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Zarankiewicz's problem for semi-algebraic hypergraphs

Thao T. Do

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, United States of America

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ABSTRACT

Zarankiewicz's problem asks for the largest possible number of edges in a graph that does not contain a $K_{u,u}$ subgraph for a fixed positive integer u. Recently, Fox, Pach, Sheffer, Sulk and Zahl [12] considered this problem for semi-algebraic graphs, where vertices are points in $\mathbb{R}^d$ and edges are defined by some semi-algebraic relations. In this paper, we extend this idea to semi-algebraic hypergraphs. For each $k \geq 2$, we find an upper bound on the number of hyperedges in a k-uniform k-partite semi-algebraic hypergraph without $K_{u_1,\ldots,u_k}$ for fixed positive integers $u_1, \ldots, u_k$. When $k = 2$, this bound matches the one of Fox et al. and when $k = 3$, it is

$$O\Big((mnp)^{\frac{2d}{2d+1}+\varepsilon} + m(np)^{\frac{d}{d+1}+\varepsilon} + n(mp)^{\frac{d}{d+1}+\varepsilon} + p(mn)^{\frac{d}{d+1}+\varepsilon} + mn + np + pm\Big),$$

where m, n, p are the sizes of the parts of the tripartite hypergraph and ε is an arbitrarily small positive constant. We then present applications of this result to a variant of the unit area problem, the unit minor problem and intersection hypergraphs.

E-mail address: thaodo@mit.edu.

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1. Introduction

The problem of Zarankiewicz [24] is a central problem in graph theory. It asks for the largest possible number of edges in an $m \times n$ bipartite graph that avoids $K_{u,u}$ for some fixed positive integer u. Here $K_{u,u}$ denotes the complete bipartite graph of size $u \times u$, and we say a graph G avoids H or G is H-free if G does not contain any subgraph congruent to H. In 1954, Kővári, Sós and Turán proved a general upper bound of form $O_u(mn^{1-1/u} + n)$, which is only known to be tight for $u = 2$ and $u = 3$.

A natural context in which such bipartite graphs emerge is in incidence geometry, where edges represent incidences between two families of geometric objects, such as points and lines. Better bounds are known in certain of these cases, i.e. points and lines in $\mathbb{R}^2$ (Szemerédi–Trotter theorem [22]), points and curves in $\mathbb{R}^2$ (Pach–Sharir [18]), and points and hyperplanes in $\mathbb{R}^d$ (Apfelbaum–Sharir [1]). Recently, Fox, Pach, Sheffer, Suk and Zahl [12] generalized these results to all *semi-algebraic graphs* (defined below).

Fixing some positive integers d_1, d_2, let $G = (P, Q, \mathcal{E})$ be a bipartite graph on sets P and Q, where we think of P as a set of n points in $\mathbb{R}^{d_1}$ and Q as a set of m points in $\mathbb{R}^{d_2}$. We say G is *semi-algebraic with description complexity* t if there are t polynomials $f_1, \dots, f_t \in \mathbb{R}[x_1, \dots, x_{d_1+d_2}]$, each of degree at most t and a Boolean function $\Phi(X_1, \dots, X_t)$ such that for any $p \in P, q \in Q$:

$$(p, q) \in \mathcal{E} \iff \Phi(f_1(p, q) \geq 0, \dots, f_t(p, q) \geq 0) = 1.$$

In other words, we can describe the incidence relation using at most t inequalities involving polynomials of degree at most t. We now restate more formally the main result of [12]:

Theorem 1.1 *(Fox, Pach, Sheffer, Suk and Zahl 2015 [12]). Given a bipartite semi-algebraic graph $G = (P, Q, \mathcal{E})$ with description complexity t, if G avoids $K_{u,u}$ then for any $\varepsilon > 0$,*

$$|\mathcal{E}(G)| = O_{t,d_1,d_2,u,\varepsilon}\left(m^{\frac{d_1 d_2 - d_2}{d_1 d_2 - 1}+\varepsilon} n^{\frac{d_1 d_2 - d_1}{d_1 d_2 - 1}} + m + n\right).$$

When $d_1 = d_2 = 2$ we can delete the ε term. Moreover, if P belongs to an irreducible variety of degree D and dimension e_1 (with $e_1 \leq d_1$) then

$$|\mathcal{E}(G)| = O_{t,d_1,d_2,e_1,D,k,\varepsilon}\left(m^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} n^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + n\right).$$

As many incidence graphs are semi-algebraic, this theorem and its proof method not only imply the incidence bounds mentioned previously (modulo the extra ε), but also imply many new ones. It is natural to ask if similar results hold for semi-algebraic hypergraphs.

1.1. Semi-algebraic hypergraphs

Fix some integer $k \geq 2$. A hypergraph H is called k-uniform if each hyperedge is a k-tuple of its vertices. It is k-partite if its vertices can be partitioned into k disjoint subset $P_1, \ldots, P_k$ and each hyperedge is some tuple $(p_1, \ldots, p_k)$ where $p_i \in P_i$ for $i = 1, \ldots, k$. We usually use $\mathcal{E}$, or $\mathcal{E}(H)$ to denote the set of hyperedges of H.

Fix some positive integers $d_1, \ldots, d_k$ and t. Let H be a k-uniform k-partite hypergraph $H = (P_1, \ldots, P_k, \mathcal{E})$ where P_i is a set of n_i points in $\mathbb{R}^{d_i}$ for $i = 1, \ldots, k$ and $\mathcal{E}$ is the set of all hyperedges. This hypergraph is said to be *semi-algebraic with description complexity* t if there are t polynomials $f_1, \ldots, f_t \in \mathbb{R}[x_1, \ldots, x_{d_1+\cdots+d_k}]$, each of degree at most t, and a Boolean function $\Phi(X_1, \ldots, X_t)$ such that for any $p_i \in P_i$, $i = 1 \ldots, k$:

$$(p_1, \ldots, p_k) \in \mathcal{E} \iff \Phi(f_1(p_1, \ldots, p_k) \geq 0, \ldots, f_t(p_1, \ldots, p_k) \geq 0) = 1.$$

Semi-algebraic hypergraphs have been the subject of much recent work (see for example [4,13,14]), the main theme of which is that many classical theorems about hypergraphs (such as Ramsey's theorem and Szemerédi's regularity lemma) can be significantly improved in the semi-algebraic setting. Since the graphs and hypergraphs arising in discrete geometry problems are often semi-algebraic (with low description complexity), such improved results have many applications there. Our paper follows this paradigm, improving upon a result of Erdős regarding Zarankiewicz's problem for semi-algebraic hypergraphs.

We first recall the classical result. Given positive integers $u_1, \ldots, u_k$, let $K_{u_1,\ldots,u_k}$ denote the complete $k-$uniform k-partite hypergraph $(U_1, \ldots, U_k, \mathcal{E})$ where $|U_i| = u_i$ and any k-tuple in $U_1 \times \cdots \times U_k$ is a hyperedge. Zarankiewicz's problem for hypergraphs asks for the maximum number of hyperedges in a k-uniform hypergraph that does not contain a copy of $K_{u_1,\ldots,u_k}$. The first statement in the following theorem was proved by Erdős in [6]. Using his proof method, we obtain the second statement whose proof can be found in appendix A.

Theorem 1.2 *(Erdős 1964 [6]). A k-uniform $K_{u,\ldots,u}$-free hypergraph H on n vertices has at most $O_{u,k}(n^{k-\frac{1}{u^{k-1}}})$ hyperedges. More generally, if the k-partite k-uniform hypergraph $H = (P_1, \ldots, P_k, \mathcal{E})$ is $K_{u_1,\ldots,u_k}$-free, then $|\mathcal{E}| = O_{u_1,\ldots,u_k,k}\Big(\Big(n_k^{-1/u_1\ldots u_{k-1}} + n_1^{-1} + \cdots + n_{k-1}^{-1}\Big) \prod_{i=1}^k n_i\Big)$, where $n_i = |P_i|$ for $i = 1, \ldots, k$.*

Remark 1.3. In the preliminary report on arXiv of [12], Fox et al. improved this bound for semi-algebraic hypergraphs (Corollary 6.11 therein), but their proof is flawed: they claimed a $K_{u,\ldots,u}$-free semi-algebraic hypergraph $H = (P_1, \ldots, P_k, \mathcal{E})$ is a semi-algebraic $K_{s,s}$-free bipartite graph between $P = \cup_{i\in S_1} P_i$ and $Q = \cup_{j\in S_2} P_j$ for any partition $S_1 \cup S_2 = [k]$ and some s that only depends on u, k. It is true that this new graph is semi-algebraic with bounded complexity; however, it may not be $K_{s,s}$-free for any fixed

s. For example, the unit minor hypergraph in $\mathbb{R}^d$ on n vertices (see 5.2) does not contain $K_{2,\ldots,2}$ but may contain $K_{1,(n-1)/d,\ldots,(n-1)/d}$.

In this paper we present a way to extend Theorem 1.1 to semi-algebraic hypergraphs. Ultimately we will prove the number of hyperedges is bounded by a function of $n_1, \ldots, n_k$, with the exponents depending on d_i, in this way resembling Theorem 1.1. However, the formulas involved are sufficiently complicated that we need to fix some notation before stating them precisely.

