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RSK correspondence and classically irreducible Kirillov–Reshetikhin crystals[☆]

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ABSTRACT

We give a new combinatorial model of the Kirillov–Reshetikhin crystals of type $A_n^{(1)}$ in terms of non-negative integral matrices based on the classical RSK algorithm, which has a simple description of the affine crystal structure without using the promotion operator. We have a similar description of the Kirillov–Reshetikhin crystals associated to exceptional nodes in the Dynkin diagrams of classical affine or non-exceptional affine type, which are called classically irreducible together with those of type $A_n^{(1)}$.

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1. Introduction

The Robinson–Schensted–Knuth (simply RSK) correspondence is a weight preserving bijection from the set $\mathcal{M}_{m \times n}$ of $m \times n$ non-negative integral matrices to the set $\mathcal{T}_{m \times n}$ of pairs of semistandard Young tableaux of the same shape with entries from m and n letters, respectively [1].

The RSK map κ has nice representation theoretic interpretations from a viewpoint of the Kashiwara’s crystal base theory [2]. In [3], Lascoux shows that $\mathcal{M}_{m \times n}$ has a $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ -crystal structure and κ is an isomorphism of crystals, where one can define a $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ -crystal structure on $\mathcal{T}_{m \times n}$ in an obvious way following [4]. As an application, a non-symmetric Cauchy kernel expansion into a sum of product of Demazure characters is obtained. In [5], the author shows that κ can be extended to an isomorphism of \mathfrak{gl}_{m+n} -crystals. Here $\mathcal{M}_{m \times n}$ or $\mathcal{T}_{m \times n}$ can be regarded as a crystal associated to a generalized Verma module over \mathfrak{gl}_{m+n} . As an application, a weight generating function of plane partitions in a bounded region is given as a Demazure character of \mathfrak{gl}_{m+n} . (See also [6] for another application of RSK to the crystal base of a modified quantized enveloping algebra of type $A_{+\infty}$ and A_{∞} .)

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The purpose of this paper is to study the RSK correspondence further in this direction and discuss its connection with affine crystals. It is motivated by the observation that $\mathcal{M}_{r \times (n-r)}$ has a natural affine crystal structure of type $A_{n-1}^{(1)}$ for $n \geq 2$ and $1 \leq r \leq n-1$ by [5] and the symmetry of the Dynkin diagram of $A_{n-1}^{(1)}$. For $s \geq 1$, we let $\mathcal{M}_{r \times (n-r)}^s$ be the set of matrices in $\mathcal{M}_{r \times (n-r)}$ such that the length of a maximal decreasing subsequence of its row or column word is no more than s . Then as the main result in this paper, we show (Theorem 3.8) that as an affine crystal of type $A_{n-1}^{(1)}$,

$$\mathcal{M}_{r \times (n-r)}^s \otimes T_{s\omega_r} \cong \mathbf{B}^{r,s}, \quad (1.1)$$

where $\mathbf{B}^{r,s}$ is a perfect crystal [7] with highest weight $s\omega_r$ or the rectangular partition (s^r) as a classical \mathfrak{gl}_n -crystal, and $T_{s\omega_r} = \{t_{s\omega_r}\}$ is a crystal with $\text{wt}(t_{s\omega_r}) = s\omega_r$, $\varepsilon_i(t_{s\omega_r}) = \varphi_i(t_{s\omega_r}) = -\infty$ for all i .

To prove (1.1), two RSK maps κ^{\nearrow} and κ^{\searrow} are considered, which map a matrix in $\mathcal{M}_{r \times (n-r)}^s$ to a pair of semistandard Young tableaux of normal and anti-normal shape, respectively. They turn out to be the projections of $\mathcal{M}_{r \times (n-r)}^s$ to a classical crystal of type A_{n-1} corresponding to maximal parabolic subalgebras obtained from $A_{n-1}^{(1)}$ by removing the simple roots α_0 and α_r respectively. These two RSK maps play an important role in proving the regularity of $\mathcal{M}_{r \times (n-r)}^s \otimes T_{s\omega_r}$ and constructing the isomorphism in (1.1). Note that $\mathcal{M}_{r \times (n-r)}$ can be regarded as a limit of the crystals $\mathbf{B}^{r,s} \otimes T_{-s\omega_r}$ as s goes to infinity.

Let \mathfrak{g} be an affine Kac–Moody algebra and let $U'_q(\mathfrak{g})$ be the quantized enveloping algebra associated to the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The finite dimensional irreducible $U'_q(\mathfrak{g})$ -modules do not have crystal bases in general. But it was conjectured by Hatayama et al. [8,9] that a certain family of finite dimensional irreducible $U'_q(\mathfrak{g})$ -modules $W^{r,s}$ called *Kirillov–Reshetikhin modules* (simply *KR modules*) [10] have crystal bases, where r denotes a simple root index of \mathfrak{g} except 0 and s is an arbitrary positive integer. The conjectured crystals $\mathbf{B}^{r,s}$ are now called *KR crystals*.

For type $A_{n-1}^{(1)}$, the KR crystals $\mathbf{B}^{r,s}$ are the perfect crystals in (1.1). In this case, a combinatorial description of $\mathbf{B}^{r,s}$ was given by Shimozono [11] using semistandard Young tableaux of a rectangular shape and the Schützenberger’s promotion operator [12]. But, the main advantage of our model using $r \times (n-r)$ integral matrices is that the description of its crystal structure is remarkably simple, where the crystal operators or Kashiwara operators corresponding to α_0 and α_r are given as adding ± 1 at the entries at southeast and northwest corners of a matrix, respectively (see Fig. 1).

Recently, the existence of KR crystals $\mathbf{B}^{r,s}$ for the other classical affine or non-exceptional affine type was proved by Okado and Schilling [13], and its combinatorial construction was given in [13,14], where the Kashiwara–Nakashima tableaux [4] were used to describe the classical crystal structure on $\mathbf{B}^{r,s}$.

We use (1.1) to obtain a new description of the KR crystals associated to so-called *exceptional nodes* in the Dynkin diagrams of classical affine type (see [14, Table 1]). These crystals together with $\mathbf{B}^{r,s}$ of type $A_{n-1}^{(1)}$ are called *classically irreducible* [15] since they are connected as a classical crystal, and they are also perfect crystals [7].

We use the Kashiwara’s method of folding crystals [16] to construct $\mathbf{B}^{n,s}$ of type $D_{n+1}^{(2)}$ and $C_n^{(1)}$ in terms of symmetric non-negative integral matrices (Theorem 4.4), and we describe $\mathbf{B}^{n-1,s}$ and $\mathbf{B}^{n,s}$ of type $D_n^{(1)}$ in terms of semistandard Young tableaux of type A_{n-1} (Theorem 5.4). (See Figs. 2 and 3.) In both cases, the affine crystal structures are given explicitly as in $A_{n-1}^{(1)}$.

It would be nice to have a similar description of arbitrary KR crystals of classical affine type, but we do not know how to generalize the method here in a natural way.

2. Preliminary

2.1. Quantum groups and crystals

Let us give a brief review on crystals (cf. [17,18]). Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix with an index set I . Consider a quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ called a Cartan datum, where P^\vee is a

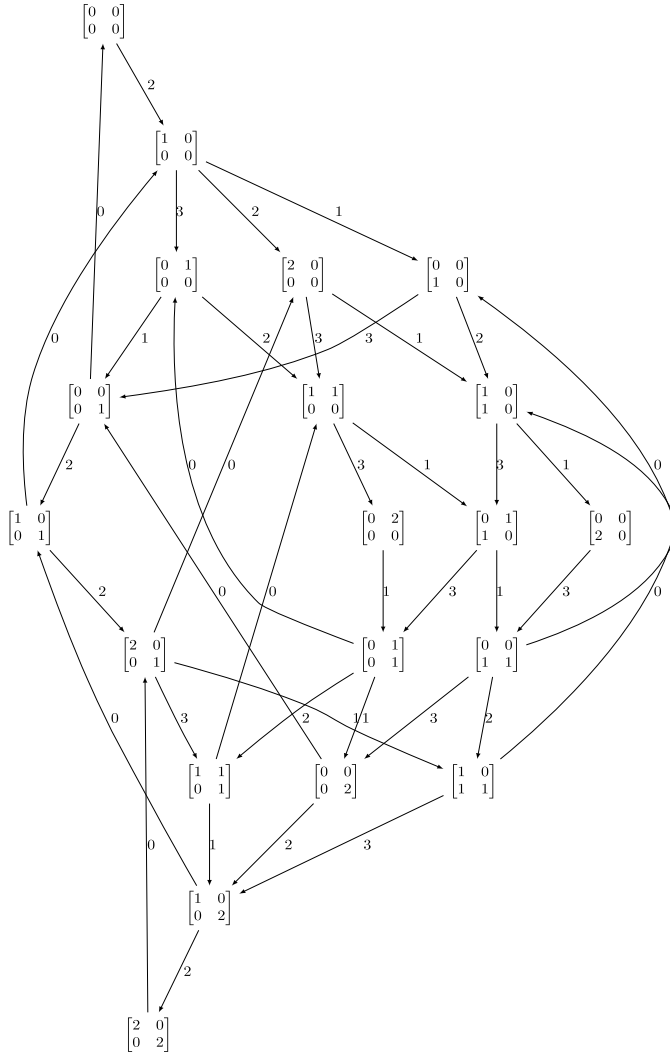


Fig. 1. The KR crystal $\mathbf{B}^{2,2}$ of type $A_3^{(1)}$ where the vertices are given in terms of non-negative integral 2×2 matrices with the length of column or row words no more than 2. This graph was implemented by SAGE.

free \mathbb{Z} -module of finite rank, $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$, $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, and $\Pi = \{\alpha_i \mid i \in I\} \subset P$ such that $\langle \alpha_j, h_i \rangle = a_{ij}$ for $i, j \in I$.

A crystal associated to $(A, P^\vee, P, \Pi^\vee, \Pi)$ is a set B together with the maps $\text{wt}: B \rightarrow P$, $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i: B \rightarrow B \cup \{\mathbf{0}\}$ ($i \in I$) such that for $b \in B$ and $i \in I$

- (1) $\varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b)$,
- (2) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq \mathbf{0}$,
- (3) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq \mathbf{0}$,
- (4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,
- (5) $\tilde{e}_i b = \tilde{f}_i b = \mathbf{0}$ if $\varphi_i(b) = -\infty$,

where $\mathbf{0}$ is a formal symbol. Here we assume that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. Note that B is equipped with an I -colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \in I$. We call B connected if it is connected as a graph, and normal if $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq \mathbf{0}\}$ for $b \in B$ and $i \in I$. The dual crystal B^\vee of B is defined to be the set $\{b^\vee \mid b \in B\}$ with $\text{wt}(b^\vee) = -\text{wt}(b)$, $\varepsilon_i(b^\vee) = \varphi_i(b)$, $\varphi_i(b^\vee) = \varepsilon_i(b)$, $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ and $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$ for $b \in B$ and $i \in I$. We assume that $\mathbf{0}^\vee = \mathbf{0}$.

Let B_1 and B_2 be crystals. A morphism $\psi : B_1 \rightarrow B_2$ is a map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$ such that

- (1) $\psi(\mathbf{0}) = \mathbf{0}$,
- (2) $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq \mathbf{0}$,
- (3) $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{e}_i b) \neq \mathbf{0}$,
- (4) $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{f}_i b) \neq \mathbf{0}$,

for $b \in B_1$ and $i \in I$. We call ψ an embedding and B_1 a subcrystal of B_2 when ψ is injective, and call ψ strict if $\psi : B_1 \cup \{\mathbf{0}\} \rightarrow B_2 \cup \{\mathbf{0}\}$ commutes with \tilde{e}_i and \tilde{f}_i for all $i \in I$, where we assume that $\tilde{e}_i \mathbf{0} = \tilde{f}_i \mathbf{0} = \mathbf{0}$. When ψ is a bijection, it is called an isomorphism. For $b_i \in B_i$ ($i = 1, 2$), we say that b_1 is equivalent to b_2 if there exists an isomorphism of crystals $C(b_1) \rightarrow C(b_2)$ sending b_1 to b_2 , where $C(b_i)$ is the connected component in B_i including b_i as an I -colored oriented graph.

A tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned}$$

for $i \in I$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is a crystal.

