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A character relationship between symmetric group and hyperoctahedral group

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ABSTRACT

We relate the character theory of the symmetric groups $\mathbb{S}_{2n}$ and $\mathbb{S}_{2n+1}$ with that of the hyperoctahedral group $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$, as part of the expectation that the character theory of reductive groups with diagram automorphism and their Weyl groups, is related to the character theory of the fixed subgroup of the diagram automorphism.

Contents

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1. Introduction

Let G be either the symmetric group $\mathbb{S}_{2n}$ or the symmetric group $\mathbb{S}_{2n+1}$, and H the hyperoctahedral group $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$, sitting naturally inside G ($\mathbb{B}_n \subset \mathbb{S}_{2n} \subset \mathbb{S}_{2n+1}$) as the centralizer of a fixed point free involution w_0 in $\mathbb{S}_{2n}$. In this paper, we take the symmetric group $\mathbb{S}_{2n}$ as acting on the set $\{\pm 1, \pm 2, \cdots, \pm n\}$ of cardinality $2n$, and the symmetric group $\mathbb{S}_{2n+1}$ as acting on the set $\{0, \pm 1, \pm 2, \cdots, \pm n\}$ of cardinality $2n+1$. We fix $w_0 = (1, -1)(2, -2) \cdots (n, -n)$. The paper proves a character relationship between the irreducible representations of the groups G and $\mathbb{B}_n$, closely related to character identities available between (finite dimensional, irreducible, algebraic) representations of the groups $\mathrm{GL}_{2n}(\mathbb{C})$, or $\mathrm{GL}_{2n+1}(\mathbb{C})$ which are self-dual, i.e., invariant under the involution $g \to {}^t g^{-1}$, with (finite dimensional, irreducible, algebraic) representations of the groups $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{Sp}_{2n}(\mathbb{C})$, for which we refer to [5]. These character identities are classically known as Shintani character identities, first observed between representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_{q^d})$, cf. [11], although for the case at hand, it would be much closer to consider irreducible unipotent representations of say $\mathrm{U}_{2n}(\mathbb{F}_q)$ corresponding to an irreducible representation of the Weyl group $\mathbb{B}_n$ and the associated basechanged representation of $\mathrm{GL}_{2n}(\mathbb{F}_{q^2})$ associated to a representation of the Weyl group $\mathbb{S}_{2n}$, and which are related by a basechange character identity, cf. [4], [1].

Observe that in all the cases above, the group G comes equipped with an automorphism, call it j, of finite order (such as conjugation by w_0 for symmetric groups), and H is either the subgroup of fixed points of this automorphism, or is closely related to the subgroup of fixed points (through a dual group construction as in [5]), and the theory of basechange relates character theory of irreducible representations of H to the character theory of irreducible representations of $G \rtimes \langle j \rangle$ which remain irreducible when restricted to G. (If the automorphism j of G is an inner automorphism of G — as is the case in this paper, representation theory of $G \rtimes \langle j \rangle$ is the same as representation theory of G.) This is what this paper achieves for G the symmetric group $\mathbb{S}_{2n}$ or the symmetric group $\mathbb{S}_{2n+1}$, and H the hyperoctahedral group $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$, in the theorem below.

Before proceeding further, we need to introduce some notation on partitions $\mathfrak{p} := \{p_1 \geq p_2 \geq \cdots \geq p_r\}$. First, recall that the conjugacy class of an element $w \in \mathbb{S}_n$ is encoded by a partition μ_w of size $|\mu_w| = n$. Indeed, μ_w is the collection of lengths of the cycles in the decomposition of w as a product of disjoint cycles.

For λ, a partition of n, a 2-*hook* in λ consists of two adjacent squares in the Young diagram of λ whose removal leaves the diagram of a partition of $(n-2)$. By repeatedly removing 2-hooks from λ, one obtains the 2-core of λ, denoted $c_2(\lambda)$. It is not obvious, but is well-known, that the 2-core of a partition is independent of the manner in which one removes the 2-hooks. The 2-core of a partition could be empty, denoted ϕ, else is the stair-case partition $\{k, k-1, \cdots, 1\}$. One can also define a pair of partitions (λ_0, λ_1), called the 2-quotient of λ, cf. section 2. The association:

$$\lambda \to (c_2(\lambda), \lambda_0, \lambda_1),$$

gives a bijective correspondence between λ and triples of partitions $(c_2(\lambda), \lambda_0, \lambda_1)$ with

$$|\lambda| = |c_2(\lambda)| + 2(|\lambda_0| + |\lambda_1|).$$

Next, recall that the irreducible complex representations of $\mathbb{S}_n$ are parameterized by partitions λ of n, to be denoted as π_λ, with character $\Theta_\lambda : \mathbb{S}_n \to \mathbb{C}$. Similarly, the irreducible complex representations of $\mathbb{B}_n$ are parameterized by pairs of partitions λ_0, λ_1 with $|\lambda_0| + |\lambda_1| = n$, to be denoted as $\pi_{(\lambda_0,\lambda_1)}$, with character $\Theta_{(\lambda_0,\lambda_1)} : \mathbb{B}_n \to \mathbb{C}$.

Let w be an element of $\mathbb{S}_n$ whose conjugacy class defines a partition μ_w of n. Let $\tilde{w}$ be any element of $\mathbb{S}_{2n}$ whose conjugacy class defines the partition $\mu_{\tilde{w}} = 2\mu_w$. In particular, $\tilde{w} \in \mathbb{S}_{2n}$ has no fixed points. We will use $\tilde{w}$ also for the element of $\mathbb{S}_{2n+1}$ under the natural embedding of $\mathbb{S}_{2n}$ inside $\mathbb{S}_{2n+1}$.

The following is the main theorem of this work proving a character relationship:

Theorem 1.1. *Let w be a conjugacy class in $\mathbb{S}_n$ treated as a conjugacy class in $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$ with $\tilde{w}$ the conjugacy classes in $\mathbb{S}_{2n}, \mathbb{S}_{2n+1}$ defined above. Let (λ_0, λ_1) be a pair of partitions with $|\lambda_0| + |\lambda_1| = n$, giving rise to a representation $\pi_{(\lambda_0,\lambda_1)}$ of $\mathbb{B}_n$ with character $\Theta_{(\lambda_0,\lambda_1)} : \mathbb{B}_n \to \mathbb{C}$. The pair (λ_0, λ_1) gives rise to a partition λ of $2n$ (resp. of $(2n+1)$) with empty 2-core (resp. with 2-core 1) and with 2-quotient (λ_0, λ_1), and defines an irreducible representation π_λ of $\mathbb{S}_{|\lambda|} = \mathbb{S}_{2n}, \mathbb{S}_{2n+1}$ with character $\Theta_\lambda : \mathbb{S}_{|\lambda|} \to \mathbb{C}$. Then we have for an $\epsilon(\lambda) = \pm 1$, the character identity:*

$$\Theta_{(\lambda_0,\lambda_1)}(w) = \epsilon(\lambda)\Theta_\lambda(\tilde{w}).$$

In particular, we have

$$\Theta_{(\lambda_0,\lambda_1)}(1) = \epsilon(\lambda)\Theta_\lambda(w_0),$$

determining $\epsilon(\lambda) = \pm 1$. Further, an irreducible representation π_λ of $\mathbb{S}_{|\lambda|}$ takes nonzero character value at w_0 if and only if the partition λ has empty 2-core if $|\lambda| = 2n$, and has 2-core 1 if $|\lambda| = 2n+1$.

After the completion of this work, Prof. G. Lusztig informed the authors that the special case of the theorem, viz. $\Theta_{(\lambda_0,\lambda_1)}(1) = \epsilon(\lambda)\Theta_\lambda(w_0)$, occurs on page 110 of his paper [8].

The above theorem was arrived at by computations done via the GAP software [3], and inspired by the hope that basechange character identities available in many situations

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involving reductive groups, also have an analogue for Weyl groups of these algebraic groups. We eventually found that it is a simple consequence of Theorem 4.6 due to Littlewood in [6] for even symmetric groups for which we provide a complete proof both for $\mathbb{S}_{2n}$ as well as $\mathbb{S}_{2n+1}$. It is surprising that the desired character identity relating symmetric groups and hyperoctahedral group is proved via Frobenius character formula for symmetric groups which is a form of Schur-Weyl duality by a "factorization" of the character formula (i.e., Schur polynomials) for irreducible representations of $\mathrm{GL}_{2n}(\mathbb{C})$ on special elements discovered by the second author in [9], which is already there in [6] written in 1940!

