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A tightness property of relatively smooth permutations

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It is well known that many geometric properties of Schubert varieties of type A (and others) can be interpreted combinatorially. Given two permutations $w, x \in S_n$ we give a combinatorial consequence of the property that the smooth locus of the Schubert variety X_w contains the Schubert cell Y_x. This provides a necessary ingredient for the interpretation of recent representation-theoretic results of the author with Mínguez in terms of identities of Kazhdan–Lusztig polynomials.

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1. Introduction

Consider the flag variety $\mathcal{F}_n$ (over $\mathbb{C}$) consisting of all flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n, \quad \dim V_i = i$$

of an n-dimensional vector space. It can be identified with $\mathrm{GL}_n(\mathbb{C})/B_n$ where $B_n \subset \mathrm{GL}_n(\mathbb{C})$ is the subgroup of upper triangular matrices, which is the stabilizer of the standard flag $V_i = \mathbb{C}^i$. The orbits of B_n on $\mathcal{F}_n$ are parameterized by permutation matrices. Let Y_w and X_w, respectively, be the Schubert cell (the orbit) and the Schubert variety (its closure) corresponding to a permutation $w \in S_n$. The dimension of Y_w (and X_w) is the length of w. The inclusion relation $Y_x \subset X_w$ gives rise to the Bruhat partial order on S_n which can be described combinatorially as $r_w \le r_x$ where $r_w : \{0, \dots, n\} \times \{0, \dots, n\} \to \mathbb{Z}_{\ge 0}$ is the rank function

$$r_w(i,j) = \#\{u = 1, \dots, i : w(u) \le j\} \le \min(i,j) = r_e(i,j).$$

Thus, X_w consists of the flags satisfying the conditions

$$\dim(V_i \cap \mathbb{C}^j) \ge r_w(i,j), \ i,j = 1, \dots, n. \tag{1}$$

In practice, many of these conditions are redundant [8].

It is well-known that smooth Schubert varieties are defined by inclusions, i.e., by relations of the form $V_i \subset \mathbb{C}^j$ or $\mathbb{C}^j \subset V_i$ [18,19] – see also [10]. In other words, in the conditions (1) above it suffices to take i, j for which $r_w(i,j) = r_e(i,j)$. This fact admits the following generalization which is our main result.

Theorem 1.1. *Suppose that Y_x is contained in the smooth locus of X_w.*[1] *Let $y \in S_n$ and assume that $r_y(i,j) = r_w(i,j)$ whenever $r_w(i,j) = r_x(i,j)$. Then $Y_y \subset X_w$.*

Of course, unlike in the case $x = e$, the condition on y is in general not necessary for the inclusion $Y_y \subset X_w$.

[1] Recall that X_w is smooth if and only if it is smooth at the point Y_e.

The condition on the pair (w, x) can be spelled out combinatorially using the well-known description of the tangent space of X_w at Y_x [15].

Theorem 1.1 provides a necessary combinatorial ingredient for the translation of the representation-theoretic result of [14] to a certain identity of Kazhdan–Lusztig polynomials with respect to the symmetric group. We refer the reader to [14, §10] for a self-contained statement of the identity and for an explanation of the role of Theorem 1.1.[2] At any rate, we hope that Theorem 1.1 is of interest in its own right.

Our proof is purely combinatorial. A key input is a simple construction due to Gasharov [9], which was introduced in the solution of the Lakshmibai–Sandhya conjectural description of the singular locus of X_w, completed independently by Billey–Warrington [4], Manivel [17], Kassel–Lascoux–Reutenauer [13] and Cortez [5]. Some other ideas from [4] are also used in the current proof. However, we are not aware of a logical implication between the singular locus theorem and our result. We will give an outline of the proof in the next section after introducing some notation. It would be desirable to give a geometric context of the statement, if not the proof, of Theorem 1.1.

2. Statement of main result

2.1. Notation and preliminaries

Let S_n be the symmetric group on $\{1, \dots, n\}$ with length function ℓ. We denote the Bruhat order on S_n by $\le$ and write $y \lhd w$ if w covers y, i.e., if $y < w$ and $\ell(w) = \ell(y) + 1$. (We refer to [2] for basic properties about the Bruhat order.) Let

$$\square_n = \{1, \dots, n-1\} \times \{1, \dots, n-1\}, \quad \boxed{}_n = \{0, \dots, n\} \times \{0, \dots, n\}$$

be the *restricted square* and the *framed square* respectively. For any $w \in S_n$ let $\Gamma_w = \{\underline{p}_w(i) : i = 1, \dots, n\} \subset \boxed{}_n$ be the graph of w where $\underline{p}_w(i) = (i, w(i))$ and let $r_w : \boxed{}_n \to \mathbb{Z}_{\ge 0}$ be the *rank function*

$$r_w(i, j) = \#\{u = 1, \dots, i : w(u) \le j\}.$$

Note that $r_w(0, i) = r_w(i, 0) = 0$ and $r_w(n, i) = r_w(i, n) = i$ for all i. Recall that for any $w, y \in S_n$ we have $y \le w$ if and only if $r_w \le r_y$ on $\boxed{}_n$ (or equivalently, on $\square_n$).

For any $p = (i, j), p' = (i', j') \in \boxed{}_n$ we write

$$\mathcal{C}(p, p') = \{(i, j'), (i', j)\}$$

and define the *difference function*

$$\Delta_w(p, p') := \sum_{q \in \{p, p'\}} r_w(q) - \sum_{q \in \mathcal{C}(p, p')} r_w(q). \tag{2}$$

[2] In fact, for this application one may assume in addition that $x \le y$ and that x is 213-avoiding.

We also write $p' > p$ (resp., $p' \geq p$) if $i' > i$ and $j' > j$ (resp., $i' \geq i$ and $j' \geq j$). (We caution that $<$ is not the strict partial order subordinate to $\leq$.) Clearly, $\Delta_w(p,p') = \Delta_w(p',p)$. If $p' \geq p$ then

$$\Delta_w(p,p') = \#\{i < u \leq i' : j < w(u) \leq j'\} = \#(\Gamma_w \cap (p,p']) \geq 0$$

where

$$(p,p'] = \{q \in \square_n : p < q \leq p'\}.$$

Similarly we also use the notation $[p,p') = \{q \in \square_n : p \leq q < p'\}$ and $(p,p') = \{q \in \square_n : p < q < p'\}$. We write $p \perp p'$ if either $p > p'$ or $p' > p$; equivalently $(i'-i)(j'-j) > 0$.

Define the set of *pairs*

$$\mathfrak{P}_n = \{(w,x) \in S_n \times S_n : x \leq w\}.$$

For any $\Sigma = (w,x) \in \mathfrak{P}_n$ we write

$$r_\Sigma = r_x - r_w : \square_n \to \mathbb{Z}_{\geq 0}$$

and for any $p, p' \in \square_n$ let

$$\Delta_\Sigma(p,p') = \Delta_x(p,p') - \Delta_w(p,p') = \sum_{q \in \{p,p'\}} r_\Sigma(q) - \sum_{q \in \mathcal{C}(p,p')} r_\Sigma(q).$$

Define the *level set* of Σ and its complement to be

$$L_\Sigma = \{p \in \square_n : r_\Sigma(p) = 0\}, \quad \tilde{L}_\Sigma = \{p \in \square_n : r_\Sigma(p) > 0\} \subset \square_n. \tag{3}$$

Note that if $x \leq y \leq w$ then

$$L_\Sigma = L_{(w,y)} \cap L_{(y,x)}.$$

We will often use the following simple fact:

$$\text{if } p, q \in L_\Sigma,\ p \perp q \text{ and } \Delta_w(p,q) = 0 \text{ then } \Delta_x(p,q) = 0 \text{ and } \mathcal{C}(p,q) \subset L_\Sigma. \tag{4}$$

Indeed, $0 \leq \Delta_x(p,q) = \Delta_\Sigma(p,q) = -\sum_{p' \in \mathcal{C}(p,q)} r_\Sigma(p') \leq 0$.

Denote by $\mathcal{R}_n$ the set of transpositions in S_n. Thus, $\mathcal{R}_n = \{t_{i,j} : 1 \leq i < j \leq n\}$ where $t_{i,j} = t_{\{i,j\}}$ is the transposition interchanging i and j. By our convention, when using the notation $t_{i,j}$ (in contrast to $t_{\{i,j\}}$) it is understood, often implicitly, that $i < j$. Note that $w < wt_{i,j}$ if and only if $w(i) < w(j)$ if and only if $\underline{p}_w(i) < \underline{p}_w(j)$ and in this case

$$\tilde{L}_{(wt_{i,j},w)} = [\underline{p}_w(i), \underline{p}_w(j));$$

otherwise $wt_{i,j} < w$. Thus,

$$\ell(w) = \#\{t \in \mathcal{R}_n : wt < w\}.$$

For $\Sigma = (w,x) \in \mathfrak{P}_n$ let $\ell(\Sigma) = \ell(w) - \ell(x) \geq 0$,

$$\mathcal{R}_\Sigma = \{t \in \mathcal{R}_n : x < xt \leq w\}, \quad \mathcal{R}_\Sigma^{\lhd} = \{t \in \mathcal{R}_n : x \lhd xt \leq w\}.$$

Thus, $\ell(\Sigma) = 0$ if and only if $w = x$ if and only if $\mathcal{R}_\Sigma^{\lhd} = \varnothing$. We have

$$t_{i,j} \in \mathcal{R}_\Sigma \iff \underline{p}_x(i) < \underline{p}_x(j) \text{ and } \tilde{L}_{(xt_{i,j},x)} = [\underline{p}_x(i), \underline{p}_x(j)) \subset \tilde{L}_\Sigma. \tag{5}$$

Hence,

$$\text{if } i < j < k < l,\ x(j) < x(i) < x(l) < x(k) \text{ and } t_{i,k}, t_{j,l} \in \mathcal{R}_\Sigma \text{ then } t_{i,l} \in \mathcal{R}_\Sigma. \tag{6}$$

If $t = t_{i,j} \in \mathcal{R}_\Sigma$ then $t \in \mathcal{R}_\Sigma^{\lhd}$ if and only if there does not exist i' such that $t_{i,i'}, t_{i',j} \in \mathcal{R}_\Sigma$.

For any $t \in \mathcal{R}_\Sigma$ let $\Sigma_t = (w, xt) \in \mathfrak{P}_n$. Thus, $\ell(\Sigma_t) < \ell(\Sigma)$.

We have (see §3.3)

$$\#\mathcal{R}_\Sigma \geq \ell(\Sigma). \tag{7}$$

We say that Σ is *smooth* if equality holds.

The geometric interpretation is as follows. As in the introduction, let Y_w be the Schubert cell pertaining to w in the flag variety of $\mathrm{GL}_n(\mathbb{C})$ and let X_w be the corresponding Schubert variety, i.e., the Zariski closure of Y_w. Then $\Sigma = (w,x) \in \mathfrak{P}_n$ if and only if $Y_x \subset X_w$ and in this case

$$\#\mathcal{R}_\Sigma + \ell(x) = \#\{t \in \mathcal{R}_n : xt \leq w\}$$

is the dimension of the tangent space of X_w at any point of Y_x [15]. Thus, Σ is smooth if and only if Y_x is contained in the smooth locus of X_w. (Other equivalent conditions are that X_w is rationally smooth at any point of Y_x or that the Kazhdan–Lusztig polynomial $P_{x,w}$ with respect to S_n is 1 [11,12,6]. We refer the reader to [3] for more details and generalizations.) We also recall that X_w is smooth (i.e., (w,e) is smooth) if and only if w is 3412 and 4231 avoiding [16].

2.2. Symmetries

For any $w \in S_n$ we write $w^* = w_0 w w_0 \in S_n$ for the *upended* permutation where w_0 is the longest element of S_n. For any $\Sigma = (w,x) \in \mathfrak{P}_n$ let $\Sigma^* = (w^*, x^*) \in \mathfrak{P}_n$ and $\Sigma^{-1} = (w^{-1}, x^{-1}) \in \mathfrak{P}_n$ be the upended and *inverted* pair respectively. For any $p = (i,j) \in \square_n$ let $p^* = (n-i, n-j) \in \square_n$ and $p^{-1} = (j,i) \in \square_n$ be the upended and inverted point respectively. Thus, $\underline{p}_{w^*}(n+1-r) = \underline{p}_w(r)^* - (1,1)$ and $\underline{p}_{w^{-1}}(w(r)) = \underline{p}_w(r)^{-1}$ for any r, $\Gamma_{w^*} = \Gamma_w^* - (1,1)$, $\Gamma_{w^{-1}} = \Gamma_w^{-1}$, $r_{w^*}(p^*) = n - i - j + r_w(p)$, $r_{\Sigma^*}(p^*) = r_\Sigma(p)$, $r_{w^{-1}}(p^{-1}) = r_w(p)$ and $r_{\Sigma^{-1}}(p^{-1}) = r_\Sigma(p)$, In particular, $L_{\Sigma^*} = L_\Sigma^*$ and $L_{\Sigma^{-1}} = L_\Sigma^{-1}$. Note that $\mathcal{R}_{\Sigma^*} = \mathcal{R}_\Sigma^*$ and $\mathcal{R}_{\Sigma^{-1}} = \mathcal{R}_\Sigma^x$ where $t^x = xtx^{-1}$; similarly for $\mathcal{R}^\triangleleft$. For any $t \in \mathcal{R}_\Sigma$ we have $(\Sigma_t)^* = (\Sigma^*)_{t^*}$ and $(\Sigma_t)^{-1} = (\Sigma^{-1})_{t^x}$.