1.2. Notation

Let $\vec{d} = (d_1, \ldots, d_k)$ and $\vec{n} = (n_1, \ldots, n_k)$ be vectors in $\mathbb{Z}_+^k$. For each $\vec{d}$, define a function $E_{\vec{d}}(\vec{n}) : \mathbb{R}^k \to R$ via:

$$E_{\vec{d}}(\vec{n}) := E_{d_1,\ldots,d_k}(n_1, \ldots, n_k) := \prod_{i=1}^{k} n_i^{1-\frac{\frac{1}{d_i-1}}{k-1+\frac{1}{d_1-1}+\cdots+\frac{1}{d_k-1}}}. \tag{1.4}$$

For example, $E_d(n) = 1$ for all d and n, and $E_{d_1,d_2}(m,n) = m^{1-\frac{d_2-1}{d_1 d_2-1}} n^{1-\frac{d_1-1}{d_1 d_2-1}}$.[1] The function $E_{\vec{d}}(\vec{n})$ satisfies many nice properties that are discussed in subsection 2.4. Let $[k]$ denote the set $\{1, \ldots, k\}$, and for a subset $I \subset [k]$ let $\vec{d_I}$ denote the vector $(d_i)_{i \in I} \in \mathbb{R}^{|I|}$, and similarly let $\vec{n_I} = (n_i)_{i \in I}$. For $i \in [k]$, let π_i be the projection of $\mathbb{R}^k$ to $\langle \mathfrak{e}_i \rangle^\perp$; i.e. for any vector $\vec{a} \in \mathbb{R}^k$, $\pi_i(\vec{a}) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$. For each $\varepsilon > 0$ and each $\vec{d}$, define a function $F_{\vec{d}}^{\varepsilon} : \mathbb{R}^k \to \mathbb{R}$ as follows:

$$F_{\vec{d}}^{\varepsilon}(\vec{n}) := \sum_{I \subset [k], |I| \geq 2} E_{\vec{d_I}}(\vec{n_I}) \prod_{i \in I} n_i^{\varepsilon} \prod_{i \notin I} n_i + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\right) \prod_{i=1}^{k} n_i \tag{1.5}$$

1.3. Our results

We first prove a more general version of Theorem 1.1 for semi-algebraic graphs, which shows that, if *both* P and Q belong to irreducible varieties of dimensions (e_1, e_2) inside of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, then we may replace both dimensions (d_1, d_2) by (e_1, e_2) in the upper bound.

Theorem 1.6. *Given a semi-algebraic bipartite graph $G = (P, Q, \mathcal{E})$ where P is a set of m points in an irreducible variety of dimension e_1 and complexity at most D in $\mathbb{R}^{d_1}$ and Q is a set of n points in an irreducible variety of dimension e_2 and complexity at most D in $\mathbb{R}^{d_2}$. If G has description complexity t and contains no $K_{u,u}$, then*

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[1] Note that here we do not require $d_i \geq 2$ because we can multiply the numerator and denominator of the exponent by $\prod_{i=1}^{k}(d_i - 1)$ to get rid of the term $1/(d_i - 1)$.

$$|\mathcal{E}| = O_{e_1,e_2,D,t,u,\varepsilon}(m^{\frac{e_1e_2-e_2}{e_1e_2-1}+\varepsilon} n^{\frac{e_1e_2-e_1}{e_1e_2-1}+\varepsilon} + m + n).$$

We then extend this result to k-uniform semi-algebraic hypergraphs for any $k \geq 3$.

Theorem 1.7 *(Main Theorem). Given a k-uniform k-partite hypergraph $H = (P_1, \ldots, P_k, \mathcal{E})$ with description complexity t which avoids $K_{u,\ldots,u}$, then*

$$|\mathcal{E}(H)| = O_{t,k,u,\vec{d},\varepsilon}\left(F^{\varepsilon}_{\vec{d}}(\vec{n})\right). \tag{1.8}$$

Moreover, if for each $i \leq k$, P_i belongs to an irreducible variety of degree D and dimension $e_i \leq d_i$, then $|\mathcal{E}(H)| = O_{t,k,u,\vec{d},D,\varepsilon}(F^{\varepsilon}_{\vec{e}}(\vec{n}))$ where $\vec{e} = (e_1, \ldots, e_k)$.

Remark 1.9.

(i) Without the ε term, function F has a nicer form $F_{\vec{d}}(\vec{n}) = \sum_{\emptyset \neq I \subset [k]} E_{\vec{d}_I}(\vec{n}_I) \prod_{i \notin I} n_i$. As mentioned in [12] the term n_1^{ε} is not necessary when $k = d_1 = d_2 = 2$, and we conjecture that this artifact of the proof can be removed in general.

(ii) When $k = 2$, this theorem implies $|\mathcal{E}(H)| \lesssim F^{\varepsilon}_{d_1,d_2}(m, n) = m^{\frac{d_1d_2-d_2}{d_1d_2-1}+\varepsilon} n^{\frac{d_1d_2-d_1}{d_1d_2-1}+\varepsilon} + m + n$. It is slightly weaker yet essentially the same as the bound in Theorem 1.1. Indeed, as we shall see in Remark 4.3, the term $m + n$ dominates unless $n^{1/d_2} \leq m \leq n^{d_1}$. Hence

$$m^{\frac{d_1d_2-d_2}{d_1d_2-1}+\varepsilon} n^{\frac{d_1d_2-d_1}{d_1d_2-1}+\varepsilon} + m + n \lesssim m^{\frac{d_1d_2-d_2}{d_1d_2-1}+\varepsilon'} n^{\frac{d_1d_2-d_1}{d_1d_2-1}} + m + n,$$

where $\varepsilon' = (d_2 + 1)\varepsilon$. In general, we can prove a stronger result where ε appears only once in each term of $F^{\varepsilon}_{\vec{d}}(\vec{n})$.

(iii) When $d_1 = d_2 = \cdots = d$, we have

$$|\mathcal{E}(H)| = O_{t,k,u,\vec{d},\varepsilon}\left(\sum_{j=2}^{k} \sum_{I \subset [k]; |I|=j} \left(\prod_{i \notin I} n_i\right)\left(\prod_{i \in I} n_i\right)^{1-\frac{1}{(j-1)d+1}+\varepsilon} + \left(\sum_{i=1}^{k} \frac{1}{n_i}\right)\prod_{i=1}^{k} n_i\right).$$

When $k = 3$ we get the formula mentioned in the abstract. Assume furthermore $n_1 = \cdots = n_k = n$ the bound becomes $n^{k-\frac{k}{(k-1)d+1}+\varepsilon}$ which is smaller than $n^{k-\frac{1}{u^{k-1}}}$ (the bound in Theorem 1.2) when $d < \frac{ku^{k-1}-1}{k-1}$.

(iv) If, on the other hand, if $u^{k-1} < d_i$ for some i, say $u^{k-1} < d_k$, we can use Theorem 1.2 instead of Proposition 4.1 in the proof to derive the bound $F^{\varepsilon}_{d_1,\ldots,d_{k-1},u^{k-1}}(\vec{n})$ where we replace d_k by u^{k-1}.

1.4. Applications

Our main result, Theorem 1.7, implies nontrivial bounds for many geometric problems. In section 5, we present several applications including a variant of the unit area problem, the unit minor problem, and intersection hypergraphs.

First we find an upper bound $O_\varepsilon(n^{12/5+\varepsilon})$ for the number of triangles with area very close to 1, say between 0.9 and 1.1, formed by n points in the plane, assuming for some fixed $u > 0$ there does not exist $3u$ points a_i, b_i, c_i, $i \in [u]$ among those given points such that the triangles formed by (a_i, b_j, c_k) have area between 0.9 and 1.1 for any $i, j, k \in [u]$.

The unit minor problem asks for the largest number of unit $d \times d$ minors in a $d \times n$ matrix with no repeated columns. This problem was considered in [8] but only for the case the matrix is totally positive, which is a much stronger assumption. In 5.2 we prove an upper bound $O_{d,\varepsilon}(n^{d-\frac{d}{d^2-d+1}+\varepsilon})$ for the number of unit minors for any matrix with no repeated columns. As a corollary, the maximum number of unit volume d-simplices formed by n points in $\mathbb{R}^d$ is $O_{d,\varepsilon}(n^{d+1-\frac{d}{d^2-d+1}+\varepsilon})$.

The last application is about intersection hypergraphs. Given a set S of geometric objects, their intersection graph $H(S)$ is defined as a graph on the vertex set S, in which two vertices are joined by an edge if and only if the corresponding elements of S have a point in common. Fox and Pach proved that if $H(S)$ is $K_{u,u}$-free for some $u > 0$, then $H(S)$ has $O(n)$ edges when S is a set of n line segments [11] or arbitrary continuous arcs [9,10] in $\mathbb{R}^2$, Mustafa and Pach [17] gave a hypergraph version of this, but only for simplices. Given a set S of geometric objects (usually of dimension $d-1$) in $\mathbb{R}^d$, their intersection hypergraph $H(S)$ is defined as a d-uniform hypergraph on the vertex set of S, in which d vertices form a hyperedge if and only if the corresponding sets in S have a point in common. Mustafa and Pach [17] proved that if $H(S)$ is $K_{u,\ldots,u}$-free then $H(S)$ has $O_{d,\epsilon}(|S|^{d-1+\epsilon})$ hyperedges given S is a set of $(d-1)$-dim simplices in $\mathbb{R}^d$. In this paper, we found a nontrivial upper bound for many other types of geometric objects such as spheres.

1.5. Organization

In section 2, we introduce several useful tools such as the Milnor–Thom's theorem, the polynomial partitioning method and a packing-type result from set system theory. We then prove Theorem 1.6 in section 3 and our main theorem in section 4. Section 5 is devoted to applications. We end with several open problems in section 6.