2. A lemma on 2-core and 2-quotient of partitions

A partition is called a 2-core partition if none of the hook lengths in its Young diagram is a multiple of 2. It is easy to see that a 2-core partition exists for a number n if and only if the number n is a triangular number:

$$n = \frac{d(d+1)}{2},$$

and in this case there is a unique 2-core partition which is the stair-case partition

$$\{d, d-1, d-2, \cdots, 2, 1\}.$$

Every partition $\mathfrak{p}$ has associated to it a 2-core partition, call it $c_2(\mathfrak{p})$, and a pair of partitions $(\mathfrak{p}_0, \mathfrak{p}_1)$ called the 2-quotient of $\mathfrak{p}$ such that

$$|\mathfrak{p}| = |c_2(\mathfrak{p})| + 2(|\mathfrak{p}_0| + |\mathfrak{p}_1|).$$

We already recalled the definition of the 2-core of a partition in the introduction. We now recall the definition of the 2-quotient partitions of any partition $\mathfrak{p}$ which is an ordered pair of partitions $(\mathfrak{p}_0, \mathfrak{p}_1)$. (More generally there is the notion of a p-core and p-quotient of a partition due to Littlewood, cf. [7], which arose there in his study of modular representations of the symmetric group.)

Define the β-set associated to a partition $\underline{\mathfrak{p}} := \{p_1 \geq p_2 \geq \cdots \geq p_r\}$ (we allow some of the p_i to be zero) to be the collection of (now strictly decreasing) numbers

$$\beta(\mathfrak{p}) = \{p_1 + (r-1) > p_2 + (r-2) > \cdots > p_r\}.$$

For $i = 0, 1$, let $\beta^i(\mathfrak{p})$ be those numbers in $\beta(\mathfrak{p}) = \{p_1 + (r-1), p_2 + (r-2), \cdots, p_r\}$ which are congruent to i mod 2. Define a new β-set $\beta_i(\mathfrak{p})$ by subtracting i from each of the numbers in $\beta^i(\mathfrak{p})$, and then dividing by 2. These β-sets $\beta_i(\mathfrak{p})$ are associated to a partition $\mathfrak{p}_i$, defining 2-quotient of $\mathfrak{p}$, an ordered pair of partitions $(\mathfrak{p}_0, \mathfrak{p}_1)$. It can be seen that to any partition $\mathfrak{p}$, its 2-core $c_2(\mathfrak{p})$, and 2-quotient $(\mathfrak{p}_0, \mathfrak{p}_1)$, determine the

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partition $\mathfrak{p}$ uniquely, and conversely, any triple of partitions $(c_2(\mathfrak{p}), \mathfrak{p}_0, \mathfrak{p}_1)$ is associated to a partition $\mathfrak{p}$.

We will now construct the 2-quotient of a partition in some detail to prove a lemma needed for our work later.

Let $\mathfrak{p} = \{p_1 \geq p_2 \geq \cdots \geq p_m\}$ be a partition of $|\mathfrak{p}|$ with associated β-set:

$$\beta(\mathfrak{p}) := \{p_1 + (m-1) > p_2 + (m-2) > \cdots > p_m\}.$$

Consider the even and odd parts of this β-set as:

$$\begin{aligned} \beta^0(\mathfrak{p}) :=\ & p_{i_1} + (m-i_1) > p_{i_2} + (m-i_2) > \cdots > p_{i_k} + (m-i_k). \\ \beta^1(\mathfrak{p}) :=\ & p_{j_1} + (m-j_1) > p_{j_2} + (m-j_2) > \cdots > p_{j_\ell} + (m-j_\ell). \end{aligned}$$

Divide the numbers appearing in $\beta^0(\mathfrak{p})$ and $\beta^1(\mathfrak{p}) - 1$ by 2 to obtain $\beta_0(\mathfrak{p}), \beta_1(\mathfrak{p})$:

$$\begin{aligned} \beta_0(\mathfrak{p}) :\ & \frac{p_{i_1} + (m-i_1)}{2} > \frac{p_{i_2} + (m-i_2)}{2} > \cdots > \frac{p_{i_k} + (m-i_k)}{2}. \\ \beta_1(\mathfrak{p}) :\ & \frac{p_{j_1} + (m-j_1-1)}{2} > \frac{p_{j_2} + (m-j_2-1)}{2} > \cdots > \frac{p_{j_\ell} + (m-j_\ell-1)}{2}. \end{aligned}$$

These sequences of strictly decreasing numbers are β-sets for the partitions:

$$\begin{aligned} \mathfrak{p}_0 :\ & \frac{p_{i_1} + (m-i_1)}{2} - (k-1) \geq \frac{p_{i_2} + (m-i_2)}{2} - (k-2) \geq \cdots \geq \frac{p_{i_k} + (m-i_k)}{2}. \\ \mathfrak{p}_1 :\ & \frac{p_{j_1} + (m-j_1-1)}{2} - (\ell-1) \geq \frac{p_{j_2} + (m-j_2-1)}{2} - (\ell-2) \geq \cdots \\ & \geq \frac{p_{j_\ell} + (m-j_\ell-1)}{2}. \end{aligned}$$

Using that $m = k + \ell$, and the fact that the set of integers comprising of $m - i_\alpha$ and $m - j_\beta$ is a permutation of the set of integers $0, 1, 2, \cdots, m-1$, it follows that:

$$\begin{aligned} |\mathfrak{p}| - 2(|\mathfrak{p}_0| + |\mathfrak{p}_1|) &= 2(k-1+k-2+\cdots+1+0) + 2(\ell-1+\ell-2+\cdots+1+0) \\ &\quad -[(m-i_1) + (m-i_2) + \cdots + (m-i_k)] \\ &\quad -[(m-j_1-1) + (m-j_2-1) + \cdots + (m-j_\ell-1)] \\ &= k(k-1) + \ell(\ell-1) - \frac{m(m-1)}{2} + \ell \\ &\overset{(\star)}{=} \frac{(k-\ell)(k-\ell-1)}{2}. \end{aligned}$$

The following lemma is an easy consequence of $(\star)$.

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Lemma 2.1. *A partition $\mathfrak{p}$ has empty 2-core (thus necessarily with $|\mathfrak{p}|$ even) if and only if $k = \ell$, or $k = \ell + 1$, i.e., in the β-set associated with $\mathfrak{p}$, either half of them are even and half of them are odd, or the even ones are one more than the odd ones.*

A partition $\mathfrak{p}$ has 2-core consisting of 1 (thus necessarily with $|\mathfrak{p}|$ odd) if and only if $k + 1 = \ell$, or $k = \ell + 2$, i.e., in the β-set associated with $\mathfrak{p}$, the odd numbers are one more in cardinality than the even numbers, or the even ones are two more than the odd ones.

In particular, if $|\mathfrak{p}|$ is even, and has an even number of parts, then $\mathfrak{p}$ has empty 2-core if and only if in the β-set associated with $\mathfrak{p}$, half of them are even and half of them are odd; whereas if $|\mathfrak{p}|$ is odd, and has an odd number of parts, then $\mathfrak{p}$ has 2-core 1 if and only if in the β-set associated with $\mathfrak{p}$, the odd ones are one more than the even ones.

3. Schur-Weyl theory and Frobenius character formula

Our main theorem about character values of representations of the symmetric group uses a theorem of Frobenius on characters of the symmetric group, as well as the Schur-Weyl duality which turns theorems about character theory of $\mathrm{GL}_n(\mathbb{C})$ to character theory for the symmetric group. In this section we recall the relationship between the two character theories: it is rather remarkable that they are the same, and give a proof of a form of the Frobenius theorem on characters of the symmetric group.