2.3. Statement

We make the following key definition.

Definition 2.1. Let $w \in S_n$.

(1) A subset $A \subset \square_n$ is called *tight* with respect to w if for any $y \in S_n$ such that $r_y\big|_A \equiv r_w\big|_A$ we have $y \le w$.
(2) We say that a pair $\Sigma = (w, x) \in \mathfrak{P}_n$ is tight if L_Σ is tight with respect to w.

The main result of the paper is the following equivalent reformulation of Theorem 1.1.

Theorem 2.2. *Every smooth pair $\Sigma = (w, x) \in \mathfrak{P}_n$ is tight.*

Remark 2.3. In the case where $x = e$, Theorem 2.2 follows from [18,19]. Moreover, it was proved by Gasharov–Reiner [10] that (w, e) is tight (namely, X_w is defined by inclusions) if and only if w avoids the patterns 4231, 35142, 42513 and 351624. Curiously, this condition occurs in other contexts as well (cf. [1]).

Remark 2.4. In [8] Fulton defined the *essential set* of w by

$$\mathcal{E}(w) = \{(i,j) \in \square_n : w(i) \le j < w(i+1) \text{ and } w^{-1}(j) \le i < w^{-1}(j+1)\}.$$

He showed, among other things, that

$$y \le w \iff r_y(p) \ge r_w(p) \text{ for all } p \in \mathcal{E}(w)$$

and that this equivalence ceases to be true if we replace $\mathcal{E}(w)$ by any proper subset. See [7] for a follow-up on these ideas.

Thus, if $L_\Sigma \supset \mathcal{E}(w)$ then Σ is tight. If $x = e$ the converse is also true [10]. However, in general this is not the case. For instance, for $n = 5$ and the smooth pair $\Sigma = (35142, 21345)$ we have

$$\mathcal{E}(w) = \{(1,3), (3,1), (3,3)\} \not\subset L_\Sigma \cap \square_n = \{(1,1), (1,3), (1,4), (3,1), (4,1)\}.$$

This is one of the reasons why the present proof of Theorem 2.2 is technically more complicated than that of [10, Theorem 4.2].

2.4. First step

Let $w \in S_n$. For any subsets $A, B \subset \square_n$ define

$$C_A^w(B) = B \cup \{p \in \square_n : \exists p' \in A \text{ such that } p \perp p', \Delta_w(p,p') = 0 \text{ and } \mathcal{C}(p,p') \subset B\}.$$

The following simple lemma will be crucial for the argument.

Lemma 2.5. *Let $w, y \in S_n$ and $A, B \subset \square_n$. Suppose that $r_y\big|_A \equiv r_w\big|_A$ and $r_y \ge r_w$ on B. Then $r_y \ge r_w$ on $C_A^w(B)$.*

Proof. Let $p \in C_A^w(B)$. If $p \in B$ there is nothing to prove. Otherwise, let $p' \in A$ be such that $p' \perp p$, $\Delta_w(p,p') = 0$ and $\mathcal{C}(p,p') \subset B$. Then,

$$r_y(p) + r_y(p') \ge \sum_{q \in \mathcal{C}(p,p')} r_y(q) \ge \sum_{q \in \mathcal{C}(p,p')} r_w(q) = r_w(p) + r_w(p') = r_w(p) + r_y(p').$$

Hence, $r_y(p) \ge r_w(p)$ as required. $\square$

For any subset $A \subset \square_n$ we define inductively $A_0 = A$ and $A_k = C_A^w(A_{k-1})$, $k > 0$. The set $\mathcal{I}^w(A) = \cup_{k \ge 0} A_k$ is called the *influence set* of A.

Clearly, if $A' \supset A$ then $\mathcal{I}^w(A') \supset \mathcal{I}^w(A)$.

Corollary 2.6. *Let $w \in S_n$. For any $y \in S_n$ we have $r_y \ge r_w$ on*

$$\mathcal{I}^w(\{(i,j) \in \square_n : r_w(i,j) = r_y(i,j)\}).$$

Definition 2.7. We say that $\Sigma = (w,x) \in \mathfrak{P}_n$ is *influential* if $\mathcal{I}^w(L_\Sigma) = \square_n$.

By Corollary 2.6, every influential pair is a tight pair.

We will in fact prove the following stronger version of Theorem 2.2.

Theorem 2.8. *Every smooth pair is influential.*

2.5. Statement of main technical result

In order to prove Theorem 2.8 we introduce more notation and terminology.

Definition 2.9. Let $\Sigma = (w,x) \in \mathfrak{P}_n$ and $p = (i,j) \in \square_n$.

(1) Define $p_\Sigma^\downarrow = (i',j) \in \square_n$ by

$$\begin{aligned} i' &= \max\{l \ge i : (l,j') \in L_\Sigma \text{ and } \Delta_w(p,(l,j')) = 0\} \\ &\quad \text{where } j' = \min\{k > j : (i,k) \in L_\Sigma\}. \end{aligned}$$

Similarly, define $p_\Sigma^\uparrow = (i', j) \in \square_n$ by

$$i' = \min\{l \le i : (l, j') \in L_\Sigma \text{ and } \Delta_w(p, (l, j')) = 0\}$$
$$\text{where } j' = \max\{k < j : (i, k) \in L_\Sigma\}.$$

(2) We say that the condition $\mathcal{P}_\Sigma^\downarrow(p)$ (resp., $\mathcal{P}_\Sigma^\uparrow(p)$) is satisfied if $r_\Sigma(p_\Sigma^\downarrow) < r_\Sigma(p)$ (resp., $r_\Sigma(p_\Sigma^\uparrow) < r_\Sigma(p)$).

(3) We write

$$\mathcal{Q}_\Sigma(p) = \mathcal{P}_\Sigma^\downarrow(p) \vee \mathcal{P}_\Sigma^\uparrow(p) \vee \mathcal{P}_\Sigma^\uparrow(p_\Sigma^\downarrow) \vee \mathcal{P}_\Sigma^\downarrow(p_\Sigma^\uparrow).$$

Thus, by definition the condition $\mathcal{Q}_\Sigma(p)$ holds when at least one of the conditions $\mathcal{P}_\Sigma^\downarrow(p)$, $\mathcal{P}_\Sigma^\uparrow(p)$, $\mathcal{P}_\Sigma^\uparrow(p_\Sigma^\downarrow)$ or $\mathcal{P}_\Sigma^\downarrow(p_\Sigma^\uparrow)$ is satisfied.

Note that the conditions $\mathcal{P}_\Sigma^\downarrow(p)$ and $\mathcal{P}_{\Sigma^*}^\uparrow(p^*)$ are equivalent, as are the conditions $\mathcal{Q}_\Sigma(p)$ and $\mathcal{Q}_{\Sigma^*}(p^*)$. However, the latter are not equivalent to $\mathcal{Q}_{\Sigma^{-1}}(p^{-1})$.

The key technical result is the following.

Proposition 2.10. *Suppose that $\Sigma = (w, x) \in \mathfrak{P}_n$ is a smooth pair. Then $\mathcal{Q}_\Sigma(p)$ holds for any $p \in \tilde{L}_\Sigma$.*

We will prove Proposition 2.10 in sections 3 and 4 below. Before that, let us explain how Proposition 2.10 implies Theorem 2.8 (and hence Theorem 2.2).

Let $\Sigma = (w, x) \in \mathfrak{P}_n$ be a smooth pair and let

$$L_\Sigma^{\le i} = \{p \in \square_n : r_\Sigma(p) \le i\}, \quad i = 0, 1, \dots, n.$$

Clearly, $L_\Sigma^{\le 0} = L_\Sigma$ and $L_\Sigma^{\le n} = \square_n$. We show by induction on i that $L_\Sigma^{\le i} \subset \mathcal{I}^w(L_\Sigma)$. For the induction step, let $p \in \square_n$ with $r_\Sigma(p) = i > 0$. By Proposition 2.10 $\mathcal{Q}_\Sigma(p)$ holds. If $\mathcal{P}_\Sigma^\downarrow(p)$ or $\mathcal{P}_\Sigma^\uparrow(p)$ is satisfied then it immediately follows from the definitions that $p \in C_{L_\Sigma}^w(L_\Sigma^{\le i-1})$. Otherwise, $\mathcal{P}_\Sigma^\uparrow(p_\Sigma^\downarrow)$ or $\mathcal{P}_\Sigma^\downarrow(p_\Sigma^\uparrow)$ holds and then respectively $p_\Sigma^\downarrow \in C_{L_\Sigma}^w(L_\Sigma^{\le i-1})$ or $p_\Sigma^\uparrow \in C_{L_\Sigma}^w(L_\Sigma^{\le i-1})$. Either way $p \in C_{L_\Sigma}^w(C_{L_\Sigma}^w(L_\Sigma^{\le i-1}))$ and the induction hypothesis implies that $p \in \mathcal{I}^w(L_\Sigma)$ as required.

We end this section by commenting on the proof of Proposition 2.10. We first consider in §3 an important special case for which a stronger statement holds. It is proved by induction on $\ell(\Sigma)$. It is here that we use the smoothness of Σ in a crucial way. The general case is proved in §4, also by induction on $\ell(\Sigma)$. The induction step is facilitated by the aforementioned special case. However, the smoothness assumption is no longer used directly – only through the induction hypothesis and the appeal to the special case. Unfortunately, the argument is complicated by the four-headed and asymmetric nature of property $\mathcal{Q}_\Sigma(p)$, which leads to a lengthy case-by-case analysis.

3. Basic case

In this section we carry out the first step of the proof of Proposition 2.10.

3.1. Statement

Definition 3.1. For any $\Sigma = (w, x) \in \mathfrak{P}_n$ and $p = (i, j) \in \tilde{L}_\Sigma$ let

$$p_\Sigma^{\searrow} = (\min\{l > i : (l, j) \in L_\Sigma\}, \min\{l > j : (i, l) \in L_\Sigma\}) > p,$$
$$p_\Sigma^{\nwarrow} = (\max\{l < i : (l, j) \in L_\Sigma\}, \max\{l < j : (i, l) \in L_\Sigma\}) < p,$$

so that $\mathcal{C}(p, p_\Sigma^{\searrow}), \mathcal{C}(p, p_\Sigma^{\nwarrow}) \subset L_\Sigma$. We say that $\mathcal{P}_\Sigma^{\searrow}(p)$ (resp., $\mathcal{P}_\Sigma^{\nwarrow}(p)$) is satisfied if

$$\Delta_x(p, p_\Sigma^{\searrow}) = 1 \text{ (resp., } \Delta_x(p, p_\Sigma^{\nwarrow}) = 1). \tag{8}$$

Finally, we write

$$\mathcal{P}_\Sigma(p) = \mathcal{P}_\Sigma^{\searrow}(p) \vee \mathcal{P}_\Sigma^{\nwarrow}(p).$$

The condition $\mathcal{P}_\Sigma^{\searrow}(p)$ turns out to be stronger than the condition $\mathcal{P}_\Sigma^{\downarrow}(p)$ defined in §2.5 (see Lemma 4.2 below). At any rate, we will not use the notation of §2.5 in this section.

Let $p \in \tilde{L}_\Sigma$ and $p' = p_\Sigma^{\searrow}$. Since $\mathcal{C}(p, p') \subset L_\Sigma$, we have

$$\Delta_x(p, p') = \Delta_w(p, p') + r_\Sigma(p) + r_\Sigma(p').$$

Thus,

$$\mathcal{P}_\Sigma^{\searrow}(p) \text{ holds if and only if } r_\Sigma(p) = 1,\ p_\Sigma^{\searrow} \in L_\Sigma \text{ and } \Delta_w(p, p_\Sigma^{\searrow}) = 0. \tag{9}$$

Similarly for $\mathcal{P}_\Sigma^{\nwarrow}(p)$ where $p_\Sigma^{\searrow}$ is replaced by $p_\Sigma^{\nwarrow}$.

Note that the conditions $\mathcal{P}_\Sigma^{\searrow}(p)$, $\mathcal{P}_{\Sigma^*}^{\nwarrow}(p^*)$ and $\mathcal{P}_{\Sigma^{-1}}^{\searrow}(p^{-1})$ are equivalent.

Consider the *critical set*

$$\mathfrak{C}_\Sigma = \bigcup_{t \in \mathcal{R}_\Sigma} (L_{\Sigma_t} \setminus L_{(xt,x)}) = \bigcup_{t \in \mathcal{R}_\Sigma} L_{\Sigma_t} \setminus L_\Sigma. \tag{10}$$

Note that if $p \in \mathfrak{C}_\Sigma$ then $r_\Sigma(p) = 1$. (However, the converse is not true in general, even if Σ is smooth.) Also, $\mathfrak{C}_{\Sigma^*} = \mathfrak{C}_\Sigma^*$ and $\mathfrak{C}_{\Sigma^{-1}} = \mathfrak{C}_\Sigma^{-1}$.

The main result of this section is the following.