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2. Preliminary

2.1. Milnor–Thom type results

Milnor–Thom's theorem [16,23] states that the zero set of a degree D polynomial f, denoted by $Z(f)$, divides $\mathbb{R}^d$ into at most $(50D)^d$ connected components (i.e. $\mathbb{R}^d \setminus Z(f)$ has at most $(50D)^d$ connected components). Basu, Pollack and Roy extended this result to the case when we restrict our attention to a variety inside $\mathbb{R}^d$.

A *sign pattern* for a set of s d-variate polynomials $\{f_1, \ldots, f_s\}$ is a vector $\sigma \in \{-1, 0, +1\}^s$. A sign pattern σ is *realizable* over a variety $V \subset \mathbb{R}^d$ if there is some $x \in V$ such that $(\mathrm{sign}(f_1(x)), \mathrm{sign}(f_2(x)), \ldots, \mathrm{sign}(f_s(x))) = \sigma$. The set of all such x is the realization space of σ in V, denoted by Ω_σ.

Theorem 2.1 *(Basu, Pollack and Roy, 1996 [2]). Given positive integers d, e, M, t, l, let V be an e-dimensional real algebraic set in $\mathbb{R}^d$ of complexity[2] at most M, and let $f_1, \ldots, f_s$ be d-variate real polynomials of degree at most t. Then the total number of connected components of Ω_σ for all realizable sign patterns σ of $\{f_1, \ldots, f_s\}$ is at most $O_{M,d,e}((ts)^e)$.*

Milnor–Thom's theorem follows from this result by taking $s = 1, V = \mathbb{R}^d$ and noting that $\mathbb{R}^d \setminus Z(f)$ is the union of two realizable spaces $\{f > 0\}$ and $\{f < 0\}$. This result implies if we restrict to a variety V with dimension e and bounded complexity in $\mathbb{R}^d$, then the number of connected components that $f_1, \ldots, f_s$ partition V grows with e instead of d. Since each realizable sign pattern has at least one connected component, we get a bound on the number of realizable sign patterns.

Corollary 2.2. *Under the same assumption as above, the number of realizable sign patterns of $(f_1, \ldots, f_s)$ in V is at most $O_{M,d,e}((ts)^e)$.*

Furthermore, a similar result holds if we replace V by $V \setminus W$ for some variety W with bounded complexity.

Theorem 2.3 *(Theorem A.2 in [21]). Given positive integers d, e, M, t such that $e \leq d$, let V and W be a real algebraic sets in $\mathbb{R}^d$ of complexity at most M such that V is e-dimensional. Then for any polynomial $P : \mathbb{R}^d \to \mathbb{R}$ of degree $t \geq 1$, the set $\{x \in V \setminus W : P(x) \neq 0\}$ has $O_{M,d,e}(t^e)$ connected components.*

2.2. Polynomial partitioning

The polynomial partitioning method was first introduced by Guth and Katz in [15] in 2010, and numerous modifications have appeared since then. In this paper we use

[2] A variety has *complexity* at most M if it can be realized as the intersection of zero-sets of at most M polynomials, each of degree at most M.

the version proved in [12]. Given a set P of points in $\mathbb{R}^d$, we say a polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ is an *r-partitioning for P* if the zero set of f (denoted $Z(f)$) divides the space into open connected components, each of which contains at most $|P|/r$ points of P.

Theorem 2.4 *(Theorem 4.2 in [12]). Let P be a set of points in $\mathbb{R}^d$, and let $V \subset \mathbb{R}^d$ be an irreducible variety of degree D and dimension e. Then for big enough r, there exists an r-partitioning polynomial g for P such that $g \notin I(V)$ and $\deg g \leq C_{part} \cdot r^{1/e}$ where C_{part} depends only on d and D.*

This theorem implies in $\mathbb{R}^d$, if we restrict our attention to points in an irreducible variety of small degree and dimension $e < d$, then we can perform a polynomial partitioning the same way as in $\mathbb{R}^e$.

2.3. A result about set systems

A *set system* $\mathcal{F}$ over a ground set P is just a collection of subsets of P (here we allow $\mathcal{F}$ to contain repeated elements). Given a bipartite graph $G = (P, Q, \mathcal{E})$, let $N(p) := \{q \in Q : (p, q) \in \mathcal{E}\}$ denote the neighbors of a vertex $p \in P$, and likewise for $q \in Q$. Then $\mathcal{F}_1 := \{N(q) : q \in Q\}$, and $\mathcal{F}_2 := \{N(p) : p \in P\}$ are two set systems with ground sets P and Q respectively. The *primal shatter function* of a set system $(\mathcal{F}, P)$ is defined as

$$\pi_{\mathcal{F}}(z) = \max_{P' \subset P, |P'| = z} |\{A \cap P' : A \in \mathcal{F}\}|.$$

Given two sets A and B, we say A *crosses* B if $A \cap B \notin \{\emptyset, B\}$. The following lemma is essential to our proof.

Lemma 2.5 *(Observation 2.6 in [12]). For the set systems $(\mathcal{F}_1, P)$ and $(\mathcal{F}_2, Q)$ defined from the graph $G = (P, Q)$, if $\pi_{\mathcal{F}_1}(z) \leq cz^d$ for all z, then for each u, there exists u points $q_1, \dots, q_u \in Q$ such that at most $O_{u,d,t,c}(|P||Q|^{-1/d})$ sets from $\mathcal{F}_2$ cross $\{q_1, \dots, q_u\}$.*

In the paper we use this result as a black box. For readers who are interested in some intuition, it follows from a packing-type result in VC dimension theory. The *Vapnik–Chervonenkis (VC) dimension* of a set system $\mathcal{F}$ is the largest D such that $\pi_F(D) = 2^D$. It is easy to see if $\pi_{\mathcal{F}}(z) \leq cz^d$ for some fixed $c, d > 0$ then the VC dimension of $\mathcal{F}$ does not excess $c'd \log d$ for some $c' > 0$. Intuitively, if a set system F has a bounded VC dimension, and its elements are well separated in the symmetric distance, then F cannot have too many elements; it is analogous to how we pack spheres in Euclidean spaces. To be precise, lemma 2.5 in [12] says that if a set system $\mathcal{F}$ on a ground set of m elements satisfies $\pi_{\mathcal{F}}(z) \leq cz^d$ for all z and $|(A_1 \cup \dots \cup A_u) \setminus (A_1 \cap \dots \cap A_u)| \geq \delta$ for all choices of $A_1, \dots, A_u \in \mathcal{F}$ and some fixed u, δ, then $|\mathcal{F}| \leq C_{pack}(m/\delta)^d$ for some constant C_{pack} that depends on c, d, u. In Lemma 2.5, assume for each set $\{q_1, \dots, q_u\}$

in Q there are more than $\delta := c_1|P||Q|^{-1/d}$ sets from $\mathbb{F}_2$ crossing it. This implies $|(N(q_1) \cup \cdots \cup N(q_u)) \setminus (N(q_1) \cap \cdots \cap N(q_u))| \geq \delta$ for all choices of $q_1, \ldots, q_u \in Q$. In other words, $\mathcal{F}_1$ is (u, δ) separated. Applying the packing lemma to $\mathcal{F}_1$ leads to a contradiction for small enough c_1.

$$|Q| = |\mathcal{F}_1| \leq c_1 \left(\frac{|P|}{\delta}\right)^d = c_1 \left(\frac{|P|}{C_{pack}|P||Q|^{-1/d}}\right)^d = c_1 C_{pack}^d |Q| < |Q|$$

2.4. Some properties of functions $E_{\vec{d}}$ and $F^{\varepsilon}_{\vec{d}}$

In this subsection, we collect some useful properties of $E_{\vec{d}}$ and $F^{\varepsilon}_{\vec{d}}$. All the proofs are quite straightforward and can be found in appendix B. Recall $E_{\vec{d}}(\vec{n}) = \prod_{i=1}^k n_i^{\alpha_i}$ where $\alpha_i = 1 - \frac{1/(d_i-1)}{k-1+\sum_l 1/(d_l-1)}$.

Lemma 2.6. *For each $i \in [k]$ we have $\alpha_i = \sum_{j \neq i} d_j(1-\alpha_j)$. In other words, the exponents $\{\alpha_i\}$ satisfy a nice system of equations:*

$$\begin{pmatrix} 1 & d_2 & \ldots & d_k \\ d_1 & 1 & \ldots & d_k \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^k d_i - d_1 \\ \sum_{i=1}^k d_i - d_2 \\ \vdots \\ \sum_{i=1}^k d_i - d_k \end{pmatrix} \tag{2.7}$$

Corollary 2.8. *For any $r > 0$ and each $i \in [k]$ we have*

$$r^{d_1+\cdots+d_{k-1}} E_{\vec{d}}\left(\frac{n_1}{r^{d_1}}, \ldots, \frac{n_{k-1}}{r^{d_{k-1}}}, \frac{n_k}{r}\right) = E_{\vec{d}}(\vec{n}).$$

Similar equalities, in which we replace the special index k by some other index, also hold.

