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Dense near polygons with hexes of type $H^{\mathrm{D}}(5, q^2)$, $Q(5, q)\times L_{q+1}$ or $Q(5, q)\otimes Q(5, q)$

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Abstract

We determine all dense near $2n$-gons, $n \geqslant 3$, in which each hex is isomorphic to either the dual polar space $H^{\mathrm{D}}(5, q^2)$, the product near hexagon $Q(5, q) \times L_{q+1}$ or a glued near hexagon of type $Q(5, q) \otimes Q(5, q)$.
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1. Introduction

Let $\Gamma = (V, E)$ be a graph. A *clique* of Γ is a set of mutually adjacent vertices. A clique is called *maximal* if it is not properly contained in another clique. We will denote the distance between two vertices x and y of Γ by $\mathrm{d}(x, y)$. If X_1 and X_2 are two nonempty sets of vertices, then we denote by $\mathrm{d}(X_1, X_2)$ the minimal distance between a vertex of X_1 and a vertex of X_2. If X_1 is a singleton $\{x\}$, then we will also write $\mathrm{d}(x, X_2)$ instead of $\mathrm{d}(\{x\}, X_2)$. For every $i \in \mathbb{N}$ and every nonempty set of vertices X, we denote by $\Gamma_i(X)$ the

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set of all vertices y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A *near* $2d$*-gon* is a connected graph of diameter d with the property that for every vertex x and every maximal clique M there exists a unique vertex x' in M nearest to x. A near 0-gon consists of one vertex and a near 2-gon is just a maximal clique with at least two vertices.

A point-line incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, is called a *partial linear space* (respectively a *linear space*) if every two different points are contained in at most (respectively exactly) one line. The *point graph* or *collinearity graph* of a point-line incidence structure $\mathcal{S}$ is the graph whose vertices are the points of $\mathcal{S}$ where two different vertices are adjacent whenever they are collinear.

There is a bijective correspondence between the class of near polygons and a class of partial linear spaces. If a graph Γ is a near polygon, then the point-line incidence structure with points, respectively lines, the vertices, respectively maximal cliques, of Γ (natural incidence) is a partial linear space $\mathcal{S}$. The graph Γ can easily be retrieved from $\mathcal{S}$: Γ is the point graph of $\mathcal{S}$. Because of this bijective correspondence, the partial linear spaces which correspond with near polygons are also called near polygons. In the sequel we will always adopt the geometric point of view. A near 0-gon is then a point and a near 2-gon a line. In the sequel we will denote by L_{s+1} any line with $s + 1$ points.

A near $2d$-gon, $d \geqslant 2$, is called a *generalized* $2d$*-gon* if for every $i \in \{1, \ldots, d-1\}$ and for every two points x and y at distance i, $|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = 1$. By definition the generalized quadrangles are precisely the near quadrangles. A generalized $2d$-gon is called *degenerate* if it does not contain ordinary $2d$-gons as subgeometries. For more background information on generalized quadrangles and generalized polygons, we refer to [15,20].

A nonempty set X of points of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *subspace* if every line meeting X in at least two points is completely contained in X. A subspace X is called *geodetically closed* or *convex* if every point on a shortest path between two points of X is contained in X as well. Having a subspace X, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in X. If X is convex, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. If a convex sub near polygon $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then X (and often also $\mathcal{S}_X$) will be called a *quad*. Sufficient conditions for the existence of quads were given in [17]. Every nonempty set X of points is contained in a unique minimal convex sub near polygon $\mathcal{C}(X)$, namely the intersection of all convex sub near polygons containing X. We define $\mathcal{C}(\emptyset) := \emptyset$. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$.

A near polygon is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points and if every point is incident with exactly $t + 1$ lines. A near polygon is called *dense* if every line is *thick*, i.e. is incident with at least three points, and if every two points at distance 2 have at least two common neighbours. (Compare with the notion "strong" in the theory of parapolar spaces, see [4,14].) Dense near polygons satisfy several nice properties. By Lemma 19 of [2], every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines; we denote this number by $t_{\mathcal{S}} + 1$. If x and y are two points of a dense near polygon then, by Theorem 4 of [2], $\mathcal{C}(x, y)$ is the unique convex sub near $[2 \cdot d(x, y)]$-gon containing x and y. Convex sub near hexagons of a dense near polygon are called *hexes*. For

every point x of a dense near polygon $\mathcal{S}$, we can define a linear space $\mathcal{L}(x, \mathcal{S})$. The points, respectively lines, of $\mathcal{L}(x, \mathcal{S})$ are the lines, respectively quads, containing x and incidence is containment. The linear space $\mathcal{L}(x, \mathcal{S})$ is called the *local space at x*. The *modified local space at x* is the partial linear space obtained from $\mathcal{L}(x, \mathcal{S})$ by removing all lines of size 2.

Lemma 1 (*Shad and Shult [16], Shult and Yanushka [17]*). *Let $\mathcal{S}$ be a dense near polygon, let x be a point of $\mathcal{S}$ and let Q be a quad of $\mathcal{S}$. Then precisely one of the following cases occurs*:

(a) *There exists a unique point x' in Q nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, x') + \mathrm{d}(x', y)$ for every point y of Q. In this case we say that x is classical with respect to Q or that Q is gated with respect to x.*
(b) *The points of Q nearest to x form an ovoid of Q, i.e. a set of points meeting each line of Q at precisely one point. In this case we say that x is ovoidal with respect to Q.*

So, if x is classical, respectively ovoidal, with respect to a quad Q, then $\mathrm{d}(x, y)$ with $y \in Q$ takes three, respectively two, values.

The main result of this paper is the determination of all dense near polygons in which each hex is isomorphic to either the dual polar space $H^{\mathrm{D}}(5, q^2)$, the product near polygon $Q(5, q) \times L_{q+1}$ or a glued near hexagon of type $Q(5, q) \otimes Q(5, q)$. (We refer to Section 2 for the definitions of product and glued near polygons.) We will show, see also Theorem 1, that any such dense near $2n$-gon is isomorphic to the dual polar space $H^{\mathrm{D}}(2n-1, q^2)$, the product near polygon $H^{\mathrm{D}}(2n-3, q^2) \times L_{q+1}$, a product near polygon $H^{\mathrm{D}}(2n_1-1, q^2) \times H^{\mathrm{D}}(2n_2-1, q^2)$ ($n_1, n_2 \geqslant 2$ such that $n_1 + n_2 = n$), or a glued near polygon of type $H^{\mathrm{D}}(2n_1-1, q^2) \otimes H^{\mathrm{D}}(2n_2-1, q^2)$ ($n_1, n_2 \geqslant 2$ such that $n_1 + n_2 = n+1$).

This result is one of the various classification results regarding near polygons recently obtained. Others are the (almost complete) classification of all dense near hexagons of order $(3, t)$ [7], the classification of all near hexagons in which at least one local space is an affine plane [9,1], the classification of all dense near octagons of order $(2, t)$ [13] and the classification of all dense near polygons of order $(2, t)$ with a so-called "nice chain" of convex sub near polygons [12]. It is the aim of the author to construct new near polygons as a by-product of his attempts to classify near polygons which satisfy certain properties. In [1], this procedure was successful and a new near hexagon was found in which exactly two local spaces are affine planes of order 3. In the present case, however, we do not find new near polygons. Instead, we find a nice characterization of certain classes of near polygons.

2. Constructions of near polygons

2.1. Product near polygons

For any near polygons $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$, a new near polygon $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. We have: $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$, the point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y\ \mathrm{I}_2\ L$, the

point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x\ \mathrm{I}_1\ M$ and $y = u$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2d_i$-gon then $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1 + d_2)$-gon. If $d_1, d_2 \geqslant 1$, then we call $\mathcal{S}_1 \times \mathcal{S}_2$ a *product near polygon*. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are dense, then also $\mathcal{S}_1 \times \mathcal{S}_2$ is dense and $t_{\mathcal{S}} = t_{\mathcal{S}_1} + t_{\mathcal{S}_2} + 1$. Product near polygons can be characterized in the following ways.

