\max_r(w^{(0)}) < n - m$. If $\max_c(w^{(0)}) = m$, then $w^{(0)}$ can be written as w_1u_J , where $w_1 \in A_{(n-m,m)}$ and J is a proper subset of $\mathbb{Z}/n\mathbb{Z}$ satisfying $\ell(w^{(0)}) = \ell(w_1) + \ell(u_J)$. Since $w^{(0)}$ is 0-Grassmannian, J must be an interval $[-m+1, 0]$. On the other hand, if $\max_r(w^{(0)}) = n - m$, $w^{(0)}$ can be written as $w_2d_{[0, n-m-1]}$ for $w_2 \in A_{(n-m,m)}$ satisfying $\ell(w^{(0)}) = \ell(w_2) + n - m$. Since for any $i \in \mathbb{Z}/n\mathbb{Z}$ except $n - m$, i appears either in $u_{[-m+1, 0]}$ or $d_{[0, n-m-1]}$, it creates a contradiction. Therefore, without loss of generality, we may assume that $\max_c(w^{(0)}) < m$.

Let $w = d_{j_p} \cdots d_{j_1}$ be the unique maximal decomposition into cyclically decreasing elements. Since w is 0-Grassmannian, J_j must be $[-j+1, \lambda_j - j]$ for any $1 \leq j \leq p$ for some integer sequences $n - m \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0$, where a proof is given at the end of Section 3 (except that now λ_1 is less than equal to $n - m$). Since $\max_c(w^{(0)}) < m$, it follows that $p < m$ and any interval $[-j+1, \lambda_j - j]$ does not contain $n - m$ since $-m < -p+1 \leq -j+1$ and $\lambda_j - j \leq \lambda_1 - 1 \leq n - m - 1$. This creates a contradiction to our assumption. \square

Theorem 7. There is a bijection ϕ between the set $A_{(n-m,m)}^0$ and the set of the cylindric shapes $\lambda/d/\emptyset$ for $d \geq 0$ and $\lambda \in P_{mn}$. Moreover, for $w, v \in A_{(n-m,m)}^0$, $D_{\phi(v)}$ contains

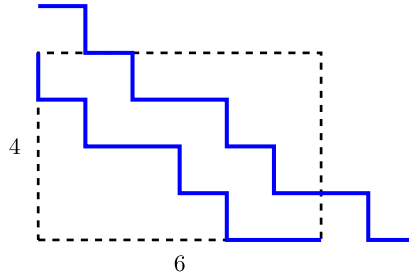


Fig. 2. $\lambda = (4, 3, 1)$ and $\lambda[1]$.

$D_{\phi(w)}$ if and only if $w < v$ where $<$ is the weak order in \tilde{S}_n . Furthermore, the affine Schur function F_w is the same as the cylindric Schur functions $s_{\phi(w)}$.

Proof. For an element $w \in A_{(n-m,m)}^0$, we define $\phi(w)$ by $A_w \cdot \emptyset/\emptyset/\emptyset$ and we will show that ϕ is a bijection. Assume that we have a ribbon decomposition $w = w^{(0)}r_m^d$ for some $w^{(0)} \in A_{(n-m,m)}^0$. When $d = 0$, the bijectivity of ϕ comes from the well-known bijection ϕ' between P_{mn} and the subset of S_n with the unique descent at m . Indeed, ϕ is $\psi \circ \phi'$, where $\psi : S_n \rightarrow \tilde{S}_n$ is defined by sending s_i to s_{i-m} . Note that the image of ψ does not contain an element whose reduced word contains s_{n-m} .

If $d > 0$, we have $w = w^{(0)}r_m^d$ for some $w^{(0)} \in A_{(n-m,m)}^0$ such that s_{n-m} is not used in any reduced word of $w^{(0)}$ and $\ell(w) = \ell(w^{(0)}) + nd$. Note that the decomposition $w = w^{(0)}r_m^d$ is unique. Then, $\phi(w)$ is $\phi(w^{(0)})/d/\emptyset$ and ϕ becomes the bijection between the set $A_{(n-m,m)}^0$ and the set of cylindric shapes $\lambda/d/\emptyset$ for $\lambda \in P_{mn}$ and $d \geq 0$.