Proof.
$$\begin{aligned} LHS &= r^{d_1+\cdots+d_{k-1}} \left(\frac{n_1}{r^{d_1}}\right)^{\alpha_1} \ldots \left(\frac{n_{k-1}}{r^{d_{k-1}}}\right)^{\alpha_{k-1}} \left(\frac{n_k}{r}\right)^{\alpha_k} \\ &= \left(\prod_i n_i^{\alpha_i}\right) r^{d_1+\cdots+d_{k-1}-\sum_{j=1}^{k-1} d_j\alpha_j - \alpha_k} = E_{\vec{d}}(\vec{n}) \end{aligned}$$

In the last step, the exponent of r becomes 0 because $\sum_{j=1}^{k-1} d_j(1-\alpha_j) = \alpha_k$ by Lemma 2.6. □

Lemma 2.9. *Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$ be the standard basis in $\mathbb{R}^k$. Then $F^{\varepsilon}_{\vec{d}-\mathfrak{e}_i}(\vec{n}) \leq F^{\varepsilon}_{\vec{d}}(\vec{n})$ assuming $n_i \geq n_j^{1/(d_j)}$ for any $j \neq i$.*

Lemma 2.10. *Assume for each $i \in [k]$ we have*

$$n_i^{-1/d_i} \prod_{j=1}^{k} n_j \geq n_i F^{\varepsilon}_{\pi_i(\vec{d})}(\pi_i(\vec{n})), \tag{2.11}$$

then $E_{\vec{d}}(\vec{n}) \prod_{i=1}^{k} n_i^{\varepsilon} \geq cF^{\varepsilon}_{\vec{d}}(\vec{n})$ for some constant c. In other words, $E_{\vec{d}}(\vec{n}) \prod_i n_i^{\varepsilon}$ is the dominant term of $F^{\varepsilon}_{\vec{d}}(\vec{n})$.

3. Proof of Theorem 1.6

In this section, we first outline the proof of the last statement in Theorem 1.1 and then carry out the modifications needed to prove Theorem 1.6. In both cases, $G = (P, Q)$ is a semi-algebraic $K_{u,u}$-free graph with description complexity t. Recall in the last statement of Theorem 1.1, we assume P is a set of m points from an irreducible variety with dimension e_1 and bounded complexity in $\mathbb{R}^{d_1}$ and Q is a set of n points in $\mathbb{R}^{d_2}$, and the conclusion is

$$|\mathcal{E}| \lesssim m^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1} + \varepsilon} n^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + n. \tag{3.1}$$

In Theorem 1.6, we assume additionally that Q belongs to an irreducible variety with bounded complexity and dimension e_2 in $\mathbb{R}^{d_2}$ and wish to prove

$$|\mathcal{E}| \lesssim m^{\frac{e_1 e_2 - e_2}{e_1 e_2 - 1} + \varepsilon} n^{\frac{e_1 e_2 - e_1}{e_1 e_2 - 1}} + m + n, \tag{3.2}$$

i.e. that we can replace d_2 by e_2. Note that this is similar in spirit to Theorem 2.1, which is precisely what we shall use.

Fox, Pach, Sheffer, Sulk and Zahl's proof of Theorem 1.1 proceeds in two steps: first, Lemma 2.5 is used to show $|\mathcal{E}(G)| \lesssim mn^{1-1/d_2} + n$, and then induction and polynomial partitioning are used to derive the desired bound.

The first step begins by proving that

$$\pi_{\mathcal{F}_1}(z) \leq c(t, d_2) z^{d_2}$$

where the set systems $(\mathcal{F}_1, P)$ and $(\mathcal{F}_2, Q)$ are defined in 2.3. For each $p \in P$, its neighbors belong to a semi-algebraic set γ_p in $\mathbb{R}^{d_2}$ defined as followed:

$$\gamma_p := \{x \in \mathbb{R}^{d_2} : \Phi(f_1(p, x) \geq 0, \dots, f_t(p, x) \geq 0) = 1\}.$$

For any z points $p_1, \dots, p_z \in P$, their semi-algebraic sets $\gamma_{p_1}, \dots, \gamma_{p_z}$ are defined by at most tz polynomials $\{f_j(p_i, x) : i \in [z], j \in [t]\}$. By the Milnor–Thom theorem, these tz polynomials have at most $c(t, d_2)z^{d_2}$ realizable sign patterns in $\mathbb{R}^{d_2}$. If two points q and q' in Q share the same sign pattern, their neighborhoods in P restricted to $\{p_1, \dots, p_z\}$ are the same. As a consequence, $\pi_{\mathcal{F}_1}(z) = \max_{p_i} |\{N(q) \cap \{p_1, \dots, p_z\} : q \in Q\}| \leq c(t, d_2)z^{d_2}$.

Applying Lemma 2.5, there exists $q_1, \ldots, q_u \in Q$ such that at most $O(|P||Q|^{-1/d_2}) = O(mn^{-1/d_2})$ sets from $\mathcal{F}_2$ cross $\{q_1, \ldots, q_u\}$. On the other hand, there are at most $u-1$ sets from $\mathcal{F}_2$ that contain $\{q_1, \ldots, q_u\}$ because the graph is $K_{u,u}$-free. Therefore, the degree of q_1 in G is at most $O(mn^{-1/d_2}) + (u-1)$. Removing q_1 and repeating this argument at most n times, we have $|\mathcal{E}| \lesssim mn^{1-1/d_2} + n$. The term n dominates unless $n < m^{d_2}$. Thus from now on we assume $n < m^{d_2}$.

In the second step, we view edges of G as incidences between P and n semi-algebraic sets $\{\gamma_q : q \in Q\}$ in $\mathbb{R}^{d_1}$ where γ_q is the set of all potential neighbors of q in $\mathbb{R}^{d_1}$, i.e.

$$\gamma_q := \{x \in \mathbb{R}^{d_1} : \Phi(f_1(x,q) \geq 0, \ldots, f_t(x,q) \geq 0) = 1\}.$$

We prove (3.1) by double induction – first on e_1 and then on $m+n$. The result is obvious to $e_1 = 0$ (since zero-dimensional irreducible varieties are just singletons). For a fixed $e_1 \geq 1$, the result is true for small $m+n$. In the induction step, by Theorem 2.4, for a parameter r to be chosen later, we can find a polynomial f not vanishing on V of small degree to partition the points in P equally with respect to the variety V of dimension e_1. More precisely, we can find f of degree at most $C_{part}r$ such that $V \setminus Z(f)$ has $s = O(r^{e_1})$ connected components, or *cells*, such that each cell contains $O(\frac{m}{r^{e_1}})$ points of P. There are 3 types of incidences between a point p and a semi-algebraic set γ_q, which we shall group into sets I_1, I_2, and I_3 respectively: (1) where p belongs to $Z(f) \cap V$; (2) where p belongs to a cell in $V \setminus Z(f)$ and γ_q contains the whole cell; and (3) where p belongs to a cell and γ_q *crosses* the cell, i.e. has non-empty intersection with the cell but does not contain the entirety of it. As $|\mathcal{E}| = |I_1| + |I_2| + |I_3|$, it suffices to bound the number of each type of incidence by the right side of (3.1).

To bound $|I_1|$, note that $V \cap Z(f)$ is a variety of dimension at most $e_1 - 1$, so we can apply the inductive hypothesis and get $I_1 \lesssim E_{e_1-1,d_2}(m,n) + m + n$. By Lemma 2.9, $E_{e_1-1,d_2}(m,n) \lesssim E_{e_1,d_2}(m,n)$ when $m > n^{1/d_2}$, which we are allowed to assume (this was the point of step one!)

The number of incidences in I_2 is bounded by $um + unr^{e_1}$ because the graph is $K_{u,u}$-free. Indeed, for each cell, either the cell contains fewer than u points, or it is contained in fewer than u semi-algebraic sets γ_q. The contribution in the first case is at most un in each cell and at most unr^{e_1} in total. The contribution in the second case is at most um. We can choose r small enough so that the nr^{e_1} is less than $m^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} n^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}}$ when $n < m^{d_2}$.

Finally, to bound $|I_3|$, suppose there are $s = O(r^{e_1})$ cells $\Omega_1, \ldots, \Omega_s$, where each Ω_i contains $m_i = O(m/r^{e_1})$ points and is crossed by n_i semi-algebraic sets. We claim that each semi-algebraic set γ_q crosses at most $O(r^{e_1-1})$ cells. Indeed, each γ_q is defined by t polynomials $f_1(x,q), \ldots, f_t(x,q)$. In order for γ_q to cross a cell in $V \setminus Z(f)$, some polynomial, say f_1, must not vanish on V. Then $Z(f_1) \cap V$ is some variety of dimension at most $e_1 - 1$. By Theorem 2.3, f partitions this variety in at most $O_{t,d_1,D}(r^{e_1-1})$ cells; this in turn implies γ_q crosses $O(r^{e_1-1})$ cells. As a result, the total number of pairs

(Ω_i, γ_q) such that γ_q crosses Ω_i is $\sum_{i=1}^{s} n_i \lesssim r^{e_1-1}n$. We apply the inductive hypothesis in each cell (for smaller $m_i + n_i$), add them up, and use Hölder's inequality:

$$
\begin{aligned}
I_3 = \sum_{i=1}^{s} I(m_i, n_i) &\lesssim \sum_{i=1}^{s} \left(m_i^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} n_i^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m_i + n_i \right) \\
&\lesssim \sum_{i=1}^{s} \left(\frac{m}{r^{e_1}}\right)^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} n_i^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + \sum_{i=1}^{s} n_i \\
&\lesssim \left(\frac{m}{r^{e_1}}\right)^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} s \left(\frac{\sum n_i}{s}\right)^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + nr^{e_1 - 1} \\
&\lesssim \left(\frac{m}{r^{e_1}}\right)^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} r^{e_1} \left(\frac{nr^{e_1-1}}{r^{e_1}}\right)^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + nr^{e_1 - 1} \\
&= r^{-\varepsilon e_1} m^{\frac{e_1 d_2 - d_2}{e_1 d_2 - 1}+\varepsilon} n^{\frac{e_1 d_2 - e_1}{e_1 d_2 - 1}} + m + nr^{e_1 - 1}
\end{aligned}
$$

We choose r small enough so that nr^{e_1-1} is bounded by the first term when $n < m^{d_2}$, but also large enough so that $r^{-\varepsilon e_1}$ is strictly smaller than the coefficient chosen in (1.8). This finishes the proof of Theorem 1.1.

