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Selberg integrals, Askey–Wilson polynomials and lozenge tilings of a hexagon with a triangular hole



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ABSTRACT

We obtain an explicit formula for a certain weighted enumeration of lozenge tilings of a hexagon with an arbitrary triangular hole. The complexity of our expression depends on the distance from the hole to the center of the hexagon. This proves and generalizes conjectures of Ciucu et al., who considered the case of plain enumeration when the triangle is located at or very near the center. Our proof uses Askey–Wilson polynomials as a tool to relate discrete and continuous Selberg-type integrals.

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1. Introduction

One of the most influential results of enumerative combinatorics is MacMahon's formula [28]

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}$$

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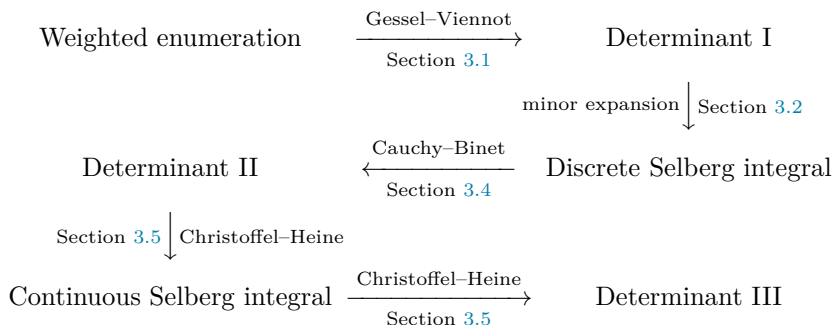
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for the number of plane partitions contained in a box of size $a \times b \times c$, where $H(n) = \prod_{k=1}^n (k-1)!$. Equivalently, this identity enumerates lozenge tilings of a hexagon with side lengths a , b and c .

There has been quite a lot of work on lozenge tilings of a hexagon with various kinds of holes [4–6, 8–15, 19, 24–26, 29–31]. In the seminal paper [31], Propp conjectured an explicit formula for the number of tilings of a hexagon H whose side lengths are almost equal, with a small triangle T removed from the center of H (more precisely, in the notation explained in Section 2.1, this is the region $H \setminus T$ with $a = b = c$, $m = 1$, $M = N = 0$). This conjecture was proved in [4, 19]. More generally, Ciucu et al. [7] enumerated the tilings when the side lengths of H and T are arbitrary, but T is still positioned at (or very near) the center of H . They also conjectured enumerations for some adjacent positions of T .

In the present paper, we consider the general case, when the position of T within H is arbitrary. Our main result, Theorem 2.1, expresses a weighted extension of the number of tilings as a determinant, whose complexity depends on the distance of T from the center of H . Thus, it is a closed formed evaluation if the position of T relative to the center of H is fixed, but the side lengths of T and H are arbitrary.

As in [7], the starting point of our proof is the Gessel–Viennot method [18], which gives an explicit determinant formula for the weighted enumeration. In [7] the determinant is computed using the method of identification of factors [22]. It seems very difficult to handle the more general determinants that we encounter in this way. Instead, we derive a chain of intermediate expressions for our weighted enumeration as indicated in the following diagram.



Here, Determinant I is obtained by the Gessel–Viennot method. It has completely factored entries and its dimension is equal to one of the side lengths of H . Applying an appropriate minor expansion leads to a multivariable basic hypergeometric series. As it contains the factor $\prod_{i < j} (q^{m_j} - q^{m_i})^2$, with m_j being summation indices, it can be considered as a discrete Selberg-type integral [16]. Special cases of this sum appear in [7], but are considered there as consequences of the enumeration rather than as a tool.

In general, Selberg-type refers to hypergeometric series or integrals containing factors like $\prod_{i < j} |x_j - x_i|^c$, where the archetypal example is the integral

$$\int_{[0,1]^n} \prod_{1 \leq i < j \leq n} |x_j - x_i|^c \prod_{j=1}^n x_j^{a-1} (1-x_j)^{b-1} dx_j.$$

The cases $c = 1$ and $c = 2$ are determinantal, in the sense that they can be expressed as determinants of one-variable integrals. In our setting, an application of the Cauchy–Binet identity leads to an alternative determinant formula for the weighted enumeration, Determinant II. It is quite different from Determinant I as its entries are Askey–Wilson polynomials and its dimension is equal to the side length of T . Using classical results on orthogonal polynomials due to Christoffel and Heine, we can rewrite Determinant II as a continuous Selberg integral, where $\prod_{i < j} (x_j - x_i)^2$ is integrated against the Askey–Wilson orthogonality measure. The key observation is now that the results of Christoffel and Heine can be applied in a different way to the same Selberg integral. This leads to our end result, Determinant III. Here, the matrix entries are again Askey–Wilson polynomials, but in base q^2 rather than q . The size of the determinant is related to the distance from T to the center of H .

The above description of our proof is not quite accurate, as we glossed over two important technical aspects. First, Determinant I only applies when the side length m of T is even. To extend our result to odd m , we need an a priori result on how our weighted enumeration behaves as a function of m , [Lemma 3.5](#). This is achieved by another application of the Gessel–Viennot method. Second, the orthogonality relation for Askey–Wilson polynomials is actually not valid for the specific parameters appearing from the tiling problem. Thus, the continuous Selberg integral mentioned above does not make sense as an expression for the weighted enumeration, but only appears after continuation to a different range of parameters.

The explicit expression given in [Theorem 2.1](#) is admittedly rather complicated, but we believe that the method of proof is more interesting than the result. It seems likely that there are other problems related to tilings and plane partitions that can be approached with similar methods. For instance, one could ask for a “dual” of our result in the sense of [\[12\]](#).

2. Main result

2.1. Weighted enumeration of tilings

Consider the triangular lattice in the plane, formed by equilateral triangles of side length 1 and height $\phi = \sqrt{3}/2$. On this lattice, we draw a convex hexagon H and remove an equilateral triangle $T \subseteq H$. We are interested in tilings of $H \setminus T$ by lozenges, that is, by quadrilaterals formed by adjoining two adjacent lattice triangles. Using the bijection

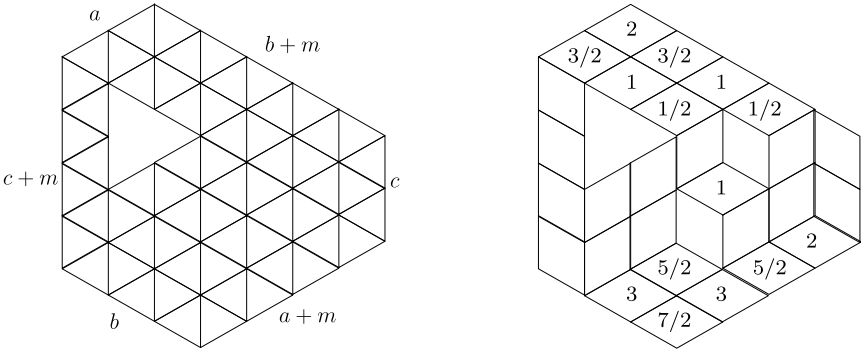


Fig. 1. The region $H \setminus T$ described by $(a, b, c, m, A, B, C) = (2, 3, 2, 2, 4, 2, 1)$. A tiling of this region, with each horizontal lozenge labeled by its height.

to lattice paths discussed in Section 3.1, it is easy to see that, for such tilings to exist, H must have consecutive side-lengths $a, b + m, c, a + m, b, c + m$, where m is the side-length of T . Moreover, the sides of T must be parallel to the long sides (of length $a + m, b + m$ and $c + m$) of H .

We will refer to the sides of H by the expression for their length; for instance, the side $b + m$ is the second side in the ordering given above. We will picture the region $H \setminus T$ as in Fig. 1. This allows us to use terminology such as “horizontal” to refer to the direction orthogonal to the side c .

To specify the position of T within H , let $A\phi, B\phi$ and $C\phi$ denote the distance from T to the line containing the side $a + m, b + m$ and $c + m$, respectively. It is easy to see that

$$A + B + C = a + b + c, \tag{2.1}$$

for instance, by applying Viviani’s theorem to the triangle formed by extending the short sides of H . The distances from T to the lines containing the short sides of H are $(b + c - A)\phi, (a + c - B)\phi$ and $(a + b - C)\phi$. Thus,

$$0 \leq A \leq b + c, \quad 0 \leq B \leq a + c, \quad 0 \leq C \leq a + b. \tag{2.2}$$

Conversely, any non-negative integers a, b, c, A, B, C and m subject to (2.1) and (2.2) describe a region $H \setminus T$. Note that we include degenerate cases when some sides of H have length zero, when no triangle is removed ($m = 0$) or when T touches the boundary of H . We will also specify the location of T by the coordinates

$$M = 2A - b - c, \quad N = 2B - a - c, \tag{2.3}$$

so that

$$A = \frac{b + c + M}{2}, \quad B = \frac{a + c + N}{2}, \quad C = \frac{a + b - M - N}{2}.$$

Then, M and N are integers of the same parity as $b + c$ and $a + c$, respectively, such that

$$|M| \leq b + c, \quad |N| \leq a + c, \quad |M + N| \leq a + b.$$

As an example, the region in Fig. 1 corresponds to $(M, N) = (3, 0)$.