Lemma 2 (*De Bruyn and Vandecasteele [11, Lemma 2]*). *Let $\mathcal{S}$ be a dense near polygon and let T_1 and T_2 be two partitions of $\mathcal{S}$ in convex sub near polygons. For every $i \in \{1, 2\}$ and every point x of $\mathcal{S}$, let $F_i(x)$ denote the unique element of T_i containing x. If $\Gamma_1(x) \subseteq F_1(x) \cup F_2(x)$ and $F_1(x) \cap F_2(x) = \{x\}$ for every point x of $\mathcal{S}$, then*

(a) *all elements of T_1 are isomorphic*,
(b) *all elements of T_2 are isomorphic*,
(c) *$\mathcal{S} \cong F_1(x) \times F_2(x)$ for every point x of $\mathcal{S}$.*

Lemma 3 (*De Bruyn and Vandecasteele [11, Lemma 3]*). *Let $\mathcal{S}$ be a dense near $2(n_1 + n_2)$-gon and let F_1 and F_2 be two convex sub near polygons such that* (i) *F_i has diameter $n_i \geqslant 1$ ($i \in \{1, 2\}$),* (ii) *F_1 intersects F_2 in a point, and* (iii) *every line through x is contained in either F_1 or F_2. Then $\mathcal{S} \cong F_1 \times F_2$.*

2.2. Glued near polygons

If K and L are two lines of a near polygon, then by Lemma 1 of [2], there are two possibilities:

- There exist unique points $k^* \in K$ and $l^* \in L$ such that $\mathrm{d}(k, l) = \mathrm{d}(k, k^*) + \mathrm{d}(k^*, l^*) + \mathrm{d}(l^*, l)$ for all points $k \in K$ and $l \in L$.
- For every point $k \in K$, there exists a unique point $l \in L$ such that $\mathrm{d}(K, L) = \mathrm{d}(k, l)$. In this case K and L are called *parallel* ($K \| L$).

A *spread* of a near polygon $\mathcal{A}$ is a set of lines partitioning the point set. If $\mathcal{A}$ is the direct product $\mathcal{B} \times L$ of a near polygon $\mathcal{B}$ and a line L, then the set $S = \{L_x | x \text{ is a point of } \mathcal{B}\}$ with $L_x := \{(x, y) | y \in L\}$ is a spread of $\mathcal{A}$. We call any such spread a *trivial spread*. A spread of $\mathcal{A}$ is called *admissible* if every two lines of it are parallel. A spread S of $\mathcal{A}$ is called *a spread of symmetry* if for every line $K \in S$ and for every two points k_1 and k_2 on K there exists an automorphism of $\mathcal{A}$ fixing each line of $\mathcal{S}$ and mapping k_1 to k_2. Every trivial spread is a spread of symmetry. Every spread of symmetry is an admissible spread, because the stabilizer of the spread acts transitively on the set of points of each line in the spread. If S is a nontrivial spread of symmetry of a near polygon $\mathcal{A}$, then the group of automorphisms of $\mathcal{A}$ fixing each line of S acts regularly on each line of S and hence has order $s + 1$, where $s + 1$ denotes the constant number of points on a line of S (see the remark following Theorem 9 of [8]).

2.2.1. Construction

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two near polygons both with constant line size $s + 1$, and suppose that their respective diameters d_1 and d_2 are at least 2. Let $S_i = \{L_1^{(i)}, \ldots, L_{\alpha_i}^{(i)}\}$, $i \in \{1, 2\}$, be

an admissible spread of $\mathcal{A}_i$. In S_i, a special line $L_1^{(i)}$ is chosen which we call the *base line* of the spread S_i. For every $i \in \{1,2\}$, for all $j,k \in \{1,\ldots,\alpha_i\}$ and for every $x \in L_j^{(i)}$, let $p_{j,k}^{(i)}(x)$ denote the unique point $L_k^{(i)}$ nearest to x. We put $\Phi_{j,k}^{(i)} := p_{k,1}^{(i)} \circ p_{j,k}^{(i)} \circ p_{1,j}^{(i)}$. For every $i \in \{1,2\}$, the group $\Pi_{S_i}(L_1^{(i)}) := \langle \Phi_{j,k}^{(i)} | 1 \leqslant j,\ k \leqslant \alpha_i \rangle$ is called the *group of projectivities of* $L_1^{(i)}$ *with respect to* S_i.

For every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$, we consider the following graph Γ_θ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two vertices $(x, L_{i_1}^{(1)}, L_{j_1}^{(2)})$ and $(y, L_{i_2}^{(1)}, L_{j_2}^{(2)})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(a) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $L_{j_1}^{(2)} = L_{j_2}^{(2)}$ and $x \neq y$,
(b) $L_{j_1}^{(2)} = L_{j_2}^{(2)}$, $\mathrm{d}(L_{i_1}^{(1)}, L_{i_2}^{(1)}) = 1$ and $\Phi_{i_1,i_2}^{(1)}(x) = y$,
(c) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $\mathrm{d}(L_{j_1}^{(2)}, L_{j_2}^{(2)}) = 1$ and $\Phi_{j_1,j_2}^{(2)} \circ \theta(x) = \theta(y)$.

By [8], the graph Γ_θ has diameter $d_1 + d_2 - 1$ and every two adjacent vertices are contained in a unique maximal clique (of size $s+1$). Considering these maximal cliques as lines, we obtain a partial linear space $\mathcal{S}_\theta$. If $\mathcal{S}_\theta$ is a near polygon, then it is called a *glued near polygon (of type* $\mathcal{A}_1 \otimes \mathcal{A}_2$*)*. The following lemma gives necessary and sufficient conditions for $\mathcal{S}_\theta$ to be a near polygon.

Lemma 4 (*De Bruyn [8, Theorem 14]*). *Let* $\theta, \mathcal{S}_\theta$ *and* $\Pi_{S_i}(L_1^{(i)})$ *be as above, then the partial linear space* $\mathcal{S}_\theta$ *is a near polygon if and only if the commutator* $[\Pi_{S_1}(L_1^{(1)}), \theta^{-1}\Pi_{S_2}(L_1^{(2)})\theta]$ *is the trivial group of permutations of* $L_1^{(1)}$.

The following two lemmas explain why the notion spread of symmetry is important in the theory of glued near polygons.

Lemma 5 (*De Bruyn [8, Theorem 16]*). *Let* $\mathcal{S}_\theta$, S_1 *and* S_2 *be as above. If* $\mathcal{S}_\theta$ *is a near polygon and if none of the spreads* S_1 *and* S_2 *is trivial, then* S_1 *and* S_2 *are spreads of symmetry*.

Lemma 6 (*De Bruyn [6, Section 3; 8, Theorem 10]*). *For every* $i \in \{1,2\}$, *let* $\mathcal{A}_i$ *be a near polygon, let* S_i *be a nontrivial spread of symmetry of* $\mathcal{A}_i$ *and let* G_i *denote the group of automorphisms of* $\mathcal{A}_i$ *fixing each line of* S_i. *Then there exists a glued near polygon arising from* $\mathcal{A}_1, \mathcal{A}_2, S_1$ *and* S_2 (*for a certain choice of the base lines and the bijection* θ *between these base lines*) *if and only if* G_1 *is isomorphic to* G_2.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are dense near polygons, then every glued near polygon $\mathcal{S}$ of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also dense and $t_{\mathcal{S}} = t_{\mathcal{A}_1} + t_{\mathcal{A}_2}$. We refer to Section 3 of [10] for a discussion of the convex sub near polygons of glued near polygons. If $\mathcal{A}_i$, $i \in \{1,2\}$, is a generalized quadrangle of order (s,t_i) and if $\mathcal{S}$ is a glued near hexagon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$, then every quad of $\mathcal{S}$ is isomorphic to $\mathcal{A}_1$, $\mathcal{A}_2$ or the $(s+1) \times (s+1)$-grid.

2.2.2. Characterization

If we consider a tuple $(\mathcal{A}_1, \mathcal{A}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ which gives rise to a glued near polygon $\mathcal{A}_1 \otimes \mathcal{A}_2$ and if we define

$$T_1 := \left\{ \{(x, L, M) | x \in L_1^{(1)}, L \in S_1\} | M \in S_2 \right\},$$

$$T_2 := \left\{ \{(x, L, M) | x \in L_1^{(1)}, M \in S_2\} | L \in S_1 \right\},$$

then T_i, $i \in \{1, 2\}$, is a partition of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in convex sub near polygons isomorphic to $\mathcal{A}_i$. Now, if we have an arbitrary near polygon $\mathcal{A}$, then with every tuple which gives rise to a glued near polygon isomorphic to $\mathcal{A}$, there corresponds a pair $\{T_1, T_2\}$ of partitions of $\mathcal{A}$ in convex sub near polygons, as described above. We denote by $\Delta_1(\mathcal{A})$ the set of all pairs $\{T_1, T_2\}$ arising in this way. If $\{T_1, T_2\} \in \Delta_1(\mathcal{A})$, then:

- all elements of T_i, $i \in \{1, 2\}$, are isomorphic;
- every element of T_1 intersects every element of T_2 in a line;
- if x denotes an arbitrary point of $\mathcal{A}$ and if F_i, $i \in \{1, 2\}$, denotes the unique element of T_i containing x, then every line through x different from $F_1 \cap F_2$ is contained in either F_1 or F_2.