Note that by the construction of the bijection ϕ , $s_i w = v$ for some $w < v$ if and only if $A_i \cdot \phi(w) = A_i A_w \cdot (\emptyset/\emptyset/\emptyset) = \phi(v)$. Note that $A_i \cdot \phi(w) = \phi(v)$ if and only if $D_{\phi(v)}$ contains $D_{\phi(w)}$ and the difference $D_{\phi(v)}/D_{\phi(w)}$ consists of a box at the i -th diagonal. This proves the second statement.

For the last statement, recall that for $v \in \tilde{S}_n$ such that A_v is nonzero and cylindric diagrams D, D' such that $A_v \cdot D = D'$, D'/D is a horizontal strip if and only if v is cyclically decreasing. Therefore, a cyclically decreasing decomposition of w corresponds to a semi-standard tableau on a cylindric diagram $D_{\phi(w)}$ with the same weight; hence, F_w is equal to the cylindric Schur functions $s_{\phi(w)}$. \square

Example 2. When $n = 10, m = 4$ and $\lambda = (4, 3, 1)$, the 0-Grassmannian element w corresponding to λ is $(s_4)(s_5 s_0 s_1)(s_3 s_2 s_1 s_0)$. Since λ is contained in 4×6 , there is no s_6 in any reduced word of w . The 0-Grassmannian element corresponding to $\lambda[1]$, depicted in Fig. 2, is

$$wr_m = (s_6 s_5 s_3 s_1 s_0 s_8)(s_7 s_9 s_2 s_4)w.$$

Corollary 5. For a cylindric shape $\lambda/d/\mu$ of type (m, n) , a cylindric skew Schur function $s_{\lambda/d/\mu}$ is the same as F_w , where $w = \phi^{-1}(\lambda/d/\emptyset) (\phi^{-1}(\mu/\emptyset))^{-1} \in A_{(n-m,m)}$.

Proof. Since $\lambda/d/\mu$ is a cylindric shape of type (m, n) , $D_{\lambda[d]}$ contains $D_{\mu[0]}$. By Theorem 7, there is $w \in \tilde{S}_n$ such that $A_w \cdot (\mu/0/\emptyset) = \lambda/d/\emptyset$ and we have $w\phi^{-1}(\mu/0/\emptyset) = \phi^{-1}(\lambda/d/\emptyset)$ such that $\ell(w) = |D_{\lambda[d]}/D_{\mu}| = |D_{\lambda/d/\mu}|$. Since $A_w \cdot (\mu/0/\emptyset)$ is not zero, w is in $A_{(n-m, m)}$. Moreover, since $A_w \cdot (\mu/0/\emptyset) = \lambda/d/\emptyset$, we have $w = \phi^{-1}(\lambda/d/\mu) (\phi^{-1}(\mu/0/\emptyset))^{-1}$.

Recall that for $v \in A_{(n-m, m)}$ and cylindric diagrams D, D' such that $A_v \cdot D = D'$, D'/D is a horizontal strip if and only if v is cyclically decreasing. Therefore, a cyclically decreasing decomposition of w corresponds to a semi-standard tableau on a cylindric diagram $D_{\lambda/d/\mu}$ with the same weight; hence, F_w is equal to the cylindric skew Schur functions $s_{\lambda/d/\mu}$. \square

Proposition 10. For $w \in A_{(n-m, m)}^0$ and for a ribbon decomposition $w = w^{(0)}r_m^d$, we have

$$\mathbf{s}_w^{(m, n)} = \mathbf{s}_{w^{(0)}}^{(m, n)} (\mathbf{s}_{r_m}^{(m, n)})^d$$

in $\mathbb{B}_{m, n}$.

Proof. We use induction on d . Note that for $v \in A_{(n-m, m)}^0$, A_v is the unique 0-Grassmannian element in $\mathbf{s}_v^{(m, n)}$, so it is sufficient to show that A_v appears in the expansion of $\mathbf{s}_{w^{(0)}}^{(m, n)} (\mathbf{s}_{r_m}^{(m, n)})^d$ for a 0-Grassmannian element v if and only if $v = w$. Since A_v appears in $(\mathbf{s}_{w^{(0)}}^{(m, n)} (\mathbf{s}_{r_m}^{(m, n)})^{d-1}) \mathbf{s}_{r_m}^{(m, n)}$, there exists $v_1 \in \tilde{S}_n$ such that $v = v_1 v_2$ with $\ell(v) = \ell(v_1) + \ell(v_2)$ and A_{v_1} is in the expansion of $\mathbf{s}_{w^{(0)}}^{(m, n)} (\mathbf{s}_{r_m}^{(m, n)})^{d-1}$ and A_{v_2} appears in $\mathbf{s}_{r_m}^{(m, n)}$. Since v_2 is also 0-Grassmannian, v_2 must be r_m . By Lemma 6, v_1 is 0-Grassmannian. By induction, we have $v_1 = w^{(0)}r_m^{d-1}$ and $v = w^{(0)}r_m^d = w$. \square