Let $\mathcal{R}_d$ be the representation ring of the symmetric group $\mathbb{S}_d$, treated here only as an abelian group. Let $\mathcal{R} = \sum_{d=0}^{\infty} \mathcal{R}_d$, together with the multiplication $\mathcal{R}_n \otimes \mathcal{R}_m \to \mathcal{R}_{n+m}$ which corresponds to induction of a representation $(V_1 \boxtimes V_2)$ of $\mathbb{S}_n \times \mathbb{S}_m$ to $\mathbb{S}_{n+m}$, turning $\mathcal{R}$ into a commutative and associative graded ring. It can be seen that as graded rings, $\mathcal{R} \cong \mathbb{Z}[H_1, \cdots, H_d, \cdots]$, the polynomial ring in infinitely many variables $H_i, i \geq 1$, where each H_i is given weight i, and corresponds to the trivial representation of $\mathbb{S}_i$.

On the other hand, let

$$\Lambda_n = \mathbb{Z}[X_1, X_2, \cdots, X_n]^{\mathbb{S}_n} = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k is the space of symmetric polynomials in $\mathbb{Z}[X_1, X_2, \cdots, X_n]$ of degree k.

Define,

$$\Lambda^k = \varprojlim \Lambda_n^k,$$

where the inverse limit is taken with respect to natural map of polynomial rings

$$\mathbb{Z}[X_1, X_2, \cdots, X_{n+1}]^{\mathbb{S}_{n+1}} \to \mathbb{Z}[X_1, X_2, \cdots, X_n]^{\mathbb{S}_n}$$

in which X_{n+1} is sent to the zero element.

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Finally, define the graded ring

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k.$$

The ring Λ, often by abuse of language (in which we too will indulge in) is called the ring of symmetric polynomials in infinitely many variables, comes equipped with surjective homomorphisms to Λ_n for all $n \geq 0$, which is in fact an isomorphism restricted to Λ^k onto Λ_n^k for $k \leq n$.

As typical elements of the ring Λ, also of paramount importance, note the following symmetric functions of degree m in infinitely many variables $X_1, X_2, X_3, \cdots$:

(1) $p_m = X_1^m + X_2^m + \cdots$ (an infinite sum),
(2) $e_m = \sum X_{i_1} X_{i_2} \cdots X_{i_m}$, where the sum is over all indices $i_1 < i_2 < \cdots < i_m$,
(3) $h_m = \sum X_{i_1} X_{i_2} \cdots X_{i_m}$, where the sum is over all indices $i_1 \leq i_2 \leq \cdots \leq i_m$.

Later, for any partition $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$ of k, we will have occasion to use the following symmetric functions (in infinite number of variables) of degree k:

(1) $p_{\underline{\lambda}} = p_{\lambda_1} \cdot p_{\lambda_2} \cdots p_{\lambda_m}$,
(2) $e_{\underline{\lambda}} = e_{\lambda_1} \cdot e_{\lambda_2} \cdots e_{\lambda_m}$,
(3) $h_{\underline{\lambda}} = h_{\lambda_1} \cdot h_{\lambda_2} \cdots h_{\lambda_m}$.

As $\underline{\lambda}$ varies over all partitions of k, $e_{\underline{\lambda}}$ form a basis of the space of symmetric polynomials (in infinitely many variables) of degree k. Similarly, $h_{\underline{\lambda}}$ form a basis of the space of symmetric polynomials of degree k, whereas $p_{\underline{\lambda}}$ forms a basis after $\mathbb{Z}$ is replaced by $\mathbb{Q}$ as the coefficient ring.

Observe that the graded rings $\mathcal{R} = \sum_{d=0}^{\infty} \mathcal{R}_d$ and $\Lambda = \sum_{k=0}^{\infty} \Lambda^k$ are isomorphic under the map Ψ which sends H_m to h_m. The map Ψ is usually called the *characteristic map*.

Note that the character of a polynomial representation of $\mathrm{GL}_m(\mathbb{C})$ at the diagonal element $(X_1, X_2, \cdots, X_m)$ in $\mathrm{GL}_m(\mathbb{C})$, is a symmetric polynomial in $\mathbb{Z}[X_1, \cdots, X_m]^{\mathbb{S}_m}$. A nice fact about irreducible polynomial representations of $\mathrm{GL}_m(\mathbb{C})$, say with highest weight $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ is that it makes sense to speak of "corresponding irreducible representations" of $\mathrm{GL}_d(\mathbb{C})$ for *all* $d \geq m$ by extending the highest weight $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ by adding a few zeros after λ_m. The characters of these representations of $\mathrm{GL}_d(\mathbb{C})$ for $d \geq m$, which are symmetric polynomials in $\mathbb{Z}[X_1, X_2, \cdots X_d]^{\mathbb{S}_d}$, correspond to each other under the maps: $\mathbb{Z}[X_1, X_2, \cdots, X_d]^{\mathbb{S}_d} \to \mathbb{Z}[X_1, X_2, \cdots, X_{d'}]^{\mathbb{S}_{d'}}$ for $d \geq d' \geq m$. Therefore, a polynomial representation of $\mathrm{GL}_m(\mathbb{C})$ has as its character an element which can be considered to belong to $\Lambda = \sum_{k=0}^{\infty} \Lambda^k$ (and not only in $\mathbb{Z}[X_1, \cdots, X_m]^{\mathbb{S}_m}$). For example, the character of the standard m-dimensional representation of $\mathrm{GL}_m(\mathbb{C})$ is the infinite sum $X_1 + X_2 + \cdots$.

So far, nothing non-obvious has been said. Now, here is a non-obvious fact, a form of the Schur-Weyl duality, that if $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$ is a partition of n, defining

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an irreducible representation of $\mathbb{S}_n$, say $\pi_{\underline{\lambda}}$, and defining at the same time an irreducible representation of $\mathrm{GL}_d(\mathbb{C})$ with highest weight $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \geq \cdots \geq 0\}$ for all $d \geq m$, with its character, the Schur function $\mathcal{S}_{\underline{\lambda}}$, then

$$\Psi(\pi_{\underline{\lambda}}) = \mathcal{S}_{\underline{\lambda}}.$$

(It is helpful to recall that under the Schur-Weyl duality, the trivial representation of $\mathbb{S}_m$ goes to the irreducible representation $\mathrm{Sym}^m(\mathbb{C}^d)$ of $\mathrm{GL}_d(\mathbb{C})$ corresponding to the partition $\{1, 1, \cdots, 1\}$ of m.)

The following proposition relates character theory of symmetric groups and $\mathrm{GL}_n(\mathbb{C})$.

Proposition 3.1. *For π any representation of a symmetric group $\mathbb{S}_n$, and $\Psi(\pi)$ the associated symmetric function of degree n in infinitely many variables arising through the isomorphism $\Psi : \mathcal{R} \to \Lambda$, we have the following identity between homogeneous polynomials of degree n:*

$$\Psi(\pi) = \sum_{\rho} \frac{\Theta_\pi(c_\rho)}{|Z(c_\rho)|} p_\rho, \tag{1}$$

where $\rho = (\rho_1, \rho_2, \cdots)$ is a partition of n, defining a conjugacy class c_ρ in $\mathbb{S}_n$, whose centralizer in $\mathbb{S}_n$ is $Z(c_\rho)$ of order $|Z(c_\rho)|$; the symmetric function p_ρ is the product of symmetric polynomials $p_{\rho_i} = X_1^{\rho_i} + X_2^{\rho_i} + X_3^{\rho_i} + \cdots$.

Proof. Since the identity (1) is linear in the representation π, it suffices to check it on a set of linear generators (over $\mathbb{Z}$, although it suffices to do it for generators over $\mathbb{Q}$ too) of the ring $\mathcal{R}$. Our proof of identity (1) will therefore be accomplished in two steps.

(1) Prove that the identity (1) holds for the trivial representation H_n of $\mathbb{S}_n$.
(2) If the identity (1) holds for a representation π_1 of $\mathbb{S}_m$ and π_2 of $\mathbb{S}_n$, then it also holds good for the representation $\pi_1 \times \pi_2$ of $\mathbb{S}_{m+n}$ induced from the representation $\pi_1 \otimes \pi_2$ of $\mathbb{S}_m \times \mathbb{S}_n \subset \mathbb{S}_{m+n}$.

We begin by proving step 1, i.e. that the identity (1) holds for the trivial representation H_n of $\mathbb{S}_n$.