Proposition 3.2. *Let $\Sigma = (w, x) \in \mathfrak{P}_n$ be a smooth pair. Then for any $p \in \mathfrak{C}_\Sigma$:*

(1) *$\mathcal{P}_\Sigma(p)$ is satisfied.*

(2) *If* $\Delta_w(p, p_\Sigma^{\searrow}) = 0$ *then* $\mathcal{P}_\Sigma^{\searrow}(p)$ *holds (or, equivalently,* $p_\Sigma^{\searrow} \in L_\Sigma$*).*
(3) *Similarly, if* $\Delta_w(p, p_\Sigma^{\nwarrow}) = 0$ *then* $\mathcal{P}_\Sigma^{\nwarrow}(p)$ *holds (or, equivalently,* $p_\Sigma^{\nwarrow} \in L_\Sigma$*).*

In the rest of the section we will prove Proposition 3.2 by induction on $\ell(\Sigma)$. The induction step reduces the statement to a special case which will be examined directly. This reduction step uses the smoothness in a crucial way. We first explain a purely formal reduction step.

3.2. Preliminary reduction

We record some notation and results from [4, §5] adjusted to the setup at hand.
Recall the following standard result (cf. proof of [2, Theorem 2.1.5]).

$$\text{Suppose that } \Sigma = (w, x) \in \mathfrak{P}_n \text{ and } \underline{p}_x(i) \in \tilde{L}_\Sigma. \text{ Then } \exists i' \text{ such that } t_{i,i'} \in \mathcal{R}_\Sigma. \tag{11}$$

For any subset $I \subset \{1, \dots, n\}$ of size m let $\sigma_I : \{1, \dots, m\} \to I$ be the monotone bijection and define $\pi_I : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\}$ by $\pi_I(i) = \max\{j : \sigma_I(j) \le i\}$ (with $\max \varnothing = 0$), a weakly monotone function.
Fix $\Sigma = (w, x) \in \mathfrak{P}_n$ and let

$$I = \{i = 1, \dots, n : \exists i' \ne i \text{ such that } t_{\{i,i'\}} \in \mathcal{R}_\Sigma\}$$

and $m = \#I$. We say that Σ is *reduced* if $I = \{1, \dots, n\}$.
By [4, Proposition 14] and (11) we have $\underline{p}_x(i) = \underline{p}_w(i) \in L_\Sigma$ for all $i \notin I$ and in particular $J := x(I) = w(I)$. Let $\overline{\Sigma} = (\pi_J \circ w \circ \sigma_I, \pi_J \circ x \circ \sigma_I) \in S_m \times S_m$ be the *reduction* of Σ. For any $t \in \mathcal{R}_n$ let $\overline{t} \in \mathcal{R}_m \cup \{e\}$ be given by $\overline{t_{i,j}} = t_{\pi_I(i),\pi_I(j)}$ if $\pi_I(i) \ne \pi_I(j)$ and $\overline{t_{i,j}} = e$ otherwise. Finally, for any $p = (i, j) \in \square_n$ we write $\overline{p} = (\pi_I(i), \pi_J(j)) \in \square_m$. (Of course, $\overline{t}$ and $\overline{p}$ depend implicitly on Σ.)
The following elementary result is essentially in [4, §5]. We omit the details.

Lemma 3.3. *Let* $\Sigma = (w, x) \in \mathfrak{P}_n$. *Then*

(1) $\overline{\Sigma} \in \mathfrak{P}_m$.
(2) $\ell(\overline{\Sigma}) = \ell(\Sigma)$.
(3) *For any* $p \in \square_n$ *we have* $r_\Sigma(p) = r_{\overline{\Sigma}}(\overline{p})$. *In particular,* $p \in L_\Sigma$ *if and only if* $\overline{p} \in L_{\overline{\Sigma}}$.
(4) *For any* $p_1, p_2 \in \square_n$ *with* $p_1 \le p_2$ *we have* $\Delta_w(p_1, p_2) \ge \Delta_{\pi_J \circ w \circ \sigma_I}(\overline{p}_1, \overline{p}_2)$ *and* $\Delta_\Sigma(p_1, p_2) = \Delta_{\overline{\Sigma}}(\overline{p}_1, \overline{p}_2)$.
(5) *The map* $t \mapsto \overline{t}$ *defines a bijection between* $\mathcal{R}_\Sigma$ *and* $\mathcal{R}_{\overline{\Sigma}}$.
(6) *For any* $t \in \mathcal{R}_\Sigma$ *and* $p \in \square_n$ *we have* $p \in L_{\Sigma_t}$ *if and only if* $\overline{p} \in L_{\overline{\Sigma}_{\overline{t}}}$.
(7) $\overline{\Sigma}$ *is smooth if and only if* Σ *is smooth.*
(8) $\overline{\Sigma}$ *is reduced.*
(9) $p \in \mathfrak{C}_\Sigma$ *if and only if* $\overline{p} \in \mathfrak{C}_{\overline{\Sigma}}$.

(10) *For any* $p \in \tilde{L}_\Sigma$ *we have* $\overline{p}^{\searrow}_{\overline{\Sigma}} = \overline{p^{\searrow}_{\Sigma}}$ *and* $\overline{p}^{\nwarrow}_{\overline{\Sigma}} = \overline{p^{\nwarrow}_{\Sigma}}$.

Lemma 3.4. *Let* $\Sigma = (w, x) \in \mathfrak{P}_n$ *and* $p \in \tilde{L}_\Sigma$ *with* $r_\Sigma(p) = 1$. *Then*

$$\mathcal{P}^{\searrow}_{\Sigma}(p) \iff \mathcal{P}^{\searrow}_{\overline{\Sigma}}(\overline{p}), \quad \mathcal{P}^{\nwarrow}_{\Sigma}(p) \iff \mathcal{P}^{\nwarrow}_{\overline{\Sigma}}(\overline{p})$$
$$\Delta_w(p, p^{\searrow}_{\Sigma}) = 0 \iff \Delta_w(\overline{p}, \overline{p}^{\searrow}_{\overline{\Sigma}}) = 0, \quad \Delta_w(p, p^{\nwarrow}_{\Sigma}) = 0 \iff \Delta_w(\overline{p}, \overline{p}^{\nwarrow}_{\overline{\Sigma}}) = 0.$$

Thus Proposition 3.2 holds for $\overline{\Sigma}$ *and* $\overline{p}$ *if and only if it holds for* Σ *and* p.

Proof. Let $p = (i, j)$, $p' = p^{\searrow}_{\Sigma} = (i', j')$, $\overline{\Sigma} = (w', x')$. Suppose that $\Delta_{w'}(\overline{p}, \overline{p}') = 0$ but $\Delta_w(p, p') > 0$. Then $\varnothing \neq \Gamma_w \cap (p, p'] \subset \Gamma_x \cap L_\Sigma$. Let $q = \underline{p}_w(i_0) \in (p, p']$ (with $i_0 \notin I$). Then $\Delta_w(p, q) \le \Delta_x(p, q)$, i.e., $\Delta_\Sigma(p, q) \ge 0$. On the other hand, $r_\Sigma(p) = 1$ and $q \in L_\Sigma$. Hence $\mathcal{C}(p, q) \cap L_\Sigma \neq \varnothing$. However, if say $(i_0, j) \in L_\Sigma$ then $(i_0 - 1, j) \in L_\Sigma$ and hence by the definition of i' we would have $i' < i_0$ in contradiction to the fact that $q \in (p, p']$. By (9) we also infer the equivalence of $\mathcal{P}^{\searrow}_{\Sigma}(p)$ and $\mathcal{P}^{\searrow}_{\overline{\Sigma}}(\overline{p})$. By symmetry we get the other equivalences. □

Thus, it suffices to prove Proposition 3.2 in the case where Σ is reduced (and smooth). We make the following simple remark.

$$\text{If } \Sigma \text{ is reduced and } x \text{ is monotone decreasing for } i < i_1 \text{ then } w(i) > x(i) \text{ for all } i < i_1. \tag{12}$$

Otherwise, if $w(i) \le x(i)$ then $r_w(\underline{p}_w(i)) \ge 1 \ge r_x(\underline{p}_w(i))$ by the assumption on x. Hence $w(i) = x(i)$ and $\underline{p}_w(i) \in L_\Sigma$. But then $t_{i,i'} \notin \mathcal{R}_\Sigma$ for all i' by (5). Since also $t_{i',i} \notin \mathcal{R}_\Sigma$ for all i' (by the assumption on x) Σ cannot be reduced.

3.3. Gasharov's map

Let $\Sigma = (w, x) \in \mathfrak{P}_n$ and $t' \in \mathcal{R}_\Sigma$. In [9] Gasharov introduced an injective map

$$\phi^{\Sigma}_{t'} : \mathcal{R}_{\Sigma_{t'}} \to \mathcal{R}_\Sigma \setminus \{t'\}.$$

We will use the following variant of [4][3]:

$$\phi^{\Sigma}_{t'}(t) = \begin{cases} t^{t'} & \text{if } t^{t'} \in \mathcal{R}_\Sigma, \\ t & \text{otherwise.} \end{cases}$$

The map $\phi^{\Sigma}_{t'}$ is well-defined and injective ([4, Theorem 20]). Taking $t' \in \mathcal{R}^{\triangleleft}_{\Sigma}$, this gives a combinatorial inductive proof of the inequality (7). It also yields that

[3] In [4] this is defined on a case-by-case basis, but it amounts to the same formula as given here.

$$\Sigma \text{ is smooth} \iff \phi_{t'}^{\Sigma} \text{ is surjective and } \Sigma_{t'} \text{ is smooth.}$$

Hence, if Σ is smooth then (w, x') is smooth for any $x \le x' \le w$. (Again, this fact is also clear from the geometric characterization.)

Corollary 3.5. *Suppose that $\Sigma = (w, x)$ is smooth, $t' \in \mathcal{R}_{\Sigma}^{\triangleleft}$ and $t \in \mathcal{R}_{\Sigma}$ with $t \neq t'$. Then at least one of t or $t^{t'}$ is in $\mathcal{R}_{\Sigma_{t'}}$.*

3.4. Main reduction

Definition 3.6. Let $\Sigma = (w, x) \in \mathfrak{P}_n$ and $t = t_{i_1,i_2} \in \mathcal{R}_{\Sigma}$. We say that Σ is *t-minimal* if for every $t_{i'_1,i'_2} \in \mathcal{R}_{\Sigma}$ we have $i_1 = i'_1$ or $i'_2 = i_2$. We say that Σ is minimal if it is t-minimal for some $t \in \mathcal{R}_{\Sigma}$.

Note that it is not excluded that Σ is both t_1-minimal and t_2-minimal for two different $t_1, t_2 \in \mathcal{R}_{\Sigma}$.

Also note that by Lemma 3.3 with its notation, Σ is t-minimal if and only if $\overline{\Sigma}$ is $\overline{t}$-minimal. In particular, Σ is minimal if and only if $\overline{\Sigma}$ is.

Definition 3.7. Let $\Sigma = (w, x) \in \mathfrak{P}_n$ be a smooth pair. For any $p \in \tilde{L}_{\Sigma}$ let

$$\mathcal{R}_{\Sigma}(p) = \{t \in \mathcal{R}_{\Sigma} : p \in L_{\Sigma_t}\}$$

so that $p \in \mathfrak{C}_{\Sigma}$ if and only if $\mathcal{R}_{\Sigma}(p) \neq \varnothing$. Let

$$\mathfrak{C}_{\Sigma}^{r} = \{p \in \mathfrak{C}_{\Sigma} : \Sigma \text{ is } t\text{-minimal for every } t \in \mathcal{R}_{\Sigma}(p)\}.$$

Note that by Lemma 3.3, given $p \in \tilde{L}_{\Sigma}$ and $t \in \mathcal{R}_{\Sigma}$ we have $t \in \mathcal{R}_{\Sigma}(p)$ if and only if $\overline{t} \in \mathcal{R}_{\overline{p}}(\overline{\Sigma})$. Hence,

$$p \in \mathfrak{C}_{\Sigma}^{r} \text{ if and only if } \overline{p} \in \mathfrak{C}_{\overline{\Sigma}}^{r}. \tag{13}$$

We will prove below that

$$\text{Proposition 3.2 holds for every } \Sigma = (w, x) \in \mathfrak{P}_n \text{ and } p \in \mathfrak{C}_{\Sigma}^{r}. \tag{14}$$

Assuming this for the moment, let us prove Proposition 3.2 by induction on $\ell(\Sigma)$.

Proof of Proposition 3.2. Let $\Sigma = (w, x) \in \mathfrak{P}_n$ be a smooth pair and $p \in \mathfrak{C}_{\Sigma}$. If $p \in \mathfrak{C}_{\Sigma}^{r}$ then Proposition 3.2 holds by assumption. Otherwise, there exists $t = t_{i_1,i_2} \in \mathcal{R}_{\Sigma}(p)$ and $t' = t_{i'_1,i'_2} \in \mathcal{R}_{\Sigma}$ such that $i'_1 \neq i_1$ and $i'_2 \neq i_2$.