Note that $(M, N) = (0, 0)$ corresponds to T being located at the center of H . This case, and a few other cases with T nearly central, were studied in [7]. To be precise, these authors enumerated the tilings when (M, N) equals $(0, 0)$ and $(0, 1)$ and conjectured enumerations when (M, N) equals $(0, 2)$ and $(0, 3)$. In the present paper, we will explain how to prove these conjectures and obtain analogous results for *any* M and N .

More generally, we will consider a weighted enumeration of tilings. We define the height h of a horizontal tile Q to be the vertical distance from the center of Q to the center of T (see Fig. 1). Our main object of study is the partition function

$$Z(q) = \sum_{\text{tilings of } H \setminus T} \prod_{\text{horizontal tiles}} \frac{q^h + q^{-h}}{2}. \quad (2.4)$$

In particular, $Z(1)$ is the total number of tilings. Our weight function is a special case of weights introduced in [2] for plane partitions and [34] for lattice paths.

Note that, when $m = 0$, $Z(q)$ is different from the volume generating function for plane partitions computed by MacMahon. Up to a power of q , the latter is equal to

$$\tilde{Z}(q) = \sum_{\text{tilings}} \prod_{\text{horizontal tiles}} q^{2\tilde{h}},$$

where \tilde{h} is the vertical distance from the center of a tile to the bottom corner of H . When $m > 0$, the function Z behaves better than \tilde{Z} , being given by completely factored expressions in situations (e.g. $(M, N) = (0, 0)$) when such expressions exist for the enumeration problem.

2.2. Notation

We will write $\text{sgn}(k) = 1$ for $k \geq 0$ and $\text{sgn}(k) = -1$ for $k < 0$. Recall also the standard notation [17]

$$\begin{aligned} (a; q)_n &= \prod_{j=0}^{n-1} (1 - aq^j), \\ (a_1, \dots, a_m; q)_n &= (a_1; q)_n \cdots (a_m; q)_n, \\ {}_{r+1}\phi_r \left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, x \right) &= \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} x^k. \end{aligned}$$

When $x = (x_1, \dots, x_m)$, we will write $\Delta(x) = \prod_{1 \leq i < j \leq m} (x_j - x_i)$ and also use notation such as $\Delta(q^k) = \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i})$. We introduce the multiple basic hypergeometric series

$${}_{r+1}\phi_r^{(m)} \left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, x \right) = \sum_{0 \leq k_1 < k_2 < \dots < k_m < \infty} \Delta(q^k)^2 \prod_{j=1}^m \frac{(a_1, \dots, a_{r+1}; q)_{k_j} x^{k_j}}{(q, b_1, \dots, b_r; q)_{k_j}}. \quad (2.5)$$

In view of the factor $\Delta(q^k)^2$, it can be thought of as a discrete Selberg-type integral.

We introduce the q -hyperfactorial

$$H_q(m) = \begin{cases} \prod_{j=1}^m \left(q^{-\frac{j}{2}} - q^{\frac{j}{2}} \right)^{m-j}, & m = 0, 1, 2, \dots, \\ \prod_{j=1}^{m+\frac{1}{2}} \left(q^{\frac{1}{4}-\frac{j}{2}} - q^{\frac{j}{2}-\frac{1}{4}} \right)^{m+\frac{1}{2}-j}, & m = -1/2, 1/2, 3/2, \dots \end{cases} \quad (2.6)$$

Equivalently,

$$H_q(m) = \begin{cases} q^{-\frac{1}{2}\binom{m+1}{3}} \prod_{j=1}^m (q; q)_{j-1}, & m = 0, 1, 2, \dots, \\ q^{-\frac{1}{16}\binom{2m+1}{3}} \prod_{j=1}^{m+\frac{1}{2}} (q^{\frac{1}{2}}; q)_{j-1}, & m = -1/2, 1/2, 3/2, \dots \end{cases}$$

Deleting the prefactor from these expressions, we will write

$$\tilde{H}_q(m) = \begin{cases} \prod_{j=1}^m (q; q)_{j-1}, & m = 0, 1, 2, \dots, \\ \prod_{j=1}^{m+\frac{1}{2}} (q^{\frac{1}{2}}; q)_{j-1}, & m = -1/2, 1/2, 3/2, \dots \end{cases}$$

We use both notations since our main result is easier to state in terms of H_q , but for the proof it is often more convenient to work with \tilde{H}_q .

We will sometimes write $H_q^+ = H_q$ and

$$H_q^-(m) = \frac{H_{q^2}(m)}{H_q(m)} = \begin{cases} \prod_{j=1}^m \left(q^{-\frac{j}{2}} + q^{\frac{j}{2}} \right)^{m-j}, & m = 0, 1, 2, \dots, \\ \prod_{j=1}^{m+\frac{1}{2}} \left(q^{\frac{1}{4}-\frac{j}{2}} + q^{\frac{j}{2}-\frac{1}{4}} \right)^{m+\frac{1}{2}-j}, & m = -1/2, 1/2, 3/2, \dots \end{cases}$$

Repeated arguments stand for a product; for instance,

$$H_q(a_1, \dots, a_m) = H_q(a_1) \cdots H_q(a_m).$$

Similar notation will be used for \tilde{H}_q . We collect some useful facts about the functions H_q and \tilde{H}_q in [Appendix A](#).

2.3. Statement of main result

Our main result is formulated in terms of determinants

$$Q^{MNn}(\alpha, \beta, \gamma; q) = \det_{1 \leq j, k \leq M+N} (Q_{jk}^{MNn}(\alpha, \beta, \gamma; q)), \quad (2.7)$$

labeled by non-negative integers M, N, n and generic parameters α, β, γ . The matrix entries $Q_{jk}^{MNn}(\alpha, \beta, \gamma; q)$ are given for $1 \leq k \leq M$ and $n+j$ odd by

$$\begin{aligned} & \frac{(\alpha^2, \alpha^2 \gamma^2; q^2)_{(n+j-1)/2} (\alpha^2 \beta^2; q^2)_{k-1}}{q^{\frac{1}{4}(n+j-1)(n+j-3) + \binom{k-1}{2}} \alpha^{n+j+k-2} \beta^{k-1} \gamma^{\frac{1}{2}(n+j-1)}} \\ & \times {}_4\phi_3 \left(\begin{matrix} q^{1-j-n}, \alpha^2 \beta^2 \gamma^2 q^{n+j-3}, \alpha^4 q^{2k-2}, q^{2-2k} \\ \alpha^2, \alpha^2 \beta^2, \alpha^2 \gamma^2 \end{matrix}; q^2, q^2 \right) \end{aligned} \quad (2.8a)$$

and for $1 \leq k \leq M$ and $n+j$ even by

$$\begin{aligned} & (\alpha q^{k-1} - \alpha^{-1} q^{1-k}) \frac{(\alpha^2 q^2, \alpha^2 \gamma^2; q^2)_{(n+j-2)/2} (\alpha^2 \beta^2; q^2)_{k-1}}{q^{\frac{1}{4}(n+j-2)^2 + \binom{k-1}{2}} \alpha^{n+j+k-3} \beta^{k-1} \gamma^{\frac{1}{2}(n+j-2)}} \\ & \times {}_4\phi_3 \left(\begin{matrix} q^{2-j-n}, \alpha^2 \beta^2 \gamma^2 q^{n+j-2}, \alpha^4 q^{2k-2}, q^{2-2k} \\ \alpha^2 q^2, \alpha^2 \beta^2, \alpha^2 \gamma^2 \end{matrix}; q^2, q^2 \right). \end{aligned} \quad (2.8b)$$

For the remaining cases ($M+1 \leq k \leq M+N$), they are determined by

$$Q_{j, M+k}^{MNn}(\alpha, \beta, \gamma; q) = Q_{j, k}^{NMn}(\beta, \alpha, \gamma; q), \quad 1 \leq k \leq N.$$

Though the structure of this determinant may seem complicated, we will see in Section 3.5 that it appears naturally in the context of Askey–Wilson polynomials. It is easy to check that

$$Q^{MNn}(\alpha, \beta, \gamma; q) = (-1)^{MN+n(M+N)} Q^{MNn}(\alpha^{-1}, \beta^{-1}, \gamma^{-1}; q^{-1}). \quad (2.9)$$

The matrix elements (2.8) are Laurent polynomials in α, β and γ . In particular, we may (and will) specialize these variables to points where the ${}_4\phi_3$ -sums without the prefactor are singular. Note also that each matrix entry is a sum of at most $\max(M, N)$ terms. Thus, the following result gives a closed form evaluation of $Z(q)$ for fixed M and N .