These properties have been used in the following characterization result.

Lemma 7 (*De Bruyn [10, Theorem 10]*). *Let $\mathcal{A}$ be a dense near polygon and let T_1 and T_2 denote two partitions of $\mathcal{A}$ into convex sub near polygons. For every point x of $\mathcal{S}$ and every $i \in \{1, 2\}$, let $F_i(x)$ denote the unique element of T_i containing x and put $F_i'(x) := \mathcal{C}(\Gamma_1(x) \setminus F_{3-i}(x))$. Suppose that* $\mathrm{diam}(F_1(x) \cap F_2(x)) = 1$ *and $\Gamma_1(x) \subseteq F_1(x) \cup F_2(x)$ for every point x of $\mathcal{A}$. Then at least one of the following cases occurs*:

- $F_1'(x) = F_1(x)$ and $F_2'(x) = F_2(x)$ for every point x of $\mathcal{A}$. Then $\mathcal{A}$ is glued and $\{T_1, T_2\} \in \Delta_1(\mathcal{A})$.
- There exists a point x^* such that $F_1'(x^*) \cap F_2(x^*) = \{x^*\}$. Then $\mathcal{A} \cong F_1'(x^*) \times F_2(x^*)$.
- There exists a point x^* such that $F_2'(x^*) \cap F_1(x^*) = \{x^*\}$. Then $\mathcal{A} \cong F_1(x^*) \times F_2'(x^*)$.

Remark. Note the similarity between Lemmas 2 and 7.

3. The classification result

Let $q = p^h$, p prime and $h \in \mathbb{N} \setminus \{0\}$, be a prime power and let $n \in \mathbb{N} \setminus \{0, 1\}$. Consider in the projective space $\mathrm{PG}(2n-1, q^2)$ a nonsingular hermitian variety $H(2n-1, q^2)$. Then the incidence structure $H^{\mathrm{D}}(2n-1, q^2)$ with points, respectively lines, the subspaces of $H(2n-1, q^2)$ of dimension $n-1$ (called generators), respectively $n-2$ (natural incidence), is a dense near $2n$-gon [3]. The near polygon $H^{\mathrm{D}}(2n-1, q^2)$ belongs to the family of the so-called dual polar spaces and is a dense near polygon of order $(q, \mu(n) - 1)$ where $\mu(n) := \frac{q^{2n}-1}{q^2-1}$. The generalized quadrangle $H^{\mathrm{D}}(3, q^2)$ is isomorphic to the generalized quadrangle $Q(5, q)$ arising from the points and lines of a nonsingular elliptic quadric in

PG$(5, q)$. If α is an $(n-3)$-dimensional subspace of $H(2n-1, q^2)$, then the $(q+1)(q^3+1)$ generators containing α form a quad α^ϕ of $H^{\mathrm{D}}(2n-1, q^2)$ isomorphic to $Q(5, q)$. Conversely, every quad is obtained in that way. Let Π be a hyperplane of PG$(2n-1, q^2)$ intersecting $H(2n-1, q^2)$ in a nonsingular hermitian variety $H(2n-2, q^2)$. Then the $(n-2)$-dimensional subspaces of $H(2n-2, q^2)$ (= the generators of $H(2n-2, q^2)$) form a spread S of $H^{\mathrm{D}}(2n-1, q^2)$. The spread S is a spread of symmetry and the group of automorphisms of $H^{\mathrm{D}}(2n-1, q^2)$ fixing each line of S is isomorphic to the cyclic group C_{q+1} of order $q+1$. (If we choose our reference system in PG$(2n-1, q^2)$ in such a way that $H(2n-1, q^2)$ has equation $X_0^{q+1} + X_1^{q+1} + \cdots + X_{2n-1}^{q+1} = 0$ and Π has equation $X_{2n-1} = 0$, then the $q+1$ automorphisms of PG$(2n-1, q^2)$ determined by $X_i \mapsto X_i$, $i \in \{0, \ldots, 2n-2\}$, $X_{2n-1} \mapsto \lambda X_{2n-1}$ ($\lambda \in \mathrm{GF}(q^2)$ such that $\lambda^{q+1} = 1$) determine a group of $q+1$ automorphisms of $H^{\mathrm{D}}(2n-1, q^2)$ fixing (and acting regularly on) each line of S.) By Lemma 6, it follows that there exist glued near polygons of type $H^{\mathrm{D}}(2n_1-1, q^2) \otimes H^{\mathrm{D}}(2n_2-1, q^2)$ for all $n_1, n_2 \in \mathbb{N} \setminus \{0, 1\}$ and all prime powers q. By Section 7.2 of [5], the number $N(q)$ of glued near polygons of type $Q(5, q) \otimes Q(5, q)$ is equal to $\frac{\phi(q+1)+R}{h}$. Here $\phi(q+1)$ is the number of elements of $\{1, 2, \ldots, q\}$ relative prime to $q+1$ (ϕ = Euler indicator) and R is the number of elements m of $\{1, \ldots, q\}$ for which $(m, q+1) = 1$ and $m^2 \equiv p^i \pmod{q+1}$ for a certain $i \in \mathbb{N}$. We list the number $N(q)$ in the following table for the smallest values of q.

q	$N(q)$	q	$N(q)$	q	$N(q)$
2	1	7	2	13	2
3	1	8	1	16	2
4	1	9	1	17	2
5	1	11	2	19	3

The following lemma is an immediate consequence of a more general study of convex sub near polygons in product and glued near polygons, see Section 3 of [10].

Lemma 8. *Let $\mathcal{S}$ be a near polygon isomorphic to $H^{\mathrm{D}}(2n-1, q^2)$, $H^{\mathrm{D}}(2n-3, q^2) \times L_{q+1}$, $H^{\mathrm{D}}(2n_1-1, q^2) \times H^{\mathrm{D}}(2n_2-1, q^2)$ or a glued near polygon of type $H^{\mathrm{D}}(2n_1-1, q^2) \otimes H^{\mathrm{D}}(2n_2-1, q^2)$. Then every hex of $\mathcal{S}$ is isomorphic to $H^{\mathrm{D}}(5, q^2)$, $Q(5, q) \times L_{q+1}$ or a glued near hexagon of type $Q(5, q) \otimes Q(5, q)$.*

(Recall that $Q(5, q) \cong H^{\mathrm{D}}(3, q^2)$.) In the present paper, we will prove that also the converse holds.

Theorem 1. *Let $\mathcal{S}$ be a dense near $2n$-gon with $n \geqslant 3$. Suppose one of the following holds for any hex H of $\mathcal{S}$:*

(i) $H \cong H^{\mathrm{D}}(5, q^2)$,
(ii) $H \cong Q(5, q) \times L_{q+1}$,
(iii) *H is of type $Q(5, q) \otimes Q(5, q)$,*

then $\mathcal{S}$ is one of the following near polygons:

(a) $H^{\mathrm{D}}(2n-1, q^2)$,
(b) $H^{\mathrm{D}}(2n-3, q^2) \times L_{q+1}$,
(c) $H^{\mathrm{D}}(2n_1-1, q^2) \times H^{\mathrm{D}}(2n_2-1, q^2)$ with $n_1, n_2 \geqslant 2$ such that $n_1 + n_2 = n$,
(d) *a glued near polygon of type* $H^{\mathrm{D}}(2n_1-1, q^2) \otimes H^{\mathrm{D}}(2n_2-1, q^2)$ *where* $n_1, n_2 \geqslant 2$ *such that* $n_1 + n_2 = n+1$.

4. Classical and big convex sub near polygons

Definition. Let $\mathcal{A}$ be a dense near polygon. A point x of $\mathcal{A}$ is called *classical* with respect to a convex sub near polygon F of $\mathcal{A}$ if there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. A convex sub near polygon F of $\mathcal{A}$ is called *classical* if every point of $\mathcal{A}$ is classical with respect to F. A proper convex sub near polygon F of $\mathcal{A}$ is called *big* if every point of $\mathcal{A}$ not contained in F is collinear with a (necessarily unique) point of F.