Let \mathfrak{g} be a symmetrizable Kac–Moody algebra associated to A . Let P^\vee be the dual weight lattice, $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{h_i \mid i \in I\}$ the set of simple coroots, and $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots of \mathfrak{g} .

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of \mathfrak{g} over $\mathbb{Q}(q)$ generated by e_i , f_i and q^h for $i \in I$ and $h \in P^\vee$. For a dominant integral weight Λ , let $\mathbf{B}(\pm\Lambda)$ be the crystal of an irreducible highest (respectively lowest) weight $U_q(\mathfrak{g})$ -module with highest (respectively lowest) weight $\pm\Lambda$. Then $\mathbf{B}(\pm\Lambda)$ is a crystal associated to $(A, P^\vee, P, \Pi^\vee, \Pi)$. We say that a crystal B is regular if it is isomorphic to the crystal of an integrable $U_q(\mathfrak{g}_J)$ -module for any $J \subset I$ with $|J| \leq 2$, where \mathfrak{g}_J is the Kac–Moody algebra associated to $A_J = (a_{ij})_{i,j \in J}$. Note that a regular crystal is normal.

For $\Lambda \in P$, we denote by $T_\Lambda = \{t_\Lambda\}$ a crystal with $\text{wt}(t_\Lambda) = \Lambda$ and $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$ for $i \in I$.

2.2. Quantum affine algebras

Assume that A is a generalized Cartan matrix of affine type with an index set $I = \{0, 1, \dots, n\}$ following [1, §4.8], and \mathfrak{g} is the associated affine Kac–Moody algebra with the Cartan subalgebra \mathfrak{h} . Let $P^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i \oplus \mathbb{Z} d \subset \mathfrak{h}$ be the dual weight lattice of \mathfrak{g} , where d is given by $\langle \alpha_j, d \rangle = \delta_{0j}$ for $j \in I$. Let $\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^*$ be the positive imaginary null root of \mathfrak{g} and let $\Lambda_i \in \mathfrak{h}^*$ ($i \in I$) be the i -th fundamental weight such that $\langle \Lambda_i, h_j \rangle = \delta_{ij}$ for $j \in I$ and $\langle \Lambda_i, d \rangle = 0$. Then the weight lattice of \mathfrak{g} is $P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \frac{1}{a_0} \delta$.

Let $P_{\text{cl}} = P/(\mathbb{Q}\delta \cap P) = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$ and $(P_{\text{cl}})^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$, where we still denote the image of Λ_i in P_{cl} by Λ_i . Then we define $U'_q(\mathfrak{g})$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i , f_i and q^h for $i \in I$ and

$h \in (P_{\text{cl}})^\vee$. We regard P_{cl} as the weight lattice of $U'_q(\mathfrak{g})$. For a proper subset $J \subset I$, let $\Pi_J^\vee = \{h_i \mid i \in J\}$ and $\Pi_J = \{\alpha_i \mid i \in J\}$, and let $U_q(\mathfrak{g}_J)$ be the subalgebra of $U'_q(\mathfrak{g})$ generated by e_i, f_i and q^h for $i \in J$ and $h \in (P_{\text{cl}})^\vee$.

From now on, we mean by a $U'_q(\mathfrak{g})$ -crystal (respectively $U_q(\mathfrak{g}_J)$ -crystal) a crystal associated to $(A, (P_{\text{cl}})^\vee, P_{\text{cl}}, \Pi^\vee, \Pi)$ (respectively $(A_J, (P_{\text{cl}})^\vee, P_{\text{cl}}, \Pi_J^\vee, \Pi_J)$). For simplicity, we will often write the type of the generalized Cartan matrix A (or A_J) instead of \mathfrak{g} (or \mathfrak{g}_J).

The following lemma plays an important role in this paper to have a combinatorial realization of KR crystals.

Lemma 2.1. (See Lemma 2.6 in [15].) *Let \mathfrak{g} be of classical affine or non-exceptional affine type. Fix $r \in I \setminus \{0\}$ and $s \geq 1$. Then any regular $U'_q(\mathfrak{g})$ -crystal that is isomorphic to the KR crystal $\mathbf{B}^{r,s}$ as a $U_q(\mathfrak{g}_{I \setminus \{0\}})$ -crystal is also isomorphic to $\mathbf{B}^{r,s}$ as a $U'_q(\mathfrak{g})$ -crystal.*

2.3. RSK algorithm

Let us recall some necessary background on semistandard tableaux following [19,20]. Let \mathcal{P} be the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i \geq 1}$ with a Young diagram. We denote the length of λ by $\ell(\lambda)$ and the conjugate of λ by $\lambda' = (\lambda'_i)_{i \geq 1}$. We let λ^π be the skew Young diagram obtained by 180°-rotation of λ . For example,

$$(5, 3, 2) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}, \quad (5, 3, 2)^\pi = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}.$$

Let \mathbb{A} be a linearly ordered set. For a skew Young diagram λ/μ , let $\text{SST}_{\mathbb{A}}(\lambda/\mu)$ be the set of all semistandard tableaux of shape λ/μ with entries in \mathbb{A} . Let $\mathcal{W}_{\mathbb{A}}$ be the set of finite words in \mathbb{A} . For $T \in \text{SST}_{\mathbb{A}}(\lambda/\mu)$, let $w(T)$ be a word in $\mathcal{W}_{\mathbb{A}}$ obtained by reading the entries of T row by row from top to bottom, and from right to left in each row.

Let $\text{sh}(T)$ denote the shape of a tableau T . If $\text{sh}(T) = \nu$ (respectively ν^π) for some $\nu \in \mathcal{P}$, then we say that T is of normal (respectively anti-normal) shape. For $T \in \text{SST}_{\mathbb{A}}(\lambda/\mu)$, let T^\curvearrowright (respectively T^\curvearrowleft) be the unique semistandard tableau of normal (respectively anti normal) shape such that $w(T^\curvearrowright)$ (respectively $w(T^\curvearrowleft)$) is Knuth equivalent to $w(T)$. Note that if $\text{sh}(T^\curvearrowright) = \nu$, then $\text{sh}(T^\curvearrowleft) = \nu^\pi$.

For $T \in \text{SST}_{\mathbb{A}}(\lambda)$ and $a \in \mathbb{A}$, let $a \rightarrow T$ be the tableau obtained by applying the Schensted's column insertion of a into T . For $w = w_1 \cdots w_r \in \mathcal{W}_{\mathbb{A}}$, we define $\mathbf{P}(w) = (w_r \rightarrow (\cdots (w_2 \rightarrow w_1) \cdots))$.

Let \mathbb{B} be another linearly ordered set. Let

$$\mathcal{M}_{\mathbb{A}, \mathbb{B}} = \left\{ M = (m_{ab})_{a \in \mathbb{A}, b \in \mathbb{B}} \mid m_{ab} \in \mathbb{Z}_{\geq 0}, \sum_{a,b} m_{ab} < \infty \right\}. \quad (2.1)$$

Let $\Omega_{\mathbb{A}, \mathbb{B}}$ be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W}_{\mathbb{A}} \times \mathcal{W}_{\mathbb{B}}$ such that (1) $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$ for some $r \geq 0$, (2) $(a_1, b_1) \leq \cdots \leq (a_r, b_r)$, where for (a, b) and $(c, d) \in \mathbb{A} \times \mathbb{B}$, $(a, b) < (c, d)$ if and only if $(b < d)$ or $(b = d \text{ and } a > c)$. Then we have a bijection from $\Omega_{\mathbb{A}, \mathbb{B}}$ to $\mathcal{M}_{\mathbb{A}, \mathbb{B}}$, where (\mathbf{a}, \mathbf{b}) is mapped to $M(\mathbf{a}, \mathbf{b}) = (m_{ab})$ with $m_{ab} = |\{k \mid (a_k, b_k) = (a, b)\}|$. Note that the pair of empty words (\emptyset, \emptyset) corresponds to zero matrix. Let $M \in \mathcal{M}_{\mathbb{A}, \mathbb{B}}$ be given. Suppose that $M = M(\mathbf{a}, \mathbf{b})$ and its transpose $M^t = M(\mathbf{c}, \mathbf{d})$ with $(\mathbf{c}, \mathbf{d}) \in \Omega_{\mathbb{B}, \mathbb{A}}$. Let $\mathbf{P}(M) = \mathbf{P}(\mathbf{a})$ and $\mathbf{Q}(M) = \mathbf{P}(\mathbf{c})$. Then we have a bijection called the RSK correspondence:

$$\kappa : \mathcal{M}_{\mathbb{A}, \mathbb{B}} \rightarrow \bigsqcup_{\lambda} \text{SST}_{\mathbb{A}}(\lambda) \times \text{SST}_{\mathbb{B}}(\lambda),$$

where M is mapped to $(\mathbf{P}(M), \mathbf{Q}(M))$, and the union is over all λ with $\text{SST}_{\mathbb{A}}(\lambda) \neq \emptyset$ and $\text{SST}_{\mathbb{B}}(\lambda) \neq \emptyset$.

3. KR crystals of type $A_{n-1}^{(1)}$

3.1. Affine algebra of type $A_{n-1}^{(1)}$

Assume that $\mathfrak{g} = A_{n-1}^{(1)}$ ($n \geq 2$) with $I = \{0, 1, \dots, n-1\}$. We put $I_r = I \setminus \{r\}$ for $r \in I$, and $I_{0,r} = I_0 \cap I_r$ for $r \in I_0$. Note that $\mathfrak{g}_{I_0} \cong \mathfrak{g}_{I_r} = A_{n-1}$ and $\mathfrak{g}_{I_{0,r}} = A_{r-1} \oplus A_{n-r-1}$.

Let $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 1, \dots, n-1$ and $\epsilon_n = \Lambda_0 - \Lambda_{n-1}$. Then $\epsilon_1 + \dots + \epsilon_n = 0$ and $\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ forms a weight lattice of \mathfrak{g}_{I_0} . Note that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I_0$ and $\alpha_0 = \epsilon_n - \epsilon_1$ in P_{cl} . The fundamental weights for \mathfrak{g}_{I_0} are $\omega_i = \Lambda_i - \Lambda_0 = \sum_{k=1}^i \epsilon_k$ for $i \in I_0$.

We regard $[n] = \{1 < \dots < n\}$ as a $U_q(\mathfrak{g}_{I_0})$ -crystal $\mathbf{B}(\omega_1)$ with $\text{wt}(k) = \epsilon_k$, and $[\bar{n}] = \{\bar{n} < \dots < \bar{1}\}$ as its dual crystal with $\text{wt}(\bar{k}) = -\epsilon_k$. Then $\mathcal{W}_{[n]}$ and $\mathcal{W}_{[\bar{n}]}$ are regular $U_q(\mathfrak{g}_{I_0})$ -crystals, where we identify $w = w_1 \dots w_r$ with $w_1 \otimes \dots \otimes w_r$.

The fundamental weights for \mathfrak{g}_{I_r} are $\omega'_i = \Lambda_i - \Lambda_r$ for $i \in I_r$. Note that $\omega_r = -\omega'_0$. In this case, we may identify a $U_q(\mathfrak{g}_{I_r})$ -crystal $\mathbf{B}(\omega'_{r+1})$, the crystal of the natural representation of $U_q(\mathfrak{g}_{I_r})$, with $[n]_{+r} = \{r+1 < \dots < n < 1 < \dots < r\}$.

3.2. Affine crystal $\mathcal{M}_{r \times (n-r)}$

For $1 \leq r \leq n-1$, let

$$\mathcal{M}_{r \times (n-r)} = \mathcal{M}_{[\bar{r}], [n] \setminus [r]} \quad (3.1)$$

(see (2.1)). First note that $\mathcal{M}_{r \times (n-r)}$ is a $U_q(A_{r-1})$ -crystal with respect to \tilde{e}_i, \tilde{f}_i ($1 \leq i \leq r-1$), where $\tilde{x}_i M = M(\tilde{x}_i \mathbf{a}, \mathbf{b})$ for $x = e, f$ and $M \in \mathcal{M}_{r \times (n-r)}$ with $M = M(\mathbf{a}, \mathbf{b})$. Here, we assume that $\tilde{x}_i M = \mathbf{0}$ if $\tilde{x}_i \mathbf{a} = \mathbf{0}$. In a similar way, we may view $\mathcal{M}_{r \times (n-r)}$ as a $U_q(A_{n-r-1})$ -crystal with respect to \tilde{e}_i, \tilde{f}_i ($r+1 \leq i \leq n-1$) by considering the transpose of $M \in \mathcal{M}_{r \times (n-r)}$ as an element in $\mathcal{M}_{[n] \setminus [r], [\bar{r}]}$. Since $\mathfrak{g}_{I_{0,r}} = A_{r-1} \oplus A_{n-r-1}$, $\mathcal{M}_{r \times (n-r)}$ is a regular $U_q(\mathfrak{g}_{I_{0,r}})$ -crystal with $\text{wt}(M) = \sum_{i,j} m_{ij}(\epsilon_j - \epsilon_i)$ for $M = (m_{ij}) \in \mathcal{M}_{r \times (n-r)}$.