Theorem 2.1. *With $\varepsilon = \operatorname{sgn}(MN)$, we have*

$$Z(q) = C Q^{|M|, |N|, b} \left(q^{\frac{1}{2}(1-b-c-m-|M|)}, -\varepsilon q^{\frac{1}{2}(1+a+c+m-|N|)}, q^{\frac{1}{2}(m+1)}; q \right), \quad (2.10)$$

where

$$\begin{aligned}
C = & \frac{(-1)^{\binom{|N|}{2}} \varepsilon^{\binom{|M|}{2} + N(b+M)}}{2^{\frac{1}{2}} m(a+b+M+N) + ab - \frac{a+b}{2} + \frac{1}{2} \max(|a-b|, |M-N|)} \frac{H_{q^2} \left(\frac{m}{2} \right)^2}{H_q(|M|, |N|)} \\
& \times H_{q^2} \left(\left[\frac{a}{2} \right], \left[\frac{a+1}{2} \right], \left[\frac{a}{2} \right] + \frac{m+1}{2}, \left[\frac{a+1}{2} \right] + \frac{m-1}{2} \right) \\
& \times H_{q^2} \left(\left[\frac{b}{2} \right], \left[\frac{b+1}{2} \right], \left[\frac{b}{2} \right] + \frac{m+1}{2}, \left[\frac{b+1}{2} \right] + \frac{m-1}{2} \right) \\
& \times \frac{H_{q^2} \left(\left[\frac{c+|M|}{2} \right], \left[\frac{c+|M|+1}{2} \right], \left[\frac{c+|M|}{2} \right] + \frac{m+1}{2}, \left[\frac{c+|M|+1}{2} \right] + \frac{m-1}{2} \right)}{H_q \left(\frac{a+b-M-N}{2}, \frac{a+b+M+N}{2} + m \right) H_{q^2} \left(\frac{a+b-|M|-|N|+m+1}{2}, \frac{a+b+|M|+|N|+m-1}{2} \right)} \\
& \times \frac{H_{q^2} \left(\frac{a+c+N}{2} + m \right)}{H_{q^2} \left(\frac{a+c+N}{2}, \frac{a+c-|N|+m-1}{2} \right) H_{q^2} \left(\frac{a+c-|N|}{2} + m \right)^2 H_{q^4} \left(\frac{a+c-|N|+m+1}{2} \right)} \\
& \times \frac{H_q^-(a+c+m)}{H_{q^2}^-\left(\frac{a+c-|N|+m}{2}, \frac{a+c+|N|+m-1}{2}, \frac{a+c+|N|+m}{2} \right)} \\
& \times \frac{H_{q^2} \left(\frac{b+c-M}{2} \right)}{H_{q^2} \left(\frac{b+c-M}{2} + m, \frac{b+c+|M|+m+1}{2} \right) H_{q^2} \left(\frac{b+c+|M|}{2} \right)^2 H_{q^4} \left(\frac{b+c+|M|+m-1}{2} \right)} \\
& \times \frac{H_q^-(b+c+m)}{H_{q^2}^-\left(\frac{b+c-|M|+m}{2}, \frac{b+c-|M|+m+1}{2}, \frac{b+c+|M|+m}{2} \right)} \frac{1}{H_q^-\left(\left| \frac{a-b+M-N}{2} \right|, \left| \frac{a-b-M+N}{2} \right| \right)} \\
& \times H_{q^2} \left(\left[\frac{a+b+c-|N|}{2} \right] + m, \left[\frac{a+b+c-|N|+1}{2} \right] + m \right) \\
& \times H_{q^2} \left(\left[\frac{a+b+c-|N|}{2} \right] + \frac{m+1}{2}, \left[\frac{a+b+c-|N|+1}{2} \right] + \frac{m-1}{2} \right) \\
& \times \frac{H_q^{-\varepsilon} \left(\frac{a+b+2c-|M|+|N|}{2} + m, \frac{a+b+2c+|M|-|N|}{2} + m \right)}{H_q^{-\varepsilon} \left(\frac{a+b+2c-|M|-|N|}{2} + m, \frac{a+b+2c+|M|+|N|}{2} + m \right)}. \tag{2.11}
\end{aligned}$$

As an example, consider the case $(M, N) = (2, 0)$. Assuming also that n is odd, we have

$$Q^{2,0,n}(\alpha, \beta, \gamma; q) = \begin{vmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{vmatrix},$$

where

$$\begin{aligned}
Q_{11} &= (\alpha - \alpha^{-1}) \frac{(\alpha^2 q^2, \alpha^2 \gamma^2; q^2)_{(n-1)/2}}{q^{\frac{1}{4}(n-1)^2} \alpha^{n-1} \gamma^{\frac{1}{2}(n-1)}}, \\
Q_{12} &= (\alpha q - \alpha^{-1} q^{-1}) \frac{(\alpha^2 q^2, \alpha^2 \gamma^2; q^2)_{(n-1)/2} (1 - \alpha^2 \beta^2)}{q^{\frac{1}{4}(n-1)^2} \alpha^n \beta \gamma^{\frac{1}{2}(n-1)}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(1 + \frac{(1 - q^{1-n})(1 - \alpha^2 \beta^2 \gamma^2 q^{n-1})(1 - \alpha^4 q^2)(1 - q^{-2})}{(1 - q^2)(1 - \alpha^2 q^2)(1 - \alpha^2 \beta^2)(1 - \alpha^2 \gamma^2)} q^2 \right), \\
Q_{21} &= \frac{(\alpha^2, \alpha^2 \gamma^2; q^2)_{(n+1)/2}}{q^{\frac{1}{4}(n+1)(n-1)} \alpha^{n+1} \gamma^{\frac{1}{2}(n+1)}}, \\
Q_{22} &= \frac{(\alpha^2, \alpha^2 \gamma^2; q^2)_{(n+1)/2} (1 - \alpha^2 \beta^2)}{q^{\frac{1}{4}(n+1)(n-1)} \alpha^{n+2} \beta \gamma^{\frac{1}{2}(n+1)}} \\
& \times \left(1 + \frac{(1 - q^{1-n})(1 - \alpha^2 \beta^2 \gamma^2 q^{n-1})(1 - \alpha^4 q^2)(1 - q^{-2})}{(1 - q^2)(1 - \alpha^2)(1 - \alpha^2 \beta^2)(1 - \alpha^2 \gamma^2)} q^2 \right).
\end{aligned}$$

This can be simplified to

$$\begin{aligned}
& Q^{2,0,n}(\alpha, \beta, \gamma; q) \\
&= \frac{(1 - q)(\alpha^2; q^2)_{(n+1)/2} (\alpha^2 q^2, \alpha^2 \gamma^2, \alpha^2 \gamma^2 q^2; q^2)_{(n-1)/2}}{q^{\binom{n}{2}+1} \alpha^{2n+2} \beta \gamma^n} \\
& \times \{ (1 + \alpha^2 q)(1 - \alpha^2 \beta^2)(1 - \alpha^2 \gamma^2) - (1 + q^{-n})(1 - \alpha^2 \beta^2 \gamma^2 q^{n-1})(1 - \alpha^4 q^2) \}.
\end{aligned}$$

If $(\alpha, \beta, \gamma) = (q^{x/2}, \pm q^{y/2}, q^{z/2})$, the leading Taylor coefficient of this function at $q = 1$ is a completely factored expression times

$$(x + y)(x + z) - 2(x + 1)(x + y + z + n - 1).$$

Substituting $(x, y, z, n) \mapsto (-b - c - m - 1, a + c + m + 1, m + 1, b)$ we find that, if a, b and c are all odd and $(M, N) = (2, 0)$, then $Z(1)$ is a completely factored expression times

$$(b - a)(b + c) + 2(b + c + m)(a + m) = (a + b)(b + c) + 2m(a + b + c + m).$$

After interchanging a and b , we recover the second half of [7, Conj. 1]. In this way, [7, Conj. 1 and Conj. 2] can both be obtained as special cases of Theorem 2.1.

An intriguing consequence of Theorem 2.1 is that $Z(q)$ is invariant under the transformation $(M, N) \mapsto (-M, -N)$, up to an elementary prefactor. This means that the position of T is reflected in the center of H , see Fig. 2. It would be interesting to have a conceptual explanation for this unexpected symmetry.

Corollary 2.2. Denoting by Z_{MN} the partition function $Z(q)$ with fixed values of a, b, c, m and q , we have

$$\begin{aligned}
\frac{Z_{MN}}{Z_{-M, -N}} &= \frac{1}{2^{m(M+N)}} \frac{H_q \left(\frac{a+b+M+N}{2}, \frac{a+b-M-N}{2} + m \right)}{H_q \left(\frac{a+b-M-N}{2}, \frac{a+b+M+N}{2} + m \right)} \\
& \times \frac{H_{q^2} \left(\frac{a+c-N}{2}, \frac{a+c+N}{2} + m, \frac{b+c-M}{2}, \frac{b+c+M}{2} + m \right)}{H_{q^2} \left(\frac{a+c+N}{2}, \frac{a+c-N}{2} + m, \frac{b+c+M}{2}, \frac{b+c-M}{2} + m \right)}.
\end{aligned}$$

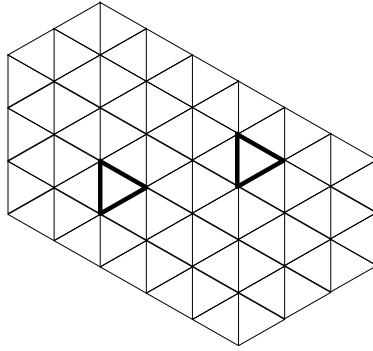


Fig. 2. Removing one of the two indicated triangles leads to partition functions related by an elementary multiplier.