For any integer m, let $V_m = \mathbb{C}^m$ be the m-dimensional vector space over $\mathbb{C}$. By definition, h_k is the character of the representation $\mathrm{Sym}^k(V_m)$ assuming that $m \geq k$; in fact the integer m will play no role as long as it is large enough, preferably infinity! Thus,

$$\sum_{n=0}^{\infty} t^n h_n = \frac{1}{(1 - tX_1)(1 - tX_2) \cdots (1 - tX_m)}.$$

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Taking logarithm on the two sides,

$$\ln(\sum_{n=0}^{\infty} t^n h_n) = -\sum_{i=1}^{m} \ln(1 - tX_i),$$
$$= t(\sum X_i) + \frac{t^2}{2}(\sum X_i^2) + \frac{t^3}{3}(\sum X_i^3) + \cdots$$

Now taking exponential of the two sides:

$$\sum_{n=0}^{\infty} t^n h_n = \exp^{t(\sum X_i)} \cdot \exp^{\frac{t^2}{2}(\sum X_i^2)} \cdot \exp^{\frac{t^3}{3}(\sum X_i^3)} \ldots$$
$$= [1 + tp_1 + \frac{(tp_1)^2}{2!} + \cdots][1 + t^2 p_2/2 + \frac{(t^2 p_2/2)^2}{2!} + \cdots]$$
$$\times [1 + t^3 p_3/3 + \frac{(t^3 p_3/3)^2}{2!} + \cdots]$$

Therefore,

$$h_n = \sum \frac{p_1^{i_1}}{i_1!} \frac{(p_2/2)^{i_2}}{i_2!} \frac{(p_3/3)^{i_3}}{i_3!} \cdots$$
$$\stackrel{*}{=} \sum \frac{p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \cdots}{i_1! 2^{i_2} i_2! 3^{i_3} i_3! \cdots}$$

where the summation is taken over $i_1 + 2i_2 + \cdots = n$. This proves the identity (1) for the trivial representation H_n of $\mathbb{S}_n$ for which $\Theta_\pi(c_\rho)$ is identically 1, and the denominators in the right hand side of the equality $(*)$ are the order of the centralizer of the conjugacy class $c_\rho \in \mathbb{S}_n$.

Next, we prove that if the identity (1) holds for a representation π_1 of $\mathbb{S}_m$ and π_2 of $\mathbb{S}_n$, then it also holds good for the representation $\pi_1 \times \pi_2$ of $\mathbb{S}_{m+n}$ induced from the representation $\pi_1 \otimes \pi_2$ of $\mathbb{S}_m \times \mathbb{S}_n$.

For this, we slightly rewrite the identity (1) as:

$$\Psi(\pi) = \frac{1}{n!} \sum_{x \in \mathbb{S}_n} \Theta_\pi(x) p_x, \tag{2}$$

where π is a representation of the symmetric group $\mathbb{S}_n$ with $\Theta_\pi(x)$ its character at an element $x \in \mathbb{S}_n$, and p_x is what was earlier denoted as $p_{\rho(x)}$ where $\rho(x)$ denotes the partition of n associated to x.

Because

$$\Psi(\pi_1 \times \pi_2) = \Psi(\pi_1) \cdot \Psi(\pi_2),$$

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assuming that (2) holds for π_1 and π_2, to prove that it also holds for $\pi_1 \times \pi_2$, we are reduced to proving:

$$\frac{1}{(n+m)!} \sum_{g \in \mathbb{S}_{m+n}} \Theta_{\pi_1 \times \pi_2}(g) p_g = \frac{1}{m!n!} \sum_{(h_1,h_2) \in \mathbb{S}_m \times \mathbb{S}_n} \Theta_{\pi_1}(h_1)\Theta_{\pi_2}(h_2) p_{h_1} p_{h_2}. \tag{3}$$

This will be a simple consequence of the character of the induced representation $\pi_1 \times \pi_2$ of $\mathbb{S}_{m+n}$ given by: $\pi_1 \times \pi_2 = \mathrm{Ind}_{\mathbb{S}_m \times \mathbb{S}_n}^{\mathbb{S}_{m+n}}(\pi_1 \otimes \pi_2)$, as we now show.

Note the well-known identity regarding the character f' of the induced representation $\mathrm{Ind}_H^G(U)$ of a representation U of H with character f (a class function on H, extended to a function on G by declaring it zero outside H) at an element $s \in G$:

$$f'(s) = \frac{1}{|H|} \sum_{t \in G} f(t^{-1}st).$$

It follows that for any class function $\lambda(s)$ on G,

$$\frac{1}{|G|} \sum_{s \in G} f'(s)\lambda(s) = \frac{1}{|H|} \sum_{t \in H} f(t)\lambda(t). \tag{4}$$

Now (3) follows from the identity (4) applied to the induced representation $\pi_1 \times \pi_2 = \mathrm{Ind}_{\mathbb{S}_m \times \mathbb{S}_n}^{\mathbb{S}_{m+n}}(\pi_1 \otimes \pi_2)$, and where $\lambda(s) = p_s$ for $s \in \mathbb{S}_{m+n}$ are the symmetric functions used before (products of $X_1^{i_1} + X_2^{i_1} + X_3^{i_1} + \cdots$). (We are thus applying the identity (4) for $\lambda(s)$ not a scalar valued function on G, but rather with values in the ring of symmetric polynomials, which we leave to the reader to think about!)

We have thus completed the proof of the Proposition. $\square$

For the statement of the following corollary, a form of the Frobenius character formula, see for example [2], Exercise 4.52(e).

Corollary 3.2. *For $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$, a partition of n, defining an irreducible representation $\pi_{\underline{\lambda}}$ of $\mathbb{S}_n$ with character $\Theta_{\underline{\lambda}}$, and defining at the same time an irreducible representation of $\mathrm{GL}_d(\mathbb{C})$ with highest weight $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \geq \cdots \geq 0\}$ for all $d \geq m$, with its character, the Schur function $\mathcal{S}_{\underline{\lambda}}$, we have the following character identity between homogeneous polynomials of degree n:*

$$\mathcal{S}_{\underline{\lambda}} = \sum_{\rho} \frac{\Theta_{\underline{\lambda}}(c_\rho)}{|Z(c_\rho)|} p_\rho, \tag{1}$$

where $\rho = (\rho_1, \rho_2, \cdots)$ is a partition of n, defining a conjugacy class c_ρ in $\mathbb{S}_n$, whose centralizer in $\mathbb{S}_n$ is $Z(c_\rho)$ of order $|Z(c_\rho)|$; the symmetric function p_ρ is the product of the symmetric polynomials $p_{\rho_i} = X_1^{\rho_i} + X_2^{\rho_i} + X_3^{\rho_i} + \cdots$.

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Proof. The proof of the corollary is a direct consequence of the Proposition 3.1 on noting the Schur-Weyl duality according to which the isomorphism of graded rings $\Psi : \mathcal{R} \to \Lambda$ which is defined by sending the trivial representation H_n of $\mathbb{S}_n$ to the symmetric polynomial h_n (in infinitely many variables) takes the representation $\pi_{\underline{\lambda}}$ to the Schur function $\mathcal{S}_{\underline{\lambda}}$. □

4. A theorem of Littlewood

This section aims to give a proof of a theorem due to Littlewood, see pages 143-146 of [6], on character values for symmetric groups at exactly the same set of conjugacy classes that we have considered in this paper: product of disjoint even cycles without fixed points — Littlewood's theorem covers only the case of $\mathbb{S}_{2n}$, which we prove also for $\mathbb{S}_{2n+1}$. Our main theorem proved in the next section is a simple consequence of the theorem of Littlewood (suitably extended to $\mathbb{S}_{2n+1}$). We have decided to include a proof of the theorem due to Littlewood since the proof (from 1940!) is hard to follow.

Before proceeding further, we need to introduce a sign $\epsilon(\mathfrak{p})$ associated to any partition with either empty 2-core or with 2-core 1.