Now, let us define two more operators \tilde{x}_0 and \tilde{x}_r ($x = e, f$) to make $\mathcal{M}_{r \times (n-r)}$ a $U_q(A_{n-1}^{(1)})$ -crystal. For $M = (m_{ij}) \in \mathcal{M}_{r \times (n-r)}$, we define

$$\begin{aligned} \tilde{e}_r M &= \begin{cases} M - E_{\bar{r}r+1}, & \text{if } m_{\bar{r}r+1} \geq 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases} & \tilde{f}_r M &= M + E_{\bar{r}r+1}, \\ \tilde{f}_0 M &= \begin{cases} M - E_{\bar{1}n}, & \text{if } m_{\bar{1}n} \geq 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases} & \tilde{e}_0 M &= M + E_{\bar{1}n}, \end{aligned} \quad (3.2)$$

where $E_{ij} \in \mathcal{M}_{r \times (n-r)}$ denotes the elementary matrix with 1 at the position (\bar{i}, j) and 0 elsewhere. Put

$$\begin{aligned} \varepsilon_r(M) &= \max\{k \mid \tilde{e}_r^k M \neq \mathbf{0}\}, & \varphi_r(M) &= \varepsilon_r(M) + \langle \text{wt}(M), h_r \rangle, \\ \varphi_0(M) &= \max\{k \mid \tilde{f}_0^k M \neq \mathbf{0}\}, & \varepsilon_0(M) &= \varphi_0(M) - \langle \text{wt}(M), h_0 \rangle. \end{aligned}$$

Then we have

Proposition 3.1. $\mathcal{M}_{r \times (n-r)}$ is a $U_q(A_{n-1}^{(1)})$ -crystal with respect to $\text{wt}, \varepsilon_i, \varphi_i$ and \tilde{e}_i, \tilde{f}_i ($i \in I$).

3.3. Young tableau description of $\mathcal{M}_{r \times (n-r)}$ as a $U_q(A_{n-1})$ -crystal

Let us give another description of $\mathcal{M}_{r \times (n-r)}$ in terms of semistandard tableaux. Consider

$$\mathcal{T}_{r \times (n-r)}^{\searrow} = \bigsqcup_{\ell(\lambda) \leq r, n-r} \text{SST}_{[\bar{r}]}(\lambda^\pi) \times \text{SST}_{[n] \setminus [r]}(\lambda^\pi). \quad (3.3)$$

By [4], $\text{SST}_{[\bar{r}]}(\lambda^\pi) \times \text{SST}_{[n] \setminus [r]}(\lambda^\pi)$ is a regular $U_q(\mathfrak{g}_{I_{0,r}})$ -crystal and so is $\mathcal{T}_{r \times (n-r)}^{\searrow}$.

We will define \tilde{e}_r, \tilde{f}_r on $\mathcal{T}_{r \times (n-r)}^\searrow$ to make $\mathcal{T}_{r \times (n-r)}^\searrow$ a $U_q(\mathfrak{gl}_0)$ -crystal. Let us first recall a combinatorial algorithm often called a signature rule, which will be used throughout the paper. Suppose that $\sigma = (\dots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \dots)$ is a sequence (not necessarily finite) with $\sigma_k \in \{+, -, \cdot\}$ such that $\sigma_k = +$ or \cdot for $k \gg 0$ and $\sigma_k = -$ or \cdot for $k \ll 0$. In σ , we replace a pair $(\sigma_s, \sigma_{s'}) = (+, -)$, where $s < s'$ and $\sigma_t = \cdot$ for $s < t < s'$, with (\cdot, \cdot) , and repeat this process as far as possible until we get a sequence with no $-$ placed to the right of $+$. Such a reduced sequence will be denoted by $\tilde{\sigma}$. When we have an infinite sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ (respectively $\sigma = (\dots, \sigma_2, \sigma_1)$), we also understand $\tilde{\sigma}$ as a reduced sequence obtained by applying the signature rule to a doubly infinite sequence $(\dots, \cdot, \cdot, \cdot, \sigma_1, \sigma_2, \dots)$ (respectively $(\dots, \sigma_2, \sigma_1, \cdot, \cdot, \cdot, \dots)$).

Now, let $(S, T) \in \mathcal{T}_{r \times (n-r)}^\searrow$ be given. For $k \geq 1$, let s_k and t_k be the entries in the top of the k -th columns of S and T (enumerated from the right), respectively. We put

$$\sigma_k = \begin{cases} +, & \text{if the } k\text{-th column is empty,} \\ +, & \text{if } s_k > \bar{r} \text{ and } t_k > r + 1, \\ -, & \text{if } s_k = \bar{r} \text{ and } t_k = r + 1, \\ \cdot, & \text{otherwise.} \end{cases}$$

Let $\tilde{\sigma}$ be the reduced sequence obtained from $\sigma = (\sigma_1, \sigma_2, \dots)$ by the signature rule. Then we define $\tilde{e}_r(S, T)$ to be the bitableaux obtained from (S, T) by removing $\boxed{\bar{r}}$ and $\boxed{r+1}$ in the columns of S and T corresponding to the right-most $-$ in $\tilde{\sigma}$. If there is no such $-$ sign, then we define $\tilde{e}_r(S, T) = \mathbf{0}$. We define $\tilde{f}_r(S, T)$ to be the bitableaux obtained from (S, T) by adding $\boxed{\bar{r}}$ and $\boxed{r+1}$ on top of the columns of S and T corresponding to the left-most $+$ in $\tilde{\sigma}$. Note that $\tilde{f}_r^k(S, T) \neq \mathbf{0}$ for all $k \geq 1$.

We put $\varepsilon_r(S, T) = \max\{k \mid \tilde{e}_r^k(S, T) \neq \mathbf{0}\}$ and $\varphi_r(S, T) = \varepsilon_r(S, T) + \langle \text{wt}(S, T), h_r \rangle$, where $\text{wt}(S, T) = \text{wt}(S) + \text{wt}(T)$. Then $\mathcal{T}_{r \times (n-r)}^\searrow$ is a $U_q(\mathfrak{gl}_0)$ -crystal with respect to wt , ε_i , φ_i and \tilde{e}_i, \tilde{f}_i ($i \in I_0$).

Example 3.2. Suppose that $n = 6$ and $r = 3$. Consider

$$(S, T) = \left(\begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{2} & \bar{2} & \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \right).$$

Then

$$\tilde{e}_3(S, T) = \left(\begin{array}{|c|c|c|c|} \hline \bar{2} & \bar{2} & & \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 4 & 4 & & \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \right),$$

and

$$\tilde{f}_3(S, T) = \left(\begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{2} & \bar{2} & \\ \hline \bar{3} & \bar{3} & \bar{2} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & \\ \hline 4 & 5 & 5 & 5 & 6 \\ \hline \end{array} \right).$$

Define

$$\kappa^\searrow : \mathcal{M}_{r \times (n-r)} \rightarrow \mathcal{T}_{r \times (n-r)}^\searrow \quad (3.4)$$

by $\kappa^\searrow(M) = (\mathbf{P}(M)^\searrow, \mathbf{Q}(M)^\searrow)$. By [5, Theorem 3.6], we have the following.

Proposition 3.3. κ^\searrow is an isomorphism of $U_q(\mathfrak{gl}_0)$ -crystals.

Example 3.4. Let (S, T) be as in Example 3.2. Then $(S, T) = \kappa^\searrow(M)$, where

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

We have

$$\tilde{e}_3 M = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

and $\kappa^{\nwarrow}(\tilde{e}_3 M) = \tilde{e}_3(S, T)$.

Next, let us consider

$$\mathcal{T}_{r \times (n-r)}^{\nwarrow} = \bigsqcup_{\ell(\lambda) \leq r, n-r} SST_{[\bar{r}]}(\lambda) \times SST_{[n] \setminus [r]}(\lambda). \quad (3.5)$$

As in $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$, $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$ is a regular $U_q(\mathfrak{gl}_{0,r})$ -crystal. Let us define \tilde{e}_0, \tilde{f}_0 on $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$ to make $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$ a $U_q(\mathfrak{gl}_r)$ -crystal. Let $(S, T) \in \mathcal{T}_{r \times (n-r)}^{\nwarrow}$ be given. For $k \geq 1$, let s_k and t_k be the entries in the bottom of the k -th columns of S and T (enumerated from the left), respectively. We put

$$\sigma_k = \begin{cases} -, & \text{if the } k\text{-th column is empty,} \\ -, & \text{if } s_k < \bar{1} \text{ and } t_k < n, \\ +, & \text{if } s_k = \bar{1} \text{ and } t_k = n, \\ \cdot, & \text{otherwise.} \end{cases}$$

Let $\tilde{\sigma}$ be the reduced sequence obtained from $\sigma = (\dots, \sigma_2, \sigma_1)$ by the signature rule. We define $\tilde{e}_0(S, T)$ to be the bitableaux obtained from (S, T) by adding $\boxed{\bar{1}}$ and \boxed{n} to the bottom of the columns of S and T corresponding to the right-most $-$ in $\tilde{\sigma}$. We define $\tilde{f}_0(S, T)$ to be the bitableaux obtained from (S, T) by removing $\boxed{\bar{1}}$ and \boxed{n} in the columns of S and T corresponding to the left-most $+$ in $\tilde{\sigma}$. If there is no such $+$ sign, then we define $\tilde{f}_0(S, T) = \mathbf{0}$. Note that $\tilde{e}_0^k(S, T) \neq \mathbf{0}$ for all $k \geq 1$.

We put $\varphi_0(S, T) = \max\{k \mid \tilde{f}_0^k(S, T) \neq \mathbf{0}\}$ and $\varepsilon_0(S, T) = \varphi_0(S, T) - \langle \text{wt}(S, T), h_0 \rangle$. Then $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$ is a $U_q(\mathfrak{gl}_r)$ -crystal with respect to wt , ε_i , φ_i and \tilde{e}_i, \tilde{f}_i ($i \in I_r$).

Define

$$\kappa^{\nwarrow} : \mathcal{M}_{r \times (n-r)} \rightarrow \mathcal{T}_{r \times (n-r)}^{\nwarrow} \quad (3.6)$$

by $\kappa^{\nwarrow}(M) = (\mathbf{P}(M)^{\nwarrow}, \mathbf{Q}(M)^{\nwarrow}) = (\mathbf{P}(M), \mathbf{Q}(M))$. By the same argument as in [5, Theorem 3.6], we have the following.

Proposition 3.5. κ^{\nwarrow} is an isomorphism of $U_q(\mathfrak{gl}_r)$ -crystals.

3.4. Main theorem

For $M \in \mathcal{M}_{r \times (n-r)}$ with $M = M(\mathbf{a}, \mathbf{b})$, let $\ell(M)$ be the maximal length of weakly decreasing subwords of \mathbf{a} . For $s \geq 1$, let

$$\mathcal{M}_{r \times (n-r)}^s = \{M \in \mathcal{M}_{r \times (n-r)} \mid \ell(M) \leq s\}. \quad (3.7)$$

Note that $\ell(M)$ is the number of columns in $\mathbf{P}(M)$ or $\mathbf{Q}(M)$ (cf. [19, §3.1]). We regard $\mathcal{M}_{r \times (n-r)}^s$ as a subcrystal of $\mathcal{M}_{r \times (n-r)}$ and define a $U_q(A_{n-1}^{(1)})$ -crystal

$$\mathcal{B}^{r,s} = \mathcal{M}_{r \times (n-r)}^s \otimes T_{s\omega_r}. \quad (3.8)$$

Lemma 3.6. $\mathcal{B}^{r,s}$ is a regular $U_q(A_{n-1}^{(1)})$ -crystal that is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{gl}_0)$ -crystal.