As an example, removing the left triangle in Fig. 2 corresponds to

$$(a, b, c, m, M, N) = (2, 5, 2, 1, 1, 2),$$

which gives

$$Z(q) = \frac{(1+q)^4(1+q^2)^5(1+q^3)^3(1+q^4)^5(1+q^5)}{2^{13}q^{28}} f(q),$$

with

$$f(q) = q^8 + q^7 + 2q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1.$$

For the right triangle, corresponding to $(a, b, c, m, M, N) = (2, 5, 2, 1, -1, -2)$,

$$Z(q) = \frac{(1+q)^3(1+q^2)^4(1+q^3)^2(1+q^4)^4(1-q^{10})}{2^{10}q^{25}(1-q)} f(q).$$

In particular, substituting $q = 1$ we find that there are $544 = 2^5 \cdot 17$ tilings in the first case and $1360 = 2^4 \cdot 5 \cdot 17$ in the second case, where the symmetry is responsible for the relatively large common prime factor $f(1) = 17$.

3. Proof of Theorem 2.1

3.1. Lattice paths

Following [7], we study the partition function $Z(q)$ by applying a bijection from lozenge tilings to families of non-intersecting paths in the square lattice. Given a tiling of $H \setminus T$, we mark the midpoints of the edges on the side $b + m$ and construct paths ending at these points by following the direction of the lozenges. This gives m paths starting

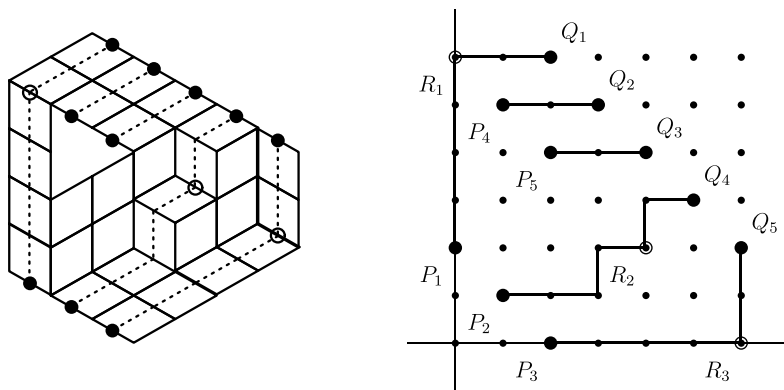


Fig. 3. Lattice paths corresponding to the tiling in Fig. 1. The circled points, denoted R_j , will appear in Section 3.3.

at the adjacent side of T and b paths starting at the side b . We then apply an affine transformation mapping the steps of the paths to edges in the square lattice.

More precisely, with the conventions illustrated in Fig. 3, tilings of $H \setminus T$ are in bijection with families of up-right non-intersecting paths starting at the points $(P_j)_{j=1}^{b+m}$ and ending at the points $(Q_j)_{j=1}^{b+m}$, where

$$P_j = \begin{cases} (j-1, b-j), & 1 \leq j \leq b, \\ (C-b+j-1, A+b+m-j), & b+1 \leq j \leq b+m, \end{cases} \quad (3.1a)$$

$$Q_j = (a+j-1, b+c+m-j), \quad 1 \leq j \leq b+m. \quad (3.1b)$$

In this setting, the weight function becomes a weight on horizontal steps, given by

$$\frac{q^{(x+2y-Z)/2} + q^{(-x-2y+Z)/2}}{2} \quad (3.2)$$

for the step from $(x-1, y)$ to (x, y) , where

$$Z = 2A + C + m - 1.$$

We now recall the Gessel–Viennot method for weighted enumeration of lattice paths [18]. Consider, in general, an arbitrary weight assigned to each horizontal edge in the square lattice. We define the weight of a family of paths to be the product of the weights of all horizontal steps in the family. Let

$$w(P_1, \dots, P_n; Q_1, \dots, Q_n)$$

denote the sum of the weight of all families of n non-intersecting up-right lattice paths, where the i -th path starts at P_i and ends at Q_i , for $1 \leq i \leq n$. We then have the following fundamental result of Lindström [27], see [23] for a historical discussion.

Lemma 3.1 (Lindström). *The following identity holds:*

$$\det_{1 \leq j, k \leq n} (w(P_j; Q_k)) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) w(P_1, \dots, P_n; Q_{\sigma(1)}, \dots, Q_{\sigma(n)}), \quad (3.3)$$

where the sum is over all permutations of $\{1, \dots, n\}$.

In other words, in the Laplace expansion of the left-hand side, the contributions from intersecting lattice paths cancel. Consider now the case when the points are given by (3.1). Since the paths starting at T will end at consecutive points on the side $b + m$, not all permutations will contribute to the sum in (3.3). In particular, if m is even, only even permutations contribute, which means that the right-hand side of (3.3) reduces to our partition function.

Corollary 3.2. *If the side length m of the triangle T is even, the partition function (2.4) has the determinant representation*

$$Z(q) = \det_{1 \leq j, k \leq b+m} (w(P_j; Q_k)). \quad (3.4)$$

If m is odd, we get instead a determinant representation for a sign-variation of the partition function, generalizing the (-1) -enumeration studied in [7]. Although our methods can be adapted to study this function, it will not be considered in the present work.

3.2. The partition function as a discrete Selberg integral

Continuing in the footsteps of Ciucu et al. [7], we rewrite the determinant of Corollary 3.2 as a discrete Selberg integral. The following result is a special case of [34, Prop. 2.1 (c)].

Lemma 3.3 (Schlosser). *One has the determinant evaluation*

$$\begin{aligned} & \det_{1 \leq j, k \leq m} (w((x_1 + j - 1, y_1 + m - j); (x_2 + l_k, y_2 - l_k))) \\ &= (-1)^{\binom{m}{2}} 2^{\binom{m}{2} - m(x_2 - x_1) - |l|} q^X \Delta(q^l) \\ & \quad \times \prod_{j=1}^m \frac{(q; q)_{x_2 + y_2 - x_1 - y_1 - m + j} (-q^{Z - x_2 - y_1 - y_2 - m + j}; q)_{x_2 - x_1 - j + 1 + l_j}}{(q; q)_{x_2 - x_1 + l_j} (q; q)_{y_2 - y_1 - l_j}}, \end{aligned}$$

where $|l| = \sum_{j=1}^m l_j$ and

$$\begin{aligned} X &= -\frac{3}{2} \binom{m}{3} + \frac{1}{2} \binom{m}{2} (-3x_1 + 2x_2 - 2y_1 + Z - 1) \\ & \quad + \frac{1}{4} m(x_2 - x_1)(x_1 + x_2 + 4y_1 - 2Z + 1) \\ & \quad + \frac{1}{2} \sum_{j=1}^m \binom{l_j}{2} + \frac{1}{2} |l|(x_2 + 2y_1 - Z + 1). \end{aligned}$$

The exponent X looks complicated, but can be determined from the fact that the weight is invariant under $q^{1/2} \mapsto q^{-1/2}$. If we would use a more symmetric notation (based on q -numbers $q^{-a/2} - q^{a/2}$ rather than $1 - q^a$), the resulting expression would be simpler.

Let us now expand the determinant in [Corollary 3.2](#) into minors according to

$$\det_{1 \leq j, k \leq b+m} (w(P_j; Q_k)) = \sum_{0 \leq l_1 < \dots < l_m \leq b+m-1} (-1)^{|l| + \binom{m}{2} + bm} \\ \times \det_{1 \leq j, k \leq b} (w(P_j; Q_{\hat{l}_k+1})) \det_{1 \leq j, k \leq m} (w(P_{b+j}; Q_{l_k+1})),$$

where $\hat{l}_1 < \dots < \hat{l}_b$ is the ordered complement of $\{l_1, \dots, l_m\}$ in $[0, b+m-1]$. The determinants on the right-hand side may be evaluated using [Lemma 3.3](#). Rewriting all factors involving the indices \hat{l}_j in terms of l_j , using in particular

$$\Delta(q^{\hat{l}}) = (-1)^{\binom{b}{2} + \binom{m}{2}} q^{\binom{b+m}{3} + (2-b-m)|l| + \sum_{j=1}^m \binom{l_j}{2}} \frac{\prod_{j=1}^{b+m} (q; q)_{j-1}}{\prod_{j=1}^m (q; q)_{l_j} (q; q)_{b+m-1-l_j}} \Delta(q^l)$$

gives

$$\det_{1 \leq j, k \leq b+m} (w(P_j; Q_k)) \\ = \frac{q^X}{2^{m(a+b-C)+ab}} \prod_{j=1}^{b+m} \frac{(q; q)_{j-1} (-q^{A-B-b+1}; q)_{a+j-1}}{(q; q)_{a+j-1} (q; q)_{c+j-1}} \\ \times \prod_{j=1}^b \frac{(q; q)_{a+c+m+j-1}}{(-q^{A-B-b+1}; q)_{j-1}} \sum (-1)^{|l| + bm + \binom{m}{2}} q^{\sum_{j=1}^m \binom{l_j}{2} + |l|(A-b-m+2)} \Delta(q^l)^2 \\ \times \prod_{j=1}^m \frac{(q; q)_{B+j-1} (q; q)_{a+l_j} (q; q)_{b+c+m-1-l_j} (-q^{-B-m+j}; q)_{a-C-j+1+l_j}}{(q; q)_{l_j} (q; q)_{a-C+l_j} (q; q)_{b+m-1-l_j} (q; q)_{b+c+m-A-1-l_j} (-q^{A-B-b+1}; q)_{a+l_j}},$$

where the sum is over indices satisfying

$$\max(0, C-a) \leq l_1 < \dots < l_m \leq \min(b+m-1, b+c+m-A-1) \quad (3.5)$$

and where

$$X = \frac{1}{2}(B-A-c) \binom{m}{2} + \frac{1}{4}((a-C)(a-C+1) + b(3b-4A-2C-1))m \\ + \frac{1}{4}ab(a+3b-4A-2C).$$

In the notation [\(2.3\)](#) and [\(2.5\)](#), this identity can be expressed as follows, where we have rewritten the prefactor in a way that will be convenient later.