Definition 4.1. (Sign of a partition) (a)Let $\mathfrak{p}$ be a partition of an even number with even number of parts $2m$ (by adding a zero at the end if necessary) with β-set $\beta(\mathfrak{p}) = \{\beta_{2m-1} > \beta_{2m-2} > \cdots > \beta_1 > \beta_0\}$ of which half are even and half are odd. Let $X_{2m} = \{2m-1, 2m-2, \cdots, 1, 0\}$, and let $i(\mathfrak{p})$ be the bijection from X_{2m} to $\beta(\mathfrak{p})$ sending $i \to \beta_i$. Let $j(\mathfrak{p})$ be the unique bijective map from $\beta(\mathfrak{p})$ to X_{2m} taking even numbers to even numbers preserving their orders, and odd numbers to odd numbers preserving their orders. The permutation $s(\mathfrak{p})$ of X_{2m} defined as the composition of the maps $X_{2m} \stackrel{i(\mathfrak{p})}{\to} \beta(\mathfrak{p}) \stackrel{j(\mathfrak{p})}{\to} X_{2m}$ will be called the *shuffle permutation* associated to $\mathfrak{p}$, and its sign $(-1)^{s(\mathfrak{p})}$ will be denoted $\epsilon(\mathfrak{p})$.

(b) Let $\mathfrak{p}$ be a partition of an odd number with odd number of parts $2m+1$ (by adding a zero at the end if necessary) with β-set $\beta(\mathfrak{p}) = \{\beta_{2m} > \beta_{2m-1} > \cdots > \beta_1 > \beta_0\}$ in which there is one more odd number than even. Let $X_{2m+1} = \{2m+1, 2m, \cdots, 1\}$. Let $i(\mathfrak{p})$ be the bijection from X_{2m+1} to $\beta(\mathfrak{p})$ sending $i \to \beta_{i-1}$. Let $j(\mathfrak{p})$ be the unique bijective map from $\beta(\mathfrak{p})$ to X_{2m+1} taking even numbers to even numbers preserving their orders, and odd numbers to odd numbers preserving their orders. The permutation $s(\mathfrak{p})$ on X_{2m+1} defined as the composition of the maps $X_{2m+1} \stackrel{i(\mathfrak{p})}{\to} \beta(\mathfrak{p}) \stackrel{j(\mathfrak{p})}{\to} X_{2m}$ will be called the *shuffle permutation* associated to $\mathfrak{p}$. Define, $\epsilon(\mathfrak{p}) = (-1)^m(-1)^{s(\mathfrak{p})}$.

The following proposition gives another, much simpler, way to interpret the sign $\epsilon(\mathfrak{p})$ just defined. We owe the proof of this proposition to Arvind Ayyer.

Proposition 4.2. *Let $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ be a partition of n, written as usual with $\mathfrak{p}_1 \geq \cdots \geq \mathfrak{p}_n$ with the β-set $\beta(\mathfrak{p}) = (\beta_1, \ldots, \beta_n)$ with $\beta_i = \mathfrak{p}_i + n - i$.*

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(1) *For* $\mathfrak{p} \vdash 2n$ *with empty* 2*-core, let* $2k$ *be the number of odd parts in* $\mathfrak{p}$*. Then* $\epsilon(\mathfrak{p}) = (-1)^k$*.*
(2) *For* $\mathfrak{p} \vdash 2n-1$ *with* 2*-core equal to* (1)*, let* $2k-1$ *be the number of odd parts in* $\mathfrak{p}$*. Then* $\epsilon(\mathfrak{p}) = (-1)^{k+n-1}$*.*

Proof. We begin with the proof of case (1). By Lemma 2.1, exactly n of these β_i's are odd. Let the integers i such that β_i is odd be labeled $o_1 < \cdots < o_n$, and the integers i such that β_i is even be labeled $e_1 < \cdots < e_n$.

Before computing the sign of the shuffle permutation, we first compute the sign ϵ_1 of the permutation that moves $o_1 < \cdots < o_n$ to $1 < \cdots < n$ (preserving their order), and moves $e_1 < \cdots < e_n$ to $(n+1) < \cdots < 2n$ (preserving their order). This can be calculated by looking at the corresponding action on a vector space V of dimension $2n$ with basis vectors $\{v_1, v_2, \cdots, v_{2n}\}$. Thus we find that ϵ_1 is given by:

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n \wedge v_{n+1} \wedge \cdots \wedge v_{2n} = \epsilon_1 v_{o_1} \wedge v_{o_2} \wedge \cdots \wedge v_{o_n} \wedge v_{e_1} \wedge \cdots \wedge v_{e_n},$$

which means that ϵ_1 has the parity of:

$$(n - e_1 + 1) + (n - e_2 + 2) + \cdots + (n - e_n + n) = (o_1 + \cdots + o_n) - (1 + \cdots + n).$$

By the computation just done, if ϵ_2 is the sign of the permutation that moves all the odd integers $1 \leq i \leq (2n-1)$ to positions $1, \ldots, n$ (preserving their order), and even integers $2 \leq i \leq 2n$ to positions $n+1, \ldots, 2n$ (preserving their order), then ϵ_2 is given by:

$$= (o_1 + \cdots + o_n) - (1 + \cdots + n) = (1 + 3 + \cdots + 2n - 1) - (1 + \cdots + n).$$

Therefore, we find that $\epsilon(\mathfrak{p}) = \epsilon_1 \epsilon_2$ has the same parity as $(o_1 + \cdots + o_n) + n$. Thus, we have to prove that the parity of $o_1 + \cdots + o_n + n$ is the same as that of k. Now, since $\beta_{o_i} = \mathfrak{p}_{o_i} + 2n - o_i$ is odd, so is $\mathfrak{p}_{o_i} - o_i$. Thus, the parity of $o_i + 1$ is the same as that of $\mathfrak{p}_{o_i}$. As a result, we have to show that the parity of $\mathfrak{p}_{o_1} + \cdots + \mathfrak{p}_{o_n}$ is the same as that of k. Among all $\mathfrak{p}_{o_i}$'s, those which are even clearly do not contribute to the parity, and those which are odd contribute a 1. We thus have to prove that the cardinality of the set $S = \{i \mid \mathfrak{p}_{o_i} \text{is odd}\}$ has the same parity as that of k.

We will prove something stronger, namely that $|S| = k$. Suppose that $|S| = j$. Partition the set $\{1, \ldots, 2n\} = O \cup E$, where $O = \{o_1, \ldots, o_n\}$. By assumption, $S \subset O$ are the positions where $\mathfrak{p}$ takes odd values. For convenience, let $\delta_i = 2n - i$ for $1 \leq i \leq 2n$ so that $\beta_i = \mathfrak{p}_i + \delta_i$. There are exactly n even and n odd δ's. Since β_i for $i \in O$ is odd, δ_i's for $i \in S$ are even and δ_i's for $i \in O \setminus S$ are odd. Which means that there are j even and $n - j$ odd δ_i's for $i \in O$. Consequently, we must have $n - j$ even and j odd δ_i's for $i \in E$. Combining all this information, we infer that the number of odd parts in $\mathfrak{p}$ is j (from S) and j (from E, since β_i's for $i \in E$ are even). But we had assumed that $\mathfrak{p}$ has $2k$ odd parts, and therefore $j = k$, proving first case of the Proposition.

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Now we consider the second case of the Proposition where the length of the partition $\mathfrak{p}$ is $2n-1$. In this case, arguing just as in case (1), we have to prove that the parity of $o_1 + \cdots + o_n$ is the same as that of $k+1$.

Now, since $\beta_{o_i} = \mathfrak{p}_{o_i} + 2n - 1 - o_i$ is odd, so is $\mathfrak{p}_{o_i} - o_i - 1$. Thus, the parity of o_i is the same as that of $\mathfrak{p}_{o_i}$. As a result, we have to show that the parity of $\mathfrak{p}_{o_1} + \cdots + \mathfrak{p}_{o_n}$ is the same as that of $k+1$. Among all $\mathfrak{p}_{o_i}$'s, those which are even clearly do not contribute to the parity, and those which are odd contribute a 1. We thus have to prove that the cardinality of the set $S = \{i \mid \mathfrak{p}_{o_i} \text{is odd}\}$ has the same parity as that of $k+1$.

We will again prove something stronger, namely that $|S| = k+1$. Suppose that $|S| = j$. Partition the set $\{1, \ldots, 2n-1\} = O \cup E$, where $O = \{o_1, \ldots, o_n\}$. By assumption, $S \subset O$ are the positions where $\mathfrak{p}$ takes odd values. For convenience, let $\delta_i = 2n - 1 - i$ for $1 \leq i \leq 2n-1$ so that $\beta_i = \mathfrak{p}_i + \delta_i$. There are n even and $n-1$ odd δ_i's.