Proof. When restricted to $\mathcal{M}_{r \times (n-r)}^s$, we have the following bijections

$$\kappa^{\nwarrow} : \mathcal{M}_{r \times (n-r)}^s \rightarrow \mathcal{T}_{r \times (n-r)}^{\nwarrow, s}, \quad \kappa^{\nwarrow} : \mathcal{M}_{r \times (n-r)}^s \rightarrow \mathcal{T}_{r \times (n-r)}^{\nwarrow, s}, \quad (3.9)$$

where

$$\begin{aligned}\mathcal{T}_{r \times (n-r)}^{\searrow, s} &= \bigsqcup_{\substack{\ell(\lambda) \leq r, n-r \\ \lambda_1 \leq s}} \text{SST}_{[\bar{r}]}(\lambda^\pi) \times \text{SST}_{[n] \setminus [r]}(\lambda^\pi), \\ \mathcal{T}_{r \times (n-r)}^{\nwarrow, s} &= \bigsqcup_{\substack{\ell(\lambda) \leq r, n-r \\ \lambda_1 \leq s}} \text{SST}_{[\bar{r}]}(\lambda) \times \text{SST}_{[n] \setminus [r]}(\lambda).\end{aligned}$$

Since $\mathcal{T}_{r \times (n-r)}^{\searrow, s}$ (respectively $\mathcal{T}_{r \times (n-r)}^{\nwarrow, s}$) can be viewed as a subcrystal of $\mathcal{T}_{r \times (n-r)}^{\searrow}$ (respectively $\mathcal{T}_{r \times (n-r)}^{\nwarrow}$), κ^{\searrow} (respectively κ^{\nwarrow}) is an isomorphism of $U_q(\mathfrak{gl}_0)$ (respectively $U_q(\mathfrak{gl}_r)$)-crystals.

First we claim that $\mathcal{T}_{r \times (n-r)}^{\searrow, s} \otimes T_{s\omega_r}$ is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{gl}_0)$ -crystal. Recall that $\mathbf{B}(s\omega_r)$ can be identified with $\text{SST}_{[n]}((s^r))$ [4].

Let $(S, T) \in \mathcal{T}_{r \times (n-r)}^{\searrow, s}$ be given where $\text{sh}(S) = \text{sh}(T) = \lambda^\pi$ for some $\lambda \in \mathcal{P}$ with $\lambda_1 \leq s$. Consider an isomorphism of $U_q(\mathfrak{gl}_{\{1, \dots, r-1\}})$ -crystals,

$$\varsigma : \text{SST}_{[\bar{r}]}(\lambda^\pi) \otimes T_{s\omega_r} \rightarrow \text{SST}_{[r]}(\lambda^c),$$

where $\lambda^c = (s^r) \setminus \lambda^\pi = (s - \lambda_r, \dots, s - \lambda_1)$ is a rectangular complement of λ^π in (s^r) (see [21, Lemma 5.8] for an explicit description of ς , which is given as σ^s). Let $S^c = \varsigma(S \otimes t_{s\omega_r})$ and let U be the semistandard tableau in $\text{SST}_{[n]}((s^r))$ obtained by gluing S^c and T . Therefore, the map sending $(S, T) \otimes t_{s\omega_r}$ to U defines a weight preserving bijection (with the same notation)

$$\varsigma : \mathcal{T}_{r \times (n-r)}^{\searrow, s} \otimes T_{s\omega_r} \rightarrow \text{SST}_{[n]}((s^r)). \quad (3.10)$$

By definition, it is straightforward to check that ς commutes with \tilde{e}_r and \tilde{f}_r , which therefore implies that it is an isomorphism of $U_q(\mathfrak{gl}_0)$ -crystals.

Next consider $\mathcal{T}_{r \times (n-r)}^{\nwarrow, s} \otimes T_{s\omega_r} = \mathcal{T}_{r \times (n-r)}^{\nwarrow, s} \otimes T_{-s\omega'_0}$. We claim that $\mathcal{T}_{r \times (n-r)}^{\nwarrow, s} \otimes T_{s\omega_r}$ is isomorphic to $\mathbf{B}(-s\omega'_0)$ as a $U_q(\mathfrak{gl}_r)$ -crystal. Since $\mathbf{B}(-s\omega'_0) = \mathbf{B}(s\omega'_t)$ where $t \equiv 2r \pmod{n}$, $\mathbf{B}(-s\omega'_0)$ can be identified with $\text{SST}_{[n]+r}((s^r))$.

Let $(S, T) \in \mathcal{T}_{r \times (n-r)}^{\nwarrow, s}$ be given where $\text{sh}(S) = \text{sh}(T) = \lambda$ for some $\lambda \in \mathcal{P}$ with $\lambda_1 \leq s$. By modifying the bijection in [21, Lemma 5.8] (exchanging k^\vee and k), we have an isomorphism of $U_q(\mathfrak{gl}_{\{1, \dots, r-1\}})$ -crystals,

$$\bar{\varsigma} : \text{SST}_{[\bar{r}]}(\lambda) \otimes T_{s\omega_r} \rightarrow \text{SST}_{[r]}((s^r)/\lambda).$$

Let $\bar{S}^c = \bar{\varsigma}(S \otimes t_{s\omega_r})$ and let U be the semistandard tableau in $\text{SST}_{[n]+r}((s^r))$ obtained by gluing \bar{S}^c and T . Then the map sending $(S, T) \otimes t_{s\omega_r}$ to U defines a weight preserving bijection (with the same notation)

$$\bar{\varsigma} : \mathcal{T}_{r \times (n-r)}^{\nwarrow, s} \otimes T_{s\omega_r} \rightarrow \text{SST}_{[n]+r}((s^r)). \quad (3.11)$$

As in (3.10), $\bar{\varsigma}$ commutes with \tilde{e}_0 and \tilde{f}_0 and it is an isomorphism of $U_q(\mathfrak{gl}_r)$ -crystals.

Now, for a proper subset $J \subset I$ with $|J| \leq 2$, we have $J \subset I_0$ or $J \subset I_r$ or $J \subset \{0, r\}$. By (3.10) and (3.11), $\mathcal{B}^{r, s}$ is a crystal of an integrable $U_q(\mathfrak{gl}_J)$ -module. Hence it is a regular $U_q'(A_{n-1}^{(1)})$ -crystal. \square

Example 3.7. Assume that $n = 6$ and $r = 3$. Consider

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \in \mathcal{M}_{3 \times 3}^4.$$

Then we have

$$\mathbf{P}(M) \searrow = \begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{2} & \bar{2} & \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \quad \mathbf{Q}(M) \searrow = \begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array}.$$

Note that as an element in a $U_q(A_2)$ -crystal, $\mathbf{P}(M) \searrow$ is equivalent to

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & & & \\ \hline \end{array}.$$

By gluing it with $\mathbf{Q}(M) \searrow$, we have

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \in \mathbf{B}(4\omega_3),$$

which is equivalent to $M \otimes t_{4\omega_3} \in \mathcal{B}^{3,4}$ as an element in a $U_q(\mathfrak{gl}_0)$ ($= U_q(A_5)$)-crystal. If we view $M \in \mathcal{M}_{4 \times 3}^5$, then $M \otimes t_{5\omega_3} \in \mathcal{B}^{3,5}$ corresponds to

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 \\ \hline 3 & 5 & 5 & 5 & 6 \\ \hline \end{array} \in \mathbf{B}(5\omega_3).$$

On the other hand, we have

$$\mathbf{P}(M) \nwarrow = \begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{3} & \bar{2} & \bar{2} \\ \hline \bar{2} & \bar{1} & \bar{1} & \\ \hline \end{array}, \quad \mathbf{Q}(M) \nwarrow = \begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & 6 \\ \hline 5 & 5 & 5 & \\ \hline \end{array}.$$

Note that as an element in a $U_q(A_2)$ -crystal, $\mathbf{P}(M) \nwarrow$ is equivalent to

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array}.$$

By gluing it with $\mathbf{Q}(M) \nwarrow$, we have

$$\begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & 6 \\ \hline 5 & 5 & 5 & 1 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array} \in \mathbf{B}(-4\omega'_0) \cong \mathbf{B}(4\omega'_0),$$

which is equivalent to $M \otimes t_{4\omega_3} \in \mathcal{B}^{3,4}$ as an element in a $U_q(\mathfrak{gl}_3)$ ($= U_q(A_5)$)-crystal.

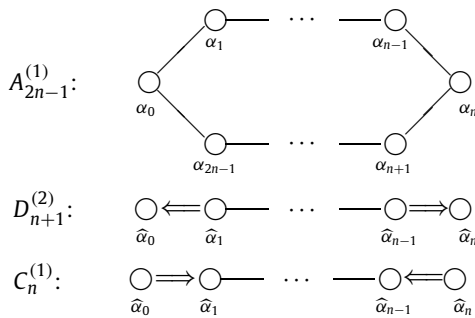
Theorem 3.8. Let $\mathbf{B}^{r,s}$ be the KR crystal of type $A_{n-1}^{(1)}$ for $1 \leq r \leq n-1$ and $s \geq 1$. Then as a $U'_q(A_{n-1}^{(1)})$ -crystal, we have $\mathcal{B}^{r,s} \cong \mathbf{B}^{r,s}$.

Proof. Note that $\mathbf{B}^{r,s}$ is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{gl}_0)$ -crystal [7]. Then it follows from Lemmas 2.1 and 3.6 that $\mathcal{B}^{r,s} \cong \mathbf{B}^{r,s}$. \square

4. Classically irreducible KR crystals of type $D_{n+1}^{(2)}$ and $C_n^{(1)}$

4.1. Affine algebras of type $D_{n+1}^{(2)}$ and $C_n^{(1)}$

Assume that $\mathfrak{g} = A_{2n-1}^{(1)}$ ($n \geq 2$) with $I = \{0, 1, \dots, 2n-1\}$ and the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, and $\widehat{\mathfrak{g}} = D_{n+1}^{(2)}$ or $C_n^{(1)}$ with $\widehat{I} = \{0, \dots, n\}$ and the Cartan datum $(\widehat{A}, \widehat{P}^\vee, \widehat{P}, \widehat{\Pi}^\vee, \widehat{\Pi})$.



Throughout this section, we assume that $\epsilon \in \{1, 2\}$ and $\widehat{\mathfrak{g}} = D_{n+1}^{(2)}$ (respectively $\widehat{\mathfrak{g}} = C_n^{(1)}$) when $\epsilon = 1$ (respectively $\epsilon = 2$). Put $\widehat{I}_r = \widehat{I} \setminus \{r\}$ ($r = 0, n$) and $\widehat{I}_{0,n} = \widehat{I}_0 \cap \widehat{I}_n$. Note that $\widehat{\mathfrak{g}}_{\widehat{I}_0} \cong \widehat{\mathfrak{g}}_{\widehat{I}_n} = B_n$ (respectively C_n) when $\epsilon = 1$ (respectively $\epsilon = 2$) and $\widehat{\mathfrak{g}}_{\widehat{I}_{0,n}} = A_{n-1}$. We may assume that

$$\begin{aligned}
 \widehat{P}^\vee &= \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d \subset P^\vee, \\
 \widehat{P} &= \left\{ \lambda \mid \frac{1}{\epsilon} \langle \lambda, h_i \rangle \in \mathbb{Z} \ (i = 0, n), \ \langle \lambda, h_i \rangle = \langle \lambda, h_{2n-i} \rangle \ (i \in \widehat{I}_{0,n}) \right\} \subset P, \\
 \widehat{\Pi}^\vee &= \{ \widehat{h}_i = h_i \ (i \in \widehat{I}) \} \subset \Pi^\vee, \\
 \widehat{\Pi} &= \{ \widehat{\alpha}_i = \epsilon \alpha_i \ (i = 0, n), \ \widehat{\alpha}_i = \alpha_i + \alpha_{2n-i} \ (i \in \widehat{I}_{0,n}) \} \subset \Pi.
 \end{aligned}$$

The classical weight lattice of $\widehat{\mathfrak{g}}$ is $\widehat{P}_{\text{cl}} = \bigoplus_{i \in \widehat{I}} \mathbb{Z} \widehat{\Lambda}_i$ and its dual classical weight lattice is $(\widehat{P}_{\text{cl}})^\vee = \bigoplus_{i \in \widehat{I}} \mathbb{Z} h_i$, where $\widehat{\Lambda}_i = \epsilon \Lambda_i$ for $i = 0, n$ and $\widehat{\Lambda}_i = \Lambda_i + \Lambda_{2n-i}$ for $i \in \widehat{I}_{0,n}$. Note that $\widehat{\alpha}_i = \widehat{\epsilon}_i - \widehat{\epsilon}_{i+1}$ ($i \in \widehat{I}_{0,n}$), where $\widehat{\epsilon}_i = \epsilon_i - \epsilon_{2n-i+1}$ for $i = 1, \dots, n$, $\widehat{\alpha}_0 = -\epsilon \widehat{\epsilon}_1$ and $\widehat{\alpha}_n = \epsilon \widehat{\epsilon}_n$ in \widehat{P}_{cl} . We denote the fundamental weights for $\widehat{\mathfrak{g}}_{\widehat{I}_0}$ by $\widehat{\omega}_i = \omega_i + \omega_{2n-i}$ for $i \in \widehat{I}_{0,n}$ and $\widehat{\omega}_n = \epsilon \omega_n$, and those for $\widehat{\mathfrak{g}}_{\widehat{I}_n}$ by $\widehat{\omega}'_i = \omega'_i + \omega'_{2n-i}$ for $i \in \widehat{I}_{0,n}$ and $\widehat{\omega}'_0 = \epsilon \omega'_0 = -\widehat{\omega}_n$.