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. By Corollary 5, $s_{\lambda/d/\mu}$ is the same as F_w , where $w = \phi^{-1}(\lambda/d/\emptyset) \times (\phi^{-1}(\mu/0/\emptyset))^{-1}$ is an element in $A_{(n-m, m)}$. Let α denote $\phi^{-1}(\lambda/d/\emptyset)$ and let v denote $\phi^{-1}(\mu/0/\emptyset)$ so that we have $wv = \alpha$ satisfying $\ell(w) + \ell(v) = \ell(\alpha)$. Note that both α and v are in $A_{(n-m, m)}^0$. By Theorem 3, we have the expansion

$$F_w = \sum_{u \in \tilde{S}_n} c_u^w F_u.$$

We first show that if c_u^w is nonzero, then u is in $A_{(n-m, m)}^0$. By Theorem 5, c_u^w is equal to $d_{u, v}^\alpha$ by comparing the coefficient of A_w on both sides of an equality $\mathbf{s}_u^{(k)} \mathbf{s}_v^{(k)} = \sum_{\alpha \in \tilde{S}_n^0} d_{u, v}^\alpha \mathbf{s}_\alpha^{(k)}$. By comparing the coefficient of A_w on both sides of an equality $\mathbf{s}_v^{(k)} \mathbf{s}_u^{(k)} = \sum_{\alpha \in \tilde{S}_n^0} d_{u, v}^\alpha \mathbf{s}_\alpha^{(k)}$, there exists $\beta \in \tilde{S}_n$ such that A_β appears in $\mathbf{s}_v^{(k)}$ and $\alpha = \beta u$ satisfying $\ell(\alpha) = \ell(\beta) + \ell(u)$ and $c_u^w = d_{u, v}^\alpha$. Note that the existence of such an element β is guaranteed if c_u^w is nonzero. Since α is in $A_{(n-m, m)}^0$, so is u .

Now, it is sufficient to show that $c_{\nu/e/\emptyset}^{\lambda/d/\mu}$ is the same as $c_{\nu/e-1/\emptyset}^{\lambda/d-1/\mu}$, where $\lambda/d-1/\mu, \nu/e-1/\emptyset$ are cylindric shapes of type (m, n) and $e \geq 1$. Let u be $\phi^{-1}(\nu/e/\emptyset)$. Then, c_u^w is equal to $c_{\nu/e/\emptyset}^{\lambda/d/\mu}$. Therefore, we can write

$$s_{\lambda/d/\mu} = F_w = \sum_{u \in A_{(n-m, m)}^0} c_u^w F_u = \sum_{\nu/e/\emptyset} c_u^w s_{\nu/e/\emptyset}$$

by Theorem 7 and Corollary 5.

The condition $e \geq 1$ implies that s_i occurs in a reduced word of $u = \phi^{-1}(\nu/e/\emptyset)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. By Lemma 8 and Theorem 6, $u = u' r_m$ for some $u' \in A_{(n-m, m)}$. By Lemma 6, u is in $A_{(n-m, m)}^0$. Therefore, by Proposition 10, we have

$$\mathbf{s}_u^{(k)} = \mathbf{s}_{u'}^{(k)} \mathbf{s}_{r_m}^{(k)}.$$

Indeed, if $u = u_0 r_m^f$ is a ribbon decomposition, a ribbon decomposition of u' is $u_0 r_m^{f-1}$. Consider a coefficient of A_w in $\mathbf{s}_{r_m}^{(k)} \mathbf{s}_{u'}^{(k)}$, which is c_u^w . By Theorem 6, the coefficient of A_w in $\mathbf{s}_{r_m}^{(k)} \mathbf{s}_{u'}^{(k)}$ is $\sum_{w=ab} c_{u'}^b$, where a is an n -connected ribbon and $w = ab$ with $\ell(w) = \ell(a) + \ell(b)$. By Lemma 8, there is a unique a , and we have $c_u^w = c_{u'}^b$. By construction, we have $b = \phi^{-1}(\lambda/d-1/\emptyset)(\phi^{-1}(\mu/0/\emptyset))^{-1}$ and $u' = \phi^{-1}(\nu/e-1/\emptyset)$. Therefore, we show that

$$c_{\nu/e/\emptyset}^{\lambda/d/\mu} = c_u^w = c_{u'}^b = c_{\nu/e-1/\emptyset}^{\lambda/d-1/\mu}.$$