Proposition 3.4.

$$\begin{aligned}
& \det_{1 \leq j, k \leq b+m} (w(P_j; Q_k)) \\
&= \frac{q^X}{2^{\frac{1}{2}m(a+b+M+N)+ab}} \prod_{j=1}^b \frac{(q; q)_{a+c+m+j-1} (q; q)_{j-1} (-q^{\frac{1}{2}(-a-b+M-N)+j}; q)_a}{(q; q)_{a+j-1} (q; q)_{c+m+j-1}} \\
&\quad \times \prod_{j=1}^m \left(\frac{(q^{1-b-c-m}, q^{1+\frac{1}{2}(a-b+M+N)}, -q^{1+\frac{1}{2}(a-b+M-N)}; q)_{b+j-1}}{(q^{1-b-m}; q)_{j-1} (q^{2-2m-b-c+M}; q^2)_{j-1} (q^{a+1}; q)_{b+j-1}} \right. \\
&\quad \times \left. \frac{(q^2; q^2)^{\frac{1}{2}(a+c+N)+j-1}}{(q; q)_{\frac{a+b+M+N}{2}+j-1} (q^2; q^2)^{\frac{b+c-M}{2}+j-1}} \right) \\
&\quad \times {}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-b-m}, q^{a+1}, q^{1-\frac{b}{2}-\frac{c}{2}+\frac{M}{2}-m}, -q^{1-\frac{b}{2}-\frac{c}{2}+\frac{M}{2}-m} \\ q^{1-b-c-m}, q^{1+\frac{a}{2}-\frac{b}{2}+\frac{M}{2}+\frac{N}{2}}, -q^{1+\frac{a}{2}-\frac{b}{2}+\frac{M}{2}+\frac{N}{2}} \end{matrix}; q, q \right), \tag{3.6}
\end{aligned}$$

where

$$\begin{aligned}
X &= -2 \binom{m}{3} - \frac{1}{4} (a - 3b + M + N + 4) \binom{m}{2} - \frac{1}{4} ab(2c + M - N) \\
&\quad - \left(\frac{1}{2} b(a - b) + \frac{1}{16} (a - 3b + M + N)(a + b + 4c - 3M + N + 2) \right) m.
\end{aligned}$$

Note that the factor $\prod_{j=1}^m (q^{1+\frac{1}{2}(a-b+M+N)}; q)_{b+j-1}$ may vanish. If this is the case, the ${}_4\phi_3^{(m)}$ in (3.6) should be interpreted as a sum over indices l_j such that this zero is canceled by $\prod_{j=1}^m (q^{1+\frac{1}{2}(a-b+M+N)}; q)_{l_j}^{-1}$. This gives the restriction $l_1 \geq (-a+b-M-N)/2 = C-a$ as in (3.5).

3.3. The partition function as a function of m

Corollary 3.2 is only valid for even values of m . In order to study the case of odd m , we will need the following fact. An analogous result for the enumeration $Z(1)$ was proved in [7, §6]. The simple proof given there seems difficult to extend to the weighted enumeration, so we use a slightly different approach.

Lemma 3.5. For fixed values of a, b, c, A, B and C ,

$$Z(q) = \prod_{j=1}^m \left(\frac{q^{\frac{j}{2}} + q^{-\frac{j}{2}}}{2} \right)^{a+b-C} f\left(q^{\frac{m}{2}}\right), \tag{3.7}$$

where f is a rational function independent of m .

Proof. Given a lozenge tiling of $H \setminus T$, we split H in two parts, separated by the line L containing the side of T parallel to the side b . Recall the bijection to lattice paths

described in Section 3.1. The paths starting at the side b cross L at b line-segments, which are marked with circles in the left part of Fig. 3. Ordering them from left to right, let x_j denote the distance from the j -th segment to T . If l is the number of crossings to the left of T , the numbers x_j are restricted by

$$0 \leq x_l < \cdots < x_1 \leq \min(C-1, b+c-A-1), \quad (3.8a)$$

$$0 \leq x_{l+1} < \cdots < x_b \leq \min(A-1, a+b-C-1). \quad (3.8b)$$

Applying the same affine map as in Section 3.1, the circled points are mapped to

$$R_j = \begin{cases} (C-x_j-1, A+x_j+m), & 1 \leq j \leq l, \\ (C+x_j+m, A-x_j-1), & l+1 \leq j \leq b. \end{cases}$$

The lattice paths split into a family of b paths starting at $(P_j)_{j=1}^b$ and ending at $(R_j)_{j=1}^b$ and a family of $b+m$ paths starting at $(S_j)_{j=1}^{b+m}$ and ending at $(Q_j)_{j=1}^{b+m}$, where

$$(S_1, \dots, S_{b+m}) = (R_1, \dots, R_l, P_{b+1}, \dots, P_{b+m}, R_{l+1}, \dots, R_b).$$

More explicitly, $S_j = (C+y_j, A+m-y_j-1)$, where

$$(y_1, \dots, y_{b+m}) = (-x_1-1, \dots, -x_l-1, 0, 1, \dots, m-1, x_{l+1}+m, \dots, x_b+m).$$

Applying Lemma 3.1 to both families, only the identity permutation contributes to (3.3), and we find that

$$Z(q) = \sum_{1 \leq j, k \leq b} \det(w(P_j; R_k))_{1 \leq j, k \leq b+m} \det(w(S_j; Q_k)), \quad (3.9)$$

where the sum is over all solutions to (3.8), for $0 \leq l \leq b$. In contrast to (3.4), (3.9) holds regardless of the parity of m .

The first determinant in (3.9) can be computed using Lemma 3.3. By the symmetry

$$w_Z((a, b); (c, d)) = w_{1-Z}((-c, -d); (-a, -b)),$$

where we indicate the Z -dependence in (3.2) by a subscript, the second determinant can be expressed as

$$(-1)^{\binom{b+m}{2}} \det_{1 \leq j, k \leq b+m} (w_{1-Z}(-a-b-m+j, 1-c-j; -C-y_k, 1-m-A+y_k)), \quad (3.10)$$

which is again computed by Lemma 3.3.

It will be convenient to write $f \sim g$ if, as a function of m , $f/g = h(q^{m/2})$, with h rational. We need to prove that

$$Z(q) \sim W^{a+b-C}, \quad (3.11)$$

where

$$W = \prod_{j=1}^m \frac{q^{\frac{j}{2}} + q^{-\frac{j}{2}}}{2} = 2^{-m} q^{-\frac{1}{2} \binom{m+1}{2}} (-q; q)_m.$$

Consider first (3.10), which is obtained by substituting

$$(m, x_1, x_2, y_1, y_2, l_k, Z) \\ \mapsto (b+m, 1-a-b-m, -C, 1-b-c-m, 1-m-A, -y_k, 2-2A-C-m)$$

in Lemma 3.3. Under this substitution, $2^{\binom{m}{2}-m(x_2-x_1)-|l|} \mapsto 2^{m(C-a-l)}$ and

$$q^X \sim q^{2\binom{m}{3} + \frac{4b-a+C-3l+2}{2} \binom{m+1}{2}}.$$

The factor $\Delta(q^l) \mapsto \prod_{1 \leq j < k \leq b+m} (q^{-y_k} - q^{-y_j})$ splits into six parts, depending on whether j and k belong to the interval $[1, l]$, $[l+1, l+m]$ or $[l+m+1, b+m]$. The three parts with neither j nor k in the middle interval are clearly rational in q^m . This leaves us with

$$\prod_{1 \leq j \leq l, 1 \leq k \leq m} (q^{1-k} - q^{x_j+1}) \prod_{1 \leq j < k \leq m} (q^{1-k} - q^{1-j}) \prod_{1 \leq j \leq m, 1 \leq k \leq b-l} (q^{-m-x_k} - q^{1-j}),$$

where the first factor can be written

$$q^{-l\binom{m}{2}} \prod_{j=1}^l (q^{x_j+1}; q)_m = q^{-l\binom{m}{2}} (q; q)_m^l \prod_{j=1}^l \frac{(q^{m+1}; q)_{x_j}}{(q; q)_{x_j}} \sim q^{-l\binom{m+1}{2}} (q; q)_m^l.$$

The second factor is equal to $q^{-2\binom{m}{3} - \binom{m+1}{2} + m} \tilde{H}_q(m)$ and the third factor equivalent to $q^{-2(b-l)\binom{m+1}{2}} (q; q)_m^{b-l}$. Next, we have

$$\prod_{j=1}^m (q; q)_{x_2+y_2-x_1-y_1-m+j} \mapsto \prod_{j=1}^{b+m} (q; q)_{B+j-1} \sim \tilde{H}_q(b+B+m), \\ \prod_{j=1}^m (q; q)_{x_2-x_1+l_j} \mapsto \prod_{j=1}^l (q; q)_{a+b-C+m+x_j} \prod_{j=1}^m (q; q)_{a+b-C+j-1} \prod_{j=l+1}^b (q; q)_{a+b-C-x_j-1} \\ \sim (q; q)_m^l \tilde{H}_q(a+b-C+m), \\ \prod_{j=1}^m (q; q)_{y_2-y_1-l_j} \mapsto \prod_{j=1}^l (q; q)_{b+c-A-x_j-1} \prod_{j=1}^m (q; q)_{b+c-A+j-1} \prod_{j=l+1}^b (q; q)_{b+c-A+x_j+m} \\ \sim (q; q)_m^{b-l} \tilde{H}_q(b+c-A+m).$$