In Theorem 1.6, since Q belongs to a variety of dimension e_2 with complexity D, by Corollary 2.2, $\pi_{\mathcal{F}_1}(z) \le c(t, d_2, e_2, D) z^{e_2}$. This is "step one" in the above proof outline, with d_2 replaced by e_2. Combining with Lemma 2.5 we get:

Lemma 3.3. *Given $G = (P, Q, \mathcal{E})$ as in the statement of Theorem 1.6. Then there exist u points $q_1, \dots, q_u \in Q$ such that at most $O_{u,e_2,d_2,t,D}(|P||Q|^{-1/e_2})$ sets from $\mathcal{F}_2$ cross $\{q_1, \dots, q_u\}$.*

We can subsequently replace d_2 by e_2 in all other steps and get Theorem 1.6. □

4. Proof of the main theorem

4.1. Overview of the proof

Our proof will proceed by induction on k – the statement clearly holds for $k = 1$, so fix some $k \ge 2$. For the inductive step, we follow the same general strategy as in the previous section: first using Lemma 2.5 to obtain a bound where the exponents only depend on $\vec{d}$, and then using polynomial partitioning to get a better bound. We highlight some differences: in the first step, we view the hypergraph as a semi-algebraic bipartite graph between P_k and $P_1 \times \cdots \times P_{k-1}$ to apply Lemma 2.5. To bound the number of edges that contain u points in P_k, we need to use the induction assumption for some $(k-1)$-uniform semi-algebraic hypergraph on $P_1, \dots, P_{k-1}$. In the second step, we reduce the problem to counting incidences between n_k semi-algebraic sets defined by P_k and the grid $P_1 \times \cdots \times P_{k-1}$ in $\mathbb{R}^{d_1+\cdots+d_{k-1}}$. If we simply apply the usual polynomial partitioning, each cell may not have the structure of a k-partite hypergraph. We overcome this by using a *product* of $k-1$ polynomials $f_1, \ldots, f_k$ of the same degree, where f_i partitions P_i equally in $\mathbb{R}^{d_i}$ for $i < k$. By doing this, we preserve the grid structure and thus can use induction on a smaller grid in each cell.

We begin now with step one.

Proposition 4.1. *Given a k-uniform k-partite semi-algebraic $K_{u,\ldots,u}$-free hypergraph $(P_1, \ldots, P_k, \mathcal{E})$ with description complexity t such that for each $i \in [k]$, A_i belongs to V_i, an irreducible variety in $\mathbb{R}^{d_i}$ of dimension $e_i \leq d_i$ and degree bounded by D, then*

$$|\mathcal{E}(H)| = O_{\vec{d},k,u,D,\varepsilon}\left(n_1 \ldots n_{k-1} n_k^{1-1/e_k} + n_k F^{\varepsilon}_{e_1,\ldots,e_{k-1}}(n_1, \ldots, n_{k-1})\right).$$

Proof. Given $p_i \in P_i$ for $i \in [k-1]$, let $N(p_1, \ldots, p_{k-1}) := \{p_k \in P_k : (p_1, \ldots, p_k) \in \mathcal{E}\}$ be their neighbors in P_k. Moreover, let $\mathcal{F} := \{N(p_1, \ldots, p_{k-1}) : p_i \in P_i\}$ be the set system on the ground set P_k.

We first claim there exist u points $q_1, \ldots, q_u$ in P_k such that the number of sets in $\mathcal{F}$ that cross $\{q_1, \ldots, q_u\}$ is at most $c_1 n_1 \ldots n_{k-1} n_k^{-1/e_k}$ for some $c_1(\vec{d}, t, u)$. Indeed, we can think of our hypergraph as a bipartite graph $(P_1 \times \cdots \times P_{k-1}, P_k)$ where there is an edge between $(p_1, \ldots, p_{k-1}) \in P_1 \times \cdots \times P_{k-1}$ and $p_k \in P_k$ iff there is a hyperedge $(p_1, \ldots, p_k)$ in $\mathcal{E}$. Clearly this bipartite graph is semi-algebraic with description complexity t, hence we can apply Lemma 3.3 for this graph where Q is P_k and obtain the desired claim.

Next, we claim that the number of sets in $\mathcal{F}$ that contain $\{q_1, \ldots, q_u\}$ is at most $c_2 F^{\varepsilon}_{d_1,\ldots,d_{k-1}}(n_1, \ldots, n_{k-1})$ for some constant c_2. Indeed, let $\mathcal{E}'$ be the sets of all $(p_1, \ldots, p_{k-1}) \in P_1 \times \cdots \times P_{k-1}$ such that their neighbor set $N(p_1, \ldots, p_{k-1})$ contains $\{q_1, \ldots, q_u\}$. Then the *induced* $(k-1)$-uniform hypergraph $H' = (P_1, \ldots, P_{k-1}, \mathcal{E}')$ is semi-algebraic with description complexity tu and contains no $K_{u,\ldots,u}$. Inductively, $|\mathcal{F}'| \leq c_2 F^{\varepsilon}_{e,\ldots,e_{k-1}}(n_1, \ldots, n_{k-1})$.

Combining these two claims, we conclude that there are at most $(c_1 + c_2)(n_1 \ldots n_{k-1} n_k^{-1/e_k} + F^{\varepsilon}_{e_1,\ldots,e_{k-1}}(n_1, \ldots, n_{k-1}))$ hyperedges in H that contain q_1 (because each such hyperedge must either cross or contain $\{q_1, \ldots, q_u\}$). Removing this vertex and repeating this argument until there are fewer than u vertices in P_k, we get

$$|\mathcal{E}(H)| \leq c_3\left(n_1 \ldots n_{k-1} n_k^{1-1/e_k} + n_k F^{\varepsilon}_{e_1,\ldots,e_{k-1}}(n_1, \ldots, n_{k-1})\right)$$

for some constant $c_3(\vec{d}, t, u)$. This finishes the proof of Proposition 4.1. $\square$

Corollary 4.2. *Theorem 1.7 holds if $n_1 \ldots n_k n_i^{-1/e_i} \leq n_i F^{\varepsilon}_{\pi_i(\vec{e})}(\pi_i(\vec{n}))$ for some $i \leq k$.*

Proof. By symmetry, Proposition 4.1 holds if we replace k by i. Hence $|\mathcal{E}| \lesssim n_1 \ldots n_k n_i^{-1/e_i} + n_i F^{\varepsilon}_{\pi_i(\vec{e})}(\pi_i(\vec{n})) \leq 2n_i F^{\varepsilon}_{\pi_i(\vec{e})}(\pi_i(\vec{n})) \lesssim F^{\varepsilon}_{\vec{e}}(\vec{n})$. The last inequality follows from the fact that $n_i F^{\varepsilon}_{\pi_i(\vec{e})}(\pi_i(\vec{n}))$ are terms that appear in the definition (1.5) of $F^{\varepsilon}_{\vec{e}}(\vec{n})$; in fact, they are precisely the terms where $I \not\ni i$. $\square$

Remark 4.3. Thus from now on we can assume $n_1 \dots n_{k-1} n_k n_i^{-1/e_i} \geq n_i F^{\varepsilon}_{\pi_i(\vec{d})}(\pi_i(\vec{n}))$ for any $i \in [k]$. In particular, we can assume $n_j \geq n_i^{1/e_i}$ for any $i \neq j$ (since $n_1 \dots n_k n_i^{-1/e_i} \geq n_1 \dots n_k n_j^{-1}$). By Lemma 2.10, we can assume $E_{\vec{e}}(\vec{n})$ is the dominant term in $F^{\varepsilon}_{\vec{e}}(\vec{n})$.

In the second step, we think of the hyperedges as incidences between n_k semi-algebraic sets from P_k with a grid $P_1 \times \cdots \times P_{k-1}$ in $\mathbb{R}^{d_1+\cdots+d_{k-1}}$. Recall there are t polynomials $f_1, \dots, f_t$ and a Boolean function ϕ such that $\vec{p} = (p_1, \dots, p_k) \in P_1 \times \cdots \times P_k$ is a hyperedge of H if and only if $\phi(f_1(\vec{p}) \geq 0, \dots, f_t(\vec{p}) \geq 0) = 1$. For each $p \in P_k$, define the set of its neighbors: $\gamma_p := \{(x_1, \dots, x_{k-1}) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{k-1}} : (x_1, \dots, x_{k-1}, p) \in \mathcal{E}(H)\}$. It is easy to see each γ_p is a semi-algebraic set in $\mathbb{R}^{d_1+\cdots+d_{k+1}}$ defined by $f_1, \dots, f_t$ and ϕ. Moreover, $\mathcal{E}$ is exactly the number of incidences between these semi-algebraic sets $\{\gamma_p\}_{p \in P_k}$ with the grid $P_1 \times \cdots \times P_{k-1}$, which is denoted by $I(P_1 \times \cdots \times P_{k-1}, P_k)$.