If $e = 0$, let us consider the following:

$$s_{\lambda/d/\mu}(x_1, \dots, x_m) = \sum c_{\nu/e/\emptyset}^{\lambda/d/\mu} s_{\nu/e/\emptyset}(x_1, \dots, x_m).$$

Postnikov [15] showed that $s_{\nu/e/\emptyset}(x_1, \dots, x_m)$ is nonzero if and only if $\nu/e/\emptyset$ is toric. The condition is equivalent to $e = 0$. Then, $s_{\nu/e/\emptyset}(x_1, \dots, x_m)$ is equal to $s_\nu(x_1, \dots, x_m)$, and by (1), $c_{\nu/0/\emptyset}^{\lambda/d/\mu}$ is equal to $C_{\mu, \nu}^{\lambda, d}$. \square

Note that we also proved the following theorem:

Theorem 8. *If $w \in \tilde{S}_n$ is 321-avoiding, c_u^w is 0 unless u is a 321-avoiding 0-Grassmannian element. Moreover, if w is 321-avoiding and satisfies $\max_c(w) \leq m$ and $\max_r(w) \leq n - m$, then $c_u^w = 0$ unless u is in $A_{(n-m, m)}^0$.*

We can use Theorem 8 to characterize the basis of $\mathbb{B}_{m, n}$. Via the projection $\mathbb{B} \rightarrow \mathbb{B}_{m, n}$, $\mathbf{s}_u^{(m, n)}$ is zero when $u \notin A_{(n-m, m)}^0$ since for every term A_w appearing in $\mathbf{s}_u^{(k)}$, we have $w \notin A_{(n-m, m)}$. On the other hand, $\mathbf{s}_u^{(m, n)}$ is not zero when u is in $A_{(n-m, m)}^0$ because $\mathbf{s}_u^{(m, n)}$ have the unique 0-Grassmannian term A_u . Therefore, the set $\{\mathbf{s}_u^{(m, n)} \mid u \in A_{(n-m, m)}^0\}$ forms a basis of $\mathbb{B}_{m, n}$.

5. Effective algorithm for the computation

In this section, we provide an effective algorithm to compute the expansion of cylindric skew Schur functions in terms of cylindric Schur functions, and the expansion of affine Stanley symmetric functions in terms of affine Schur functions. The strategy is to use the following dual Pieri rule of the affine Stanley symmetric functions.

Theorem 9. [8] *For any $m \leq n$, we have*

$$h_m^\perp(F_w) = \sum_{w=uv} F_v = \sum_{w=vu} F_v,$$

where the summations run over all cyclically decreasing elements u of length m satisfying $\ell(w) = \ell(u) + \ell(v)$.

To describe the algorithm, we define a total order on a set of partitions. For two partitions μ, ν , we denote $\mu < \nu$ when either $|\mu| < |\nu|$ or $|\mu| = |\nu|$ and μ is less than or equal to ν with respect to the reverse colexicographic ordering, i.e., there exists $a > 0$ such that $\mu_a > \nu_a$ and $\mu_b = \nu_b$ for all $b > a$. Here, for a partition λ , we set $\lambda_b = 0$ if $b > \ell(\lambda)$.

For $w \in \tilde{S}_n$ and $v \in \tilde{S}_n$ such that wv is p -Grassmannian for some $p \in \mathbb{Z}/n\mathbb{Z}$, consider a maximal cyclically decreasing decomposition $d_{J_\ell} \cdots d_{J_1}$ of v , and we denote the partition $(|J_1|, \dots, |J_\ell|)$ by $\lambda(v)$. We will show that F_w can be written as a summation $\sum_u \pm F_u$ such that for any $u \in \tilde{S}_n$, there exists $v_u \in \tilde{S}_n$ such that uv_u is p -Grassmannian and $\lambda(v_u) < \lambda(v)$. Then, by repeating the process on each F_u when $\lambda(v_u)$ is not an empty partition, F_w is equal to a linear combination of F_u with $\lambda(u) = \emptyset$, which means that all u is p -Grassmannian and the proof is completed. The algorithm preserves the 321-avoiding condition, i.e., if w is in $A_{(n-m,m)}$ for some m , then $u \in A_{(n-m,m)}$ for all u appearing in the summation.