4.2. Crystals of symmetric matrices

Put

$$\widehat{\mathcal{M}}_n = \{ M = (m_{ij}) \in \mathcal{M}_{n \times n} \mid m_{ij} = m_{ji} \text{ and } \epsilon | m_{ii} \text{ for } i, j \in [n] \}. \quad (4.1)$$

Define

$$\widehat{e}_i = \begin{cases} (\widetilde{e}_i)^\epsilon, & \text{for } i = 0, n, \\ \widetilde{e}_i \widetilde{e}_{2n-i}, & \text{for } i \in \widehat{I}_{0,n}, \end{cases} \quad \widehat{f}_i = \begin{cases} (\widetilde{f}_i)^\epsilon, & \text{for } i = 0, n, \\ \widetilde{f}_i \widetilde{f}_{2n-i}, & \text{for } i \in \widehat{I}_{0,n}. \end{cases}$$

Note that $\mathcal{M}_{n \times n}$ is a $U'_q(A_{2n-1}^{(1)})$ -crystal with respect to wt , ε_i , φ_i and \widetilde{e}_i , \widetilde{f}_i ($i \in I$) by Proposition 3.1. Then it is not difficult to see that $\widehat{\mathcal{M}}_n \cup \{\mathbf{0}\}$ is invariant under \widehat{e}_i and \widehat{f}_i for $i \in \widehat{I}$ (cf. [5, Proposition 5.14]). For $M \in \widehat{\mathcal{M}}_n$, define $\widehat{\text{wt}}(M) = \text{wt}(M)$,

$$\widehat{\varepsilon}_i(M) = \begin{cases} \frac{1}{\epsilon} \varepsilon_i(M), & \text{if } i = 0, n, \\ \varepsilon_i(M), & \text{if } i \in \widehat{I}_{0,n}, \end{cases} \quad \widehat{\varphi}_i(M) = \begin{cases} \frac{1}{\epsilon} \varphi_i(M), & \text{if } i = 0, n, \\ \varphi_i(M), & \text{if } i \in \widehat{I}_{0,n}. \end{cases}$$

Hence $\widehat{\mathcal{M}}_n$ is a $U'_q(\widehat{\mathfrak{g}})$ -crystal with respect to $\widehat{\text{wt}}$, $\widehat{\varepsilon}_i$, $\widehat{\varphi}_i$, \widehat{e}_i , \widehat{f}_i ($i \in \widehat{I}$).

Consider

$$\widehat{\mathcal{T}}_n^{\searrow \lambda} = \bigsqcup_{\ell(\lambda) \leq n} \text{SST}_{[\overline{n}]}(\epsilon \lambda^\pi), \quad \widehat{\mathcal{T}}_n^{\nwarrow} = \bigsqcup_{\ell(\lambda) \leq n} \text{SST}_{[\overline{n}]}(\epsilon \lambda), \quad (4.2)$$

where $2\lambda = (2\lambda_i)_{i \geq 1}$ for $\lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P}$. They are regular $U_q(\widehat{\mathfrak{gl}}_{0,n})$ -crystals with respect to \tilde{e}_i, \tilde{f}_i ($i \in \widehat{I}_{0,n}$). Here $\text{wt}(T) = -\sum_{i \in [n]} m_i \tilde{e}_i$, for $T \in \widehat{\mathcal{T}}_n^{\searrow}$ or $\widehat{\mathcal{T}}_n^{\swarrow}$, where m_i is the number of i 's appearing in T .

Let us define \tilde{e}_n, \tilde{f}_n on $\widehat{\mathcal{T}}_n^{\searrow}$ corresponding to $\widehat{\alpha}_n$ as follows: Let $T \in \widehat{\mathcal{T}}_n^{\searrow}$ be given. Suppose that $\epsilon = 1$. For $k \geq 1$, let t_k be the entry in the top of the k -th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \dots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_k > \bar{n} \text{ or the } k\text{-th column is empty,} \\ -, & \text{if } t_k = \bar{n}. \end{cases}$$

Then we define $\tilde{e}_n T$ to be the tableau obtained from T by removing $\boxed{\bar{n}}$ in the column corresponding to the right-most $-$ in $\tilde{\sigma}$. If there is no such $-$ sign, then we define $\tilde{e}_n T = \mathbf{0}$. We define $\tilde{f}_n T$ to be the tableau obtained from T by adding $\boxed{\bar{n}}$ on top of the column corresponding to the left-most $+$ in $\tilde{\sigma}$. Suppose that $\epsilon = 2$. For each $k \geq 1$, let (t_{2k}, t_{2k-1}) the pair of entries in the top of the $2k$ -th and $(2k-1)$ -st columns of T (from the right), respectively. Note that t_{2k} and t_{2k-1} are placed in the same row and $t_{2k} \leq t_{2k-1}$. Consider $\sigma = (\sigma_1, \sigma_2, \dots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_{2k}, t_{2k-1} > \bar{n} \text{ or the } (2k-1)\text{-st column is empty,} \\ -, & \text{if } t_{2k} = t_{2k-1} = \bar{n}, \\ \cdot, & \text{otherwise.} \end{cases}$$

Then we define $\tilde{e}_n T$ and $\tilde{f}_n T$ in the same way as in $\epsilon = 1$ with $\boxed{\bar{n}}$ replaced by $\boxed{\bar{n} \bar{n}}$.

Hence $\widehat{\mathcal{T}}_n^{\searrow}$ is a $U'_q(\widehat{\mathfrak{gl}}_0)$ -crystal with respect to $\text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ ($i \in \widehat{I}_0$), where $\varepsilon_n(T) = \max\{k \mid \tilde{e}_n^k T \neq \mathbf{0}\}$ and $\varphi_n(T) = \varepsilon_n(T) + \langle \text{wt}(T), \widehat{h}_n \rangle$.

Proposition 4.1. The map $\widehat{\kappa}^{\searrow} : \widehat{\mathcal{M}}_n \rightarrow \widehat{\mathcal{T}}_n^{\searrow}$ given by $\widehat{\kappa}^{\searrow}(M) = \mathbf{P}(M)^{\searrow}$ is an isomorphism of $U_q(\widehat{\mathfrak{gl}}_0)$ -crystals.

Proof. It follows from [22, Propositions 3.5 and 6.5]. \square

Next, let us define \tilde{e}_0, \tilde{f}_0 on $\widehat{\mathcal{T}}_n^{\swarrow}$ corresponding to $\widehat{\alpha}_0$ as follows: Let $T \in \widehat{\mathcal{T}}_n^{\swarrow}$ be given. Suppose that $\epsilon = 1$. For $k \geq 1$, let t_k be the entry in the bottom of the k -th column of T (enumerated from the left). Consider $\sigma = (\dots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_k < \bar{1} \text{ or the } k\text{-th column is empty,} \\ +, & \text{if } t_k = \bar{1}. \end{cases}$$

Then we define $\tilde{e}_0 T$ to be the tableau obtained from T by adding $\boxed{\bar{1}}$ to the bottom of the column corresponding to the right-most $-$ in $\tilde{\sigma}$. We define $\tilde{f}_0 T$ to be the tableau obtained from T by removing $\boxed{\bar{1}}$ in the column corresponding to the left-most $+$ in $\tilde{\sigma}$. If there is no such $+$ sign, then we define $\tilde{f}_0 T = \mathbf{0}$. Suppose that $\epsilon = 2$. For $k \geq 1$, let (t_{2k-1}, t_{2k}) be the pair of entries in the bottom boxes of the $(2k-1)$ -st and $2k$ -th columns of T (from the left), respectively. Note that t_{2k-1} and t_{2k} are placed in the same row and $t_{2k-1} \geq t_{2k}$. Consider $\sigma = (\dots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_{2k-1}, t_{2k} < \bar{1} \text{ or the } (2k-1)\text{-st column is empty,} \\ +, & \text{if } t_{2k-1} = t_{2k} = \bar{1}, \\ \cdot, & \text{otherwise.} \end{cases}$$

Then we define $\tilde{e}_n T$ and $\tilde{f}_n T$ in the same way as in $\epsilon = 1$ with $\boxed{\bar{1}}$ replaced by $\boxed{\bar{1} \bar{1}}$.

Hence $\widehat{\mathcal{T}}_n^{\swarrow}$ is a $U'_q(\widehat{\mathfrak{gl}}_n)$ -crystal with respect to $\text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ ($i \in \widehat{I}_n$), where $\varphi_0(T) = \max\{k \mid \tilde{f}_0^k T \neq \mathbf{0}\}$ and $\varepsilon_0(T) = \varphi_0(T) - \langle \text{wt}(T), \widehat{h}_0 \rangle$. Then we have

Proposition 4.2. The map $\widehat{\kappa}^{\swarrow} : \widehat{\mathcal{M}}_n \rightarrow \widehat{\mathcal{T}}_n^{\swarrow}$ given by $\widehat{\kappa}^{\swarrow}(M) = \mathbf{P}(M)^{\swarrow}$ is an isomorphism of $U_q(\widehat{\mathfrak{gl}}_n)$ -crystals.

4.3. KR crystals $\mathbf{B}^{n,s}$

For $s \geq 1$, let $\widehat{\mathcal{M}}_n^s = \widehat{\mathcal{M}}_n \cap \mathcal{M}_{n \times n}^{\epsilon s}$. We regard $\widehat{\mathcal{M}}_n^s$ as a subcrystal of $\widehat{\mathcal{M}}_n$ and consider a $U'_q(\widehat{\mathfrak{g}})$ -crystal

$$\mathcal{B}^{n,s} = \widehat{\mathcal{M}}_n^s \otimes T_{s\widehat{\omega}_n}. \quad (4.3)$$

Lemma 4.3. $\mathcal{B}^{n,s}$ is a regular $U'_q(\widehat{\mathfrak{g}})$ -crystal that is isomorphic to $\mathbf{B}(s\widehat{\omega}_n)$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$ -crystal.