We now prove $I(P_1 \times \cdots \times P_{k-1}, P_k) \lesssim F^{\varepsilon}_{\vec{e}}(\vec{n})$ by induction on $\sum_{i=1}^{k-1} e_i$ and then on $\sum_{i=1}^{k-1} n_i$. The statement is vacuous when $\sum e_i = 0$. Fix some $e_1, \dots, e_k$ and assume our inequality (1.8) holds whenever $\sum \dim(V_i) < \sum e_i$. We now use induction on $\sum_{i=1}^{k-1} n_i$. We can choose the coefficient big enough so that it holds for small $n_1, \dots, n_k$. For the induction step, fix some $n_1, \dots, n_k$ and assume (1.8) holds whenever $\sum |P_i| < \sum n_i$. Let $r > 0$ be a constant to be chosen later. By Theorem 2.4 for each $i < k$, there exists an r^{e_i}-partitioning polynomial $f_i \in \mathbb{R}[x_1, \dots, x_{d_i}]$ with respect to V_i of degree at most $C_{part} r$. The polynomial we use to partition the grid $P_1 \times \cdots \times P_{k-1}$ is

$$h(x_1, \dots, x_{d_1+\cdots+d_{k-1}}) = f_1(x_1, \dots, x_{d_1}) f_2(x_{d_1+1}, \dots, x_{d_1+d_2}) \dots$$
$$f_{k-1}(x_{d_1+\cdots+d_{k-2}+1}, \dots, x_{d_1+\cdots+d_{k-1}}).$$

By Theorem 2.3, for each $i < k$, f_i divides $\mathbb{R}^{d_i}$ into $s_i = O(r^{e_i})$ cells. Therefore $\mathbb{R}^{d_1+\cdots+d_{k-1}} \setminus Z(h)$ consists of $s_1 \dots s_{k-1} = O(r^{e_1+\cdots+e_{k-1}})$ cells, each cell contains a sub-grid of $P_1 \times \cdots \times P_{k-1}$ of size at most $\frac{n_1}{r^{e_1}} \times \cdots \times \frac{n_{k-1}}{r^{e_{k-1}}}$. We consider 3 types of incidences in $I(P_1 \times \cdots \times P_{k-1}, P_k)$:

- I_1 consists of the incidences $(p_1, \dots, p_k)$ where $p_i \in V_i \cap Z(f_i)$ for some $i < k$.
- I_2 consists of the incidences $(p_1, \dots, p_k)$ where $(p_1, \dots, p_{k-1})$ is in a cell Ω of the partitioning of h and the semi-algebraic set γ_{p_k} fully contains Ω.
- I_3 consists of the incidences $(p_1, \dots, p_k)$ where $(p_1, \dots, p_{k-1})$ is in a cell Ω of the partitioning of h and the semi-algebraic set γ_{p_k} intersects Ω but does not fully contain Ω (in other words, γ_{p_k} crosses Ω, or γ_{p_k} properly intersects Ω).

Then we have $I(P_1 \times \cdots \times P_{k-1}, P_k) = I_1 + I_2 + I_3$.

Bounding I_1: Let I_1^i denote the number of incidences $(p_1, \dots, p_k)$ where p_i belongs to $V_i \cap Z(f_i)$. Since $I_1 \leq \sum_i I_1^i$, it is enough to bound I_1^1 (the bound for I_1^i for $i > 2$ is similar). The points of P_1 participating in I_1^1 belong to $Z(f_1) \cap V_1$. Assume $V' := Z(f_1) \cap V_1$ has dimension e'_1. Since V_1 is irreducible and $f \notin I(V_1)$, we have $e'_1 < \dim V_1 = e_1$.

We can partition this new variety into at most c_1 irreducible varieties, each of dimension at most $e_1 - 1$ and degree at most c_2 where c_1, c_2 only depend on D_1, C_{part}, d_1 and r. Since $e_1' + e_2 + \cdots + e_{k-1} < e_1 + \cdots + e_{k-1}$, we can apply induction hypothesis for each irreducible component and add together to get $I_1^1 \lesssim F^{\varepsilon}_{e_1', e_2, \ldots, e_k}(\vec{n})$. By applying Lemma 2.9 repeatedly and by Remark 4.3, we have $F^{\varepsilon}_{e_1', \ldots, e_k}(\vec{n}) \le F_{e_1, \ldots, e_k}(\vec{n})$. Thus $I_1^1 \lesssim F^{\varepsilon}_{\vec{e}}(\vec{n})$.

Bounding I_2: Any cell in the partitioning using h has form $\Omega = \Omega^{f_1} \cap \cdots \cap \Omega^{f_{k-1}}$ where Ω^{f_i} is some cell in the decomposition by f_i. For each $i < k$, the contribution to I_2 from all cells that satisfy $|\Omega^{f_i} \cap P_i| < u$ is bounded by $u \prod_{j \neq i} n_j$. Hence we only need to bound the contribution from cells that contain a grid of size at least $u \times \cdots \times u$ (there are $k - 1$ $u's$). For such a cell Ω, the number of semi-algebraic sets γ_{p_k} that contain Ω is bounded by u, because otherwise the hypergraph would contain $K_{u,\ldots,u}$. Thus the contribution to I_2 from this last cell type is at most $u \prod_{i<k} n_i$. In conclusion, $I_2 \lesssim \prod n_i (\sum \frac{1}{n_i}) < F^{\varepsilon}_{\vec{e}}(\vec{n})$.

Bounding I_3: For each $i < k$, and each $j \le s_j$, let $n_{i,j}$ denote the number of points of P_i that lies in a cell $\Omega^{f_i}_j$. Then clearly for each $i < k$, $\sum_{j=1}^{s_i} n_{i,j} \le n_i$. For $j_i \le s_i$, let $n_{k,j_1,\ldots,j_{k-1}}$ denote the number of semi-algebraic sets γ_{p_k} that cross the cell $\Omega^{f_1}_{j_1} \cap \cdots \cap \Omega^{f_{k-1}}_{j_{k-1}}$. Using a similar argument with the previous section, by Theorem 2.3, each semi-algebraic set γ_{p_k} crosses at most $O(r^{e_1+\cdots+e_{k-1}-1})$ cells, and hence $\sum n_{k,j_1,\ldots,j_{k-1}} \le r^{e_1+\cdots+e_{k-1}-1} n_k$. Hence

$$
\begin{aligned}
I_3 &= \sum_{j_1,\ldots,j_{k-1}} I(n_{1,j_1}, \ldots, n_{k-1,j_{k-1}}, n_{k,j_1,\ldots,j_{k-1}}) \\
&\lesssim \sum_{j_1,\ldots,j_{k-1}} F^{\varepsilon}_{\vec{e}}(n_{1,j_1}, \ldots, n_{k-1,j_{k-1}}, n_{k,j_1,\ldots,j_{k-1}}) \qquad (4.4) \\
&\le r^{e_1+\cdots+e_{k-1}} F^{\varepsilon}_{\vec{e}}\left(\frac{n_1}{r^{e_1}}, \ldots, \frac{n_{k-1}}{r^{e_{k-1}}}, \frac{n_k}{r}\right) \qquad (4.5) \\
&\le r^{-\varepsilon} F^{\varepsilon}_{\vec{e}}(\vec{n}). \qquad (4.6)
\end{aligned}
$$

Here (4.4) follows by the induction assumption for smaller $\sum n_i$, (4.5) follows by Hölder's inequality (as all the exponents are either 1 or less than 1) and the last step (4.6) follows from Corollary 2.8 (note that by Remark 4.3, we only need to care about the dominant term $E_{\vec{d}}(\vec{n}) \prod_i n_i^{\varepsilon}$).

Adding $I = I_1 + I_2 + I_3$, choosing appropriate r and coefficients, we get the desired bound. This finishes the proof of our main theorem. □

5. Applications

5.1. Almost-unit-area triangle problem

Many geometric problems can be viewed as counting edges in a certain semi-algebraic hypergraph. Let us start with the unit area triangle problem: namely, given n points in $\mathbb{R}^2$, how many triangles of area 1 can they form? Given such a point set P, we can construct the 3-uniform 3-partite hypergraph $(P, P, P, \mathcal{E})$, where three points form a hyperedge iff they define a triangle of unit area. This is a semi-algebraic hypergraph with bounded complexity that contains no $K_{1,2,2}$ (see Lemma 5.5). Hence the number of unit area triangles, which is the number of hyperedges, is $O(n^{12/5+o(1)})$ by Theorem 1.7. This is weaker than the best bound known $O(n^{9/4})$ (see [19]), but more robust. For example, if we are interested in counting the number of triangles with area close to 1, say between 0.9 and 1.1 with an additional condition of not containing some $K_{u,u,u}$, Theorem 1.7 gives us a nice bound. The hypergraph formed by all triangles with areas between 0.9 and 1.1 is still semi-algebraic with bounded complexity, and hence Theorem 1.7 has the following corollary:

Corollary 5.1. *Given a set S of n points in $\mathbb{R}^2$ and some $u \geq 1$. Assume the hypergraph formed by all triangles with areas between 0.9 and 1.1 contains no $K_{u,u,u}$ (i.e. there do not exist disjoint sets $A, B, C \subset S$ such that $|A|, |B|, |C| \geq u$ and each $a \in A, b \in B, c \in C$ forms a triangle of area between 0.9 and 1.1). Then the number of those triangles is $O_\varepsilon(n^{12/5+\varepsilon})$ for arbitrarily small $\varepsilon > 0$.*

5.2. Unit-minor problem

The previous application seems somewhat artificial as we had to impose the $K_{u,\ldots,u}$-free condition. In our next application, that condition is automatically satisfied.