By Lemma A.1,

$$\frac{\tilde{H}_q(m) \tilde{H}_q(b+B+m)}{\tilde{H}_q(a+b-C+m) \tilde{H}_q(b+c-A+m)} \sim 1.$$

Finally, we have

$$\prod_{j=1}^m (-q^{Z-x_2-y_1-y_2-m+j}; q)_{x_2-x_1-j+1+l_j} \mapsto \prod_{j=1}^l (-q^{c-A+j}; q)_{a+b-C+x_j+1+m-j} \\ \times \prod_{j=1}^m (-q^{c-A+l+j}; q)_{a+b-C+m-l+1-2j} \prod_{j=l+1}^b (-q^{c-A+m+j}; q)_{a+b-C-m-x_j-j},$$

where the first factor is equivalent to $(-q; q)_m^l$ and the third factor to $(-q; q)_m^{l-b}$. If $a+b \geq C+l$, the second factor can be expressed as

$$\prod_{j=1}^m (-q^{c-A+l+j}; q)_{a+b-C-l} = \prod_{j=1}^{a+b-C-l} (-q^{c-A+l+j}; q)_m \sim (-q; q)_m^{a+b-C-l}.$$

By a similar computation, this holds also for $a+b < C+l$. In conclusion, (3.10) is equivalent to W^{a+l-C} . Similarly, though with less effort, we find that the first determinant in (3.9) is equivalent to W^{b-l} , which gives (3.11). \square

3.4. Discrete Selberg integrals and Askey–Wilson polynomials

We recall some basic facts on Askey–Wilson polynomials [1]. Normalizing them to be monic (which is not the standard choice in the literature), they are given by

$$P_n(x) = P_n(x; a, b, c, d; q) \\ = \frac{(ab, ac, ad; q)_n}{2^n a^n (abcdq^{n-1}; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, a\xi, a/\xi \\ ab, ac, ad \end{matrix}; q, q \right), \quad (3.12)$$

where $x = (\xi + \xi^{-1})/2$. When $\max(|a|, |b|, |c|, |d|, |q|) < 1$, they satisfy the orthogonality relation

$$\int_{-1}^1 P_m(x) P_n(x) w(x) dx = C_n \delta_{mn},$$

where, using standard notation such as $(a\xi^\pm; q)_\infty = (a\xi; q)_\infty (a\xi^{-1}; q)_\infty$,

$$w(x) = w(x; a, b, c, d; q) = \frac{(\xi^{\pm 2}; q)_\infty}{(a\xi^\pm, b\xi^\pm, c\xi^\pm, d\xi^\pm; q)_\infty \sqrt{1-x^2}} \quad (3.13a)$$

and

$$C_n = C_n(a, b, c, d; q) \\ = \frac{2\pi (abcdq^{2n-1}, abcdq^{2n}; q)_\infty}{4^n (q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n, abcdq^{n-1}; q)_\infty}. \quad (3.13b)$$

The polynomial P_n is symmetric in the parameters a, b, c, d .

To link Askey–Wilson polynomials to discrete Selberg integrals, we will need the Cauchy–Binet identity

$$\det_{1 \leq j, k \leq m} \left(\sum_{l=0}^N A_{jl} B_{lk} \right) = \sum_{0 \leq l_1 < \dots < l_m \leq N} \det_{1 \leq j, k \leq m} (A_{j, l_k}) \det_{1 \leq j, k \leq m} (B_{l_k, j}) \quad (3.14)$$

as well as the determinant evaluation

$$\det_{1 \leq j, k \leq m} ((aq^{j-1}, bq^{m-j}; q)_{l_k}) = q^{\binom{m}{3}} b^{\binom{m}{2}} \prod_{j=1}^m \frac{(a, b; q)_{l_j} (q^{j-m} a/b; q)_{j-1}}{(a, b; q)_{j-1}} \Delta(q^l). \quad (3.15)$$

To prove the latter, we write the determinant as

$$\begin{aligned} & \det_{1 \leq j, k \leq m} \left(\frac{(a, b; q)_{l_k}}{(a; q)_{j-1} (b; q)_{m-j}} (aq^{l_k}; q)_{j-1} (bq^{l_k}; q)_{m-j} \right) \\ &= \prod_{j=1}^m \frac{(a, b; q)_{l_j}}{(a, b; q)_{j-1}} \det_{1 \leq j, k \leq m} ((aq^{l_k}; q)_{j-1} (bq^{l_k}; q)_{m-j}), \end{aligned}$$

which can be evaluated using [21, Lemma 2.2] or [33, Lemma A.1].

Proposition 3.6. *Let*

$$x_k = \frac{\xi q^{k-1} + \xi^{-1} q^{1-k}}{2}, \quad k = 1, \dots, m$$

and let $P_n(x)$ be the monic Askey–Wilson polynomial (3.12). Then,

$$\begin{aligned} \frac{\det_{1 \leq j, k \leq m} (P_{n+j-1}(x_k))}{\Delta(x)} &= \frac{1}{q^{2\binom{m}{3} + (n+1)\binom{m}{2}} (2a)^{mn}} \\ &\times \prod_{j=1}^m \frac{(ab, ac, ad; q)_{n+j-1}}{(abcdq^{n-1}; q)_{n+j-1} (q, q^{1-m-n}, a\xi, a\xi^{-1} q^{1-m}; q)_{j-1}} \\ &\times {}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, abcdq^{n-1}, a\xi, aq^{1-m}\xi^{-1} \\ ab, ac, ad \end{matrix}; q, q \right). \quad (3.16) \end{aligned}$$

Proof. Since

$$\begin{aligned} P_{n+j-1}(x_k) &= \frac{(ab, ac, ad; q)_{n+j-1}}{(2a)^{n+j-1} (abcdq^{n+j-2}; q)_{n+j-1}} \\ &\times \sum_{l \geq 0} \frac{(q^{1-j-n}, abcdq^{n+j-2}, a\xi q^{k-1}, a\xi^{-1} q^{1-k}; q)_l}{(q, ab, ac, ad; q)_l} q^l, \end{aligned}$$

expanding the left-hand side of (3.16) using (3.14) gives

$$\prod_{j=1}^m \frac{(ab, ac, ad; q)_{n+j-1}}{(2a)^{n+j-1}(abcdq^{n+j-2}; q)_{n+j-1}} \sum_{0 \leq l_1 < \dots < l_m \leq n+m-1} \prod_{j=1}^m \frac{q^{l_j}}{(q, ab, ac, ad; q)_{l_j}} \\ \times \det_{1 \leq j, k \leq m} ((q^{1-j-n}, abcdq^{n+j-2}; q)_{l_k}) \det_{1 \leq j, k \leq m} ((a\xi q^{j-1}, a\xi^{-1}q^{1-j}; q)_{l_k}).$$

Applying (3.15) and simplifying, using also

$$\Delta(x) = \frac{\prod_{j=1}^m (q, q^{j-1}\xi^2; q)_{j-1}}{q^{2\binom{m}{3} + \binom{m}{2}} (2\xi)^{\binom{m}{2}}},$$

completes the proof. \square

We note that, since the Askey–Wilson polynomial is symmetric in its parameters, the right-hand side of (3.16) is invariant under interchanging a and b . This proves the following multiple Sears' transformation, which is a very special case of a transformation for discrete elliptic Selberg integrals conjectured by Warnaar [35] and proved by Rains [32]. We will use this transformation in Section 3.5.

Corollary 3.7 (Rains). *If $q^{1-n}abc = def$, then*

$${}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, a, b, c \\ dq^{m-1}, e, f \end{matrix}; q, q \right) = \left(\frac{bc}{d} \right)^{mn} \prod_{j=1}^m \frac{(b, c; a)_{j-1} (de/bc, df/bc; q)_{n+j-1}}{(d/b, d/c; q)_{j-1} (e, f; q)_{n+j-1}} \\ \times {}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, a, d/b, d/c \\ dq^{m-1}, de/bc, df/bc \end{matrix}; q, q \right).$$

3.5. Continuous Selberg integrals

Let μ be a linear functional on $\mathbb{C}[x]$, which we write as a formal integral

$$\mu(p) = \int p(x) d\mu(x).$$

We will assume that μ is non-degenerate in the sense that there exist monic polynomials p_n of degree n such that

$$\mu(p_m p_n) = C_n \delta_{mn}. \quad (3.17)$$

We do not require any positivity condition for μ . We then have the identity

$$\int \Delta(x)^2 \prod_{j=1}^n \prod_{k=1}^m (y_k - x_j) d\mu(x_1) \cdots d\mu(x_n) \\ = n! C_0 \cdots C_{n-1} \frac{\det_{1 \leq j, k \leq m} (p_{n+j-1}(y_k))}{\Delta(y)}, \quad (3.18)$$

relating a Selberg-type integral to a determinant of orthogonal polynomials. This identity can be obtained by combining two classical results [20, Thm. 2.1.2 and Thm. 2.7.1] due to Heine and Christoffel. More explicitly, it appears in [3]. A direct proof is very easy; simply write the integrand as

$$\frac{\Delta(x)\Delta(x,y)}{\Delta(y)},$$

expand both factors in the numerator using $\Delta(x) = \det(p_{j-1}(x_i))$ and then integrate using (3.17). We can apply (3.18) to prove the following quadratic transformation formula for determinants of Askey–Wilson polynomials.