Lemma 9. *Let $\mathcal{A}$ be a dense near polygon, let F denote a convex sub near polygon of $\mathcal{A}$ and let x be a point of $\mathcal{A}$ such that* $\mathrm{d}(x, F) = 1$. *Then x is classical with respect to F.*

Proof. Let x' denote the unique point of F collinear with x. Suppose there exists a point y in F such that $\mathrm{d}(y, x) \neq 1 + \mathrm{d}(y, x')$. Then $\mathrm{d}(y, x) \leqslant \mathrm{d}(y, x')$. Hence, the unique point z of the line xx' nearest to y lies at distance $\mathrm{d}(y, x') - 1$ from y. So, z is contained on a shortest path between the points y and x' of F. This would imply that $z \in F$ and $x'z \subseteq F$, contradicting $x \notin F$. □

Lemma 10. *A convex sub near $2(m-1)$-gon F of a dense near $2m$-gon $\mathcal{A}$ is big if and only if it is classical.*

Proof. If F is big, then F is also classical by Lemma 9. Conversely, suppose that F is classical and let x denote a point of $\mathcal{A}$ not contained in F. Let y denote a point of F at maximal distance $m-1$ from $\pi_F(x)$. Then $\mathrm{d}(x, \pi_F(x)) = \mathrm{d}(x, y) - \mathrm{d}(\pi_F(x), y) \leqslant m - (m-1) = 1$. Hence x is collinear with $\pi_F(x)$. This proves that F is big. □

Lemma 11. *Let F be a big convex sub near polygon of a dense near polygon $\mathcal{A}$. Then every convex sub near polygon F' which meets F is either contained in F or intersects F in a big convex sub near polygon of F'.*

Proof. Suppose that F' is not contained in F and let x denote a common point of F and F'. Let y denote an arbitrary point of F' not contained in $F \cap F'$. Then y is collinear with a unique point y' of F. Since F is classical, $\mathrm{d}(y, x) = \mathrm{d}(y, y') + \mathrm{d}(y', x')$. Hence, the point y', which lies on a shortest path between the points x and y of F', also belongs to F'. So, every point of F' not contained in $F \cap F'$ is collinear with a unique point of $F \cap F'$. This proves the lemma. □

Lemma 12. *Let $\mathcal{A}$ be a dense near $2n$-gon of order (s,t). If $\mathcal{A}$ has a big convex sub near polygon F of order (s,t') with v_F points, then every convex sub near polygon of order (s,t') with v_F points is big in $\mathcal{A}$.*

Proof. The number of points at distance at most 1 from F is equal to $v_F + v_F(t-t')s$ and this is precisely the number $v_{\mathcal{A}}$ of points of $\mathcal{A}$. If G is another convex sub near polygon of $\mathcal{A}$ of order (s,t') with v_F points, then the number of points at distance at most 1 from G is also $v_{\mathcal{A}} = v_F + v_F(t-t')s$. This proves that G is big. □

Lemma 13. *Let $\mathcal{A}$ be a dense near $2n$-gon of order (s,t) with the property that every two parallel lines at distance 1 from each other are contained in a subgrid of order $(s,1)$. Let F_1 and F_2 be two disjoint big convex sub near polygons of $\mathcal{A}$. Then*

(a) *F_1 and F_2 are isomorphic,*
(b) *there exists a set $\{F_1, F_2, \ldots, F_{s+1}\}$ of mutually disjoint big convex sub near polygons of $\mathcal{S}$ such that every line that meets F_1 and F_2 also meets every F_i, $i \in \{3, \ldots, s+1\}$.*

Proof. (a) For every $i \in \{1,2\}$ and every point x of F_i, let $\pi_{i,3-i}(x)$ denote the unique point of F_{3-i} collinear with x. Since $\pi_{3-i,i} \circ \pi_{i,3-i}(x) = x$ for every point x of F_i, $\pi_{1,2}$ and $\pi_{2,1}$ are bijections and $\pi_{2,1} = \pi_{1,2}^{-1}$. We will now show that $\pi_{1,2}$ is an isomorphism. It suffices to show that $\pi_{1,2}$ maps collinear points x and y of F_1 to collinear points $\pi_{1,2}(x)$ and $\pi_{1,2}(y)$ of F_2. Since the point $\pi_{1,2}(y)$ does not lie on the line xy, $\mathrm{d}(x, \pi_{1,2}(y)) = 2$. Now, $2 = \mathrm{d}(x, \pi_{1,2}(y)) = \mathrm{d}(x, \pi_{1,2}(x)) + \mathrm{d}(\pi_{1,2}(x), \pi_{1,2}(y)) = 1 + \mathrm{d}(\pi_{1,2}(x), \pi_{1,2}(y))$. So, $\pi_{1,2}(x)$ and $\pi_{1,2}$ are collinear.

(b) Let x_1 and y_1 denote two points of F_1 at distance $n-1$ from each other. Let x_2, respectively y_2, denote the unique point of F_2 collinear with x_1, respectively y_1. Put $L = x_1x_2$ and $M = y_1y_2$.

- We will show that L and M are parallel lines at distance $n-1$ from each other. If y_1 is not the unique point of M nearest to x_1, then the line M would be contained in F_1, a contradiction. Hence, y_1 is the unique point of M nearest to x_1. Similarly, y_2 is the unique point of M nearest to x_2. Since $\pi_M(x_1) \neq \pi_M(x_2)$, the lines L and M are parallel. Also, $\mathrm{d}(L,M) = \mathrm{d}(x_1, \pi_M(x_1)) = \mathrm{d}(x_1, y_1) = n-1$.
- Put $L = \{x_1, x_2, \ldots, x_{s+1}\}$ and $M = \{y_1, y_2, \ldots, y_{s+1}\}$ such that $\mathrm{d}(x_i, y_i) = n-1$ for every $i \in \{1, \ldots, s+1\}$. Put $F_i := \mathcal{C}(x_i, y_i)$ for every $i \in \{1, \ldots, s+1\}$. We will now show that $F_i \cap F_j = \emptyset$ for all $i, j \in \{1, \ldots, s+1\}$ with $i \neq j$. Suppose that u is a common point of F_i and F_j. Since x_i is classical with respect to F_j (see Lemma 9), $\mathrm{d}(x_i, u) = \mathrm{d}(x_i, x_j) + \mathrm{d}(x_j, u)$. So, x_j is on a shortest path between x_i and u. Since $x_i, u \in F_i$, also $x_j \in F_i$, a contradiction.
- Let (s,t') denote the order of F_1 (or of F_2) and let $N_1, \ldots, N_{t'+1}$ denote the $t'+1$ lines of F_1 through the point y_1. We will now show that each quad $\mathcal{C}(M, N_i)$, $i \in \{1, \ldots, t'+1\}$, intersects each F_j, $j \in \{1, \ldots, s+1\}$, in a line. Obviously, this holds for $j = 1$. It also holds for $j = 2$ by Lemma 11. Suppose therefore that $j \geqslant 3$. It suffices to show that $\mathcal{C}(M, N_i)$ contains a point of $\Gamma_1(y_j) \cap \Gamma_{n-2}(x_j)$. Let z_1 denote the unique point on N_i at distance $n-2$ from x_1 and let z_2 denote the unique point of F_2 collinear with z_1. Since $\pi_{1,2}$ is an isomorphism, $\mathrm{d}(x_2, z_2) = \mathrm{d}(x_1, z_1) = n-2$. Since $\mathrm{d}(x_j, z_1) =$

$d(x_j, x_1) + d(x_1, z_1) = n-1$, $d(x_j, y_1) = n$, $d(x_j, z_2) = d(x_j, x_2) + d(x_2, z_2) = n-1$ and $d(x_j, y_2) = n$, it follows that the point x_j cannot be ovoidal with respect to $\mathcal{C}(M, N_i)$. So, x_j is classical with respect to $\mathcal{C}(M, N_i)$ and there exists a unique point z_j in $\mathcal{C}(M, N_i)$ at distance $n-2$ from x_j. The point y_j has distance $n-1$ from x_j and hence is collinear with z_j. This proves the statement. (Remark: Since z_1 and z_2 are also collinear with z_j, the point z_j necessarily lies on the line z_1z_2.)