Since β_i for $i \in O$ is odd, δ_i's for $i \in S$ are even and δ_i's for $i \in O \setminus S$ are odd. Which means that there are j even and $n-j$ odd δ_i's for $i \in O$. Consequently, we must have $n-j$ even and $j-1$ odd δ_i's for $i \in E$. Combining all this information, we infer that the number of odd parts in $\mathfrak{p}$ is j (from S) and $j-1$ (from E, since β_i's for $i \in E$ are even), giving a total of $2j-1$. But we had assumed that $\mathfrak{p}$ has $2k+1$ odd parts, and therefore $j = k+1$.

This completes the proof of both the cases of the Proposition. $\square$

Remark 4.3. The sign defined here associated to a partition $\mathfrak{p}$ with empty core or with core 1, has thus three different ways of looking at it:

(1) In Definition 4.1.
(2) Through Proposition 4.2.
(3) As a particular case of our main theorem as the sign of the character of the representation $\pi_{\mathfrak{p}}$ of $\mathbb{S}_{|\mathfrak{p}|}$ at the element $w_0 = (12)(34)\cdots(2n-1, 2n)$ — where it takes nonzero value — inside $\mathbb{S}_{2n}$ or $\mathbb{S}_{2n+1}$.

Next, we recall the following theorem from [6] where it is contained in equations 7.3.1 and 7.3.2 of page 132; it is also contained in [9] as Theorem 2. Both [6] as well as [9] deal with a more general theorem involving $\mathrm{GL}_{mn}(\mathbb{C})$. This theorem as well as the next theorem should be considered as the $\mathrm{GL}_n(\mathbb{C})$ analogues of theorems on character values that we strive to prove here. It is in this theorem that the notion of p-core and p-quotients make their appearance (stated here only for $p = 2$, although stated more generally in [9]).

Theorem 4.4. *Let $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2m}\}$ be the highest weight of an irreducible polynomial representation $V_{\underline{\lambda}}$ of $\mathrm{GL}_{2m}(\mathbb{C})$ with character $\mathcal{S}_{\underline{\lambda}}$. Let*

$$X = (x_1, x_2, \cdots, x_m, -x_1, -x_2, \cdots, -x_m) = (\underline{X}, -\underline{X}),$$

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be a diagonal matrix in $\mathrm{GL}_{2m}(\mathbb{C})$ *with* $x_i \in \mathbb{C}^\times$ *arbitrary. Then if* $\mathcal{S}_{\underline{\lambda}}(X)$ *is not identically zero, its 2-core* $c_2(\underline{\lambda})$ *must be empty, in which case, if* $\{\underline{\lambda}_0, \underline{\lambda}_1\}$ *is the 2-quotient of* $\underline{\lambda}$*, then,*

$$\mathcal{S}_{\underline{\lambda}}(X) = \epsilon(\underline{\lambda})\mathcal{S}_{\underline{\lambda}_0}(\underline{X}^2)\mathcal{S}_{\underline{\lambda}_1}(\underline{X}^2),$$

where $\mathcal{S}_{\underline{\lambda}_0}$ *and* $\mathcal{S}_{\underline{\lambda}_1}$ *are the characters of the corresponding highest weight modules of* $\mathrm{GL}_m(\mathbb{C})$*,* $\underline{X}$ *is the diagonal matrix* $(x_1, x_2, \cdots, x_m)$*, and* $\underline{X}^2$ *its square.*

Proof. For the sake of completeness, we give the proof. Write the matrix whose determinant represents the numerator of the Weyl character formula as (where $\beta_i = \lambda_i + 2m - i$):

$$\begin{pmatrix} x_1^{\beta_1} & x_2^{\beta_1} & \cdots & x_m^{\beta_1} & (-x_1)^{\beta_1} & \cdots & (-x_m)^{\beta_1} \\ x_1^{\beta_2} & x_2^{\beta_2} & \cdots & x_m^{\beta_2} & (-x_1)^{\beta_2} & \cdots & (-x_m)^{\beta_2} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ x_1^{\beta_{2m}} & x_2^{\beta_{2m}} & \cdots & x_m^{\beta_{2m}} & (-x_1)^{\beta_{2m}} & \cdots & (-x_m)^{\beta_{2m}} \end{pmatrix}.$$

In this $2m \times 2m$-matrix, adding the first m columns of the matrix to the last m columns, we find that all the rows of the last m columns with β_i odd become zero, and those rows with β_i even get multiplied by 2. In the new matrix, subtracting the half of last m columns to the first m columns, makes all rows in the first m columns with β_i even to be zero. Let d be the number of β_i which are odd, and therefore $2m - d$ is the number of β_i which are even. Thus we get a matrix in which in the first m columns, there are exactly d nonzero rows, and in the last m columns, there are exactly $2m - d$ complementary rows which are nonzero.

The determinant of such a matrix is nonzero only for $d = 2m - d$, i.e., $d = m$. By Lemma 2.1, the condition $d = m$ is equivalent to 2-core of $\underline{\lambda}$ being empty. Assuming which, by a permutation of rows, we come to a block diagonal matrix which looks like (where γ_k, δ_ℓ are the odd and even numbers among β_i written in decreasing order):

$$\begin{pmatrix} x_1^{\gamma_1} & x_2^{\gamma_1} & \cdots & x_m^{\gamma_1} & 0 & 0 & \cdots & 0 \\ x_1^{\gamma_2} & x_2^{\gamma_2} & \cdots & x_m^{\gamma_2} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ x_1^{\gamma_m} & x_2^{\gamma_m} & \cdots & x_m^{\gamma_m} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_1^{\delta_1} & x_2^{\delta_1} & \cdots & x_m^{\delta_1} \\ 0 & 0 & \cdots & 0 & x_1^{\delta_2} & x_2^{\delta_2} & \cdots & x_m^{\delta_2} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & x_1^{\delta_m} & x_2^{\delta_m} & \cdots & x_m^{\delta_m} \end{pmatrix}.$$

The determinant of this matrix is the product of two Weyl numerators for $\mathrm{GL}_m(\mathbb{C})$, in which δ_i being even, we write $x_j^{\delta_i}$ as $(x_j^2)^{\delta_i/2}$, and γ_i being odd, we write $x_j^{\gamma_i}$ as $(x_j^2)^{(\gamma_i-1)/2} \cdot x_j$. We have a similar factorization of the Weyl denominator.

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Next we observe that the determinant of a matrix in which the rows have been shuffled using a permutation σ of the rows, changes by multiplication by the sign $(-1)^\sigma$. This needs to be done both for the numerator which involves the 'shuffle permutation' introduced earlier in Definition 4.1, and the denominator for which the shuffle permutation is identity matrix; to be more precise, the numerator needs the permutation $s(\mathfrak{p})s_0$, and the denominator needs s_0 where s_0 is the permutation of $X_{2m} = \{2m-1, 2m-2, \cdots, 1, 0\}$ sending odd numbers in X_{2m} consecutively to $\{2m-1, 2m-2, \cdots, m\}$ and even numbers in X_{2m} consecutively to $\{m-1, \cdots, 1, 0\}$.

This completes the proof of the theorem. $\square$

We will also need the following variant of Theorem 4.4 which is proved as this theorem by manipulations with the explicit character formula for $\mathrm{GL}_{2m+1}(\mathbb{C})$ as quotients of two determinants; we will not give a proof of this theorem.