Proof. By (3.9), we have bijections

$$\widehat{\kappa}^{\searrow} : \widehat{\mathcal{M}}_n^s \rightarrow \widehat{\mathcal{T}}_n^{\searrow,s}, \quad \widehat{\kappa}^{\nwarrow} : \widehat{\mathcal{M}}_n^s \rightarrow \widehat{\mathcal{T}}_n^{\nwarrow,s}, \quad (4.4)$$

where $\widehat{\mathcal{T}}_n^{\searrow,s}$ (respectively $\widehat{\mathcal{T}}_n^{\nwarrow,s}$) is the set of tableaux $T \in \widehat{\mathcal{T}}_n^{\searrow}$ of $\text{sh}(T) = \epsilon \lambda^\pi$ (respectively $\epsilon \lambda$) with $\lambda \subset (\epsilon s^n)$. We may regard $\widehat{\mathcal{T}}_n^{\searrow,s}$ and $\widehat{\mathcal{T}}_n^{\nwarrow,s}$ as subcrystals of $\widehat{\mathcal{T}}_n^{\searrow}$ and $\widehat{\mathcal{T}}_n^{\nwarrow}$, respectively. Then by Propositions 4.1 and 4.2, the bijections in (4.4) are isomorphisms of $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$ and $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_n})$ -crystals, respectively. On the other hand, by [5, Remark 5.16] (or as a special case of [22, Theorem 6.4] when λ is the empty partition), we have $\mathcal{B}^{n,s} \cong \widehat{\mathcal{T}}_n^{\searrow,s} \otimes T_{s\widehat{\omega}_n} \cong \mathbf{B}(s\widehat{\omega}_n)$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$ -crystal, and $\mathcal{B}^{n,s} \cong \widehat{\mathcal{T}}_n^{\nwarrow,s} \otimes T_{s\widehat{\omega}_n} \cong \mathbf{B}(-s\widehat{\omega}'_0) \cong \mathbf{B}(s\widehat{\omega}'_0)$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_n})$ -crystal. This implies that $\mathcal{B}^{n,s}$ is regular. \square

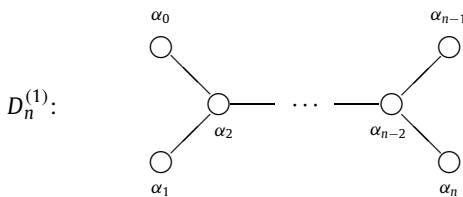
Theorem 4.4. Let $\mathbf{B}^{n,s}$ be the KR crystal of type $\widehat{\mathfrak{g}}$ for $s \geq 1$. Then as a $U'_q(\widehat{\mathfrak{g}})$ -crystal, we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$.

Proof. Since $\mathbf{B}^{n,s} \cong \mathbf{B}(s\widehat{\omega}_n)$ as an $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$ -crystal (cf. [14]), we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$ by Lemmas 2.1 and 4.3. \square

5. Classically irreducible KR crystals of type $D_n^{(1)}$

5.1. Affine algebra of type $D_n^{(1)}$

Assume that $\mathfrak{g} = D_n^{(1)}$ ($n \geq 4$) with $I = \{0, 1, \dots, n\}$. Put $I_r = I \setminus \{r\}$ ($r = 0, n$), and $I_{0,n} = I_0 \cap I_n$. Note that $\mathfrak{g}_{I_0} \cong \mathfrak{g}_{I_n} = D_n$ and $\mathfrak{g}_{I_{0,n}} = A_{n-1}$.



Let $\epsilon_1 = \Lambda_1 - \Lambda_0$, $\epsilon_2 = \Lambda_2 - \Lambda_1 - \Lambda_0$, $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 3, \dots, n-2$, $\epsilon_{n-1} = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-2}$ and $\epsilon_n = \Lambda_n - \Lambda_{n-1}$. Then $\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ forms a weight lattice of \mathfrak{g}_{I_0} . Note that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I_{0,n}$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$, and $\alpha_0 = -\epsilon_1 - \epsilon_2$ in P_{cl} . The fundamental weights for \mathfrak{g}_{I_0} are $\omega_i = \sum_{k=1}^i \epsilon_k$ for $i = 1, \dots, n-2$, $\omega_{n-1} = (\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n)/2$ and $\omega_n = (\epsilon_1 + \dots + \epsilon_{n-1} + \epsilon_n)/2$. We denote the fundamental weights for \mathfrak{g}_{I_n} by ω'_i for $i \in I_n$, where $\omega'_i = \omega_i$ for $i \in I_{0,n}$ and $\omega'_0 = -\omega_n$.

5.2. Young tableau descriptions of $\mathbf{B}(s\omega_n)$ and $\mathbf{B}(-s\omega'_0)$

Consider

$$\mathcal{T}_n^{\searrow} = \bigsqcup_{\substack{\lambda'_i: \text{ even} \\ \ell(\lambda) \leq n}} \text{SST}_{[\bar{n}]}(\lambda^\pi). \quad (5.1)$$

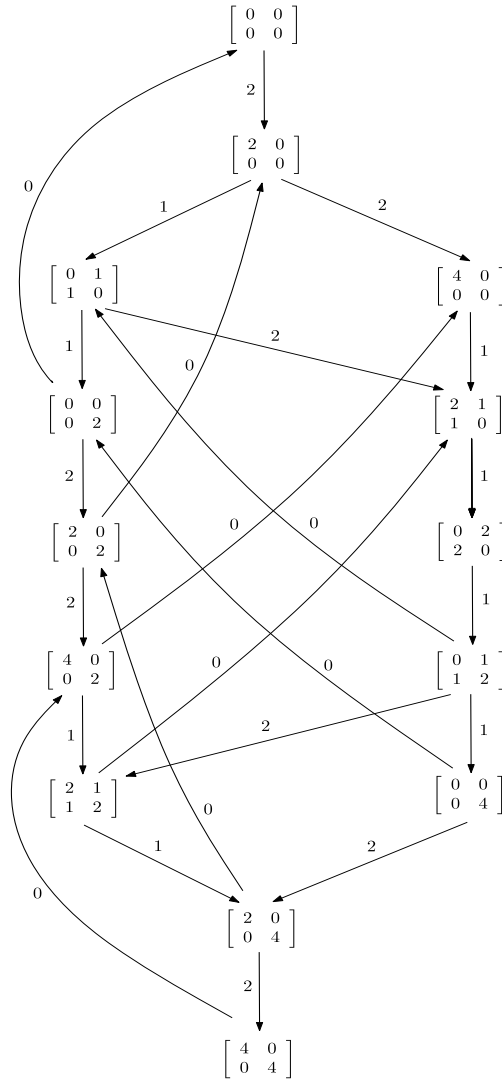


Fig. 2. The KR crystal graph $\mathbf{B}^{2,2}$ of type $C_2^{(1)}$.

It is a regular $U_q(\mathfrak{gl}_{0,n})$ -crystal with respect to \tilde{e}_i and \tilde{f}_i for $i \in I_{0,n}$, where $\text{wt}(T) = -\sum_{i \in [n]} m_i \epsilon_i$ (m_i is the number of i 's in T) for $T \in \mathcal{T}_n^{\searrow}$.

Let $T \in \mathcal{T}_n^{\searrow}$ be given. For $k \geq 1$, let t_k be the entry in the top of the k -th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \dots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_k > \overline{n-1} \text{ or the } k\text{-th column is empty,} \\ -, & \text{if the } k\text{-th column has both } \overline{n-1} \text{ and } \bar{n} \text{ as its entries,} \\ \cdot, & \text{otherwise.} \end{cases}$$

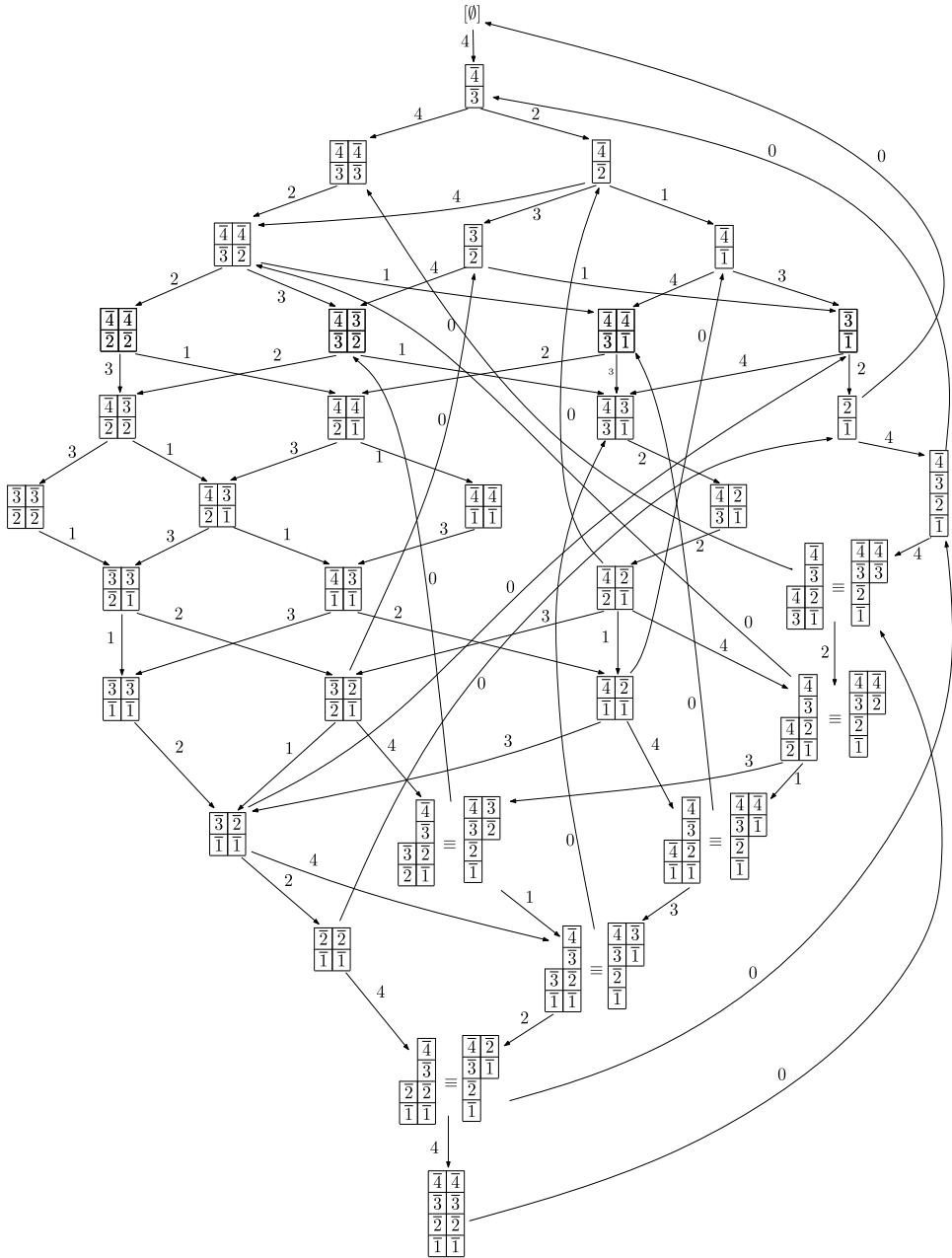


Fig. 3. The KR crystal graph $B^{4,2}$ of type $D_4^{(1)}$. Here \equiv denotes the Knuth equivalence or $U_q(A_3)$ -crystal equivalence.

Define $\tilde{e}_n T$ and $\tilde{f}_n T$ as in the case of $\widehat{\mathcal{T}}_n^{\searrow}$ (see Section 4) with \overline{n} replaced by $\frac{\overline{n}}{n-1}$. Then \mathcal{T}_n^{\searrow} is a $U_q(\mathfrak{g}_{I_0})$ -crystal with respect to wt , ε_i , φ_i , \tilde{e}_i , \tilde{f}_i ($i \in I_0$), where $\varepsilon_n(T) = \max\{k \mid e_n^k T \neq \mathbf{0}\}$ and $\varphi_n(T) = \varepsilon_n(T) + \langle \text{wt}(T), h_n \rangle$.

For $s \geq 1$, let $\mathcal{T}_n^{\searrow, s}$ be the set of tableaux $T \in \mathcal{T}_n^{\searrow}$ of shape λ^π with $\lambda \subset (s^n)$, and consider $\mathcal{T}_n^{\searrow, s}$ as a subcrystal of \mathcal{T}_n^{\searrow} .

Lemma 5.1. $\mathcal{T}_n^{\searrow, s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{gl}_0)$ -crystal.