- We will now show that every F_j, $j \in \{3, \ldots, s+1\}$, is isomorphic to F_1. By Lemma 12, it then follows that every element of $\{F_1, \ldots, F_{s+1}\}$ is big. Since every quad $\mathcal{C}(M, N_i)$, $i \in \{1, \ldots, t'+1\}$, intersects F_j in a line, F_j is a dense near polygon of order (s, t'_j) with $t'_j \geqslant t'$. Now, $\pi_{F_1}(F_j)$ is a (not necessarily convex) sub near polygon of F_1 isomorphic to F_j. Hence, $t'_j = t'$ and $F_j \cong \pi_{F_1}(F_j) = F_1$.
- We will now show that every point of F_j, $j \in \{3, \ldots, s+1\}$, lies on a line through a point of F_1 and a point of F_2. Let X_j denote the set of points of F_j which lie on a line through a point of F_1 and a point of F_2. Let $\tilde{X}_j$ denote the set of points $u \in X_j$ such that $\Gamma_{n-1}(u) \cap X_j \neq \emptyset$. By the previous reasoning, we know that if $u \in \tilde{X}_j$, then every point of $\Gamma_1(u) \cap F_j$ belongs to X_j. (The point y_j belongs to $\tilde{X}_j$ since $x_j \in \Gamma_{n-1}(y_j) \cap F_j \cap X_j$. If $u \in \Gamma_1(y_j) \cap F_j$, then the point u is contained in one of the quads $\mathcal{C}(M, N_i)$, $i \in \{1, \ldots, t'+1\}$, and hence belongs to X_j.) We will now show that if $u \in \tilde{X}_j$, then every point u' of $\Gamma_1(u) \cap F_j$ belongs to $\tilde{X}_j$. Let v be a point of $\Gamma_{n-1}(u) \cap F_j \cap X_j$ and let u'' denote the unique point of the line uu' nearest to v. Let T denote a line of F_j through v not contained in $\mathcal{C}(u'', v)$. Then one easily sees that $T \| uu'$. So, there exists a point v' on T at distance $n-1$ from u'. Since $v \in \tilde{X}_j$, $v' \in X_j$ and hence $u' \in \tilde{X}_j$. By the connectedness of F_j, it follows that every point of F_j belongs to $\tilde{X}_j$. This completes the proof of the lemma. □

5. Proof of Theorem 1

Let $\mathcal{S}$ be a dense near $2n$-gon satisfying the conditions of Theorem 1.

Lemma 14. *Every quad of $\mathcal{S}$ is isomorphic to either $Q(5, q)$ or the $(q+1) \times (q+1)$-grid.*

Proof. Every quad is contained in a hex and each of the possible hexes $H^D(5, q^2)$, $Q(5, q) \times L_{q+1}$, $Q(5, q) \otimes Q(5, q)$ contains only quads isomorphic to $Q(5, q)$ or the $(q+1) \times (q+1)$-grid. □

In the sequel we will talk about $Q(5, q)$-*quads* or *grid-quads*.

Lemma 15. (a) *Every point is contained in a $Q(5, q)$-quad.*
(b) *If $\mathcal{S}$ contains a grid-quad, then every point is contained in a grid-quad.*
(c) *If $\mathcal{S}$ does not contain grid-quads, then $\mathcal{S} \cong H^D(2n-1, q^2)$.*

Proof. (a) Every point x is contained in a hex H. In this hex, there always is a $Q(5, q)$-quad containing x.

(b) By connectedness of $\mathcal{S}$, it suffices to show that if a point x is contained in a grid-quad Q_x, then every point y collinear with x is contained in a grid-quad. This is easily seen by considering a hex containing y and Q_x.

(c) If $\mathcal{S}$ does not contain grid-quads, then every quad is isomorphic to $Q(5,q)$. Since $Q(5,q)$ has no ovoids [16,18], every point is classical with respect to every quad. Such a near polygon in which each point–quad relation is classical is called *classical*. By [3], $\mathcal{S}$ is isomorphic to a dual polar space P^{D}. (The points, respectively lines, of P^{D} are the maximal, respectively next-to-maximal, totally isotropic subspaces of a polar space P (natural incidence).) By the classification of polar spaces [19], the polar space P is associated with a nonsingular hermitian variety in a projective space of odd dimension. Since the diameter of $\mathcal{S}$ is n, we have $P^{\mathrm{D}} \cong H^{\mathrm{D}}(2n-1,q^2)$. □

Lemma 16. *If a convex sub near polygon F of $\mathcal{S}$ is isomorphic to $H^{\mathrm{D}}(2m-1,q^2)$, $m \geqslant 2$, then F is classical in $\mathcal{S}$.*

Proof. Let x denote an arbitrary point of $\mathcal{S}$ and let x' denote a point of F nearest to x. Then every neighbour of x' in F has distance $\mathrm{d}(x,x')+1$ from x. Let $\delta \in \{0,\ldots,m\}$ denote the biggest integer with the property that every point of $\Gamma_\delta(x') \cap F$ has distance $\mathrm{d}(x,x')+\delta$ from x. Clearly $\delta \geqslant 1$. If $\delta = m$, then F is classical. Suppose that $\delta \in \{1,\ldots,m-1\}$ and let u be a point of $\Gamma_{\delta+1}(x') \cap F$ for which $\mathrm{d}(x,u) \neq \mathrm{d}(x,x')+(\delta+1)$. Let $v \in \Gamma_1(u) \cap \Gamma_\delta(x')$ and $w \in \Gamma_1(v) \cap \Gamma_{\delta-1}(x')$ and consider the quad $Q := \mathcal{C}(u,w)$. There are two possibilities:

(i) x is classical with respect to Q. Let $\tilde{x}$ denote the unique point of Q nearest to x. Since $\mathrm{d}(x,w) = \mathrm{d}(x,x')+\delta-1$ and $\mathrm{d}(x,v) = \mathrm{d}(x,x')+\delta$, we have $\mathrm{d}(x,\tilde{x}) \in \{\mathrm{d}(x,x')+\delta-2, \mathrm{d}(x,x')+\delta-1\}$. If $\mathrm{d}(x,\tilde{x}) = \mathrm{d}(x,x')+\delta-2$, then $\tilde{x}$ is collinear with w. But every point of Q collinear with w has distance δ from x' and hence distance $\mathrm{d}(x,x')+\delta$ from x, a contradiction. So, $\mathrm{d}(x,\tilde{x}) = \mathrm{d}(x,x')+\delta-1$ or $\tilde{x} = w$. But then $\mathrm{d}(x,u) = \mathrm{d}(x,\tilde{x})+\mathrm{d}(\tilde{x},u) = \mathrm{d}(x,x')+(\delta-1)+2 = \mathrm{d}(x,x')+\delta+1$, again a contradiction.

(ii) x is ovoidal with respect to Q. Then the points in Q nearest to x form an ovoid of Q. Now, $Q \cong Q(5,q)$ and $Q(5,q)$ has no ovoids, a contradiction. □

From Lemmas 10 and 16, the following corollary immediately follows.

Corollary 1. *If a convex sub near polygon F of $\mathcal{S}$ is isomorphic to $H^{\mathrm{D}}(2n-3,q^2)$, then F is big in $\mathcal{S}$.*

We now distinguish two possibilities.

Case I: *$\mathcal{S}$ contains a (big) convex sub near polygon isomorphic to $H^{\mathrm{D}}(2n-3,q^2)$*
In this case, Theorem 1 is equivalent to the following result.

Proposition 1. *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geqslant 3$, that satisfies the following properties*:

- *every hex is isomorphic to either $H^{\mathrm{D}}(5,q^2)$, $Q(5,q) \times L_{q+1}$ or a glued near hexagon of type $Q(5,q) \otimes Q(5,q)$;*
- *$\mathcal{S}$ has a big convex sub near polygon isomorphic to $H^{\mathrm{D}}(2n-3,q^2)$.*

Then $\mathcal{S}$ is isomorphic to either $H^{\mathrm{D}}(2n-1,q^2)$, $H^{\mathrm{D}}(2n-3,q^2)\times L_{q+1}$ or a glued near polygon of type $H^{\mathrm{D}}(2n-3,q^2)\otimes Q(5,q)$.

The proof of Proposition 1 is divided into several parts.

Lemma 17. *Let $\mathcal{L}$ be the linear space whose points are the points of $H := H(2n-1,q^2)$ and whose lines are the lines of $\mathrm{PG}(2n-1,q^2)$ that intersect H in at least two points. Then the subspaces of $\mathcal{L}$ are precisely the intersections of H with subspaces of $\mathrm{PG}(2n-1,q^2)$.*

Proof. Every set X of points of H generates on the one hand a subspace $\langle X\rangle$ of $\mathrm{PG}(2n-1,q^2)$ and on the other hand a subspace $\overline{X}$ of $\mathcal{L}$. Since $\langle X\rangle\cap H$ is a subspace of $\mathcal{L}$, we necessarily have that $\overline{X}\subseteq\langle X\rangle\cap H$. We will now prove by induction on the dimension m of $\langle X\rangle$ that $\overline{X}=\langle X\rangle\cap H$ $(*)$. Obviously, this property holds if $m\leqslant 1$. If $m=2$, then $\langle X\rangle\cap H$ is either the plane $\langle X\rangle$, a unital or a cone pB with p a point and B a Baer subline of $\langle X\rangle$. One easily sees that Property $(*)$ holds for each of these cases. So, suppose that $m\geqslant 3$ and consider a subset X' of X generating an $(m-1)$-dimensional subspace. By the induction hypothesis, we know that $\langle X'\rangle\cap H=\overline{X'}$ is a subset of $\overline{X}$. Now consider a fixed point x in $X\setminus\langle X'\rangle$. For every point y of $\langle X\rangle\cap H$ different from x, let y' be the intersection point of the line xy with the subspace $\langle X'\rangle$. We can distinguish the following two possibilities.