Theorem 4.5. *Let $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2m+1}\}$ be the highest weight of an irreducible polynomial representation $V_{\underline{\lambda}}$ of $\mathrm{GL}_{2m+1}(\mathbb{C})$ with character $\mathcal{S}_{\underline{\lambda}}$. Let*

$$X = (x_1, x_2, \cdots, x_m, -x_1, -x_2, \cdots, -x_m, x) = (\underline{X}, -\underline{X}, x),$$

be a diagonal matrix in $\mathrm{GL}_{2m+1}(\mathbb{C})$ with $x_i, x \in \mathbb{C}^\times$ arbitrary. Then, if $\mathcal{S}_{\underline{\lambda}}(X)$ is not identically zero, the number of even entries and the number of odd entries in the β-sequence $\beta(\underline{\lambda}) = \{\beta_1 = \lambda_1 + 2m > \beta_2 = \lambda_2 + 2m - 1 > \cdots > \lambda_{2m+1}\}$, differ by one. If the number of odd entries in the β-sequence is one more than the number of even entries, then its 2-core is $\{1\}$, whereas if the number of even entries in the β-sequence is one more than the number of odd entries, its 2-core is empty. In either case, let $\{\underline{\lambda}_0, \underline{\lambda}_1\}$ be the 2-quotient of $\underline{\lambda}$. Then,

$$\mathcal{S}_{\underline{\lambda}}(X) = \epsilon(\underline{\lambda}) \cdot x \cdot \mathcal{S}_{\underline{\lambda}_0}(\underline{X}^2)\mathcal{S}_{\underline{\lambda}_1}(\underline{X}^2, x^2) \tag{a}$$

if the number of odd entries in the β-sequence is one more than the number of even entries; whereas

$$\mathcal{S}_{\underline{\lambda}}(X) = \epsilon(\underline{\lambda})\mathcal{S}_{\underline{\lambda}_1}(\underline{X}^2)\mathcal{S}_{\underline{\lambda}_0}(\underline{X}^2, x^2) \tag{b}$$

if the number of even entries in the β-sequence is one more than the number of odd entries. Here $\mathcal{S}_{\underline{\lambda}_0}$ and $\mathcal{S}_{\underline{\lambda}_1}$ are the characters of the corresponding highest weight modules of $\mathrm{GL}_m(\mathbb{C})$, and $\mathrm{GL}_{m+1}(\mathbb{C})$ in case (a), and of $\mathrm{GL}_{m+1}(\mathbb{C})$, and $\mathrm{GL}_m(\mathbb{C})$ in case (b).

The following theorem is due to Littlewood [6], pages 143-146, for even symmetric groups. Forms of this theorem seem to be available in the literature more generally for arbitrary wreath product $(\mathbb{Z}/p)^n \rtimes \mathbb{S}_n$ contained in $\mathbb{S}_{pn}$, such as in Theorem 4.56 of [10] which also follows from Theorem 2 of [9] and the corresponding analogue of Theorem 4.5 for general p.

Theorem 4.6. *Let $\underline{\lambda} \to \pi_{\underline{\lambda}}$ be the natural correspondence between the partitions of $|\underline{\lambda}|$ and the irreducible representations of the symmetric group $\mathbb{S}_{|\underline{\lambda}|}$ with $\Theta_{\underline{\lambda}}$, the character of $\pi_{\underline{\lambda}}$. Then, if the representation $\pi_{\underline{\lambda}}$ is to have nonzero character value at some element $\tilde{w}$ in $\mathbb{S}_{|\underline{\lambda}|}$ which is a product of disjoint even cycles and with at most one fixed point, the 2-core of $\underline{\lambda}$ must be empty if $|\underline{\lambda}|$ is even, and 2-core must be 1 if $|\underline{\lambda}|$ is odd, which we now assume is the case, and $(\underline{\lambda}_0, \underline{\lambda}_1)$ is the 2-quotient of $\underline{\lambda}$. Then, if $\tilde{w} = \tilde{w}_1 \cdots \tilde{w}_d \in \mathbb{S}_{|\underline{\lambda}|}$, is a product of disjoint cycles $\tilde{w}_i$ of even lengths $2\ell_i$ with at most one fixed point, we have the following character relationship:*

$$\Theta_{\underline{\lambda}}(\tilde{w}) = \epsilon(\underline{\lambda})\Theta_{\underline{\lambda}_0 \times \underline{\lambda}_1}(w),$$

where $w = w_1 \cdots w_d$ is an element in $\mathbb{S}_n$, $n = [|\underline{\lambda}|/2]$ which is a product of disjoint cycles of length ℓ_i, and where $\Theta_{\underline{\lambda}_0 \times \underline{\lambda}_1}$ is the character of the representation (usually reducible) of $\mathbb{S}_n$

$$\pi_{\underline{\lambda}_0 \times \underline{\lambda}_1} = \mathrm{Ind}_{\mathbb{S}_{|\underline{\lambda}_0|} \times \mathbb{S}_{|\underline{\lambda}_1|}}^{\mathbb{S}_n} (\pi_{\underline{\lambda}_0} \boxtimes \pi_{\underline{\lambda}_1}).$$

Proof. By Corollary 3.2,

$$\mathcal{S}_{\underline{\lambda}} = \sum_{\rho} \frac{\Theta_{\underline{\lambda}}(c_\rho)}{|Z(c_\rho)|} p_\rho, \tag{1}$$

where $\rho = (\rho_1, \rho_2, \cdots)$ is a partition of k, defining a conjugacy class c_ρ in $\mathbb{S}_k$, whose centralizer in $\mathbb{S}_k$ is $Z(c_\rho)$ of order $|Z(c_\rho)|$; the symmetric function p_ρ is the product of symmetric polynomials $p_{\rho_i} = X_1^{\rho_i} + X_2^{\rho_i} + X_3^{\rho_i} + \cdots$

In the standard notation, if a conjugacy class c in $\mathbb{S}_k$ is $(1^{i_1}, 2^{i_2}, \cdots, k^{i_k})$, the order of the centralizer of any element in c is:

$$(i_1)!2^{i_2}(i_2)! \cdots k^{i_k}(i_k)!.$$

For a partition $\rho = (\rho_1, \rho_2, \cdots)$ of k, we will be using the notation 2ρ for the partition $2\rho = (2\rho_1, 2\rho_2, \cdots)$ of $2k$, defining a conjugacy class $c_{2\rho}$ in $\mathbb{S}_{2k}$ for which the order of the centralizer of any element in $c_{2\rho}$ is:

$$2^{i_1}(i_1)!4^{i_2}(i_2)! \cdots (2k)^{i_k}(i_k)!,$$

therefore,

$$|Z(c_{2\rho})| = 2^p |Z(c_\rho)|, \tag{2}$$

where $p = i_1 + i_2 + \cdots + i_k$.

Now we split the proof of the theorem into two cases.

Case 1: $|\underline{\lambda}| = 2n$.

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We will use the identity expressed by equation (1) at the diagonal matrices in $\mathrm{GL}_{2m}(\mathbb{C})$ of the form $X = (x_1, x_2, \cdots, x_m, -x_1, -x_2, \cdots, -x_m) = (\underline{X}, -\underline{X})$. An important observation is that p_ρ which is the product of the symmetric polynomials $p_{\rho_i} = (X_1^{\rho_i} + X_2^{\rho_i} + X_2^{\rho_i} + \cdots)$ must be identically zero on such elements unless all the entries in ρ are even.

Using Theorem 4.4, we write equation (1) as:

$$\epsilon(\underline{\lambda})\mathcal{S}_{\underline{\lambda}_0}(\underline{X}^2)\mathcal{S}_{\underline{\lambda}_1}(\underline{X}^2) = \sum_\rho \frac{\Theta_{\underline{\lambda}}(c_{2\rho})}{|Z(c_{2\rho})|} p_{2\rho}(X).$$

Since $p_{2\rho}(X) = 2^p p_\rho(\underline{X}^2)$, where $p = i_1 + i_2 + \cdots + i_n$, we can rewrite this equation using (2) as:

$$\epsilon(\underline{\lambda})\mathcal{S}_{\underline{\lambda}_0}(\underline{X})\mathcal{S}_{\underline{\lambda}_1}(\underline{X}) = \sum_\rho \frac{\Theta_{\underline{\lambda}}(c_{2\rho})}{|Z(c_\rho)|} p_\rho(\underline{X}). \tag{3}$$

Since $\Psi : \mathcal{R} \to \Lambda$ is an isomorphism of rings, the element $\mathcal{S}_{\underline{\lambda}_0}(\underline{X})\mathcal{S}_{\underline{\lambda}_1}(\underline{X})$ of Λ arises from the image under Ψ of the representation $\pi_{\underline{\lambda}_0 \times \underline{\lambda}_1}$ of $\mathbb{S}_n$, therefore by Proposition 3.1, we have,

$$\epsilon(\underline{\lambda}) \sum_\rho \frac{\Theta_{\underline{\lambda}_0 \times \underline{\lambda}_1}(c_\rho)}{|Z(c_\rho)|} p_\rho(\underline{X}) = \sum_\rho \frac{\Theta_{\underline{\lambda}}(c_{2\rho})}{|Z(c_\rho)|} p_\rho(\underline{X}). \tag{4}$$

Since the polynomials $p_\rho(\underline{X})$ are linearly independent, we can equate the coefficients of $p_\rho(\underline{X})$ on the two sides of equation (4) to prove the theorem when $|\underline{\lambda}|$ is even. By Theorem 4.4, if $\underline{\lambda}$ has non-empty 2-core, then $\mathcal{S}_{\underline{\lambda}}$ is identically zero on the set of diagonal elements of the form $X = (x_1, x_2, \cdots, x_m, -x_1, -x_2, \cdots, -x_m) = (\underline{X}, -\underline{X})$. By the linear independence of the polynomials $p_\rho(\underline{X})$, we deduce that $\Theta_{\underline{\lambda}}(c_{2\rho}) \equiv 0$.