A natural generalization of the unit area triangle problem is to ask for the maximum number of unit-volume d-dimensional simplices formed by n points in $\mathbb{R}^d$ for some fixed positive integer d (see [7]). The best known bound when $d = 3$ is $O(n^{7/2})$ in [5]. For general d, we can bound that number by $nf_d(n)$ where $f_d(n)$ is the number of unit-volume simplices with a fixed vertex, say the origin. Interestingly, this is equivalent to the unit-minor problem: What is the maximum number of unit $d \times d$ minors that appear in a $d \times n$ matrix M without repeated columns?[3] Indeed, if we regard the column vectors of M as points in $\mathbb{R}^d$, then the $d+1$ points $0, v_1, \ldots, v_d$ form a unit-volume simplex if and only if $\det(v_1, \ldots, v_d) = \pm 1$. When M is *totally positive,* that is, when all minors of M are strictly positive, the best known upper bounds on $f_d(n)$ are given by the following theorem:

[3] If we allow M to have repeated columns, the answer is $\Theta(n^d)$, trivially.

Theorem 5.2 *(Farber, Ray and Smorodinsky [8], 2014). Let $f_d^+(n)$ denote the maximum number of $d \times d$ unit minors in a totally positive $d \times n$ matrix, then*

$$f_d^+(n) = \begin{cases} \Theta(n^{4/3}) & \text{if } d = 2 \\ O(n^{11/5}) & \text{if } d = 3 \\ O(n^{d-\frac{d}{d+1}}) & \text{if } d \geq 4. \end{cases}$$

Their proof uses point-hyperplane incidences: fix $d-1$ points, then the set of all points that form a minor 1 with those vectors is a hyperplane. In general, those hyperplanes may not be distinct, and the point-hyperplane graph can contain large complete bipartite graphs $K_{u,u}$. Farber et al. avoid these issues by imposing the total positivity constraint. Our Theorem 1.7 provides another way around these issues (without requiring total positivity) to obtain non-trivial bounds on $f_d(n)$.

Theorem 5.3. *Let $f_d(n)$ denote the maximum number of unit $d \times d$ minors in a $d \times n$ matrix without repeated columns, then*[4] $n^{d-2/3} \lesssim_d f_d(n) \lesssim_{d,\varepsilon} n^{d-\frac{d}{d^2-d+1}+\varepsilon}$.

Corollary 5.4. *The maximum number of unit volume d-simplices formed by n points in $\mathbb{R}^d$ is $O(nf_d(n)) = O_{d,\varepsilon}(n^{d+1-\frac{d}{d^2-d+1}+\varepsilon})$.*

Proof of Theorem 5.3. Consider the d-uniform d-partite hypergraph H where each part is the set of n column vectors of M, and d-tuple $(v_1, \ldots, v_d)$ forms a hyperedge iff $\det(v_1, \ldots, v_d) = 1$. Clearly this is a semi-algebraic hypergraph with bounded complexity. The upper bound is a simple application of Theorem 1.7 for $k = d$ and the following lemma:

Lemma 5.5. *The hypergraph H does not contain $K_{2,\ldots,2}$.*

Proof. Assume there exist $2d$ distinct points (or vectors) $\{v_i^+\}_{i=1}^d$ and $\{v_i^-\}_{i=1}^d$ in $\mathbb{R}^d$ such that $\det(v_1^{\sigma_1}, \ldots, v_d^{\sigma_d}) = 1$ for any choice of $\sigma_i \in \{+, -\}$. By multilinearity of the determinant we have for any choice of $\sigma_2, \ldots, \sigma_d$ that

$$\det(x - y, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}) = \det(x, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}) - \det(y, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}).$$

In particular, $\det(v_1^+ - v_1^-, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}) = 1 - 1 = 0$. Take any x_1 on the line that passes through v_1^+ and v_1^-, then $\det(x_1 - v_1^-, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}) = 0$ because $x_1 - v_1^-$ is parallel to $v_1^+ - v_1^-$. Thus $\det(x_1, v_2^{\sigma_2}, \ldots, v_d^{\sigma_d}) = 1$. Similar statements hold for other indices. Therefore we must have $\det(x_1, \ldots, x_d) = 1$ for any x_i on the line through v_i^+ and v_i^-.

Take a generic hyperplane through 0 that intersects all the lines $v_i^+ v_i^-$ for $i \in [d]$ (such a hyperplane always exists because we only require the hyperplane does not contain d

[4] Note that this lower bound together with the upper bound from Theorem 5.2 imply that $f_d(n)$ and $f_d^+(n)$ are fundamentally different.

fixed lines). Let x_i be the intersection of this hyperplane with the line through v_i^+ and v_i^-. By the above argument $\det(x_1, \ldots, x_d) = 1$, but as the vectors $x_1, \ldots, x_d$ are linearly dependent, this is a contradiction. □

Note that H can contain $K_{1,u,\ldots,u}$, for $u = n/d$ by choosing $P_1 = \{(1, 0, \ldots, 0)\}$, $P_2 = \{(x_2, 1, 0, \ldots, 0)\}$, $P_3 = \{(0, x_3, 1, 0, \ldots, 0)\}$ and $P_d = \{(0, \ldots, 0, x_d, 1)\}$ for $x_i \in [u]$. This suggests we cannot directly apply results for graphs or even l-uniform hypergraphs for $l < d$.

To obtain the lower bound of $f_d(M)$: pick the tight example $P_1 \times P_2$ in [8] for $d = 2$ on the x_1x_2 plane (which comes from the tight example of Szemerédi–Trotter theorem). Then choose $P_3, \ldots, P_d$ as above. We have at least $\sim n^{4/3} \times n^{d-2} = n^{d-2/3}$ unit minors of form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & x_3 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & x_d \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

This finishes the proof of Theorem 5.3. □

5.3. Intersection hypergraphs

Recall the intersection hypergraph $H(S)$ of a set S of geometric objects in $\mathbb{R}^d$ is the d-uniform hypergraph on the vertex set S where d vertices form a hyperedge if and only if the corresponding sets have nonempty intersection. In this subsection, we find a nontrivial upper bound for the number of hyperedges in $H(S)$ given it is $K_{u,\ldots,u}$-free and S is taken from some s-dim family of semi-algebraic sets.

We say F is an *s-dimensional family of semi-algebraic sets with description complexity t in $\mathbb{R}^d$* if each object in F is a semi-algebraic set in $\mathbb{R}^d$ determined by at most t polynomials $f_1, \ldots, f_t$, each of degree at most t and the coefficients of $f_1, \ldots, f_t$ belong to a s-dim variety with degree at most t in $\mathbb{R}^{t\binom{d+t}{t}}$. Informally, each semi-algebraic set in F is determined by s parameters. For example, hyperplanes or half-spaces in $\mathbb{R}^d$ form a d-dimensional family with description complexity 1, spheres in $\mathbb{R}^d$ form a $(d+1)$-dimensional family with description complexity 2 and $(d-1)$-dim simplices in $\mathbb{R}^d$ form a d^2-dimensional family with description complexity $t+1$ (since each simplex is determined by d points in $\mathbb{R}^d$ and by $t+1$ linear inequalities).

Let S be a set of n semi-algebraic sets taken from a s-dimensional family with degree t. It is not difficult to see that the intersection hypergraph $H(S)$ is a semi-algebraic hypergraph with bounded complexity (which depends on t, s and d). Indeed, $(S_1, \ldots, S_d) \in S^d$ forms a hyperedge if and only if there exists some $y \in R^d$ such that $y \in S_i$ for all $i \in [d]$. This is the projection of the semi-algebraic set $T = \{(y, S_1, \ldots, S_d) : y \in S_i, \forall i \in [d]\}$

along the first axis, hence remains a semi-algebraic set with description complexity only depending on that of T and d by Theorem 2.2.1 in [3]. Applying Theorem 1.7 to this hypergraph we get the following bound on the number of hyperedges in $H(S)$:

Corollary 5.6. *Let S be n semi-algebraic sets taken from an s-dimensional family with degree t in $\mathbb{R}^d$. If $H(S)$ is $K_{u,\ldots,u}$-free for some u, then $H(S)$ has $O_{t,d,s,u,\varepsilon}(n^{d-\frac{d}{(d-1)s+1}+\varepsilon})$ hyperedges. In particular, n spheres in $\mathbb{R}^d$ form at most $O_{d,u,\varepsilon}(n^{d-1/d+\varepsilon})$ intersections if their intersection hypergraph is $K_{u,\ldots,u}$-free.*

More generally, we can extend this result to counting intersections among different families of sets, such as intersections between triangles and line segments in $\mathbb{R}^3$. More precisely, for $i = 1, \ldots, k$, let P_i be n_i semi-algebraic sets taken from a s_i-dim family in $\mathbb{R}^d$. Their intersection hypergraph is defined on $P_1 \times \cdots \times P_k$ where $(p_1, \ldots, p_k)$ form a hyperedge if and only if they have a nonempty intersection. If this hypergraph is $K_{u,\ldots,u}$-free for some $u > 0$ then the number of intersections is $O(F^{\varepsilon}_{s_1,\ldots,s_k}(n_1, \ldots, n_k))$.