Assume first that either $i'_1 = i_2$ or $i'_2 = i_1$. Upon upending Σ, p, t and t' if necessary we may assume without loss of generality that $i'_1 = i_2$. Also, by changing t' if necessary (without changing i'_1), we may assume that $t' \in \mathcal{R}^{\lhd}_{\Sigma}$. Note that by our condition on t, t' and p we have $p^{\searrow}_{\Sigma_{t'}} = p^{\searrow}_{\Sigma}$ and $p^{\nwarrow}_{\Sigma_{t'}} = p^{\nwarrow}_{\Sigma}$. In particular,

$$\Delta_w(p, p^{\searrow}_{\Sigma_{t'}}) = 0 \iff \Delta_w(p, p^{\searrow}_{\Sigma}) = 0 \text{ and } \Delta_w(p, p^{\nwarrow}_{\Sigma_{t'}}) = 0 \iff \Delta_w(p, p^{\nwarrow}_{\Sigma}) = 0.$$

Also, the conditions $\mathcal{P}^{\nwarrow}_{\Sigma_{t'}}(p)$ and $\mathcal{P}^{\nwarrow}_{\Sigma}(p)$ are clearly equivalent. Thus, to conclude Proposition 3.2 for (Σ, p) we will show that by the induction hypothesis Proposition 3.2 holds for $(\Sigma_{t'}, p)$ and that the conditions $\mathcal{P}^{\searrow}_{\Sigma_{t'}}(p)$ and $\mathcal{P}^{\searrow}_{\Sigma}(p)$ are equivalent. It is at this point that we use the smoothness of Σ. Namely, by Corollary 3.5 we have $t \in \mathcal{R}_{\Sigma_{t'}}$ or $t^{t'} \in \mathcal{R}_{\Sigma_{t'}}$. Upon inverting Σ and p and conjugating t and t' by x if necessary we may assume without loss of generality that $t \in \mathcal{R}_{\Sigma_{t'}}$. (Note that $t^{t'} \in \mathcal{R}_{\Sigma_{t'}}$ if and only if $t^x \in \mathcal{R}_{(\Sigma_{t'})^{-1}} = \mathcal{R}_{(\Sigma^{-1})_{t'^x}}$.) Clearly, $p \in L_{(\Sigma_{t'})_t} \setminus L_{\Sigma_{t'}}$. Thus, we may apply the induction hypothesis to $\Sigma_{t'}$. Let $p' = p^{\searrow}_{\Sigma} = p^{\searrow}_{\Sigma_{t'}} = (i', j')$. Since $t \in \mathcal{R}_{\Sigma_{t'}}$ we must have $j' \geq x(i'_2)$. Hence, $p' \in L_{(xt',x)}$ and thus $p' \in L_{\Sigma_{t'}} \iff p' \in L_{\Sigma}$. By (9) the conditions $\mathcal{P}^{\searrow}_{\Sigma_{t'}}(p)$ and $\mathcal{P}^{\searrow}_{\Sigma}(p)$ are equivalent as required.

By (11) and the case considered above, we may assume for the rest of the proof that $q := \underline{p}_x(i_2) \in L_{\Sigma}$ (and symmetrically $\underline{p}_x(i_1) - (1,1) \in L_{\Sigma}$).

We now show that $\Delta_w(p, p^{\searrow}_{\Sigma}) = 0$ implies $\mathcal{P}^{\searrow}_{\Sigma}(p)$. In fact, we show that $\Delta_w(p, q) = 0$ implies $\mathcal{P}^{\searrow}_{\Sigma}(p)$. (Note that $q \leq p^{\searrow}_{\Sigma}$.) Indeed, if $\Delta_w(p, q) = 0$ then by (4) (applied to Σ_t) $\mathcal{C}(p, q) \subset L_{\Sigma_t} \cap L_{(xt,x)} = L_{\Sigma}$ and $\Delta_x(p, q) = 1$. Hence, from the definition, $p^{\searrow}_{\Sigma} \leq q$. Thus, $p^{\searrow}_{\Sigma} = q$ and $\mathcal{P}^{\searrow}_{\Sigma}(p)$ is satisfied.

By a similar reasoning (or by symmetry) $\Delta_w(p, p^{\nwarrow}_{\Sigma}) = 0$ implies $\mathcal{P}^{\nwarrow}_{\Sigma}(p)$.

It remains to show that $\mathcal{P}_{\Sigma}(p)$ is satisfied in the case where t and t' commute (and $t' \neq t$). Once again, by changing t' if necessary, and using the previous case, we may assume that $t' \in \mathcal{R}^{\lhd}_{\Sigma}$. Then by Corollary 3.5 $t \in \mathcal{R}_{\Sigma_{t'}}$. Note that $\tilde{L}_{(xt,x)} = \tilde{L}_{(xt't,xt')}$ since t and t' commute. Hence

$$p \in L_{\Sigma_t} \cap \tilde{L}_{(xt,x)} \subset L_{(\Sigma_{t'})_t} \cap \tilde{L}_{(xt't,xt')} = L_{(\Sigma_{t'})_t} \setminus L_{\Sigma_{t'}}.$$

Thus, by induction hypothesis $\mathcal{P}_{\Sigma_{t'}}(p)$ holds. By symmetry, we can assume without loss of generality that $\mathcal{P}^{\searrow}_{\Sigma_{t'}}(p)$ is satisfied. Hence, $\Delta_w(p, p^{\searrow}_{\Sigma_{t'}}) = 0$. On the other hand, since $t \in \mathcal{R}_{\Sigma_{t'}}$ we have $q \leq p^{\searrow}_{\Sigma_{t'}}$. Therefore $\Delta_w(p, q) = 0$. By the previous paragraph this implies $\mathcal{P}^{\searrow}_{\Sigma}(p)$. $\square$

3.5. Minimal reduced case

It remains to consider the minimal reduced case, which we can explicate as follows.

Lemma 3.8. *Suppose that $\Sigma = (w, x) \in \mathfrak{P}_n$ is reduced and t-minimal. Write $t = t_{i_1,i_2}$ and $t^x = t_{j_1,j_2}$. Then $j_2 - j_1 + i_2 - i_1 \geq n$ and Σ is given by*

$$w(i) = \begin{cases} j_2 + 1 - i & i < i_1, \\ n + i_1 - i & i_1 \le i < n + i_1 - j_2, \\ n + 1 - i & n + i_1 - j_2 \le i \le i_2 - j_1 + 1, \\ i_2 + 1 - i & i_2 - j_1 + 1 < i \le i_2, \\ n + j_1 - i & i > i_2, \end{cases} \tag{15a}$$

$$x(i) = \begin{cases} j_2 - i & i < i_1, \\ j_1 & i = i_1, \\ n + i_1 + 1 - i & i_1 < i \le n + i_1 - j_2, \\ n + 1 - i & n + i_1 - j_2 < i \le i_2 - j_1, \\ i_2 - i & i_2 - j_1 < i < i_2, \\ j_2 & i = i_2, \\ n + j_1 + 1 - i & i > i_2. \end{cases} \tag{15b}$$

Proof. Let $t = t_{i_1,i_2}$. We proceed with the following steps, first dealing with x.

(1) $t_{i,i_2} \in \mathcal{R}_\Sigma$ for every $i < i_1$. In particular, $x(i) < j_2$ for all $i < i_1$.
Since Σ is t-minimal and $i < i_1$ we cannot have $t_{i',i} \in \mathcal{R}_\Sigma$ for any i'. Therefore, since Σ is reduced, there exists i' such that $t_{i,i'} \in \mathcal{R}_\Sigma$. Since Σ is t-minimal, $i' = i_2$.

(2) $x(i) > x(i+1)$ for all $i < i_1$.
Otherwise, the relation $t_{i,i_2} \in \mathcal{R}_\Sigma$ would imply $t_{i,i+1} \in \mathcal{R}_\Sigma$, contradicting the assumption on Σ.

(3) In a similar vein, for all $i > i_2$ we have $t_{i_1,i} \in \mathcal{R}_\Sigma$, $j_1 < x(i)$ and $x(i) < x(i-1)$.

(4) For any $i < i_1$ and $i' > i_2$ we have $x(i) > x(i')$.
Otherwise, the relations $t_{i,i_2}, t_{i_1,i'} \in \mathcal{R}_\Sigma$ would give $t_{i,i'} \in \mathcal{R}_\Sigma$ by (6) which denies the assumption on Σ.

(5) By passing to Σ^{-1} we similarly have $t_{x^{-1}(j),i_2} \in \mathcal{R}_\Sigma$ and $x^{-1}(j) > x^{-1}(j+1)$ for all $j < j_1$ while $t_{i_1,x^{-1}(j)} \in \mathcal{R}_\Sigma$ and $x^{-1}(j) < x^{-1}(j-1)$ for all $j > j_2$. Moreover, $x^{-1}(j) > x^{-1}(j')$ for all $j < j_1$ and $j' > j_2$.

(6) Suppose that $i_1 < i < i_2$ and $j_1 < x(i) < j_2$. Then
   (a) $x(i') > x(i)$ for all $i' < i_1$.
   Otherwise, $t_{i',i} \in \mathcal{R}_\Sigma$ (since $t_{i',i_2} \in \mathcal{R}_\Sigma$) in violation of the t-minimality of Σ.
   (b) Similarly, $x(i') < x(i)$ for all $i' > i_2$ and $x^{-1}(j') < i$ (resp., $x^{-1}(j') > i$) for all $j' > j_2$ (resp., $j' < j_1$).

(7) Suppose that $i_1 < i < i' < i_2$ and $j_1 < x(i), x(i') < j_2$. Then $x(i) > x(i')$.
Otherwise $t_{i,i'} \in \mathcal{R}_\Sigma$, refuting the t-minimality of Σ.

In conclusion, $n + i_1 - j_2 \le i_2 - j_1$ and x is given by (15b)

Next we deal with w.

(1) By (12) $w(i) > x(i) = j_2 - i$ for all $i < i_1$.

(2) $r_\Sigma(i_1-1,j)>0$ for all $j_2-i_1+1\le j<j_2$.
Indeed, $r_x(i_1-1,j)=j-j_2+i_1$ while $r_w(i_1-1,j)<j-j_2+i_1$ by the previous part.
(3) $r_w(i_1-1,j_2)=i_1-1$.
Otherwise, we would have $r_\Sigma(i_1-1,j_2)>0$. In view of the previous part and the fact that t_{i_1-1,i_2} and $t_{i_1,n+i_1-j_2}$ are in $\mathcal{R}_\Sigma$, this would imply that $t_{i_1-1,n+i_1-j_2}\in\mathcal{R}_\Sigma$, rebutting the t-minimality of Σ.
(4) We conclude that $w(i)=x(i)+1$ for all $i<i_1$.
(5) By symmetry $w(i)=x(i)-1$ for all $i>i_2$, $w^{-1}(j)=x^{-1}(j)+1$ for all $j<j_1$ and $w^{-1}(j)=x^{-1}(j)-1$ for all $j>j_2$.
(6) $w(n+i_1-j_2)=j_2-i_1+1$.
Let $j=w(n+i_1-j_2)$. We already have $j_1\le j\le j_2-i_1+1$. If $j\le j_2-i_1$ then $r_\Sigma(n+i_1-j_2,j)=0$, gainsaying that $t_{i_1,i_2}\in\mathcal{R}_\Sigma$.
(7) Similarly $w(i_2-j_1+1)=n-i_2+j_1$.
(8) $w(i)=x(i)$ for all $n+i_1-j_2<i\le i_2-j_1$.
This follows now again from the fact that $t_{i_1,i_2}\in\mathcal{R}_\Sigma$.

The result follows. $\square$

The following assertion is straightforward.

Lemma 3.9. *Let $1\le i_1<i_2\le n$ and $1\le j_1<j_2\le n$ be such that $j_2-j_1+i_2-i_1\ge n$ and let Σ be given by (15a), (15b).*[4] *Then*

(1) *If $i_1=j_1=1$ and $i_2+j_2=n+2$ then $w(i)=n+1-i$ for all i and $x(i)=n+2-i$ for all $i>1$. We have*

$$\mathfrak{C}^r_\Sigma=\{(i,j)\in\square_n : i+j\le n\}$$

and for any $p=(i,j)\in\mathfrak{C}^r_\Sigma$
(a) $p^{\nwarrow}_\Sigma=(0,0)$.
(b) $\mathcal{P}^{\nwarrow}_\Sigma(p)$ *is satisfied.*
(c) $p^{\searrow}_\Sigma=(n+1-j,n+1-i)$.
(d) $\mathcal{P}^{\searrow}_\Sigma(p)\iff\Delta_w(p,p^{\searrow}_\Sigma)=0\iff i+j=n$.
(2) *Similarly, if $i_2=j_2=n$ and $i_1+j_1=n$ then $w(i)=n+1-i$ for all i and $x(i)=n-i$ for all $i<n$. We have*

$$\mathfrak{C}^r_\Sigma=\{(i,j)\in\square_n : n\le i+j\}$$

and for any $p=(i,j)\in\mathfrak{C}^r_\Sigma$
(a) $p^{\searrow}_\Sigma=(n,n)$.

[4] It can be easily shown that Σ is smooth, reduced and t_{i_1,i_2}-minimal. We will not use this fact explicitly.

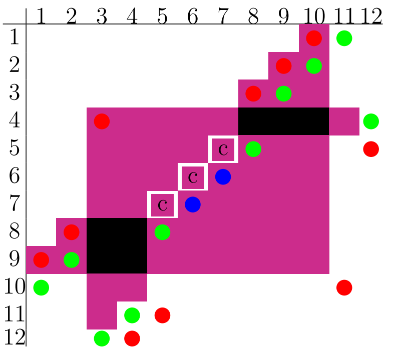
(b) $\mathcal{P}_{\Sigma}^{\searrow}(p)$ *is satisfied.*
(c) $p_{\Sigma}^{\nwarrow} = (n-1-j, n-1-i)$.
(d) $\mathcal{P}_{\Sigma}^{\nwarrow}(p) \iff \Delta_w(p, p_{\Sigma}^{\nwarrow}) = 0 \iff i + j = n$.