Theorem 3.8. *Let m, n, M and N be non-negative integers, with m even, and let a, b and q be generic parameters. Let p_n and q_n denote the monic polynomials*

$$p_n(x) = P_n(x; aq^M, -a, bq^N, -b; q),$$

$$q_n(x) = \begin{cases} 2^{-n/2} P_{n/2}(2x^2 - 1; -1, -q^{m+1}, a^2, b^2; q^2), & n \text{ even,} \\ 2^{-(n-1)/2} x P_{(n-1)/2}(2x^2 - 1; -q^2, -q^{m+1}, a^2, b^2; q^2), & n \text{ odd} \end{cases}$$

and let

$$y_k = \frac{\eta_k + \eta_k^{-1}}{2}, \quad \eta_k = iq^{k - \frac{m+1}{2}}, \quad k = 1, \dots, m,$$

$$z_k = \frac{\zeta_k + \zeta_k^{-1}}{2}, \quad \zeta_k = \begin{cases} aq^{k-1}, & k = 1, \dots, M, \\ bq^{k-M-1}, & k = M+1, \dots, M+N. \end{cases}$$

Then,

$$\frac{\det_{1 \leq j, k \leq m} (p_{n+j-1}(y_k))}{\Delta(y)} = C \frac{\det_{1 \leq j, k \leq M+N} (q_{n+j-1}(z_k))}{\Delta(z)}, \quad (3.19)$$

where

$$C = \left(2^{M+N-m} q^{\binom{M}{2} + \binom{N}{2} - \frac{m^2}{4}} a^M b^N \right)^n$$

$$\times \prod_{j=1}^n \left(\frac{(q^{2[j/2]+1}, -a^2 q^{2[(j-1)/2]+1}, -b^2 q^{2[(j-1)/2]+1}, a^2 b^2 q^{2[j/2]-1}; q^2)_{\frac{m}{2}}}{(a^2 b^2 q^{2j-3}, a^2 b^2 q^{2j-1}; q^2)_{\frac{m}{2}}} \right)$$

$$\times \frac{(a^2 b^2 q^{2j-3}, a^2 b^2 q^{2j-2}; q)_{M+N}}{(abq^{j-1}, a^2 b^2 q^{j-2}; q)_{M+N} (-a^2 q^{j-1}, -abq^{j-1}; q)_M (-b^2 q^{j-1}, -abq^{j-1}; q)_N}.$$

Proof. Let L denote the left-hand side of (3.19). Since (3.19) is a rational identity, we may assume that $\max(|a|, |b|, |q|) < 1$. We then apply (3.18) to write

$$L = \frac{1}{n! \prod_{k=0}^{n-1} C_k} \int_{[-1,1]^n} \Delta(x)^2 \prod_{j=1}^n \prod_{k=1}^m (y_k - x_j) \prod_{j=1}^n w(x_j) dx_j, \quad (3.20)$$

where w and C_k are obtained by substituting $(a, b, c, d) \mapsto (aq^M, -a, bq^N, -b)$ in (3.13).

We will now rewrite (3.20) in such a way that the roles of m and $M + N$ are interchanged. In the orthogonality measure, we write

$$\frac{(\xi^{\pm 2}; q)_{\infty}}{(aq^M \xi^{\pm}, -a\xi^{\pm}, bq^N \xi^{\pm}, -b\xi^{\pm}; q)_{\infty}} = \frac{(a\xi^{\pm}; q)_M (b\xi^{\pm}; q)_N (\xi^{\pm 4}; q^2)_{\infty}}{(-\xi^{\pm 2}, -q\xi^{\pm 2}, a^2\xi^{\pm 2}, b^2\xi^{\pm 2}; q^2)_{\infty}}$$

and observe that

$$(a\xi^{\pm}; q)_M (b\xi^{\pm}; q)_N = 2^{M+N} q^{\binom{M}{2} + \binom{N}{2}} a^M b^N \prod_{k=1}^{M+N} (z_k - x).$$

We also write

$$\prod_{k=1}^m (y_k - x) = \frac{(iq^{\frac{1-m}{2}} \xi^{\pm}; q)_m}{2^m i^m} = 2^{-m} q^{-\frac{m^2}{4}} \frac{(-q\xi^{\pm 2}; q^2)_{\infty}}{(-q^{m+1}\xi^{\pm 2}; q^2)_{\infty}}.$$

Combining these facts, we find that

$$L = \frac{D}{n! \prod_{k=0}^{n-1} C_k} \int_{[-1,1]^n} \Delta(x)^2 \prod_{j=1}^n \prod_{k=1}^{M+N} (z_k - x_j) \prod_{j=1}^n \tilde{w}(x_j) dx_j, \quad (3.21)$$

where

$$\tilde{w}(x) = \frac{(\xi^{\pm 4}; q^2)_{\infty}}{(-\xi^{\pm 2}, -q^{m+1}\xi^{\pm 2}, a^2\xi^{\pm 2}, b^2\xi^{\pm 2}; q^2)_{\infty} \sqrt{1-x^2}}$$

and

$$D = \left(2^{M+N-m} q^{\binom{M}{2} + \binom{N}{2} - \frac{m^2}{4}} a^M b^N \right)^n.$$

We now apply (3.18) to the integral (3.21). Let

$$\mu(p) = \int_{-1}^1 p(x) \tilde{w}(x) dx.$$

Then,

$$\begin{aligned} \tilde{w}(x) dx &= \frac{1}{2} w(y; -1, -q^{m+1}, a^2, b^2; q^2) dy, \\ x^2 \tilde{w}(x) dx &= \frac{1}{8} w(y; -q^2, -q^{m+1}, a^2, b^2; q^2) dy, \end{aligned}$$

where $y = 2x^2 - 1$. It follows that the polynomials $q_n(x)$ satisfy $\mu(q_m q_n) = \tilde{C}_n \delta_{mn}$, with

$$\tilde{C}_n = \begin{cases} 4^{-k} C_k(-1, -q^{m+1}, a^2, b^2; q^2), & n = 2k, \\ 4^{-k-1} C_k(-q^2, -q^{m+1}, a^2, b^2; q^2), & n = 2k + 1. \end{cases}$$

Thus, (3.18) gives

$$L = D \prod_{j=1}^n \frac{\tilde{C}_{j-1}}{C_{j-1}} \frac{\det_{1 \leq j, k \leq M+N} (q_{n+j-1}(z_k))}{\Delta(z)}.$$

Simplifying the expression for \tilde{C}_{j-1}/C_{j-1} , we arrive at the desired result. \square

We will now combine (3.16) and (3.19). Let us substitute $(a, b, c, d, \xi) \mapsto (i\alpha q^M, -i\alpha, i\beta q^N, -i\beta, iq^{(1-m)/2})$ in (3.16), where M and N are non-negative integers. Then, the left-hand side of (3.16) equals the left-hand side of (3.19), under the substitutions $(a, b) \mapsto (i\alpha, i\beta)$. We rewrite the matrix entries $q_{n+j-1}(z_k)$ in terms of ${}_4\phi_3$ series, using (3.12) with the distinguished parameter a chosen as $-\alpha^2$ for $k \leq M$ and as $-\beta^2$ for $k \geq M + 1$. Then, the matrix entries in (3.19) can be identified with those in (2.7). More precisely,

$$q_{n+j-1}(z_k) = C_j D_k Q_{jk}^{MNn}(\alpha, \beta, q^{\frac{m+1}{2}}; q),$$

where

$$C_j = \frac{(i/2)^{n+j-1} q^{\frac{m+1}{2} \left[\frac{n+j-1}{2} \right] + \left[\frac{(n+j-2)^2}{4} \right]}{(\alpha^2 \beta^2 q^2; q^2)_{[(n+j-1)/2]}},$$

$$D_k = \begin{cases} \frac{(\alpha\beta)^{k-1} q^{\binom{k-1}{2}}}{(\alpha^2 \beta^2; q^2)_{k-1}}, & k \leq M, \\ \frac{(\alpha\beta)^{k-M-1} q^{\binom{k-M-1}{2}}}{(\alpha^2 \beta^2; q^2)_{k-M-1}}, & k \geq M + 1. \end{cases}$$