The first step is to show that for each $w \in \tilde{S}_n$, we find a small $v \in \tilde{S}_n$ such that wv is p -Grassmannian for some $p \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 11. *For any affine permutation w , there exists v such that $\ell(wv) = \ell(w) + \ell(v)$, $\ell(v) \leq (k-1) \cdot 1 + (k-2) \cdot 2 + \cdots + 1 \cdot (k-1)$, and wv is p -Grassmannian for some $p \in \mathbb{Z}/n\mathbb{Z}$.*

Remark 3. When we omit the condition $\ell(v) \leq (k-1) \cdot 1 + (k-2) \cdot 2 + \cdots + 1 \cdot (k-1)$, the theorem follows from [13]. However, in their proof, $\ell(v)$ is always greater than or equal to $(k-1) \cdot 1 + (k-2) \cdot 2 + \cdots + 1 \cdot (k-1)$.

When w is in $A_{(n-m,m)}$ for some m , the upper bound on $\ell(v)$ is much smaller.

Lemma 12. *If w is in $A_{(n-m,m)}$, there exists $v \in A_{(n-m,m)}^0$ such that $\ell(wv) = \ell(w) + \ell(v)$, $\ell(v) \leq (n-m)(m-1)/2$, and wv is p -Grassmannian for some $p \in \mathbb{Z}/n\mathbb{Z}$.*

We postpone the proof of two lemmas to the end of the section and first describe the algorithm assuming the lemmas.

For an affine permutation w , assume that we have $v \in \tilde{S}_n$ such that wv is p -Grassmannian for some p . First, since F_w is equal to $F_{f(w)}$, where $f : \tilde{S}_n \rightarrow \tilde{S}_n$ is the map sending s_i to s_{i+1} , we can assume that p is equal to zero. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be $\lambda(v)$. Since v is 0-Grassmannian, v is uniquely determined by λ . Let v' be the 0-Grassmannian element corresponding to $(\lambda_1, \dots, \lambda_{\ell-1})$, so that we have

$$v = d_{[-\ell+1, \lambda_\ell - \ell]} v'$$

Now, we apply Theorem 9 to $w' = wd_{[-\ell+1, \lambda_\ell - \ell]}$ and $m = \lambda_\ell$. Let $B_-(w')$ be the set of $w'u_J$ for a subset $J \subset \mathbb{Z}/n\mathbb{Z}$ with $|J| = \lambda_\ell$, $\ell(w'u_J) = \ell(w') - |J|$, and $J \neq [-\ell+1, \lambda_\ell - \ell]$ and $B_+(w')$ be the set of $u_J w'$ for a subset $J \subset \mathbb{Z}/n\mathbb{Z}$ with $|J| = \lambda_\ell$ and $\ell(u_J w') = \ell(w') - |J|$. Then, we have

$$F_w = \sum_{u \in B_+(w')} F_u - \sum_{u \in B_-(w')} F_u.$$

Here, we use the identity $(d_J)^{-1} = u_J$.

Note that for $u = u_J w' \in B_+(w')$, uv' is 0-Grassmannian, since $wv = w'v' = d_J u v'$ is 0-Grassmannian and $\ell(d_J u v') = |J| + \ell(u) + \ell(v')$. Clearly, we have $\lambda(v) > \lambda(v')$. For $u = w'u_J \in B_-(w')$, we know that $wv = w'v' = u d_J v'$ and $d_J v'$ are 0-Grassmannian. For $J \subset \mathbb{Z}/n\mathbb{Z}$ with $|J| = \lambda_\ell$, we have $\lambda(d_J v') \leq \lambda(v)$ by the definition of v' and the equality holds for the unique J , which is $[-\ell+1, \lambda_\ell - \ell]$, and in this case, $d_J v' = v$. Therefore, we have $\lambda(d_J v') < \lambda(v)$ for other J , and we obtain an algorithm to compute the expansion of the affine Stanley symmetric function in terms of affine Schur functions by repeating the procedure.