(3) *In all other cases*

$$\mathfrak{C}_{\Sigma}^{r} = \{(i, n-i) : n + i_1 - j_2 \le i \le i_2 - j_1\}$$

and for any $p \in \mathfrak{C}_{\Sigma}^{r}$ *we have* $p_{\Sigma}^{\searrow} = (i_2, j_2)$, $p_{\Sigma}^{\nwarrow} = (i_1 - 1, j_1 - 1)$ *and both conditions* $\mathcal{P}_{\Sigma}^{\searrow}(p)$ *and* $\mathcal{P}_{\Sigma}^{\nwarrow}(p)$ *are satisfied.*

Example 3.10. In the diagram below we draw the example for $n = 12$, $i_1 = 4$, $j_1 = 3$, $i_2 = 10$, $j_2 = 11$. The green (resp., red, blue) dots represent the points of $\Gamma_w \setminus \Gamma_x$ (resp., $\Gamma_x \setminus \Gamma_w$, $\Gamma_x \cap \Gamma_w$). The background color of a point p is white (resp., magenta, black) if $r_{\Sigma}(p) = 0$ (resp., 1, 2). The points in $\mathfrak{C}_{\Sigma}^{r}$ are the framed boxes denoted by "c".



We now deduce assertion (14) to complete the proof of Proposition 3.2. By Lemma 3.4 and (13) it is enough to prove (14) in the case where Σ is reduced. This case follows from Lemmas 3.8 and 3.9.

4. Proof of Proposition 2.10

4.1. Notation and auxiliary results

We now go back to the conditions $\mathcal{P}_{\Sigma}^{\downarrow}(p)$, $\mathcal{P}_{\Sigma}^{\uparrow}(p)$, $\mathcal{Q}_{\Sigma}(p)$ defined in §2.5 and set some more notation.

Let $\Sigma = (w, x) \in \mathfrak{P}_n$ and $p = (i, j) \in \square_n$. Recall that $p_{\Sigma}^{\downarrow} = (i', j) \in \blacksquare_n$ was defined by

$$i' = \max\{l \ge i : (l, j') \in L_{\Sigma} \text{ and } \Delta_w(p, (l, j')) = 0\} \text{ where } j' = \min\{k > j : (i, k) \in L_{\Sigma}\}.$$

We will also write $\vec{p}_{\Sigma} = (i, j') \in L_{\Sigma}$ and $p_{\Sigma}^{\curvearrowright} = (i', j') \in L_{\Sigma}$, so that $\Delta_w(p, p_{\Sigma}^{\curvearrowright}) = 0$.

Similarly, define $\overleftarrow{p}_\Sigma = (i, j') \in L_\Sigma$ and $p_\Sigma^{\curvearrowleft} = (i', j') \in L_\Sigma$ by

$$j' = \max\{k < j : (i,k) \in L_\Sigma\}, \quad i' = \min\{l \le i : (l, j') \in L_\Sigma \text{ and } \Delta_w(p, (l, j')) = 0\}$$

so that $\Delta_w(p, p_\Sigma^{\curvearrowleft}) = 0$ and recall that $p_\Sigma^{\uparrow} = (i', j) \in \square_n$.

We caution that $p_\Sigma^{\curvearrowright}$ and $p_\Sigma^{\curvearrowleft}$ should not be confused with $p_\Sigma^{\searrow}$ and $p_\Sigma^{\nwarrow}$ defined in §3.1.

Note that for any $p \in \tilde{L}_\Sigma$ we have $\Delta_x(p, p_\Sigma^{\curvearrowright}) = \Delta_\Sigma(p, p_\Sigma^{\curvearrowright})$. Thus

$$\mathcal{P}_\Sigma^{\downarrow}(p) \iff \Delta_\Sigma(p, p_\Sigma^{\curvearrowright}) > 0 \iff \Delta_x(p, p_\Sigma^{\curvearrowright}) > 0. \tag{16}$$

Definition 4.1.

(1) We say that p is $\uparrow$-maximal (resp., $\downarrow$-maximal) with respect to Σ if $p_\Sigma^{\uparrow} = p$ (resp., $p_\Sigma^{\downarrow} = p$).

(2) We write

$$\mathcal{Q}_\Sigma^{\downarrow}(p) = \mathcal{P}_\Sigma^{\downarrow}(p) \vee \mathcal{P}_\Sigma^{\uparrow}(p_\Sigma^{\downarrow}), \quad \mathcal{Q}_\Sigma^{\uparrow}(p) = \mathcal{P}_\Sigma^{\uparrow}(p) \vee \mathcal{P}_\Sigma^{\downarrow}(p_\Sigma^{\uparrow})$$

so that

$$\mathcal{Q}_\Sigma(p) = \mathcal{Q}_\Sigma^{\downarrow}(p) \vee \mathcal{Q}_\Sigma^{\uparrow}(p).$$

It is clear that

$$\text{if } p \text{ is } \downarrow\text{-maximal (resp., } \uparrow\text{-maximal) then } \mathcal{Q}_p(\Sigma) \text{ implies } \mathcal{Q}_p^{\uparrow}(\Sigma) \text{ (resp., } \mathcal{Q}_p^{\downarrow}(\Sigma)). \tag{17}$$

Note that the conditions $\mathcal{Q}_\Sigma^{\downarrow}(p)$ and $\mathcal{Q}_{\Sigma^*}^{\uparrow}(p^*)$ are equivalent.

In the rest of this subsection we give some simple properties of the interplay between the various conditions $\mathcal{P}_\Sigma^{\downarrow}(p)$, $\mathcal{P}_\Sigma^{\uparrow}(p)$, $\mathcal{Q}_\Sigma^{\downarrow}(p)$, $\mathcal{Q}_\Sigma^{\uparrow}(p)$, $\mathcal{P}_\Sigma^{\searrow}(p)$, $\mathcal{P}_\Sigma^{\nwarrow}(p)$. They will be used in the induction step of Proposition 2.10. Throughout let $\Sigma = (w, x) \in \mathfrak{P}_n$.

Lemma 4.2. *Let $\overrightarrow{p}_\Sigma = (i, j')$ (resp., $\overleftarrow{p}_\Sigma = (i, j')$). Then $\mathcal{P}_\Sigma^{\downarrow}(p)$ (resp., $\mathcal{P}_\Sigma^{\uparrow}(p)$) is satisfied if and only if there exists $i' > i$ (resp., $i' < i$) such that $p' := (i', j') \in L_\Sigma$ and $\Delta_w(p, p') = 0 < \Delta_x(p, p')$. In particular, $\mathcal{P}_\Sigma^{\searrow}(p)$ (see Definition 3.1) implies $\mathcal{P}_\Sigma^{\downarrow}(p)$ and similarly $\mathcal{P}_\Sigma^{\nwarrow}(p)$ implies $\mathcal{P}_\Sigma^{\uparrow}(p)$.*

Proof. The "only if" direction clear. For the "if" direction note that $p'' := p_\Sigma^{\curvearrowright} = (i'', j')$ with $i'' \ge i'$. Therefore $\Delta_\Sigma(p, p'') = \Delta_x(p, p'') \ge \Delta_x(p, p') = \Delta_\Sigma(p, p') > 0$ and hence $r_\Sigma(p_\Sigma^{\downarrow}) < r_\Sigma(p)$. □

Remark 4.3. In fact, it is easy to see that the condition $\mathcal{P}_\Sigma^{\searrow}(p)$ is equivalent to $r_\Sigma(p) = 1$ and $\mathcal{P}_\Sigma^{\downarrow}(p)$. We will not use this fact.

Lemma 4.4. *Let $p = (i, j) \in \tilde{L}_\Sigma$, $p_\Sigma^{\Lsh} = (i', j')$ and $p' = p_\Sigma^{\uparrow} = (i', j)$. Assume that $\mathcal{P}_\Sigma^{\uparrow}(p)$ is not satisfied. Then*

(1) *p' is $\uparrow$-maximal with respect to Σ.*
(2) *For any $p'' = (i_1, j)$ with $i' \le i_1 \le i$ we have $r_\Sigma(p'') \ge r_\Sigma(p)$ and if equality holds then $p''^{\uparrow}_\Sigma = p'$, $\overleftarrow{p''}_\Sigma = (i_1, j')$ and $\mathcal{P}_\Sigma^{\uparrow}(p'')$ is not satisfied.*
(3) *In particular, $\overleftarrow{p'}_\Sigma = p_\Sigma^{\Lsh}$.*
(4) *If $\mathcal{Q}_\Sigma^{\uparrow}(p)$ is satisfied then $p'^{\downarrow}_\Sigma = (i'', j)$ with $i'' > i$.*

Proof. Let $q = p_\Sigma^{\Lsh}$ and $\overleftarrow{p'}_\Sigma = (i', j_1)$. Then $j_1 \ge j'$ since $q \in L_\Sigma$. Assume on the contrary that $j_1 > j'$. Since $\mathcal{P}_\Sigma^{\uparrow}(p)$ is not satisfied, we have $\Delta_x(q, p) = \Delta_w(q, p) = 0$. Hence $\Delta_x(q, (i, j_1)) = \Delta_w(q, (i, j_1)) = 0$ and since $(i', j'), (i, j'), (i', j_1) \in L_\Sigma$ we would get $(i, j_1) \in L_\Sigma$ which contradicts the definition of j'. It is now clear that p' is $\uparrow$-maximal.

Now let $p'' = (i_1, j)$ with $i' \le i_1 \le i$. Then $\Delta_x((i_1, j'), p) = \Delta_w((i_1, j'), p) = 0$ and $\overleftarrow{p}_\Sigma = (i, j') \in L_\Sigma$ and hence $r_\Sigma(p'') = r_\Sigma(p) + r_\Sigma(i_1, j')$. Thus, $r_\Sigma(p'') \ge r_\Sigma(p)$ with an equality if and only if $(i_1, j') \in L_\Sigma$. In this case, by the same reasoning as before we have $\overleftarrow{p''}_\Sigma = (i_1, j')$, and hence $p''^{\uparrow}_\Sigma = p'$. In particular, $\mathcal{P}_\Sigma^{\uparrow}(p'')$ is not satisfied.

The last part is now clear from the definitions. □

For convenience we also write down the symmetric version of Lemma 4.4.

Lemma 4.5. *Let $p = (i, j) \in \tilde{L}_\Sigma$, $p_\Sigma^{\Rsh} = (i', j')$ and $p' = p_\Sigma^{\downarrow} = (i', j)$. Assume that $\mathcal{P}_\Sigma^{\downarrow}(p)$ is not satisfied. Then*

(1) *p' is $\downarrow$-maximal with respect to Σ.*
(2) *For any $p'' = (i_1, j)$ with $i \le i_1 \le i'$ we have $r_\Sigma(p'') \ge r_\Sigma(p)$ and if equality holds then $p''^{\downarrow}_\Sigma = p'$, $\overrightarrow{p''}_\Sigma = (i_1, j')$ and $\mathcal{P}_\Sigma^{\downarrow}(p'')$ is not satisfied.*
(3) *In particular, $\overrightarrow{p'}_\Sigma = p_\Sigma^{\Rsh}$.*
(4) *If $\mathcal{Q}_\Sigma^{\downarrow}(p)$ is satisfied then $p'^{\uparrow}_\Sigma = (i'', j)$ with $i'' < i$.*

The following lemma is straightforward.

Lemma 4.6. *Let $p = (i, j) \in \tilde{L}_\Sigma$ and $t = t_{i_1, i_2} \in \mathcal{R}_\Sigma$. Assume that $q = \overrightarrow{p}_{\Sigma_t} \in \tilde{L}_\Sigma$ (i.e., $q \in \tilde{L}_{(xt, x)}$). Then $\overleftarrow{q}_\Sigma = \overleftarrow{p}_\Sigma$. Assume that $\mathcal{P}_\Sigma^{\nwarrow}(q)$ holds. Then $p_\Sigma^{\uparrow} = (i', j)$ with $i' < i_1$ and in fact $i' \le i''$ where $q_\Sigma^{\nwarrow} = (i'', j'')$. In particular, p is not $\uparrow$-maximal with respect to Σ. If in addition $p \in \tilde{L}_{(xt, x)}$ then $\underline{p}_x(i_1) \in (q_\Sigma^{\nwarrow}, p]$ and hence by Lemma 4.2 $\mathcal{P}_\Sigma^{\uparrow}(p)$ holds.*

Lemma 4.7. *Let $t \in \mathcal{R}_\Sigma$, $p \in \tilde{L}_{\Sigma_t}$, $p_\Sigma^{\downarrow} = (i', j)$ and $p_{\Sigma_t}^{\downarrow} = (i'', j)$. Then $i'' \ge i'$. In particular, if p is $\downarrow$-maximal with respect to Σ_t then it is also $\downarrow$-maximal with respect to Σ.*

Proof. Write $q = p_\Sigma^{\searrow} = (i', j')$, $p' = \vec{p}_\Sigma = (i, j') \in L_\Sigma$, $p_{\Sigma_t}^{\searrow} = (i'', j'')$, $p'' = \vec{p}_{\Sigma_t} = (i, j'') \in L_{\Sigma_t}$. If $p' = p''$ the lemma is clear since $L_\Sigma \subset L_{\Sigma_t}$. Otherwise, $j < j'' < j'$. We have $\Delta_w(q, p'') = 0$ and $q, p'' \in L_{\Sigma_t}$. Thus, by (4) $(i', j'') \in L_{\Sigma_t}$ and hence $i'' \ge i'$. □

Lemma 4.8. *Let $t \in \mathcal{R}_\Sigma$, $p = (i, j) \in \tilde{L}_\Sigma$, $q = \vec{p}_{\Sigma_t}$. Assume that $q \in \tilde{L}_\Sigma$ and that $\mathcal{P}_\Sigma^{\searrow}(q)$ is satisfied. If $p_{\Sigma_t}^{\searrow} \in L_\Sigma$ then $\mathcal{P}_\Sigma^{\downarrow}(p)$ holds. Otherwise, $p_\Sigma^{\downarrow} = p_{\Sigma_t}^{\downarrow}$ and the conditions $\mathcal{P}_\Sigma^{\downarrow}(p)$ and $\mathcal{P}_{\Sigma_t}^{\downarrow}(p)$ are equivalent.*

Proof. Let $q = (i, j')$, $q' = q_\Sigma^{\searrow} = (i', j'')$ and $p' = p_{\Sigma_t}^{\searrow} = (i'', j')$. Note that $\vec{p}_\Sigma = \vec{q}_\Sigma$.