Case 2: $|\underline{\lambda}| = 2n + 1$.

In this case, we will use Frobenius character relationship contained in Corollary 3.2:

$$\mathcal{S}_{\underline{\lambda}} = \sum_\rho \frac{\Theta_{\underline{\lambda}}(c_\rho)}{|Z(c_\rho)|} p_\rho, \tag{5}$$

at diagonal elements of $\mathrm{GL}_{2m+1}(\mathbb{C})$ of the form:

$$X = (x_1, x_2, \cdots, x_m, -x_1, -x_2, \cdots, -x_m, x) = (\underline{X}, -\underline{X}, x),$$

with $x_i, x \in \mathbb{C}^\times$ arbitrary. Observe that if ℓ is an odd integer, then for X as above, $p_\ell(X) = x^\ell$, whereas for ℓ an even integer $p_\ell(X) = 2p_\ell(\underline{X}) + x^\ell$. It follows that for any conjugacy class ρ in $\mathbb{S}_{2n+1}$, p_ρ, and hence each term in the right hand side of equation (5) is divisible by x, and all terms except those which correspond to those ρ which are product of disjoint even cycles together with exactly one fixed point, contribute a term which is divisible by x and no higher power, and all the other terms are divisible by higher powers of x. Furthermore, in the case $|\underline{\lambda}| = 2n+1$, each of the term $p_\rho(X)/x$ is an even function of x, hence by equation (5), $\mathcal{S}_{\underline{\lambda}}(X)$ as a function of x is an odd polynomial function of x. Thus, if $\mathcal{S}_{\underline{\lambda}}(X)$ is nonzero, we must be in case (a) of Theorem 4.5 (since in case (b) $\mathcal{S}_{\underline{\lambda}}(X)$ is an even function of x), therefore if $\mathcal{S}_{\underline{\lambda}}(X)$ is a nonzero function of x, $\underline{\lambda}$ must have 2-core $\{1\}$.

By Theorem 4.5, the left hand side of the equation (5) is also divisible by x which after dividing by x gives, $\epsilon(\underline{\lambda})\mathcal{S}_{\underline{\lambda}_0}(\underline{X}^2)\mathcal{S}_{\underline{\lambda}_1}(\underline{X}^2, x^2)$.

Thus after dividing both the sides of the Frobenius character relationship in equation (5) by x, and then putting $x = 0$, we are in exactly the same situation as in the proof of the theorem for $\mathbb{S}_{2n}$, for which we do not need to repeat the previous argument. □

5. The theorem

The following theorem is the main result of this paper (recalled from the Introduction for reader's convenience) and is a simple consequence of Theorem 4.6 of the last section.

Theorem 5.1. *Let w be a conjugacy class in $\mathbb{S}_n$ treated as a conjugacy class in $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$ with $\tilde{w}$ the conjugacy classes in $\mathbb{S}_{2n}, \mathbb{S}_{2n+1}$ defined on page 2. Let (λ_0, λ_1) be a pair of partitions with $|\lambda_0|+|\lambda_1| = n$, giving rise to a representation $\pi_{(\lambda_0,\lambda_1)}$ of $\mathbb{B}_n$. The pair (λ_0, λ_1) gives rise to a partition λ of $2n$ (resp. of $(2n+1)$) with empty 2-core (resp. with 2-core 1) and with 2-quotient (λ_0, λ_1), and defines an irreducible representation π_λ of $\mathbb{S}_{|\lambda|} = \mathbb{S}_{2n}, \mathbb{S}_{2n+1}$. Then we have for an $\epsilon(\lambda) = \pm 1$, the character identity:*

$$\Theta_{(\lambda_0,\lambda_1)}(w) = \epsilon(\lambda)\Theta_\lambda(\tilde{w}).$$

In particular, we have

$$\Theta_{(\lambda_0,\lambda_1)}(1) = \epsilon(\lambda)\Theta_\lambda(w_0),$$

determining $\epsilon(\lambda) = \pm 1$. Further, an irreducible representation π_λ of $\mathbb{S}_{|\lambda|}$ takes nonzero character value at w_0 if and only if the partition λ has empty 2-core if $|\lambda| = 2n$, and has 2-core 1 if $|\lambda| = 2n+1$.

Proof. Recall that we have used the notation $\pi_{\lambda_0} \times \pi_{\lambda_1}$ for the representation

$$\pi_{\lambda_0} \times \pi_{\lambda_1} = \mathrm{Ind}_{\mathbb{S}_{|\lambda_0|}\times\mathbb{S}_{|\lambda_1|}}^{\mathbb{S}_n}(\pi_{\lambda_0} \boxtimes \pi_{\lambda_1}).$$

Of course the representation $\pi_{\lambda_0} \times \pi_{\lambda_1}$ of $\mathbb{S}_n$ is a complicated sum of irreducible representations for which there is the Littlewood-Richardson rule. However, this complication has no role to play for us since instead of $\mathbb{S}_n$ we are dealing with the larger group,

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$\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$ and the representation $\pi_{\lambda_0} \times \pi_{\lambda_1}$ of $\mathbb{S}_n$ is the restriction to $\mathbb{S}_n$ of an irreducible representation of $\mathbb{B}_n$ that we are denoting by $\pi_{(\lambda_0,\lambda_1)}$ as we now show.

Observe that the representation $\pi_{(\lambda_0,\lambda_1)}$ of $\mathbb{B}_n = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$ is

$$\mathrm{Ind}_A^{\mathbb{B}_n}(V),$$

where A is the subgroup of $\mathbb{B}_n$ which is $A = (\mathbb{Z}/2)^n \rtimes (\mathbb{S}_{|\lambda_0|} \times \mathbb{S}_{|\lambda_1|})$, and V is the representation of A on which

$$(\mathbb{Z}/2)^n = (\mathbb{Z}/2)^{|\lambda_0|+|\lambda_1|} = (\mathbb{Z}/2)^{|\lambda_0|} \times (\mathbb{Z}/2)^{|\lambda_1|}$$

acts by the trivial character on the first factor $(\mathbb{Z}/2)^{|\lambda_0|}$, and by the non-trivial character on each of the $\mathbb{Z}/2$ factors in $(\mathbb{Z}/2)^{|\lambda_1|}$; the subgroup $\mathbb{S}_{|\lambda_0|} \times \mathbb{S}_{|\lambda_1|}$ acts by $\pi_{\lambda_0} \boxtimes \pi_{\lambda_1}$.

By standard application of Mackey theory (restriction to $\mathbb{S}_n$ of an induced representation of $\mathbb{B}_n$), we find that

$$\mathrm{Res}_{\mathbb{S}_n}(\mathrm{Ind}_A^{\mathbb{B}_n}(V)) = \mathrm{Ind}_{\mathbb{S}_{|\lambda_0|} \times \mathbb{S}_{|\lambda_1|}}^{\mathbb{S}_n}(\pi_{\lambda_0} \boxtimes \pi_{\lambda_1}) = \pi_{\lambda_0} \times \pi_{\lambda_1},$$

therefore Theorem 4.6 of the last section proves the theorem. □

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