- Suppose that there exists a Baer subline L in $\langle X'\rangle$ through the point y'. Since Property $(*)$ holds in the plane $\langle L,x\rangle$, we see that $y\in\overline{X}$.
- Suppose that every line of $\langle X'\rangle$ through y' is a tangent line. Then every point of X' is contained in the hyperplane y'^{ζ}, where ζ denotes the hermitian polarity of $\mathrm{PG}(2n-1,q^2)$ associated with H. So, we have that $y'\in\langle X'\rangle\subseteq y'^{\zeta}$. Hence $y'\in H$ and the point $y\in xy'$ belongs to $\overline{X}$.

Since also the point x belongs to $\overline{X}$, we see that $\langle X\rangle\cap H\subseteq\overline{X}$ and hence $\overline{X}=\langle X\rangle\cap H$. The lemma now easily follows. □

Let F denote a big convex sub near polygon of $\mathcal{S}$ isomorphic to $H^{\mathrm{D}}(2n-3,q^2)$. For every line L intersecting F in a point x, let $\mathcal{A}_L$ be the set of lines of F through x which are contained in a $Q(5,q)$-quad together with L. If L' is a line through x different from L and not contained in F, then by Lemmas 11 and 14, $\mathcal{C}(L,L')\cong Q(5,q)$. Hence $t_{\mathcal{S}}-t_F=(q^2-1)|\mathcal{A}_L|+1$. As a consequence, $|\mathcal{A}_L|$ is independent of the choice of L and equal to $\alpha:=\frac{t_{\mathcal{S}}-t_F-1}{q^2-1}$.

Lemma 18. *$\mathcal{A}_L$ is a subspace of $\mathcal{L}(x,F)$.*

Proof. Let K_1 and K_2 be two different lines of $\mathcal{A}_L$ and let K_3 be an arbitrary line through x contained in $\mathcal{C}(K_1,K_2)$. Since the hex $\mathcal{C}(L,K_1,K_2)$ has at least three $Q(5,q)$-quads containing x, it must be isomorphic to $H^{\mathrm{D}}(5,q^2)$. Hence, $\mathcal{C}(L,K_3)\cong Q(5,q)$ and $K_3\in\mathcal{A}_L$. This proves the lemma. □

Lemma 19. $\alpha \in \{0, 1, t_F + 1\}$.

Proof. We suppose that $2 \leqslant \alpha \leqslant t_F$ and derive a contradiction.

(a) Suppose first that $n = 4$, so $F \cong H^{\mathrm{D}}(5, q^2)$. Let x be an arbitrary point of F and let L be an arbitrary line through x not contained in F. Since $\mathcal{A}_L$ is a subspace and $2 \leqslant |\mathcal{A}_L| \leqslant t_F = q^4 + q^2$, there exists a $Q(5, q)$-quad Q_x containing x such that $\mathcal{A}_L$ is the set of $q^2 + 1$ lines of Q_x through x. Moreover, $\alpha = q^2 + 1$, $t_{\mathcal{S}} = t_F + (q^2 - 1)\alpha + 1 = 2q^4 + q^2$ and $\mathcal{C}(L, Q_x) \cong H^{\mathrm{D}}(5, q^2)$. If L' is a line through x different from L and not contained in F, then $\mathcal{C}(L, L')$ is isomorphic to $Q(5, q)$ and hence intersects F in a line of Q_x. It follows that every line through x not contained in F is contained in the hex $H_x := \mathcal{C}(L, Q_x)$. Since $H_x \cong H^{\mathrm{D}}(5, q^2)$, we have $H_x = \mathcal{C}(\Gamma_1(x) \setminus F)$. So, H_x and $Q_x = H_x \cap F$ only depend on x and not on the particular line L through x. If $y \in Q_x$, then $H_y = H_x$ and $Q_y = Q_x$. Hence, the quads Q_x, $x \in F$, partition the point set of F, and the hexes H_x, $x \in F$, partition the point set of $\mathcal{S}$. By Lemma 12, every hex isomorphic to F is big. In particular, each of the hexes H_x, $x \in F$, is big. The total number of quads Q_x, $x \in F$, equals $\frac{|H^{\mathrm{D}}(5,q^2)|}{|Q(5,q)|} = q^5 + 1$. Let X denote the set of $q^5 + 1$ points of $H(5, q^2)$ corresponding to the quads Q_x, $x \in F$. If u is a point of X, then we denote the $Q(5, q)$-quad of F corresponding with it by u^ϕ and the unique hex of $\mathcal{S}$ intersecting F in u^ϕ by $H(u)$. We will now show that X is a subspace of the linear space $\mathcal{L}$ defined in Lemma 17. Consider two different points u_1 and u_2 in X. Since u_1^ϕ and u_2^ϕ are disjoint, $u_1u_2 \cap H(5, q^2)$ is a Baer subline. By Lemma 13, there exists a unique set $\{u_1^\phi, u_2^\phi, u_3^\phi, \ldots, u_{q+1}^\phi\}$ of mutually disjoint big quads of F such that every line meeting u_1^ϕ and u_2^ϕ meets each of the quads of the set. Since every quad u^ϕ which meets u_1^ϕ and u_2^ϕ also meets u_i^ϕ, $i \in \{3, \ldots, q + 1\}$, every point u of $H(5, q^2)$ which is $H(5, q^2)$-collinear with u_1 and u_2 is also $H(5, q^2)$-collinear with u_i, $i \in \{3, \ldots, q + 1\}$. This proves that $\{u_1, u_2, \ldots, u_{q+1}\}$ is the Baer subline $u_1u_2 \cap H(5, q^2)$. Now, put $H_1 := H(u_1)$ and $H_2 := H(u_2)$. By Lemma 13, there exists a unique set $\{H_1, H_2, \ldots, H_{q+1}\}$ of mutually disjoint big hexes of $\mathcal{S}$ such that every line which meets H_1 and H_2 also meets H_i, $i \in \{3, \ldots, q+1\}$. Since $H_1 \cap F = u_1^\phi$ and $H_2 \cap F = u_2^\phi$, every H_i ($i \in \{3, \ldots, q+1\}$) will intersect F in a quad of the set $\{u_3^\phi, \ldots, u_{q+1}^\phi\}$. Without loss of generality, we may suppose that $H_i \cap F = u_i^\phi$. It is now clear that $u_i \in X$ for every $i \in \{3, \ldots, q + 1\}$. Hence, X is a subspace of the linear space $\mathcal{L}$. Hence, there exists a subspace π such that $X = \pi \cap H(5, q^2)$. Since no two points of X are collinear on $H(5, q^2)$, we have $|X| \leqslant |H(2, q^2)| = q^3 + 1$, contradicting $|X| = q^5 + 1$.

(b) We suppose that $n \geqslant 5$. Let x denote an arbitrary point of F and let L denote an arbitrary line through x not contained in F. Since $2 \leqslant \alpha \leqslant t_F$, there exist lines K_1, K_2 and K_3 through x such that $K_1, K_2 \in \mathcal{A}_L$, $K_1 \neq K_2$ and $K_3 \notin \mathcal{A}_L$. Now, the near octagon $\mathcal{C}(L, K_1, K_2, K_3)$ contradicts (a). □

Lemma 20. *If $\alpha = 0$, then $\mathcal{S} \cong H^{\mathrm{D}}(2n - 3, q^2) \times L_{q+1}$. If $\alpha = 1$, then $\mathcal{S}$ is a glued near polygon of type $H^{\mathrm{D}}(2n - 3, q^2) \otimes Q(5, q)$. If $\alpha = t_F + 1$, then $\mathcal{S} \cong H^{\mathrm{D}}(2n - 1, q^2)$.*

Proof. If $\alpha = 0$, then $t_{\mathcal{S}} - t_F = 1$ and hence $\mathcal{S} \cong H^{\mathrm{D}}(2n - 3, 4) \times L_{q+1}$ by Lemma 3. If $\alpha = t_F + 1$, then no grid-quad intersects F and hence $F \cong H^{\mathrm{D}}(2n - 1, q^2)$ by Lemma 15.