Suppose first that $p' \in L_\Sigma$. Then $i'' \ge i'$. By the condition $\mathcal{P}_\Sigma^{\searrow}(q)$ we get $\Delta_w(p, q') = \Delta_w(p, (i', j')) + \Delta_w(q, q') = 0$ and $\Delta_x(p, q') \ge \Delta_x(q, q') = 1$. Thus, the lemma follows from Lemma 4.2.

Suppose now that $p' \in \tilde{L}_\Sigma$. Then $i'' < i_2 \le i'$. We have $\Delta_w(p, (i'', j'')) = \Delta_w(p, p') + \Delta_w(q, (i'', j'')) = 0$. Also $(i'', j'') \in L_\Sigma$ since $(i, j'') \in L_\Sigma$, $q, p' \in L_{\Sigma_t}$ and $\Delta_x(q, (i'', j'')) = 0$. Hence $p_\Sigma^{\downarrow} = (i''', j)$ with $i''' \ge i''$. On the other hand by Lemma 4.7 we have $i''' \le i''$. Thus, $p_\Sigma^{\downarrow} = p_{\Sigma_t}^{\downarrow}$ the conditions $\mathcal{P}_\Sigma^{\downarrow}(p)$ and $\mathcal{P}_{\Sigma_t}^{\downarrow}(p)$ are equivalent since $i_1 \le i \le i'' < i_2$. □

For convenience, we record the symmetric (equivalent) forms of Lemmas 4.6, 4.7 and 4.8.

Lemma 4.9. *Let $p = (i, j) \in \tilde{L}_\Sigma$ and $t = t_{i_1, i_2} \in \mathcal{R}_\Sigma$. Assume that $q = \overleftarrow{p}_{\Sigma_t} \in \tilde{L}_\Sigma$. Then $\vec{q}_\Sigma = \vec{p}_\Sigma$. Assume that $\mathcal{P}_\Sigma^{\searrow}(q)$ holds. Then $p_\Sigma^{\downarrow} = (i', j)$ with $i' \ge i_2$ and in fact $i' \ge i''$ where $q_\Sigma^{\searrow} = (i'', j'')$. In particular, p is not $\downarrow$-maximal with respect to Σ. If in addition $p \in \tilde{L}_{(xt, x)}$ then $\underline{p}_x(i_2) \in (p, q_\Sigma^{\searrow}]$ and hence by Lemma 4.2 $\mathcal{P}_\Sigma^{\downarrow}(p)$ holds.*

Lemma 4.10. *Let $t \in \mathcal{R}_\Sigma$, $p \in \tilde{L}_{\Sigma_t}$, $p_\Sigma^{\uparrow} = (i', j)$ and $p_{\Sigma_t}^{\uparrow} = (i'', j)$. Then $i'' \le i'$. In particular, if p is $\uparrow$-maximal with respect to Σ_t then it is also $\uparrow$-maximal with respect to Σ.*

Lemma 4.11. *Let $t \in \mathcal{R}_\Sigma$, $p = (i, j) \in \tilde{L}_\Sigma$, $q = \overleftarrow{p}_{\Sigma_t}$. Assume that $q \in \tilde{L}_\Sigma$ and that $\mathcal{P}_\Sigma^{\nwarrow}(q)$ is satisfied. If $p_{\Sigma_t}^{\nwarrow} \in L_\Sigma$ then $\mathcal{P}_\Sigma^{\uparrow}(p)$ holds. Otherwise, $p_\Sigma^{\uparrow} = p_{\Sigma_t}^{\uparrow}$ and the conditions $\mathcal{P}_\Sigma^{\uparrow}(p)$ and $\mathcal{P}_{\Sigma_t}^{\uparrow}(p)$ are equivalent.*

We need a couple of more lemmas.

Lemma 4.12. *Let $p \in \tilde{L}_\Sigma$, $q = p_\Sigma^{\uparrow}$ and $t \in \mathcal{R}_\Sigma$. Suppose that $q \in \tilde{L}_{(xt, x)}$ but $p \in L_{(xt, x)}$. Assume that $\mathcal{P}_\Sigma^{\uparrow}(p)$ is not satisfied but $\mathcal{Q}_\Sigma^{\downarrow}(q)$ is satisfied. Then $\mathcal{P}_\Sigma^{\downarrow}(q)$ is satisfied.*

Proof. Let $p = (i, j)$, $q = (i', j)$ and $q' = q_\Sigma^{\downarrow} = (i'', j)$. Write $t = t_{i_1, i_2}$ and $t^x = t_{j_1, j_2}$. By assumption $i \ge i_2 > i'$ and $j_1 \le j < j_2$. Assume on the contrary that $\mathcal{P}_\Sigma^{\downarrow}(q)$ is not satisfied. Then $i'' < i_2 \le i$, for otherwise $\underline{p}_x(i_2) \in (q, q_\Sigma^{\searrow}]$ in contradiction with (16). Since $r_\Sigma(q) = r_\Sigma(q')$, the condition $\mathcal{P}_\Sigma^{\uparrow}(q')$, guaranteed by $\mathcal{Q}_\Sigma^{\downarrow}(q)$, is inconsistent with Lemma 4.4, part (2). □

Lemma 4.13. *Let $t = t_{i_1,i_2} \in \mathcal{R}_\Sigma$, $p = (i,j) \in \tilde{L}_{\Sigma_t}$ and $q = p_{\Sigma_t}^{\uparrow} = (i',j)$. Assume that $q \in \tilde{L}_{(xt,x)}$ and $\overleftarrow{p}_\Sigma = (i,j')$ with $j' \geq j_1 = x(i_1)$. Then q is $\uparrow$-maximal with respect to Σ.*

Proof. Assume on the contrary that q is not $\uparrow$-maximal with respect to Σ and let $q'' = q_\Sigma^{\curvearrowleft} = (i'',j'')$, $i'' < i'$ and $q' = p_{\Sigma_t}^{\curvearrowleft} = (i',j')$. Then $j'' < j_1 \leq j'$ and $\Delta_w(q'',q') = 0$. Since $q'',q' \in L_{\Sigma_t}$ we must have $(i'',j') \in L_{\Sigma_t}$ by (4). This violates the definition of i' since $\Delta_w((i'',j'),q) = 0$. □

4.2. Induction step

We will prove Proposition 2.10 by induction on $\ell(\Sigma)$. Unfortunately, the induction step splits into many cases and subcases. We analyze the different cases in Lemmas 4.14.A–4.14.E below.

For the rest of this (long) subsection we fix a smooth pair $\Sigma = (w,x) \in \mathfrak{P}_n$ and $t \in \mathcal{R}_\Sigma^{\triangleleft}$ and assume that

$$\mathcal{Q}_{\Sigma_t}(p) \text{ is satisfied for any } p \in \tilde{L}_{\Sigma_t}. \tag{IH}$$

We will write $t = t_{i_1,i_2}$ and $t^x = t_{j_1,j_2}$.

Lemma 4.14.A. *$\mathcal{Q}_\Sigma(p)$ holds for any $p = (i,j) \in \tilde{L}_{(xt,x)}$.*

Proof. Suppose first that $r_\Sigma(p) = 1$. Then p is in the critical set $\mathfrak{C}_\Sigma$ (10). In view of Lemma 4.2, this case follows from Proposition 3.2 part (1).

For the rest of the proof we assume that $r_\Sigma(p) > 1$.

Note that $r_{\Sigma_t}(p) = r_\Sigma(p) - 1 > 0$. By (IH) $\mathcal{Q}_{\Sigma_t}(p)$ holds. By passing to Σ^*, p^* and t^* if necessary we may assume without loss of generality that $\mathcal{Q}_{\Sigma_t}^{\downarrow}(p)$ is satisfied.

Write $p_{\Sigma_t}^{\curvearrowright} = (i',j')$ and $p' = p_{\Sigma_t}^{\downarrow} = (i',j)$.

Suppose first that $i' \geq i_2$.

(1) If also $j' \geq j_2$ then $p_\Sigma^{\curvearrowright} = p_{\Sigma_t}^{\curvearrowright}$ and $\mathcal{P}_\Sigma^{\downarrow}(p)$ is satisfied (even if $\mathcal{P}_{\Sigma_t}^{\downarrow}(p)$ is not satisfied) since $\underline{p}_x(i_2) \in (p, p_\Sigma^{\curvearrowright}]$.
(2) Suppose that $j' < j_2$.
We have $p_{\Sigma_t}^{\curvearrowright} \in L_\Sigma$. Let $q = \overrightarrow{p}_{\Sigma_t} = (i,j') \in L_{\Sigma_t} \setminus L_\Sigma$. Then $\mathcal{P}_\Sigma^{\nwarrow}(q)$ implies $\mathcal{P}_\Sigma^{\uparrow}(p)$ by Lemma 4.6 while $\mathcal{P}_\Sigma^{\searrow}(q)$ implies $\mathcal{P}_\Sigma^{\downarrow}(p)$ by Lemma 4.8. We conclude by Proposition 3.2 part (1).

For the rest of the proof we assume that $i' < i_2$ so that $p' = (i',j) \in \tilde{L}_{(xt,x)}$. We also assume, as we may, that $\mathcal{P}_\Sigma^{\downarrow}(p)$ is not satisfied.

If $j' < j_2$ then we can assume by Proposition 3.2 part (1), applied to $(i,j') \in L_{\Sigma_t} \setminus L_\Sigma$, that $\mathcal{P}_\Sigma^{\searrow}(i,j')$ is satisfied since by Lemma 4.6 $\mathcal{P}_\Sigma^{\nwarrow}(i,j')$ implies $\mathcal{P}_\Sigma^{\uparrow}(p)$. By Lemma 4.8 and our assumption,

$$p' = p_\Sigma^{\downarrow} \text{ and } \mathcal{P}_{\Sigma_t}^{\downarrow}(p) \text{ is not satisfied.}$$

The last line is also valid if $j' \geq j_2$, in which case $p_\Sigma^{\curvearrowright} = p_{\Sigma_t}^{\curvearrowright}$.

Thus, $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$ holds and p' is $\downarrow$-maximal with respect to Σ.

Let $p'^{\curvearrowleft}_{\Sigma_t} = (i'', j'')$ and $p'' = \overleftarrow{p'}_{\Sigma_t} = (i', j'')$. We separate the argument according to cases.

(1) Suppose that $j'' < j_1$. Then $p'^{\curvearrowleft}_{\Sigma} = p'^{\curvearrowleft}_{\Sigma_t}$ and hence $\mathcal{P}_\Sigma^{\uparrow}(p')$ follows from $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$, so that $\mathcal{Q}_\Sigma^{\downarrow}(p)$ holds.

(2) Suppose that $j'' \geq j_1$ and $i'' < i_1$.
By Lemma 4.9 $\mathcal{P}_\Sigma^{\searrow}(p'')$ doesn't hold. Thus by Proposition 3.2 part (1), $\mathcal{P}_\Sigma^{\nwarrow}(p'')$ is satisfied which by Lemma 4.11 implies $\mathcal{P}_\Sigma^{\uparrow}(p')$, hence $\mathcal{Q}_\Sigma^{\downarrow}(p)$.

(3) Finally we cannot have both $i'' \geq i_1$ and $j'' \geq j_1$ since otherwise, as $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$ holds, there would exist i_3 such that $\underline{p}_x(i_3) \in (p'^{\curvearrowleft}_{\Sigma_t}, p'] \subset (\underline{p}_x(i_1), \underline{p}_x(i_2))$ violating the assumption that $\tau \in \mathcal{R}_\Sigma^{\triangleleft}$.