Proof. First we prove the case when $s = 1$. Recall that $\mathbf{B}(\omega_n)$ is the crystal of the spin representation of $U_q(\mathfrak{gl}_0)$, and by [4] it can be identified with $\{v = (i_1, \dots, i_n) \mid i_k = \pm 1, i_1 \cdots i_n = 1\}$, where $\text{wt}(v) = \frac{1}{2} \sum_{k=1}^n i_k \epsilon_k$ and

$$\tilde{e}_k v = \begin{cases} (\dots, -i_k, -i_{k+1}, \dots), & \text{if } k \in I_{0,n} \text{ and } (i_k, i_{k+1}) = (-1, 1), \\ (\dots, -i_{n-1}, -i_n), & \text{if } k = n \text{ and } (i_{n-1}, i_n) = (-1, -1), \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$\tilde{f}_k v = \begin{cases} (\dots, -i_k, -i_{k+1}, \dots), & \text{if } k \in I_{0,n} \text{ and } (i_k, i_{k+1}) = (1, -1), \\ (\dots, -i_{n-1}, -i_n), & \text{if } k = n \text{ and } (i_{n-1}, i_n) = (1, 1), \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{T}_n^{\searrow, 1}$ is the set of semistandard tableaux with a single column of even length no more than n . Define $\rho: \mathcal{T}_n^{\searrow, 1} \otimes T_{\omega_n} \rightarrow \mathbf{B}(\omega_n)$ by $\rho(T \otimes t_{\omega_n}) = (i_1, \dots, i_n)$, where $i_k = -1$ if and only if \bar{k} appears in T . Note that the empty tableau is mapped to $(1, \dots, 1)$ of weight ω_n . Then ρ is an isomorphism of $U_q(\mathfrak{gl}_0)$ -crystals.

For $s \geq 1$, consider the map

$$\iota_s: \mathcal{T}_n^{\searrow, s} \otimes T_{s\omega_n} \rightarrow (\mathcal{T}_n^{\searrow, 1})^{\otimes s} \otimes T_{s\omega_n} \cong (\mathcal{T}_n^{\searrow, 1} \otimes T_{\omega_n})^{\otimes s} \cong \mathbf{B}(\omega_n)^{\otimes s},$$

where for $\iota_s(T \otimes t_{s\omega_n}) = T^1 \otimes \cdots \otimes T^s \otimes t_{s\omega_n}$ (T^i is the i -th column of T from the right). Then it is straightforward to check that ι_s is a strict embedding of $U_q(\mathfrak{gl}_0)$ -crystals, and its image is isomorphic to the connected component of $\emptyset^{\otimes s} \otimes t_{s\omega_n}$, where \emptyset is the empty tableau. Since $\emptyset^{\otimes s} \otimes t_{s\omega_n}$ is a highest weight element of weight $s\omega_n$ in $\mathbf{B}(\omega_n)^{\otimes s}$, $\mathcal{T}_n^{\searrow, s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(s\omega_n)$. \square

Next, consider

$$\mathcal{T}_n^{\nwarrow} = \bigsqcup_{\substack{\lambda'_i: \text{even} \\ \ell(\lambda) \leq n}} \text{SST}_{[\bar{n}]}(\lambda). \quad (5.2)$$

As in \mathcal{T}_n^{\searrow} , it is a regular $U_q(\mathfrak{gl}_{0,n})$ -crystal. Let $T \in \mathcal{T}_n^{\nwarrow}$ be given. For $k \geq 1$, let t_k be the entry in the bottom of the k -th column of T (enumerated from the left). Consider $\sigma = (\dots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_k < \bar{2} \text{ or the } k\text{-th column is empty,} \\ +, & \text{if the } k\text{-th column has both } \bar{1} \text{ and } \bar{2} \text{ as its entries,} \\ \cdot, & \text{otherwise.} \end{cases}$$

Define $\tilde{e}_0 T$ and $\tilde{f}_0 T$ as in the case of $\hat{\mathcal{T}}_n^{\nwarrow}$ (see Section 4) with $\boxed{1}$ replaced by $\boxed{\bar{2}}$. Then \mathcal{T}_n^{\nwarrow} is a $U_q(\mathfrak{gl}_n)$ -crystal with respect to wt , ε_i , φ_i , \tilde{e}_i , \tilde{f}_i ($i \in I_n$), where $\varphi_0(T) = \max\{k \mid \tilde{f}_0^k T \neq \mathbf{0}\}$ and $\varepsilon_0(T) = \varphi_0(T) - \langle \text{wt}(T), h_0 \rangle$.

For $s \geq 1$, let $\mathcal{T}_n^{\nwarrow, s}$ be the set of tableaux $T \in \mathcal{T}_n^{\nwarrow}$ of shape λ with $\lambda \subset (s^n)$ consider $\mathcal{T}_n^{\nwarrow, s}$ as a subcrystal of \mathcal{T}_n^{\nwarrow} .

Lemma 5.2. $\mathcal{T}_n^{\nwarrow, s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(-s\omega'_0)$ as a $U_q(\mathfrak{gl}_n)$ -crystal.

Proof. The proof is similar to that of Lemma 5.1. \square

5.3. KR crystals $\mathbf{B}^{n,s}$

For a semistandard tableau T of skew shape, let $[T]$ denote the equivalence class of T with respect to Knuth equivalence. For $n \geq 4$, let

$$\mathcal{T}_n = \{[T] \mid T \in \mathcal{T}_n^{\searrow}\} = \{[T] \mid T \in \mathcal{T}_n^{\nwarrow}\}. \quad (5.3)$$

Recall that under \tilde{e}_i and \tilde{f}_i for $i \in I_{0,n}$, any $T' \in [T]$ generates the same crystal as T . Hence, \mathcal{T}_n has a well-defined $U_q(\mathfrak{gl}_{0,n})$ -crystal structure. Now, for $i = 0, n$ and $x = e, f$, we define

$$\tilde{x}_i[T] = \begin{cases} [\tilde{x}_0 T^{\nwarrow}], & \text{if } i = 0, \\ [\tilde{x}_n T^{\searrow}], & \text{if } i = n, \end{cases} \quad (5.4)$$

where we assume that $[0] = 0$. Put

$$\begin{aligned} \text{wt}([T]) &= \text{wt}(T), & \varepsilon_i([T]) &= \varepsilon_i(T), & \varphi_i([T]) &= \varphi_i(T) \quad (i \in I_{0,n}), \\ \varepsilon_n([T]) &= \varepsilon_n(T^{\searrow}), & \varphi_n([T]) &= \varphi_n(T^{\searrow}), \\ \varepsilon_0([T]) &= \varepsilon_n(T^{\nwarrow}), & \varphi_0([T]) &= \varphi_n(T^{\nwarrow}). \end{aligned} \quad (5.5)$$

Then, \mathcal{T}_n is a $U'_q(\mathfrak{g})$ -crystal with respect to wt , ε_i , φ_i , \tilde{e}_i , \tilde{f}_i ($i \in I$).

Now, for $s \geq 1$, we put $\mathcal{T}_n^s = \{[T] \mid T \in \mathcal{T}_n^{\searrow, s}\} = \{[T] \mid T \in \mathcal{T}_n^{\nwarrow, s}\}$, which is a subcrystal of \mathcal{T}_n , and then define

$$\mathcal{B}^{n,s} = \mathcal{T}_n^s \otimes T_{s\omega_n}. \quad (5.6)$$

Lemma 5.3. $\mathcal{B}^{n,s}$ is a regular $U'_q(\mathfrak{g})$ -crystal that is isomorphic to $\mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{gl}_0)$ -crystal.

Proof. By definition of $\mathcal{B}^{n,s}$ and Lemmas 5.1 and 5.2, we have $\mathcal{B}^{n,s} \cong \mathcal{T}_n^{\searrow, s} \otimes T_{s\omega_n} \cong \mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{gl}_0)$ -crystal, and $\mathcal{B}^{n,s} \cong \mathcal{T}_n^{\nwarrow, s} \otimes T_{s\omega_n} \cong \mathbf{B}(-s\omega'_0)$ as a $U_q(\mathfrak{gl}_n)$ -crystal. This implies that $\mathcal{B}^{n,s}$ is regular. \square

Theorem 5.4. Let $\mathbf{B}^{n,s}$ be the KR crystal of type $\mathfrak{g} = D_n^{(1)}$ for $s \geq 1$. Then as a $U'_q(\mathfrak{g})$ -crystal, we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$.

Proof. Since $\mathbf{B}^{n,s} \cong \mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{gl}_0)$ -crystal (cf. [14]), we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$ by Lemmas 2.1 and 5.3. \square

Remark 5.5. One may expect a matrix realization of $\mathbf{B}^{n,s}$ as in the cases of $A_{n-1}^{(1)}$, $D_{n+1}^{(2)}$ and $C_n^{(1)}$. In fact, there is a variation of RSK map which is a bijection from \mathcal{T}_n to a set of symmetric non-negative integral matrices with trace zero and also an isomorphism of $U_q(A_{n-1})$ -crystals (see [21, Proposition 3.13] when $m = 0$). But there does not seem to be a natural extension to an isomorphism of $U_q(D_n)$ -crystals (and hence $U_q(D_n^{(1)})$ -crystals).

5.4. KR crystals $\mathbf{B}^{n-1,s}$

Let us give a combinatorial description of $\mathbf{B}^{n-1,s}$ to complete the list of KR crystals associated to exceptional nodes in the Dynkin diagram of classical affine type. In this case, we put

$$\mathcal{B}^{n-1,s} = \tilde{\mathcal{T}}_n^s \otimes T_{s\omega_n}, \quad (5.7)$$

where $\tilde{\mathcal{T}}_n^s$ is defined in the same way as \mathcal{T}_n^s in Section 5.3 with λ'_i being odd for all i (see (5.2)). Then

$$\mathcal{B}^{n-1,s} \cong \mathbf{B}^{n-1,s}, \quad (5.8)$$

where $\mathbf{B}^{n-1,s}$ is the KR crystal isomorphic to $\mathbf{B}(s\omega_{n-1})$ as a $U_q(\mathfrak{gl}_0)$ -crystal. The proof is almost identical to that of Theorem 5.4. So we leave the details to the reader.

6. Remarks on \tilde{e}_0 and \tilde{f}_0

6.1. Lusztig involution

Let η be the involutive automorphism of $U_q(A_{n-1})$ given by $\eta(e_i) = f_{n-i}$, $\eta(f_i) = e_{n-i}$, and $\eta(q^{h_i}) = q^{-h_{n-i}}$ ($i = 1, \dots, n-1$). Let w_0 be the longest element in the Weyl group of A_{n-1} . Recall that $w_0(\alpha_i) = -\alpha_{n-i}$ for $i = 1, \dots, n-1$. Let B be a crystal of a finite dimensional $U_q(A_{n-1})$ -module. Then by [23, Proposition 21.1.2], we have an induced map

$$\eta : B \rightarrow B \quad (6.1)$$

such that $\eta^2(b) = b$, $\text{wt}(\eta(b)) = w_0(\text{wt}(b))$, $\eta(\tilde{e}_i(b)) = \tilde{f}_{n-i}\eta(b)$ and $\eta(\tilde{f}_i b) = \tilde{e}_{n-i}\eta(b)$ for $b \in B$ and $i = 1, \dots, n-1$. Similarly, one can define η on a crystal of a finite dimensional $U_q(A_{m-1} \oplus A_{n-1})$ -module for $m, n \geq 2$.

In [24], it is shown that η coincides with the Schützenberger's involution (see e.g. [19]) when $B = \text{SST}_{[n]}(\lambda)$ for $\lambda \in \mathcal{P}$ with $\ell(\lambda) \leq n$. Indeed, for $T \in \text{SST}_{[n]}(\lambda)$, let T' be the tableau obtained by 180° -rotation of T and replacing i with $n-i+1$. Then $\eta(T) = (T')^\vee$.

Based on our combinatorial descriptions, we have the following characterization of \tilde{e}_0 and \tilde{f}_0 on classically irreducible KR crystals in terms of η on an underlying classical crystal of type A .

Proposition 6.1. *Let $\mathbf{B}^{r,s}$ be a classically irreducible KR crystal of type \mathfrak{g} ($s \geq 1$) (that is, for $r = 1, \dots, n-1$ when $\mathfrak{g} = A_{n-1}^{(1)}$, $r = n$ when $\mathfrak{g} = D_{n+1}^{(2)}$, $C_n^{(1)}$, $r = n, n-1$ when $\mathfrak{g} = D_n^{(1)}$, and $s \geq 1$). Let η denote the involution (6.1) on $\mathbf{B}^{r,s}$ as a crystal of type \mathfrak{g}_J with $J = I \setminus \{0, r\}$. Then we have on $\mathbf{B}^{r,s}$*

$$\tilde{e}_0 = \eta \circ \tilde{f}_r \circ \eta, \quad \tilde{f}_0 = \eta \circ \tilde{e}_r \circ \eta.$$

Proof. We assume that $x = e$ (respectively f) when $y = f$ (respectively e) throughout the proof.