When w is in $A_{(n-m, m)}$ for some m , we can find $v \in A_{(n-m, m)}$ such that wv is p -Grassmannian by Lemma 12. Since every element in $B_-(w')$ and $B_+(w')$ is of the form $u_J w'$ or $w'u_J$ with length $\ell(w') - |J|$, these elements are also in $A_{(n-m, m)}$, and by repeating the procedure, we get the expansion of F_w in terms of F_u , where $u \in A_{(n-m, m)}$ is 0-Grassmannian. By Theorem 7, F_u is equal to the cylindric Schur function $s_{\phi(u)}$ and the proof is completed. See Example 2 for an example of the algorithm.

Now, we prove Lemma 11 and Lemma 12.

Proof of Lemma 11. For $w \in \tilde{S}_n$ and $i \in \mathbb{Z}$, let $c_i(w)$ be the number of $j < i$ such that $w(j) > w(i)$. Note that w is p -Grassmannian if and only if $w(p+1) < w(p+2) < \dots < w(p+n)$. This condition is equivalent to $c_{p+1}(w) \geq c_{p+2}(w) \geq \dots \geq c_{p+n}(w)$, and in this case, $c_{p+n}(w) = 0$ and a partition $(c_{p+1}(w), c_{p+2}(w), \dots, c_{p+n}(w))$ is the same as the transpose of $\lambda(w)$. When w is a general element in \tilde{S}_n , we always have $c_j = c_{j+n}$ for any $j \in \mathbb{Z}$, and one can show that there exists j such that $c_j = 0$. See [2] for example.

For given $w \in \tilde{S}_n$, we inductively construct $v_i \in \tilde{S}_n$ for $i = 1, \dots, k-2$ such that $w_i = v_1 v_2 \dots v_i$ satisfy $\ell(w w_i) = \ell(w) + \sum_{j=1}^i \ell(v_j)$, $c_{q_i+1}(w w_i) \geq c_{q_i+2}(w w_i) \geq \dots \geq$

$c_{q_i+i+1}(ww_i)$ for some $q_i \in \mathbb{Z}$, and $c_{q_i+j}(ww_i)$ is less than or equal to $c_{q_i+i+1}(ww_i)$ for $i+1 < j \leq n$. We also require that $\ell(v_i) \leq i(k-i)$, so that ww_{k-1} is q_{k-1} -Grassmannian and $\ell(w_{k-1}) \leq \sum_{i=1}^{k-1} i(k-i)$, which proves the lemma.

For $w \in \tilde{S}_n$, choose an integer i_1 with maximal $c_{i_1}(w)$, and choose $i_1 < j_1 < i_1 + n$ such that $c_{j_1}(w)$ is the largest number in a set $\{c_p(w) \mid i_1 < p < i_1 + n\}$. If there are multiple choices, we just choose one of them. Define v_1 to be $s_{i_1}s_{i_1+1} \dots s_{j_1-2} = d_{[i_1, j_1-2]}$ if $j_1 > i_1 + 1$, and the identity if $j_1 = i_1 + 1$. One can show that $c_p(wv_1) = c_p(w)$ if p is not in $[i_1, j_1 - 1]$ modulo n , $c_p(wv_1) = c_{p-1}(w)$ for p in $[i_1, j_1 - 2]$, and $c_{j_1-1}(wv_1) = c_{i_1}(w) + j_1 - i_1 - 1$. Since c_{i_1} is the largest number among c_j 's, $\ell(wv_1) = \ell(w) + \ell(v_1)$. Therefore, we take $q_1 = j_1 - 2$ so that $c_{q_1+1}(wv_1) > c_{q_1+2}(wv_2)$ and other c_j 's are less than or equal to $c_{q_1+2}(wv_2)$. Note that $\ell(v_1)$ is at most $k - 1$.