Suppose now that $\alpha = 1$, then $t_{\mathcal{S}} = t_F + q^2$. If x is an arbitrary point of F and if L and M are two lines through x not contained in F, then by Lemmas 11 and 14, $\mathcal{C}(K, L) \cong Q(5, q)$. Hence, every point $x \in F$ is contained in a unique $Q(5, q)$-quad Q_x which intersects F in a line L_x. The intersection lines L_x, $x \in F$, determine a spread of F and the set $T_1 := \{Q_x | x \in F\}$ determines a partition of $\mathcal{S}$ in $Q(5, q)$-quads.

Now, consider a point x of $\mathcal{S}$ not contained in F and let y be the unique point of F collinear with x. Put $A := xy$ and $Q := Q_y$. Let L be a line intersecting Q in x and consider the hex $H := \mathcal{C}(L, Q)$. By Lemma 11, $H \cap F$ is a $Q(5, q)$-quad and $\mathcal{C}(L, A) \cap F$ is a line L'. So, $\mathcal{C}(L, A) = \mathcal{C}(L', A)$ is a grid. Since the hex H contains a grid-quad and two $Q(5, q)$-quads containing the line $Q \cap F$, it is a glued near polygon of type $Q(5, q) \otimes Q(5, q)$. Hence, exactly one line of Q through x, say B, is contained in a $Q(5, q)$-quad together with L.

Now, consider a convex sub near $2(n-1)$-gon F' of $\mathcal{S}$ containing $Q' := \mathcal{C}(B, L)$ and intersecting Q in B. We will show that every quad of F' is isomorphic to $Q(5, q)$. Let L_1 and L_2 be two lines of F' through x. If $L_1 \neq B \neq L_2$, then the hex $H' := \mathcal{C}(A, L_1, L_2)$ contains grid-quads (namely $\mathcal{C}(A, L_1)$ and $\mathcal{C}(A, L_2)$) and a $Q(5, q)$-quad (namely $\mathcal{C}(A, L_1, L_2) \cap F$). Hence H' must be isomorphic to either the product near hexagon $Q(5, q) \times L_{q+1}$ or a glued near hexagon of type $Q(5, q) \otimes Q(5, q)$. In any case, $\mathcal{C}(L_1, L_2)$ is isomorphic to $Q(5, q)$. If $L_1 = B$ and if L_2 is not contained in Q', then $\mathcal{C}(L_2, M) \cong Q(5, q)$ for every line $M \neq B$ through x contained in Q'. This is only possible if $\mathcal{C}(L_2, Q')$ is isomorphic to $H^{\mathrm{D}}(5, q^2)$. Hence, also $\mathcal{C}(L_1, L_2) = \mathcal{C}(B, L_2)$ is isomorphic to $Q(5, q)$. It now follows that every quad of F' containing x is isomorphic to $Q(5, q)$. By Lemma 15 and the fact that the diameter of F' is equal to $n-1$, it follows that $F' \cong H^{\mathrm{D}}(2n-3, q^2)$. Obviously, F' is the only convex sub near polygon containing x isomorphic to $H^{\mathrm{D}}(2n-3, q^2)$.

Repeating the above construction for every point x outside F, we obtain a partition T_2 of $\mathcal{S}$ in convex sub near polygons isomorphic to $H^{\mathrm{D}}(2n-3, q^2)$. We can now apply Lemma 7 and conclude that $\mathcal{S}$ is a glued near polygon of type $H^{\mathrm{D}}(2n-3, q^2) \otimes Q(5, q)$. □

Case II: *$\mathcal{S}$ does not contain convex sub near polygons isomorphic to $H^{\mathrm{D}}(2n-3, q^2)$*

Since $\mathcal{S}$ has $Q(5, q)$-quads, it follows that $n \geqslant 4$. We may suppose that Theorem 1 is valid for every dense near polygon of diameter at most $n-1$ (= Induction Hypothesis) and hence for every proper convex sub near polygon of $\mathcal{S}$.

Let Υ denote the set of all convex sub near polygons of $\mathcal{S}$ which are isomorphic to an $H^{\mathrm{D}}(2m-1, q^2)$ for a certain $m \in \{2, \ldots, n\}$ and let Υ' denote the set of all maximal elements of Υ, i.e. those elements of Υ which are not properly contained in another element of Υ. By our assumption every element of Υ' has diameter at most $n-2$.

Lemma 21. *Let F denote an element of Υ' isomorphic to $H^{\mathrm{D}}(2m-1, q^2)$. Then there are no grid-quads intersecting F in just a point.*

Proof. Suppose the contrary. Let Q denote a grid-quad intersecting F in a point x and let L_1 and L_2 denote the two lines of Q through x. By the Induction Hypothesis ($m+1 < n$), $\mathcal{C}(L_i, F)$ is isomorphic to either $L_{q+1} \times H^{\mathrm{D}}(2m-1, q^2)$ or a glued near polygon of type $Q(5, q) \otimes H^{\mathrm{D}}(2m-1, q^2)$. So, there is at most one line K_i of F through x for which $\mathcal{C}(L_i, K_i)$ is not a grid. As a consequence, there exists a line M of F through x for which $\mathcal{C}(L_1, L_2)$, $\mathcal{C}(L_1, M)$ and $\mathcal{C}(L_2, M)$ are grids. This contradicts the fact that the hex

$\mathcal{C}(L_1, L_2, M)$ is isomorphic to either $H^{\mathrm{D}}(5, q^2)$, $Q(5, q) \times L_{q+1}$ or a glued near polygon of type $Q(5, q) \otimes Q(5, q)$. □

Lemma 22. *Let F denote an element of Υ' isomorphic to $H^{\mathrm{D}}(2m-1, q^2)$ and let x denote an element of $\mathcal{S}$ at distance $n-m$ from F. Then $\mathcal{C}(x, \pi_F(x)) \cong H^{\mathrm{D}}(2(n-m)-1, q^2)$.*

Proof. Every point of F collinear with $\pi_F(x)$ has distance $\mathrm{d}(x, \pi_F(x)) + 1 = n - m + 1$ from x. So, the convex sub near $2(n-m)$-gon $\mathcal{C}(x, \pi_F(x))$ intersects F in only the point $\pi_F(x)$. If $\mathcal{C}(x, \pi_F(x))$ were not isomorphic to $H^{\mathrm{D}}(2(n-m)-1, q^2)$, then by Lemma 15, $\mathcal{C}(x, \pi_F(x))$ would contain a grid-quad Q containing $\pi_F(x)$, contradicting Lemma 21. □

Lemma 23. *Let F denote an element of Υ' with diameter m and let x denote an arbitrary point of F. Then there exists a point y at distance $n-m$ from F such that $\pi_F(y) = x$.*

Proof. It suffices to prove that for every point u of $\mathcal{S}$ with $\mathrm{d}(u, F) < n-m$ and $\pi_F(u) = x$, there exists a point $u' \in \Gamma_1(u)$ such that $\mathrm{d}(u', F) = \mathrm{d}(u, F) + 1$ and $\pi_F(u') = x$. Then starting with the point x of F we can construct a point y that satisfies all required conditions. So, consider such a point u and let v be the point of F at distance m from $x = \pi_F(u)$. Since x lies on a shortest path between u and v, $F = \mathcal{C}(x, v)$ is contained in $\mathcal{C}(u, v)$. Now, $\mathrm{d}(u, v) = \mathrm{d}(u, x) + \mathrm{d}(x, v) = \mathrm{d}(u, F) + m < n$ and hence $\mathcal{C}(u, v) \neq \mathcal{S}$. So, there exists a point $u' \in \Gamma_1(u)$ not contained in $\mathcal{C}(u, v)$. By Lemma 9, u' is classical with respect to $\mathcal{C}(u, v)$ and hence $\mathrm{d}(u', F) = \mathrm{d}(u', u) + \mathrm{d}(u, F) = 1 + \mathrm{d}(u, F)$. Since $\mathrm{d}(u', x) = \mathrm{d}(u', u) + \mathrm{d}(u, x) = \mathrm{d}(u, F) + 1$, $\pi_F(u') = x$. This proves the lemma. □

Proposition 2. *Let F denote an element of Υ' isomorphic to $H^{\mathrm{D}}(2m-1, q^2)$ and let F' denote a convex sub near $(2n-2m)$-gon intersecting F in just one point a (so $F' \cong H^{\mathrm{D}}(2n-2m-1, q^2)$) and let G denote a convex sub near $(2n-2m+2)$-gon containing F'. Then*