The proof of the lemma is complete. $\square$

Lemma 4.14.B. *$\mathcal{Q}_\Sigma(p)$ holds for any $p = (i, j) \in \tilde{L}_\Sigma$ such that $i_2 \leq i$ and $j_1 \leq j < j_2$.*

Proof. Note that $r_{\Sigma_t}(p) = r_\Sigma(p)$ and hence by (IH), $\mathcal{Q}_{\Sigma_t}(p)$ is satisfied. Under our condition on t we have $p_{\Sigma_t}^{\curvearrowright} = p_\Sigma^{\curvearrowright}$, and hence the conditions $\mathcal{P}_{\Sigma_t}^{\downarrow}(p)$ and $\mathcal{P}_\Sigma^{\downarrow}(p)$ are equivalent. Therefore, we may assume that they are not satisfied.

Suppose first that $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$ holds where $p' = p$ or $p' = p_\Sigma^{\downarrow} = p_{\Sigma_t}^{\downarrow}$. Write $p' = (i', j)$ and recall that $r_\Sigma(p') = r_\Sigma(p)$. Let $q = p'^{\curvearrowleft}_{\Sigma_t} = (i'', j'') \in L_{\Sigma_t}$ and $p'' = p'^{\uparrow}_{\Sigma_t} = (i'', j)$ so that $\overleftarrow{p'}_\Sigma = (i', j'')$ and $i'' < i \leq i'$. Note that $\overleftarrow{p'}_{\Sigma_t} = \overleftarrow{p'}_\Sigma$ by the condition on t. If $p'' \in L_{(xt,x)}$ (which means that either $i'' < i_1$ or $i'' \geq i_2$) then $q \in L_\Sigma$, $p'^{\uparrow}_\Sigma = p'^{\uparrow}_{\Sigma_t}$, $\Delta_x(q, p') = \Delta_{xt}(q, p')$ and hence $\mathcal{P}_\Sigma^{\uparrow}(p')$ follows from $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$.

Thus, we may assume that $i_1 \leq i'' < i_2$, i.e. that $p'' \in \tilde{L}_{(xt,x)}$.

(1) Assume first that $q \in L_\Sigma$ (i.e., that $q \in L_{(xt,x)}$, or equivalently that $j_1 > j''$).
Then $p'^{\uparrow}_\Sigma = p''$. We may assume that $\mathcal{P}_\Sigma^{\uparrow}(p')$ is not satisfied, for otherwise $\mathcal{Q}_\Sigma^{\downarrow}(p)$ holds. By Lemma 4.4 p'' is $\uparrow$-maximal with respect to Σ, $p_\Sigma^{\uparrow} = p'^{\uparrow}_\Sigma = p''$ and $\mathcal{P}_\Sigma^{\uparrow}(p)$ is not satisfied. It follows from Lemma 4.14.A and (17) that $\mathcal{Q}_\Sigma^{\downarrow}(p'')$ holds. It follows from Lemma 4.12 that $\mathcal{P}_\Sigma^{\downarrow}(p'')$ is satisfied and therefore so does $\mathcal{Q}_\Sigma^{\uparrow}(p)$.

(2) Now assume that $q \in \tilde{L}_\Sigma$, that is $j_1 \leq j''$.
By Lemma 4.13 p'' is $\uparrow$-maximal with respect to Σ. Once again, by Lemma 4.14.A and (17) $\mathcal{Q}_\Sigma^{\downarrow}(p'')$ holds. Let $p''^{\curvearrowright}_\Sigma = (i''', j''')$ and $p''' = p''^{\downarrow}_\Sigma = (i''', j)$. Note that $j''' \geq j_2$. Also, since $\mathcal{P}_{\Sigma_t}^{\uparrow}(p')$ is satisfied we have $r_\Sigma(p'') = r_{\Sigma_t}(p'') + 1 \leq r_{\Sigma_t}(p) = r_\Sigma(p)$.
   (a) Suppose that $\mathcal{P}_\Sigma^{\downarrow}(p'')$ holds.
      (i) Suppose that $i''' \leq i'$.

Then $\Delta_w(q,p''') = 0$ and hence $\Delta_\Sigma(q,p''') \ge 0$. On the other hand by assumption, $r_\Sigma(p''') < r_\Sigma(p'')$ and $q \in L_{\Sigma_t} \setminus L_\Sigma$. Therefore, $(i''',j'') \in L_\Sigma$ and $r_\Sigma(p''') = r_\Sigma(p'') - 1$. By Lemma 4.2 this implies that $\mathcal{P}^{\uparrow}_\Sigma(p')$ is satisfied.

(ii) Suppose that $i''' > i'$. Let

$$i_3 = \min\{k > i'' : (k,j'') \in L_\Sigma\}.$$

Note that $i_2 \le i_3 \le i' < i'''$ and that $\vec{q}_\Sigma = \vec{p''}_\Sigma$. Let $q' = (i_3,j''')$. Since $\Delta_w(q,q') = \Delta_w(q,(i_3,j)) + \Delta_w(p'',q') = 0$ we may apply Proposition 3.2 part (2) to q to infer that $q' \in L_\Sigma$ and $\Delta_x(q,q') = 1$, hence $\Delta_x(p'',q') = 1$ as $\underline{p}_x(i_2) \in (p'',q']$. Hence, $r_\Sigma(i_3,j) = r_\Sigma(p'') - 1 < r_\Sigma(p')$. As before, it follows from Lemma 4.2 that $\mathcal{P}^{\uparrow}_\Sigma(p')$ is satisfied.

(b) Suppose that $\mathcal{P}^{\downarrow}_\Sigma(p'')$ is not satisfied.
Then $r_\Sigma(p''') = r_\Sigma(p'')$, $i''' < i_2$ (for otherwise $\underline{p}_x(i_2) \in (p'',p''^{\curvearrowright}_\Sigma]$) and $r_\Sigma(i_4,j) \ge r_\Sigma(p'')$ for any $i'' \le i_4 \le i'''$. Let $q'' := p'''^{\curvearrowleft}_\Sigma = (i_5,j_5) \in L_\Sigma$ so that $p'''^{\uparrow}_\Sigma = (i_5,j)$. Then $j_5 < j_1 \le j''$. Since $\mathcal{Q}^{\downarrow}_\Sigma(p'')$ holds, $\mathcal{P}^{\uparrow}_\Sigma(p''')$ is satisfied and therefore $r_\Sigma(p'''^{\uparrow}_\Sigma) < r_\Sigma(p'')$. Hence $i_5 < i''$ and therefore $q'' < q$. Since $\Delta_w(q'',q) = 0$ and $q \in L_{\Sigma_t}$ we necessarily have $(i_5,j'') \in L_{\Sigma_t}$. Since $\Delta_w(q'',p'') = 0$ we would get a contradiction to the fact that $p'' = p'^{\uparrow}_{\Sigma_t}$.

It remains to consider the case where $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$ does not hold but $\mathcal{P}^{\downarrow}_{\Sigma_t}(p')$ holds for $p' = p^{\uparrow}_{\Sigma_t}$.

Write $q = p^{\curvearrowleft}_{\Sigma_t} = (i',j') \in L_{\Sigma_t}$ and $p' = p^{\uparrow}_{\Sigma_t} = (i',j)$ and note that $\overleftarrow{p}_\Sigma = \overleftarrow{p}_{\Sigma_t} = (i,j')$.

As before, if $p' \in L_{(xt,x)}$, i.e., if either $i' < i_1$ or $i' \ge i_2$ then $p' = p^{\uparrow}_\Sigma$, $q = p^{\curvearrowleft}_\Sigma$ and $\Delta_x(q,p) = \Delta_{xt}(q,p)$. Moreover, since $p'^{\downarrow}_{\Sigma_t} = (i'',j)$ with $i'' > i$ we have $p'^{\curvearrowright}_{\Sigma_t} = p'^{\curvearrowright}_\Sigma$ and $\Delta_x(p',p'^{\curvearrowright}_\Sigma) = \Delta_{xt}(p',p'^{\curvearrowright}_\Sigma)$. Hence $\mathcal{Q}^{\uparrow}_\Sigma(p)$ follows from $\mathcal{Q}^{\uparrow}_{\Sigma_t}(p)$.

Assume therefore that $i_1 \le i' < i_2$, i.e., $p' \in \tilde{L}_{(xt,x)}$. Since $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$ does not hold (by assumption), we necessarily have $j' \ge j_1$ for otherwise $\underline{p}_{xt}(i_2) \in (p^{\curvearrowleft}_{\Sigma_t},p]$.

It follows from the definitions that $p'' := p^{\uparrow}_\Sigma = (i_3,j)$ where

$$i_3 = \min\{k > i' : (k,j') \in L_\Sigma\}$$

(with $i_2 \le i_3 \le i$). By Lemma 4.4 we have $r_\Sigma(p'') = r_{\Sigma_t}(p'') = r_{\Sigma_t}(p') = r_\Sigma(p') - 1 = r_\Sigma(p) = r_{\Sigma_t}(p)$.

Write $p'^{\curvearrowright}_{\Sigma_t} = (i'',j'')$ so that $\vec{p'}_{\Sigma_t} = (i',j'') \in L_{\Sigma_t}$ and recall that $i'' > i$.

(1) Suppose that $j'' \ge j_2$.
Then $\vec{q}_\Sigma = \vec{p'}_\Sigma = \vec{p'}_{\Sigma_t}$. By Proposition 3.2 part (2) (applied to q) we have $(i_3,j'') \in L_\Sigma$ and $\Delta_{xt}(q,(i_3,j'')) = 0$. Thus, $\vec{p''}_\Sigma = (i_3,j_3)$ with $j_3 \le j''$. We claim that in fact $j_3 = j''$. Indeed, if $j_3 < j''$ then necessarily $(i',j_3) \in L_{\Sigma_t}$ since $\Delta_{xt}((i',j_3),(i_3,j'')) =$

$\Delta_w((i',j_3),(i_3,j'')) = 0$ and $(i_3,j_3),(i_3,j''),(i',j'') \in L_{\Sigma_t}$, and this contradicts the definition of j''.
We conclude from the condition $\mathcal{P}^{\downarrow}_{\Sigma_t}(p')$ that $\mathcal{P}^{\downarrow}_{\Sigma}(p'')$ holds. Thus $\mathcal{Q}^{\uparrow}_{\Sigma}(p)$ or $\mathcal{P}^{\downarrow}_{\Sigma}(p)$ is satisfied depending on whether $i_3 < i$ or $i_3 = i$.

(2) Suppose that $j'' < j_2$.
Since p' is $\uparrow$-maximal with respect to Σ_t (Lemma 4.4), it is also $\uparrow$-maximal with respect to Σ (Lemma 4.10). Let $q' = \overrightarrow{p'}_{\Sigma_t} \in L_{\Sigma_t} \setminus L_{\Sigma}$. By Lemma 4.6 we cannot have $\mathcal{P}^{\nwarrow}_{\Sigma}(q')$. Hence, by Proposition 3.2 part (1) we have $\mathcal{P}^{\searrow}_{\Sigma}(q')$. By Lemma 4.8 we conclude from $\mathcal{P}^{\downarrow}_{\Sigma_t}(p')$ that $\mathcal{P}^{\downarrow}_{\Sigma}(p')$ holds.

This finishes the proof of the lemma. □

Lemma 4.14.C. *$\mathcal{Q}_{\Sigma}(p)$ holds for any $p = (i,j) \in \tilde{L}_{\Sigma}$ such that $i < i_1 < i_2 \le i_0$ and $j_1 \le j < j_2$ where $p^{\downarrow}_{\Sigma} = (i_0,j)$.*

Proof. Note that $r_{\Sigma_t}(p) = r_{\Sigma}(p)$ and in particular $p \in \tilde{L}_{\Sigma_t}$. Hence by (IH), $\mathcal{Q}_{\Sigma_t}(p)$ is satisfied. Under our condition on t we have $p^{\curvearrowright}_{\Sigma_t} = p^{\curvearrowright}_{\Sigma}$, $\Delta_x(p,p^{\curvearrowright}_{\Sigma}) = \Delta_{xt}(p,p^{\curvearrowright}_{\Sigma})$ and the conditions $\mathcal{P}^{\downarrow}_{\Sigma_t}(p)$ and $\mathcal{P}^{\downarrow}_{\Sigma}(p)$ are equivalent. Therefore, we may assume that $\mathcal{P}^{\downarrow}_{\Sigma_t}(p)$ (and $\mathcal{P}^{\downarrow}_{\Sigma}(p)$) are not satisfied.

Suppose first that $\mathcal{P}^{\uparrow}_{\Sigma_t}(p')$ holds where $p' = p$ or $p' = p^{\downarrow}_{\Sigma} = p^{\downarrow}_{\Sigma_t}$. Write $p' = (i',j)$ and note that $r_{\Sigma}(p') = r_{\Sigma}(p)$. Let $q = p'^{\curvearrowleft}_{\Sigma_t} = (i'',j'') \in L_{\Sigma_t}$ and $p'' = p'^{\uparrow}_{\Sigma_t} = (i'',j)$. By the condition on t we have $\overleftarrow{p'}_{\Sigma_t} = \overleftarrow{p'}_{\Sigma}$. On the other hand, by Lemma 4.5 (applied to Σ_t) we have $i'' < i \le i'$. Hence $p'' \in L_{(xt,x)}$, $q \in L_{\Sigma}$, $p'^{\curvearrowleft}_{\Sigma_t} = p^{\curvearrowleft}_{\Sigma}$ and the condition $\mathcal{P}^{\uparrow}_{\Sigma}(p')$ follows from $\mathcal{P}^{\uparrow}_{\Sigma}(p)$.