We will construct v_{i+1} similarly for $i < k - 1$, assuming that we are given v_1, \dots, v_{i-1} . Note that there exists $q_i \in \mathbb{Z}$ such that $c_{q_i+1}(ww_i) \geq c_{q_i+2}(ww_i) \geq \dots \geq c_{q_i+i+1}(ww_i)$. Let $c_j(ww_i)$ be the largest number in a set $\{c_p(ww_i) \mid q_i + i + 1 < p \leq q_i + n\}$. If j is $q_i + i + 2$, then we can take v_{i+1} to be the identity and the proof is completed. If $j > q_i + i + 2$, then we take

$$v_{i+1} = d_{[q_i+i+1, j-2]} d_{[q_i+i, j-3]} \cdots d_{[q_i+1, j-i-2]}.$$

Since $c_j(ww_i)$ is less than or equal to $c_{q_i+1}(ww_i), \dots, c_{q_i+i+1}(ww_i)$, $\ell(ww_i v_{i+1}) = \ell(ww_i) + \ell(v_{i+1})$. One can check that $c_p(ww_{i+1}) = c_p(ww_i)$ if p is not in $[q_i+1, j-1]$ modulo n , $c_p(wv_1) = c_{p-i-1}(w)$ for p in $[q_i+1, j-i-2]$, and $c_p(wv_1) = c_{p-(j-q_i-i-2)}(w) + (j-q_i-i-2)$ for p in $[j-i-1, j-1]$. Further, note that $\ell(v_{i+1}) \leq (i+1)(j-q_i-i-2) \leq (i+1)(k-i-1)$. Therefore, one can take $q_{i+1} = j-i-1$ and the proof is completed. \square

Proof of Lemma 12. Although the lemma can be proved similarly, it is much better to visualize it using cylindric shapes. Lam [4] showed that for a 321-avoiding affine permutation w , there exists $q < n$ such that F_w is equal to $F_{\lambda[r]/\mu[s]}$, where $\lambda[r]/\mu[s]$ is a cylindric shape of type (q, n) for some q . If s_i appears at least once in $w \in A_{(n-m, m)}$ for all i , such q must be unique, which is the same as m . Indeed, by Lemma 8 and Lemma 7, w can be written as uv , where $\ell(w) = \ell(u) + \ell(v)$ and v is an n -connected ribbon $u_J c d_J$. Then, the n -connected ribbon determines q by $|J^c| = m$.

When s_i does not appear in $w \in \tilde{S}_n$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, one can still find a cylindric shape of type (m, n) , but we do not really need to consider this since F_w is equal to a single skew Schur function [1] and the coefficients in the Schur expansion of a skew Schur function are simply Littlewood-Richardson coefficients.

To prove the lemma, it is sufficient to find a bi-infinite increasing (m, n) -periodic sequence α such that $D_\alpha \subset D_\mu[s]$ and $\alpha_{i+1} = \alpha_{i+2} = \dots = \alpha_{i+m}$ for some i , and $|D_{\mu[s]/\alpha}| \leq (n-m)(m-1)/2$. If this happens, for $w \in A_{(n-m, m)}$ satisfying $A_w \cdot (\alpha/\alpha) = (\mu[s]/\alpha)$, w is $(\alpha_{i+m} - i - m)$ -Grassmannian, which proves the lemma. There are many ways to define α satisfying the above-mentioned conditions. Let β be $\mu[s]$. For any $a = 1, \dots, m$, we can define a (m, n) -periodic bi-infinite sequence $\alpha(a)$ to be $\alpha(a)_i = \beta_a$

for $a \leq i < a + m$. It is clear that $\alpha(a)_i = \alpha(a)_a = \beta_a < \beta_i$ for $a \leq i < a + m$ so that β/α is a cylindric shape. Moreover, $|D_{\beta/\alpha(a)}|$ is equal to

$$\sum_{i=a}^{a+m-1} (\beta_i - \beta_a) = \sum_{i=1}^m \beta_i - m \cdot \beta_a + (n-m)(a-1).$$

Therefore, the average of $|D_{\beta/\alpha(a)}|$ for $a = 1, \dots, m$ is

$$\frac{\sum_{a=1}^m (n-m)(a-1)}{m} = \frac{(n-m)(m-1)}{2};$$

hence, there exists a such that

$$|D_{\beta/\alpha(a)}| \leq \frac{(n-m)(m-1)}{2}$$

and the proof is completed. \square

Example 3. Let $n = 6, m = 3$, and $w = s_5 s_3 s_1 s_4 s_2 s_0$. We write $w = 531420$ for the sake of simplicity. One can take v to be 510 so that wv is 0-Grassmannian. Let $\lambda/r/\mu$ be $\phi(wv)$. Then, one can easily check that $\lambda = (2, 1), r = 1$ and μ is an empty partition. Since v corresponds to a partition $(2, 1)$ via the bijection between P_{mn} and a partition contained in $(n-m)^m$, w corresponds to a cylindric shape $(2, 1)/1/(2, 1)$.