(a) *G intersects F in a line; so, there are precisely $\mu(m)$ convex sub near $(2n-2m+2)$-gons containing F';*
(b) *G is isomorphic to either $H^{\mathrm{D}}(2n-2m-1, q^2) \times L_{q+1}$ or $H^{\mathrm{D}}(2n-2m+1, q^2)$;*
(c) *there is at most one convex sub near polygon isomorphic to $H^{\mathrm{D}}(2n-2m+1, q^2)$ containing F';*
(d) *$t_{\mathcal{S}} + 1$ is equal to either $\mu(m) + \mu(n-m)$ or $\mu(m) + \mu(n-m+1) - 1$;*
(e) *if $t_{\mathcal{S}} + 1 = \mu(m) + \mu(n-m)$, then $\mathcal{S}$ is isomorphic to $H^{\mathrm{D}}(2m-1, q^2) \times H^{\mathrm{D}}(2n-2m-1, q^2)$.*

If $t_{\mathcal{S}} + 1 = \mu(m) + \mu(n-m+1) - 1$, then

(f) *$\mathcal{S}$ is not a product near polygon;*
(g) *every point x of F is contained in a unique convex sub near $(2n-2m+2)$-gon $G(x)$ isomorphic to $H^{\mathrm{D}}(2n-2m+1, q^2)$ intersecting F in a line;*
(h) *every element $G(x)$, $x \in F$, belongs to Υ';*
(i) *the convex sub near polygons $G(x)$, $x \in F$, form a partition T_1 of $\mathcal{S}$ in convex sub near polygons;*
(j) *$\mathcal{S}$ is isomorphic to a glued near polygon of type $H^{\mathrm{D}}(2m-1, q^2) \otimes H^{\mathrm{D}}(2n-2m+1, q^2)$.*

Proof. (a) Let b denote a point of F' at distance $n-m$ from a. If c is a point of $G \cap F$, then $n-m+1 \geqslant \mathrm{d}(b,c) = \mathrm{d}(b,a) + \mathrm{d}(a,c) = n-m+\mathrm{d}(a,c)$. Hence $\mathrm{d}(a,c) \leqslant 1$ for every point c of $G \cap F$. So, $G \cap F$ is either a point or a line (there exists a point in $G \cap F$ at distance $\mathrm{diam}(F \cap G)$ from a). Let f be a point of G at distance $n-m+1$ from a. Then $n-m+1 = \mathrm{d}(f,a) = \mathrm{d}(f,\pi_F(f)) + \mathrm{d}(\pi_F(f),a)$. Since F is classical, $\mathrm{d}(f,\pi_F(f)) \leqslant n-m$. The point $\pi_F(f)$ lies on a shortest path between the points f and a of G and hence belongs to $G \cap F$. So, $\mathrm{d}(\pi_F(f),a) \leqslant 1$. As a consequence, $\mathrm{d}(\pi_F(f),a) = 1$ and $G \cap F$ is a line.

(b) By the Induction Hypothesis, G is isomorphic to either $H^{\mathrm{D}}(2n-2m-1,q^2) \times L_{q+1}$, $H^{\mathrm{D}}(2n-2m+1,q^2)$ or a glued near polygon of type $H^{\mathrm{D}}(2n-2m-1,q^2) \otimes Q(5,q)$. If G were isomorphic to a glued near polygon of type $H^{\mathrm{D}}(2n-2m-1,q^2) \otimes Q(5,q)$, then there would exist a grid-quad containing a intersecting F at only the point a, contradicting Lemma 21.

(c) Suppose that G_1 and G_2 are two different convex sub near polygons containing F' isomorphic to $H^{\mathrm{D}}(2n-2m+1,q^2)$. Put $L_i := G_i \cap F$, $i \in \{1,2\}$, and let L be a line of F' through a. Then $\mathcal{C}(L,L_1) \cong \mathcal{C}(L,L_2) \cong Q(5,q)$. Now, by the Induction Hypothesis ($m+1 < n$), $\mathcal{C}(L,F)$ is isomorphic to either $L_{q+1} \times H^{\mathrm{D}}(2m-1,q^2)$ or a glued near polygon of type $Q(5,q) \otimes H^{\mathrm{D}}(2m-1,q^2)$. In any case, we must have that at least one of the quads $\mathcal{C}(L,L_1)$, $\mathcal{C}(L,L_2)$ is a grid, a contradiction. This proves the statement.

(d) The convex sub near $(2n-2m+2)$-gons containing F' determine a partition of the set of lines through a which are not contained in F'. If one of these sub near polygons is isomorphic to $H^{\mathrm{D}}(2n-2m+1,q^2)$, then $t_{\mathcal{S}}$ is equal to $\mu(m)+\mu(n-m+1)-1$. If none of these sub near polygons are isomorphic to $H^{\mathrm{D}}(2n-2m+1,q^2)$, then $t_{\mathcal{S}}+1$ is isomorphic to $\mu(m)+\mu(n-m)$.

(e) This follows from Lemma 3.

(f) If $\mathcal{S}$ were a product near polygon, then every modified local space is disconnected. In particular, the modified local space in the point a would be disconnected, a contradiction.

(g) Let x denote an arbitrary point of F and let y denote a point of $\mathcal{S}$ at distance $n-m$ from F such that $\pi_F(y) = x$ (see Lemma 23). Then $\mathcal{C}(x,y)$ is isomorphic to $H^{\mathrm{D}}(2n-2m-1,q^2)$ and intersects F only in the point x. By (a) and (b), every convex sub near $(2n-2m+2)$-gon containing $\mathcal{C}(x,y)$ is isomorphic to either $H^{\mathrm{D}}(2n-2m-1,q^2) \times L_{q+1}$ or $H^{\mathrm{D}}(2n-2m+1,q^2)$ and intersects F in a line. If every such sub near polygon is isomorphic to $H^{\mathrm{D}}(2n-2m-1,q^2) \times L_{q+1}$, then $t_{\mathcal{S}}+1 = \mu(m)+\mu(n-m)$ and $\mathcal{S} \cong H^{\mathrm{D}}(2m-1,q^2) \times H^{\mathrm{D}}(2n-2m-1,q^2)$, contradicting (f). Hence, by (c) there exists a unique convex sub near $(2n-2m+2)$-gon $G(x)$ containing $\mathcal{C}(x,y)$ isomorphic to $H^{\mathrm{D}}(2n-2m+1,q^2)$. Using the fact that every line through x is contained in $G(x)$ or F, it is easily seen that $G(x)$ is the unique convex sub near $(2n-2m+2)$-gon containing x isomorphic to $H^{\mathrm{D}}(2n-2m+1,q^2)$ that intersects F in a line.

(h) Every line through x is contained in $G(x)$ or F. So, if L_1 is a line of $G(x)$ through x different from $G(x) \cap F$ and if L_2 is a line of F through x different from $G(x) \cap F$, then $\mathcal{C}(L_1,L_2)$ is a grid. Hence, every convex sub near polygon containing $G(x)$ different from $G(x)$ contains grid-quads. This proves the statement.

(i) Let y denote an arbitrary point of $\mathcal{S}$. Then $\mathcal{C}(y,\pi_F(y))$ intersects F in only the point $\pi_F(y)$. So, $\mathcal{C}(y,\pi_F(y))$ and hence also y is contained in $G(\pi_F(y))$. Suppose now that $G(x_1)$ and $G(x_2)$ have a point y in common. Since $\pi_F(y)$ is on a shortest path between the points

y and x_1 of $G(x_1)$, $\pi_F(y) \in G(x_1)$ and hence $G(x_1) = G(\pi_F(y))$. Similarly, we have $G(x_2) = G(\pi_F(y))$. So, $G(x_1) = G(x_2)$. This proves the statement.

(j) Let x^* denote an arbitrary element of F. Since $G(x^*) \in \Upsilon'$ and $\mathcal{S}$ is not a product near polygon, every element y of $G(x^*)$ is contained in a unique convex sub near $2m$-gon $G'(y)$ isomorphic to $H^{\mathrm{D}}(2m-1, q^2)$ which intersects $G(x^*)$ in a line, see property (g). Moreover, the convex sub near $2m$-gons $G'(y)$, $y \in G(x^*)$, determine a partition T_2 of $\mathcal{S}$ in convex sub near $2m$-gons by (i). Now, for every point y of $\mathcal{S}$ and every $i \in \{1, 2\}$, let $F_i(y)$ denote the unique element of T_i containing y. Then $F_1(y)$ is a convex sub near $(2n-2m+2)$-gon belonging to Υ'. Similarly, $F_2(y)$ is a convex sub near $2m$-gon belonging to Υ'. Obviously, $F_1(y)$ intersects $F_2(y)$ in a line and every line through y is contained in $F_1(y)$ or $F_2(y)$. By Lemma 7 it now follows that $\mathcal{S}$ is a glued near polygon of type $H^{\mathrm{D}}(2m-1, q^2) \otimes H^{\mathrm{D}}(2n-2m+1, q^2)$. □

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