It remains to consider the case where $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$ does not hold but $\mathcal{P}^{\downarrow}_{\Sigma_t}(p')$ holds for $p' = p^{\uparrow}_{\Sigma_t} = (i'',j)$. In particular, $r_{\Sigma}(p') = r_{\Sigma}(p)$. (Of course, $r_{\Sigma_t}(p') = r_{\Sigma}(p')$ and $r_{\Sigma_t}(p) = r_{\Sigma}(p)$ since $i < i_1$.)

Write $q' = \vec{p}_{\Sigma} = (i,j_0) \in L_{\Sigma}$ and $q = p'^{\curvearrowright}_{\Sigma_t} = (i',j')$. Since $i < i_1$ we have $p' = p^{\uparrow}_{\Sigma}$ and $\overrightarrow{p'}_{\Sigma} = \overrightarrow{p'}_{\Sigma_t}$. Also, $j_0 < j_2$ for otherwise $\underline{p}_x(i_2) \in (p, p^{\curvearrowright}_{\Sigma}]$ since $i_2 \le i_0$, and $\mathcal{P}^{\downarrow}_{\Sigma}(p)$ would be satisfied. By Lemma 4.5 (applied to Σ_t) we have $i' > i$. Also, $j_0 \ge j'$ for otherwise $\Delta_{\Sigma}(p',q') = \Delta_x(p',q') \ge 0$ and $(i'',j_0) \in \tilde{L}_{\Sigma}$ and this contradicts the fact that $q' \in L_{\Sigma}$ and $r_{\Sigma}(p') = r_{\Sigma}(p)$. If $q \in L_{\Sigma}$ then $\mathcal{P}^{\downarrow}_{\Sigma}(p')$ is satisfied by Lemma 4.2, and hence $\mathcal{Q}^{\downarrow}_{\Sigma}(p)$ holds. Otherwise, $q \in L_{\Sigma_t} \setminus L_{\Sigma}$ and $(i',j_0) \in \tilde{L}_{\Sigma}$ (since $j_0 < j_2$). Since $\Delta_x(q, p^{\curvearrowright}_{\Sigma}) = 0$ and $p^{\curvearrowright}_{\Sigma} \in L_{\Sigma}$, we must have $(i_0,j') \in L_{\Sigma}$. Hence, $\mathcal{P}^{\downarrow}_{\Sigma}(p')$ is satisfied by Lemma 4.2 (and in fact $j' = j_0$). □

Lemma 4.14.D. *$\mathcal{Q}_{\Sigma}(p)$ holds for any $p = (i,j) \in \tilde{L}_{\Sigma}$ such that $i < i_1$ and $p^{\downarrow}_{\Sigma} \in \tilde{L}_{(xt,x)}$.*

Proof. Write $p^{\curvearrowright}_{\Sigma} = (i_0,j_0) \in L_{\Sigma}$, $p' = p^{\downarrow}_{\Sigma} = (i_0,j)$ and note that $j_0 \ge j_2$. Also note that the condition $\mathcal{P}^{\downarrow}_{\Sigma_t}(p)$ is satisfied.

We may assume of course that $\mathcal{P}^{\downarrow}_{\Sigma}(p)$ is not satisfied, so that $r_\Sigma(p') = r_\Sigma(p)$.

We apply Lemma 4.14.A to p'. Recall that p' is $\downarrow$-maximal with respect to Σ. Thus, by (17) $\mathcal{Q}^{\uparrow}_{\Sigma}(p')$ is satisfied. Since $\mathcal{P}^{\uparrow}_{\Sigma}(p')$ implies $\mathcal{Q}^{\downarrow}_{\Sigma}(p)$, we may assume that $\mathcal{P}^{\uparrow}_{\Sigma}(p')$ is not satisfied. Let $p'' = p'^{\uparrow}_{\Sigma} = (i', j)$. Then $r_\Sigma(p'') = r_\Sigma(p') = r_\Sigma(p)$ and $\mathcal{P}^{\downarrow}_{\Sigma}(p'')$ holds. Let $p'^{\curvearrowleft}_{\Sigma} = (i', j')$ so that $\overleftarrow{p'}_{\Sigma} = (i_0, j')$. Then $j' < j_1$. Also $i' \ge i_1 > i$ for otherwise $\underline{p}_x(i_1) \in (p'^{\curvearrowleft}_{\Sigma}, p']$ which would imply $\mathcal{P}^{\uparrow}_{\Sigma}(p')$. By Lemma 4.5 this would contradict $\mathcal{P}^{\downarrow}_{\Sigma}(p'')$. $\square$

Lemma 4.14.E. *$\mathcal{Q}_\Sigma(p)$ holds for any $p = (i, j) \in \tilde{L}_\Sigma$ such that $i_0 < i_2$ and $j_2 \le j$ where $p^{\downarrow}_{\Sigma} = (i_0, j)$.*

Proof. We have $r_{\Sigma_t}(p) = r_\Sigma(p)$ and in particular $p \in \tilde{L}_{\Sigma_t}$. By (IH), $\mathcal{Q}_{\Sigma_t}(p)$ is satisfied. Note that the conditions $\mathcal{P}^{\downarrow}_{\Sigma}(p)$ and $\mathcal{P}^{\downarrow}_{\Sigma_t}(p)$ are equivalent. We may therefore assume that they do not hold. We separate according to cases.

(1) Suppose that $\mathcal{P}^{\uparrow}_{\Sigma_t}(p')$ holds for $p' = p^{\downarrow}_{\Sigma} = p^{\downarrow}_{\Sigma_t} = (i_0, j)$.
   Let $q = p'^{\curvearrowleft}_{\Sigma_t} = (i', j')$ and $p'' = \overleftarrow{p'}_{\Sigma_t} = (i_0, j') \in L_{\Sigma_t}$.
   (a) Suppose that $p'' \in L_\Sigma$.
      Then $p'' = \overleftarrow{p'}_{\Sigma}$ and since $i_2 > i_0$ we have $p'^{\curvearrowleft}_{\Sigma} = q$, $p'^{\uparrow}_{\Sigma} \in L_{(xt,x)}$. The condition $\mathcal{P}^{\uparrow}_{\Sigma}(p')$ (and hence, $\mathcal{Q}^{\downarrow}_{\Sigma}(p)$) follows from $\mathcal{P}^{\uparrow}_{\Sigma_t}(p')$.
   (b) Assume that $p'' \in \tilde{L}_\Sigma$.
      By Lemma 4.9 the condition $\mathcal{P}^{\searrow}_{\Sigma}(p'')$ would contradict the fact that p' is $\downarrow$-maximal with respect to Σ.
      Hence by Proposition 3.2 part (1) $\mathcal{P}^{\nwarrow}_{\Sigma}(p'')$ holds. By Lemma 4.11 we have $\mathcal{P}^{\uparrow}_{\Sigma}(p')$ and hence $\mathcal{Q}^{\downarrow}_{\Sigma}(p)$.

(2) Suppose that $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$ holds.
   Let $q = p^{\curvearrowleft}_{\Sigma_t} = (i', j')$ and $p' = \overleftarrow{p}_{\Sigma_t} = (i, j')$.
   (a) Suppose that $p' \in L_\Sigma$.
      Then $\overleftarrow{p}_{\Sigma} = p'$, $p^{\curvearrowleft}_{\Sigma} = q$, $p^{\uparrow}_{\Sigma} \in L_{(xt,x)}$. Hence $\mathcal{P}^{\uparrow}_{\Sigma}(p)$ follows from $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$.
   (b) Suppose that $p' \in \tilde{L}_\Sigma$.
      By Lemma 4.9 we cannot have $\mathcal{P}^{\searrow}_{\Sigma}(p')$ since $i_2 > i_0$.
      Hence, by Proposition 3.2 part (1) $\mathcal{P}^{\nwarrow}_{\Sigma}(p')$ holds. By Lemma 4.11 we have $\mathcal{P}^{\uparrow}_{\Sigma}(p)$.

(3) Suppose that $\mathcal{P}^{\uparrow}_{\Sigma_t}(p)$ does not hold but $\mathcal{P}^{\downarrow}_{\Sigma_t}(p^{\uparrow}_{\Sigma_t})$ holds.
   Again, let $q = p^{\curvearrowleft}_{\Sigma_t} = (i', j')$ and $p' = \overleftarrow{p}_{\Sigma_t} = (i, j')$.
   (a) As before, if $p' \in L_\Sigma$ then $\overleftarrow{p}_{\Sigma} = p'$, $p^{\curvearrowleft}_{\Sigma} = q$, $p^{\uparrow}_{\Sigma} = p^{\uparrow}_{\Sigma_t}$ and $\mathcal{Q}^{\uparrow}_{\Sigma}(p)$ follows from $\mathcal{Q}^{\uparrow}_{\Sigma_t}(p)$.
   (b) Suppose that $p' \in \tilde{L}_\Sigma$. By Lemma 4.9 we cannot have $\mathcal{P}^{\searrow}_{\Sigma}(p')$ since $i_2 > i_0$.
      Hence, by Proposition 3.2 part (1) $\mathcal{P}^{\nwarrow}_{\Sigma}(p')$ holds. By Lemma 4.11 if $q \in L_\Sigma$ then $\mathcal{P}^{\uparrow}_{\Sigma}(p)$ holds, or otherwise, $p^{\uparrow}_{\Sigma} = p^{\uparrow}_{\Sigma_t}$, and hence $\mathcal{Q}^{\uparrow}_{\Sigma}(p)$ is satisfied. $\square$

4.3. Conclusion of proof

In order to conclude the proof of 2.10 we need to show that the cases covered in the previous subsection are exhaustive.

Let

$$\mathcal{A}_\Sigma(p) = \{t_{i_1,i_2} \in \mathcal{R}_\Sigma : \underline{p}_x(i_1) \le p \text{ but } \underline{p}_x(i_2) \nleq p\}$$

and $\mathcal{A}_\Sigma^\triangleleft(p) = \mathcal{A}_\Sigma(p) \cap \mathcal{R}_\Sigma^\triangleleft$.

Lemma 4.15. *$\mathcal{A}_\Sigma^\triangleleft(p) \neq \varnothing$ for any $p \in \tilde{L}_\Sigma$.*

Proof. We first remark that it suffices to show that $\mathcal{A}_\Sigma(p) \neq \varnothing$ for any $p \in \tilde{L}_\Sigma$ since if $t_{i_1,i_2} \in \mathcal{A}_\Sigma(p)$ then for any i' such that $t_{i_1,i'}, t_{i',i_2} \in \mathcal{R}_\Sigma$ we have either $t_{i_1,i'} \in \mathcal{A}_\Sigma(p)$ or $t_{i',i_2} \in \mathcal{A}_\Sigma(p)$. We prove the statement by induction on $\ell(\Sigma)$. The statement is empty if $\ell(\Sigma) = 0$. For the induction step, assume that $\ell(\Sigma) > 0$ and observe that if $t \in \mathcal{R}_\Sigma$ and $p \in \tilde{L}_{(xt,x)}$ then $t \in \mathcal{A}_\Sigma(p)$. Let $t' \in \mathcal{R}_\Sigma^\triangleleft$ and $p \in \tilde{L}_\Sigma$. If $t' \in \mathcal{A}_\Sigma(p)$ we are done. Otherwise, $p \in \tilde{L}_{\Sigma_{t'}}$ and by induction hypothesis $\mathcal{A}_{\Sigma_{t'}}(p) \neq \varnothing$. However, it is straightforward to check that the map $\phi_{t'}^\Sigma$ of §3.3 satisfies $\phi_{t'}^\Sigma(\mathcal{A}_{\Sigma_{t'}}(p)) \subset \mathcal{A}_\Sigma(p)$. □

Proof of Proposition 2.10. We will prove the proposition by induction on $\ell(\Sigma) = \ell(x) - \ell(w) \ge 0$. The case $\ell(w) = \ell(x)$ (i.e., $w = x$) is of course trivial. Let $p = (i, j) \in \tilde{L}_\Sigma$. We may assume of course that $\mathcal{P}_\Sigma^\downarrow(p)$ is not satisfied. Write $p' = p_\Sigma^\downarrow = (i_0, j)$. Since $r_\Sigma(p') = r_\Sigma(p)$ we have $p' \in \tilde{L}_\Sigma$. By Lemma 4.15 $\mathcal{A}_\Sigma^\triangleleft(p') \neq \varnothing$. Let $t \in \mathcal{A}_\Sigma^\triangleleft(p')$. The induction hypothesis implies (IH). Writing $t = t_{i_1,i_2}$ and $t^x = t_{j_1,j_2}$, exactly one of the following conditions hold.

(1) $p \in \tilde{L}_{(xt,x)} (= [\underline{p}_x(i_1), \underline{p}_x(i_2)))$.
(2) $i_2 \le i$ and $j_1 \le j < j_2$.
(3) $i < i_1 < i_2 \le i_0$ and $j_1 \le j < j_2$.
(4) $i < i_1$ and $p' \in \tilde{L}_{(xt,x)}$.
(5) $i_1 \le i_0 < i_2$ and $j_2 \le j$.

We apply Lemmas 4.14.A, 4.14.B, 4.14.C, 4.14.D and 4.14.E respectively to conclude in each case that $\mathcal{Q}_\Sigma(p)$ holds as required. □

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