In this setup, we apply Theorem 9 to $w' = ws_5$ and $m = 1$ to obtain

$$F_w = \sum_{u \in B_+(w')} F_u - \sum_{u \in B_-(w')} F_u,$$

where $B_+(w') = \{541052, 341052, 354052\}$ and $B_-(w') = \{354105\}$. See Fig. 3 for cylindric shapes corresponding to $w, B_+(w')$ and $B_-(w')$.

Note that except for w and 341052, other elements are missing s_i for some i . For example, a reduced word 541052 does not contain s_3 . Therefore, F_{541052} is equal to skew Schur function $s_{(3,3,2)/(2)}$, which is immediate from Fig. 3 and Corollary 5. Similarly, we have $F_{354052} = s_{(3,3,2)/(1,1)}$ and $F_{354105} = s_{(3,3,2,1)/(3)} = s_{(3,2,1)}$. Although we do not perform this step in the algorithm in general, it clearly reduces the computational complexity.

In summary, we have

$$\begin{aligned} F_w &= F_{541052} + F_{341052} + F_{354052} - F_{354105} \\ &= F_{341052} + s_{(3,3,2)/(1,1)} + s_{(3,3,2)/(2)} - s_{(3,2,1)}. \end{aligned}$$

Now, we can repeat the procedure for F_{341052} . We know that $(341052)(10)$ is 0-Grassmannian and $\lambda(s_1 s_0) < \lambda(s_5 s_1 s_0) = \lambda(v)$, where the order $<$ is defined in Section 5. By applying Theorem 9 to 34105210 and $m = 2$, we obtain

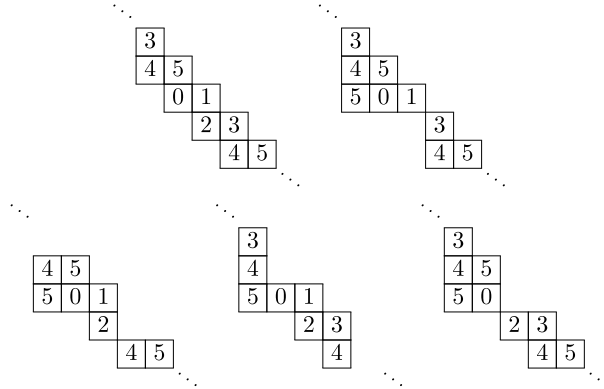


Fig. 3. Cylindric shapes for $w, 354105, 541052, 341052, 354052$.

$$F_{341052} = F_{345210} + F_{405210}$$

and both 345210 and 405210 are 0-Grassmannian. Therefore, we have

$$\begin{aligned} F_w &= F_{345210} + F_{405210} + s_{(3,3,2)/(1,1)} + s_{(3,3,2)/(2)} - s_{(3,2,1)} \\ &= F_{345210} + F_{405210} + s_{(2,2,2)} + s_{(3,3)} + s_{(3,2,1)} \\ &= F_{345210} + F_{405210} + F_{540510} + F_{105210} + F_{405210} \\ &= F_{345210} + 2F_{405210} + F_{540510} + F_{105210}. \end{aligned}$$

6. Concluding remark

It would be remarkable to find a manifestly positive formula for the expansion of cylindric skew Schur functions in terms of cylindric Schur functions, or the expansion of affine Stanley symmetric functions in terms of affine Schur functions. Although the latter problem is a challenge at this time because it includes well-known problems, such as 3-point Gromov-Witten invariants of the flag variety, the formal problem clearly has more combinatorics so that one may hope that some classical combinatorics on skew shapes, such as jeu de taquin, can be generalized. However, naive generalization of jeu de taquin is problematic. See Yoo’s thesis [16]. It would be interesting to develop combinatorics related to cylindric shapes to tackle such a problem.

In addition, it would be interesting to define equivariant versions of cylindric skew Schur functions. Although double affine Stanley symmetric functions have been defined by Lam and Shimozono [7], their definition does not seem to include much combinatorics. For example, there is no known tableau definition of double affine Stanley symmetric functions when w is 321-avoiding, whereas there is a tableau definition for double skew Schur functions [12].

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