6. Discussion

Perhaps the most important question raised but left unanswered by this paper is whether Theorem 1.7 is tight. When $k = 2$, or H is a graph, it is known to be tight when $d_1 = d_2 = 2$ and almost tight for $d_1 = d_2 = d \geq 3$ (for example see [20]). No tight example is known for hypergraphs. Another interesting question is to know the dependency of u in the expression: does the theorem say anything meaningful when u increases with n, such as $u = \Theta(\log n)$?

Furthermore, we feel that the applications of our main theorem are still largely unexplored. While it immediately gives bounds for many geometric problems, they are usually not the best ones (e.g. the unit-triangle problem). It would be interesting to find an instance where hypergraphs are more effective than graphs. Such an example should share in common with the unit-minor problem that the constructed hypergraph contains $K_{1,u,\ldots,u}$ for large u but does not contain $K_{u',\ldots,u'}$ for some fixed u'.

In the almost-unit-area problem, without the condition no $K_{u,u,u}$ we can have $\Theta(n^3)$ triangles of area in the range $[0.9, 1.1]$ by choosing 3 points reasonably far apart that form a unit area and then dividing the remaining $n-3$ points equally into small neighborhoods of those 3 points. Would we get a better result by imposing an upper bound on the ratio between the maximum and the minimum distances among the points?

Finally, we would like to improve the bound for the minor problem since the gap between the lower and upper bounds in Theorem 5.3 is quite large for big d. To improve the lower bound, instead of building from the grid example in two dimensions, we can choose points from a grid in $\mathbb{R}^d$, or a multiple of grids.

Appendix A. Proof of generalized Erdős's result

We suspect the second statement of Theorem 1.2 may be stated somewhere but since we cannot find a reference, its proof is included here for completeness. We use induction by k. It clearly holds for $k = 1$. For $k \geq 2$, assume it holds for $k - 1$. Let

$$Q := \#\{(y, x_1, \ldots, x_{u_1}) : y \in P_2 \times \cdots \times P_k, x_i \in P_1, (x_i, y) \in \mathcal{E} \quad \forall i \in [u]\}.$$

We count Q by two ways. For each choice of $(x_1, \ldots, x_u)$ define a new hypergraph on $P_2 \times \cdots \times P_k$ where y is a hyperedge iff $(x_i, y) \in \mathcal{E}$ for all $i \in [u]$. This $(k-1)$-uniform hypergraph does not contain $K_{u_2,\ldots,u_k}$, hence by induction assumption:

$$E \lesssim \binom{n_1}{u_1} n_2 \ldots n_k \left(n_k^{-1/u_2 \ldots u_{k-1}} + \sum n_i^{-1} \right). \tag{A.1}$$

On the other hand, for each $y \in P_2 \times \cdots \times P_k$, let N_y denote the number of $x \in P_1$ such that $(x, y) \in \mathcal{E}$. Then by Hölder inequality:

$$E = \sum_y \binom{N_y}{u_1} \gtrsim \frac{(\sum_{y: N_y \geq u_1} N_y)^{u_1}}{(n_2 \ldots n_k)^{s_1 - 1}} \gtrsim \frac{(|\mathcal{E}| - n_2 \ldots n_k)^{u_1}}{(n_2 \ldots n_k)^{s_1 - 1}}. \tag{A.2}$$

Combining (A.1) and (A.2) we get the desired bound for $|\mathcal{E}|$. □

Appendix B. Proofs of lemmas in 2.4

Proof of Lemma 2.6. A direct calculation yields

$$\begin{aligned}\sum_{j \neq i} d_j (1 - \alpha_j) &= \sum_{j \neq i} \frac{\frac{d_j}{d_j - 1}}{k - 1 + \sum_l \frac{1}{d_l - 1}} = \frac{\sum_{j \neq i} \left[1 + \frac{1}{d_j - 1}\right]}{k - 1 + \sum_l \frac{1}{d_l - 1}} \\ &= \frac{k - 1 + \sum_{j \neq i} \frac{1}{d_j - 1}}{k - 1 + \sum_l 1/(d_l - 1)} = 1 - \frac{\frac{1}{d_i - 1}}{k - 1 + \sum_l 1/(d_l - 1)} = \alpha_i.\end{aligned}$$

Rearranging we get $\sum_{j \neq i} d_j \alpha_j + \alpha_i = \sum_{j \neq i} d_j$, true for all $i = 1, \ldots, k$, hence (2.7) holds. □

Proof of Lemma 2.9. By examining the formula of $F^{\varepsilon}_{\vec{d}}(\vec{n})$ in (1.5), we realize it is enough to prove $E_{\vec{d}_I - \mathfrak{e}_i}(\vec{n}) \leq E_{\vec{d}_I}(\vec{n})$ whenever $i \in I$. Without loss of generality we can assume $I = [k]$ and $i = 1$. In other words, we only need to prove $E_{d_1 - 1, d_2, \ldots, d_k}(\vec{n}) \leq E_{d_1, d_2, \ldots, d_k}(\vec{n})$ given $n_i \geq n_j^{1/(d_j)}$ for any $j \neq i$. Let $M_1 = k - 1 + \frac{1}{d_1 - 2} + \sum_{i=2}^k \frac{1}{d_i - 1}$ and $M_2 = k - 1 + \frac{1}{d_1 - 2} + \sum_{i=2}^k \frac{1}{d_i - 1}$, we can write the inequality $E_{d_1 - 1, d_2, \ldots, d_k}(\vec{n}) \leq E_{\vec{d}}(\vec{n})$ as

$$n_1^{1-\frac{1/(d_1-2)}{M_1}} \prod_{i=2}^{k} n_i^{1-\frac{1/(d_i-1)}{M_1}} \leq n_1^{1-\frac{1/(d_1-1)}{M_2}} \prod_{i=2}^{k} n_i^{1-\frac{1/(d_i-1)}{M_2}}$$

$$\iff \prod_{i=2}^{k} n_i^{\frac{1}{d_i-1}\left(\frac{1}{M_2}-\frac{1}{M_1}\right)} \leq n_1^{\frac{1}{M_1(d_1-2)}-\frac{1}{M_2(d_1-1)}}$$

$$\iff \prod_{i=2}^{k} n_i^{\frac{1}{(d_i-1)M_1M_2(d_1-1)(d_1-2)}} \leq n_1^{\frac{k-1+\sum i>1 \frac{1}{d_i-1}}{M_1M_2(d_1-1)(d_1-2)}} \tag{B.1}$$

$$\iff \prod_{i\geq 2} n_i^{1/(d_i-1)} \leq n_1^{\sum_{i\geq 2} d_i/(d_i-1)} \tag{B.2}$$

In (B.1) we use $M_1 - M_2 = \frac{1}{d_1-2} - \frac{1}{d_1-1} = \frac{1}{(d_1-1)(d_2-1)}$ and $M_2(d_1-1) - M_1(d_1-2) = k - 1 + \sum_{i=2}^{k} \frac{1}{d_i-1}$. Take both sides to the power of $M_1M_2(d_1-1)(d_1-2)$ we get (B.2), which holds because $n_i \leq n_1^{d_i}$ for any $i \geq 2$ by assumption. $\square$

Proof of Lemma 2.10. To prove $E_{\vec{d}}(\vec{n}) \prod n_i^{\varepsilon}$ is the dominant term, it is enough to prove for any non-empty $I \subset [k]$:

$$E_{\vec{d}}(\vec{n}) \geq E_{\vec{d}_I}(\vec{n}_I) \prod_{i\notin I} n_i. \tag{B.3}$$

Claim. *For each $i \in [k]$, if $n_i^{-1/d_i} \prod_j n_j \geq n_i E_{\pi_i(\vec{d})}(\pi_i(\vec{n}))$ then $E_{\vec{d}}(\vec{n}) \geq n_i E_{\pi_i(\vec{d})}(\pi_i(\vec{n}))$.*

Proof of Claim. Both inequalities are equivalent to $\prod_{j\neq i} n_j^{\frac{1/(d_j-1)}{k-2+\sum_{l\neq i} 1/(d_l-1)}} \geq n_i^{1/d_i}$ by rearrangement. $\square$

We now prove (B.3) via induction by $|I|$. By the claim, it holds whenever $|I| = k - 1$ because our assumption (2.11) implies $n_i^{-1/d_i} \prod_i n_i \geq n_i E_{\pi_i(\vec{d})}(\pi_i(\vec{n}))$. Assume (B.3) holds for any $I \subset [k]$ with $|I| = l$, we will prove it holds for any $I \subset [k]$ such that $|I| = l - 1$. Take some $i \notin I$ and let $J = I \cup \{i\}$. By induction assumption $E_{\vec{d}}(\vec{n}) \geq E_{\vec{d}_J}(\vec{n}_J) \prod_{j\notin J} n_j$. By our assumption (2.11), $n_i^{-1/d_i} \prod_j n_j \geq E_{\vec{d}_I}(\vec{n}_I) \prod_{j\notin I} n_j$; dividing both sides by $\prod_{j\neq J} n_j$ we get $n_i^{-1/d_i} \prod_{j\in J} n_j \geq E_{\vec{d}_I}(\vec{n}_I) n_i$. Applying the above claim for J instead of $[k]$, we get $E_{\vec{d}_J}(\vec{n}_J) \geq n_i E_{\vec{d}_I}(\vec{n}_I)$ and thus $E_{\vec{d}}(\vec{n}) \geq E_{\vec{d}_J}(\vec{n}_J) \prod_{j\notin J} n_j \geq E_{\vec{d}_I}(\vec{n}_I) \prod_{j\notin I} n_j$. This means (B.3) also holds for I. $\square$

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