CASE 1. $\mathbf{B}^{r,s}$ of type $A_{n-1}^{(1)}$ for $r = 1, \dots, n-1$ and $s \geq 1$. Note that $\mathfrak{g}_J = A_{r-1} \oplus A_{n-r-1}$. Consider $\pi : \mathcal{M}_{r \times (n-r)} \rightarrow \mathcal{M}_{r \times (n-r)}$, where $\pi(M)$ is obtained by 180° -rotation of M . By definition of \tilde{e}_0 and \tilde{f}_0 on $\mathcal{M}_{r \times (n-r)}$, we have $\tilde{x}_0 = \pi \circ \tilde{y}_r \circ \pi$.

Let $M = M(\mathbf{a}, \mathbf{b})$ be given with $\mathbf{a} = \bar{i}_1 \cdots \bar{i}_k$. Then $\pi(M) = M(\mathbf{a}^\pi, \mathbf{b}^\pi)$ with $\mathbf{a}^\pi = \overline{r-i_k+1} \cdots \overline{r-i_1+1}$. Also, if $M^t = M(\mathbf{c}, \mathbf{d})$ with $\mathbf{c} = j_1 \cdots j_l$, then $\pi(M^t) = M(\mathbf{c}^\pi, \mathbf{d}^\pi)$ with $\mathbf{c}^\pi = (n-j_l+r+1) \cdots (n-j_1+r+1)$. This implies that

$$\tilde{x}_i M \neq \mathbf{0} \iff \tilde{y}_{n-i+r} \pi(M) \neq \mathbf{0}, \quad (6.2)$$

for $i \in I_{0,r}$, where the indices are assumed to be in \mathbb{Z}_n . On the other hand, we have

$$\tilde{x}_i M \neq \mathbf{0} \iff \tilde{y}_{n-i+r} \eta(M) \neq \mathbf{0}, \quad (6.3)$$

for $i \in I_{0,r}$.

Let $M = (m_{ij})$ be a \mathfrak{g}_J -highest weight element in $\mathcal{M}_{r \times (n-r)}$, where $m_{ij} = 0$ unless $i = j$, and $m_{\overline{r+1}} \geq m_{\overline{r-1}+2} \geq m_{\overline{r-2}+2} \geq \cdots$. It is easy to see that $\pi(M) = \eta(M)$. Then it follows from (6.2) and (6.3) that $\pi = \eta$ and hence $\tilde{x}_0 = \eta \circ \tilde{y}_r \circ \eta$ on $\mathcal{M}_{r \times (n-r)}$. Since $\mathbf{B}^{r,s}$ is a subcrystal of $\mathcal{M}_{r \times (n-r)} \otimes T_{s\omega_r}$, we have $\tilde{x}_0 = \eta \circ \tilde{y}_r \circ \eta$ on $\mathbf{B}^{r,s}$.

CASE 2. $\mathbf{B}^{n,s}$ of type $D_{n+1}^{(2)}$, $C_n^{(1)}$ for $s \geq 1$. The proof is similar to CASE 1.

CASE 3. $\mathbf{B}^{r,s}$ of type $D_n^{(1)}$ for $r = n, n-1$ and $s \geq 1$. Let us prove the case $\mathbf{B}^{n,s}$. The proof for $\mathbf{B}^{n-1,s}$ is almost the same.

Let $[T] \in \mathcal{T}_n$ be given. Define a map $\pi : \mathcal{T}_n \rightarrow \mathcal{T}_n$, where $\pi([T]) = [T']$ and T' is obtained by 180° -rotation of T and replacing each entry i in T with $n-i+1$. By definition, $\tilde{x}_i T \neq \mathbf{0}$ if and only if $\tilde{y}_{n-i} T' \neq \mathbf{0}$ ($i = 1, \dots, n-1$). This implies that $[T'] = [\eta(T)]$. Moreover, if T is of normal shape, then we have by definition of \tilde{x}_0 and \tilde{y}_n (see Section 5.2) $\tilde{x}_0([T]) = (\pi \circ \tilde{y}_n \circ \pi)([T])$. Since the action of η is also well-defined on \mathcal{T}_n (that is, $\eta([T]) = [\eta(T)]$), we conclude that $\tilde{x}_0 = \eta \circ \tilde{y}_n \circ \eta$. Since $\mathbf{B}^{n,s}$ is a subcrystal of $\mathcal{T}_n \otimes T_{s\omega_n}$, we have $\tilde{x}_0 = \eta \circ \tilde{y}_n \circ \eta$ on $\mathbf{B}^{n,s}$. \square

6.2. A connection with the Schützenberger's promotion operator

Let \mathbf{pr} be the Schützenberger's promotion operator on $SST_{[n]}(\lambda)$ for $\lambda \in \mathcal{P}$ with $\ell(\lambda) \leq n$ [12], which satisfies for $T \in SST_{[n]}(\lambda)$ with $\text{wt}(T) = m_1\epsilon_1 + m_2\epsilon_2 + \cdots + m_n\epsilon_n$

- (1) $\text{wt}(\mathbf{pr}(T)) = m_n\epsilon_1 + m_1\epsilon_2 + \cdots + m_{n-1}\epsilon_n$,
- (2) $\mathbf{pr}(\tilde{e}_i T) = \tilde{e}_{i+1}(\mathbf{pr}(T))$ and $\mathbf{pr}(\tilde{f}_i T) = \tilde{f}_{i+1}(\mathbf{pr}(T))$ for $i = 1, \dots, n-2$.

Note that \mathbf{pr} is the unique map on $SST_{[n]}(\lambda)$ satisfying (1) and (2), and \mathbf{pr} is of order n if and only if λ is a rectangle (see [15, Proposition 3.2]). It is shown in [11] that on $\mathbf{B}^{r,s}$ of type $A_{n-1}^{(1)}$ ($r = 1, \dots, n-1$, $s \geq 1$)

$$\tilde{e}_0 = \mathbf{pr}^{-1} \circ \tilde{e}_1 \circ \mathbf{pr}, \quad \tilde{f}_0 = \mathbf{pr}^{-1} \circ \tilde{f}_1 \circ \mathbf{pr}.$$

Suppose that $\mathfrak{g} = A_{n-1}^{(1)}$. For $k \in I$, let η_k denote the involution (6.1) on crystals of type $\mathfrak{g}_{I_{0,k}}$. Here $\mathfrak{g}_{I_{0,0}} = \mathfrak{g}_{I_0}$. Let $\lambda \in \mathcal{P}$ be given with $\ell(\lambda) \leq n$. Put $\xi = \eta_1 \circ \eta_0$. By definition of ξ , it is straightforward to check that

- (1) $\text{wt}(\xi(T)) = m_n\epsilon_1 + m_1\epsilon_2 + \cdots + m_{n-1}\epsilon_n$,
- (2) $\xi(\tilde{e}_i T) = \tilde{e}_{i+1}(\xi(T))$ and $\xi(\tilde{f}_i T) = \tilde{f}_{i+1}(\xi(T))$ for $i = 1, \dots, n-2$.

By the uniqueness of \mathbf{pr} , we have $\mathbf{pr} = \eta_1 \circ \eta_0$ on $SST_{[n]}(\lambda)$.

Lemma 6.2. *We have $\eta_0 \circ \tilde{e}_0 = \tilde{f}_0 \circ \eta_0$ on $\mathbf{B}^{r,s}$.*

Proof. First, we claim that

$$\tilde{e}_0 = \eta_1 \circ \tilde{f}_1 \circ \eta_1, \quad \tilde{f}_0 = \eta_1 \circ \tilde{e}_1 \circ \eta_1. \quad (6.4)$$

Note that $\mathbf{pr}^n = \text{id}_{\mathbf{B}^{r,s}}$. We have $\mathbf{pr} \circ \tilde{e}_{n-1} = \mathbf{pr}^{n-1} \circ \tilde{e}_1 \circ \mathbf{pr}^{-n+2} = \mathbf{pr}^{-1} \circ \tilde{e}_1 \circ \mathbf{pr}^2 = \tilde{e}_0 \circ \mathbf{pr}$. Since $\mathbf{pr} = \eta_1 \circ \eta_0$, we have $\tilde{e}_0 = \eta_1 \circ \eta_0 \circ \tilde{e}_{n-1} \circ \eta_0 \circ \eta_1 = \eta_1 \circ \eta_0 \circ \eta_0 \circ \tilde{f}_1 \circ \eta_1 = \eta_1 \circ \tilde{f}_1 \circ \eta_1$. Similarly, we have $\tilde{f}_0 = \eta_1 \circ \tilde{e}_1 \circ \eta_1$. Now, by (6.4), we have

$$\eta_0 \circ \tilde{e}_0 = \eta_0 \circ \mathbf{pr}^{-1} \circ \tilde{e}_1 \circ \mathbf{pr} = \eta_0 \circ \eta_0 \circ \eta_1 \circ \tilde{e}_1 \circ \eta_1 \circ \eta_0 = \tilde{f}_0 \circ \eta_0. \quad \square$$

Proposition 6.3. *Let $\mathbf{B}^{r,s}$ be a KR crystal of type $A_{n-1}^{(1)}$ for $1 \leq r \leq n-1$ and $s \geq 1$. Then we have $\mathbf{pr}^k = \eta_k \circ \eta_0$, on $\mathbf{B}^{r,s}$ for $1 \leq k \leq n-1$.*

Proof. It is not difficult to see that the highest (respectively lowest) weight elements in $\mathbf{B}^{r,s}$ as a $U_q(\mathfrak{g}_{I_{0,k}})$ -crystal are parametrized by the partitions $\lambda \subset (s^r)$, say $b_\lambda^{\text{h.w.}}$ (respectively $b_\lambda^{\text{l.w.}}$). Note that $\eta_k \circ \tilde{x}_i = \tilde{y}_{n+k-i} \circ \eta_k$ for $i \in I_{0,k}$ and $\eta_k(b_\lambda^{\text{h.w.}}) = b_\lambda^{\text{l.w.}}$ for $\lambda \subset (s^r)$. Here $x = e$ (respectively f) when $y = f$ (respectively e), and the indices are assumed to be in \mathbb{Z}_n .

Let $\xi_k = \mathbf{pr}^k \circ \eta_0$. It is straightforward to check that $\xi_k \circ \tilde{x}_i = \tilde{y}_{n+k-i} \circ \xi_k$ for $i \in I_{0,k}$. This implies that $\xi_k(b_\lambda^{\text{h.w.}})$ is a lowest weight element as a $U_q(\mathfrak{g}_{I_{0,k}})$ -crystal and $\text{wt}(\xi_k(b_\lambda^{\text{h.w.}})) = \text{wt}(b_\lambda^{\text{l.w.}})$. Hence, we have $\xi_k(b_\lambda^{\text{h.w.}}) = b_\lambda^{\text{l.w.}}$, and $\xi_k(b) = \eta_k(b)$ for $b \in \mathbf{B}^{r,s}$. \square

Corollary 6.4. *Under the above hypothesis, we have $\tilde{e}_0 = \eta_k \circ \tilde{f}_k \circ \eta_k$ and $\tilde{f}_0 = \eta_k \circ \tilde{e}_k \circ \eta_k$ on $\mathbf{B}^{r,s}$ for $1 \leq k \leq n-1$.*

Proof. Since $\mathbf{pr}^{-k} \circ \tilde{e}_k \circ \mathbf{pr}^k = \tilde{e}_0$, we have $\eta_0 \circ \eta_k \circ \tilde{e}_k \circ \eta_k \circ \eta_0 = \tilde{e}_0$ by Proposition 6.3. Hence, we have $\eta_k \circ \tilde{e}_k \circ \eta_k = \eta_0 \circ \tilde{e}_0 \circ \eta_0 = \tilde{f}_0$ by Lemma 6.2. Similarly, we have $\tilde{f}_0 = \eta_k \circ \tilde{e}_k \circ \eta_k$. \square

Remark 6.5. By Proposition 6.3, η_0 and η_1 on $\mathbf{B}^{r,s}$ generate the action of the dihedral group of order $2n$. When $k = r$, Corollary 6.4 also implies Proposition 6.1 for type $A_{n-1}^{(1)}$.

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