We compute

$$\Delta(z) = \frac{\prod_{k=1}^M (q, -q^{j-1}\alpha^2; q)_{j-1} (q^{j-M}\beta/\alpha, -q^{j-1}\alpha\beta; q)_N \prod_{j=1}^N (q, -q^{j-1}\beta^2; q)_{j-1}}{2^{\binom{M+N}{2}} i^{\binom{M+N}{2}} q^{2\binom{M}{3} + 2\binom{N}{3} + \binom{M}{2} + (M+1)\binom{N}{2}} (\alpha^2)^{\binom{M}{2}} \beta^{\binom{N}{2} + MN}}$$

and simplify the factors involving $\alpha\beta$ using

$$\frac{1}{\prod_{j=1}^M (-\alpha\beta q^{j-1}; q)_N (\alpha^2 \beta^2; q^2)_{j-1} \prod_{j=1}^N (\alpha^2 \beta^2; q^2)_{j-1}} \\ \times \prod_{j=1}^{M+N} \frac{(\alpha^2 \beta^2; q^2)_{[(n+j-1)/2]}}{(\alpha^2 \beta^2 q^{m+2[(n+j)/2]-1}; q^2)_{[(n+j-1)/2]}} \prod_{j=1}^n \frac{(\alpha^2 \beta^2 q^{2j-3}, \alpha^2 \beta^2 q^{2j-2}; q)_{M+N}}{(\alpha^2 \beta^2 q^{2j-1}, \alpha^2 \beta^2 q^{2j-3}; q^2)^{\frac{m}{2}}} \\ \times \prod_{j=1}^n \frac{(\alpha^2 \beta^2 q^{2[j/2]-1}; q^2)^{\frac{m}{2}}}{(-\alpha\beta q^{j-1}, \alpha^2 \beta^2 q^{j-2}; q)_{M+N} (\alpha\beta q^{j-1}; q)_M (\alpha\beta q^{j-1}; q)_N}$$

$$\begin{aligned}
&= \frac{\prod_{j=1}^M (\alpha\beta q^{n+j-1}; q)_N}{\prod_{j=1}^{M+N} (\alpha^2\beta^2 q^{2n+j-2}; q)_{j-1}} \\
&\quad \times \prod_{j=1}^{m/2} \frac{(\alpha^2\beta^2 q^{2j-3}; q^2)_{\lfloor \frac{M+N+n+1}{2} \rfloor} (\alpha^2\beta^2 q^{2j-1}; q^2)_{\lfloor \frac{M+N+n}{2} \rfloor}}{(\alpha^2\beta^2 q^{2j-3}; q^2)_{M+N+n} (\alpha^2\beta^2 q^{2j-1}; q^2)_n}.
\end{aligned}$$

In the exponent of q , we use

$$\begin{aligned}
&\sum_{j=1}^{M+N} \left(\frac{m+1}{2} \left\lfloor \frac{n+j-1}{2} \right\rfloor + \left\lfloor \frac{(n+j-2)^2}{4} \right\rfloor \right) \\
&= \frac{1}{2} \left(\binom{M+N+n}{3} - \binom{n}{3} \right) + \frac{m}{2} \left(\left\lfloor \frac{(M+N+n-1)^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \right).
\end{aligned}$$

This yields the following result.

Corollary 3.9. For m, n, M and N non-negative integers, with m even,

$$\begin{aligned}
&{}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, \alpha^2\beta^2 q^{M+N+n-1}, \alpha q^{M-\frac{m}{2}+\frac{1}{2}}, -\alpha q^{M-\frac{m}{2}+\frac{1}{2}} \\ \alpha^2 q^M, \alpha\beta q^M, -\alpha\beta q^{M+N} \end{matrix}; q, q \right) \\
&= (-1)^{\binom{M+N}{2}+n(M+N+\frac{m}{2})} \alpha^{(M+m)n+2\binom{M}{2}+\binom{N}{2}} \beta^{(M+n)N+\binom{M}{2}+2\binom{N}{2}} q^X \\
&\quad \times \prod_{j=1}^M \frac{(\alpha\beta q^{j+n-1}; q)_N}{(q, -\alpha^2 q^{j-1}; q)_{j-1} (q^{j-M}\beta/\alpha; q)_N} \prod_{j=1}^N \frac{1}{(q, -\beta^2 q^{j-1}; q)_{j-1}} \\
&\quad \times \prod_{j=1}^{m/2} \frac{(\alpha^2\beta^2 q^{2j-3}; q^2)_{\lfloor (M+N+n+1)/2 \rfloor} (\alpha^2\beta^2 q^{2j-1}; q^2)_{\lfloor (M+N+n)/2 \rfloor}}{(\alpha^2\beta^2 q^{2j-3}; q^2)_{M+N+n} (\alpha^2\beta^2 q^{2j-1}; q^2)_n} \\
&\quad \times \prod_{j=1}^m \frac{(\alpha^2\beta^2 q^{M+N+n-1}; q)_{n+j-1} (q, q^{1-m-n}; q)_{j-1} (\alpha^2 q^{2M-m+1}; q^2)_{j-1}}{(\alpha^2 q^M, \alpha\beta q^M, -\alpha\beta q^{M+N}; q)_{n+j-1}} \\
&\quad \times \prod_{j=1}^n \frac{(q^{2\lfloor j/2 \rfloor+1}, \alpha^2 q^{2\lfloor (j-1)/2 \rfloor+1}, \beta^2 q^{2\lfloor (j-1)/2 \rfloor+1}; q^2)_{\frac{m}{2}}}{(\alpha^2 q^{j-1}; q)_M (\beta^2 q^{j-1}; q)_N} \prod_{j=1}^{M+N} \frac{1}{(\alpha^2\beta^2 q^{j+2n-2}; q)_{j-1}} \\
&\quad \times Q^{MNn}(\alpha, \beta, q^{\frac{m+1}{2}}; q),
\end{aligned}$$

where

$$\begin{aligned}
X &= 2\binom{m}{3} + \binom{m}{2} + 3\binom{M}{3} + 3\binom{N}{3} + \binom{M}{2} + (M+1)\binom{N}{2} \\
&\quad + \left(\binom{M}{2} + \binom{N}{2} + m \left(M + \frac{m}{4} - \frac{1}{2} \right) \right) n + \frac{1}{2} \left(\binom{M+N+n}{3} - \binom{n}{3} \right) \\
&\quad + \frac{m}{2} \left(\left\lfloor \frac{(M+N+n-1)^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \right).
\end{aligned}$$

We now reformulate [Corollary 3.9](#) in a way that will be adapted to our purpose. We first apply the identity

$$\begin{aligned} & \prod_{j=1}^M \frac{1}{(-\alpha^2 q^{j-1}; q)_{j-1}} \prod_{j=1}^n \frac{(\alpha^2 q^{2[(j-1)/2]+1}; q^2)_{\frac{m}{2}}}{(\alpha^2 q^{j-1}; q)_M} \\ &= (-1)^{n(M+\frac{m}{2})} \alpha^{mn-M(M-1+2n)} q^{\frac{nm^2}{4}+m[\frac{(n-1)^2}{4}]-3\binom{M}{3}-(n+1)\binom{M}{2}-M\binom{n}{2}} \\ & \quad \times \prod_{j=1}^M \frac{1}{(-q^{j+1-2M}/\alpha^2; q)_{j-1}} \prod_{j=1}^n \frac{(q^{-2[(j-1)/2]+1-m}/\alpha^2; q^2)_{\frac{m}{2}}}{(q^{j+1-M-n}/\alpha^2; q)_M}. \end{aligned}$$

We then multiply the left-hand side by

$$q^{-\frac{1}{2}mnM} \prod_{j=1}^m \frac{(\alpha\beta q^M, -\alpha\beta q^{M+N}; q)_{n+j-1}}{(\alpha^2 q^{2M-m+1}; q^2)_{j-1}}$$

and make the change of variables

$$(M, N, \alpha, \beta) \mapsto \left(|M|, |N|, \alpha q^{-\frac{|M|}{2}}, -\operatorname{sgn}(MN)\beta q^{-\frac{|N|}{2}} \right),$$

where we no longer require M and N to be non-negative. Thus, for $M \geq 0$ we consider

$$\begin{aligned} & q^{-\frac{1}{2}mnM} \prod_{j=1}^m \frac{(\alpha\beta q^{\frac{M+N}{2}}, -\alpha\beta q^{\frac{M-N}{2}}; q)_{n+j-1}}{(\alpha^2 q^{M-m+1}; q^2)_{j-1}} \\ & \quad \times {}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, \alpha^2 \beta^2 q^{n-1}, \alpha q^{\frac{M}{2}-\frac{m}{2}+\frac{1}{2}}, -\alpha q^{\frac{M}{2}-\frac{m}{2}+\frac{1}{2}} \\ \alpha^2, \alpha\beta q^{\frac{M}{2}+\frac{N}{2}}, -\alpha\beta q^{\frac{M}{2}-\frac{N}{2}} \end{matrix}; q, q \right) \end{aligned} \quad (3.22)$$

and for $M < 0$ the same quantity with $(M, \beta) \mapsto (-M, -\beta)$. However, by [Corollary 3.7](#), [\(3.22\)](#) is invariant under this transformation. Thus, we may take the left-hand side as [\(3.22\)](#) regardless of the sign of M . This leads to the following result.

Corollary 3.10. *For m, n, M and N integers, with m and n non-negative and m even, and $\varepsilon = \operatorname{sgn}(MN)$,*

$$\begin{aligned} & {}_4\phi_3^{(m)} \left(\begin{matrix} q^{1-m-n}, \alpha^2 \beta^2 q^{n-1}, \alpha q^{\frac{M}{2}-\frac{m}{2}+\frac{1}{2}}, -\alpha q^{\frac{M}{2}-\frac{m}{2}+\frac{1}{2}} \\ \alpha^2, \alpha\beta q^{\frac{M}{2}+\frac{N}{2}}, -\alpha\beta q^{\frac{M}{2}-\frac{N}{2}} \end{matrix}; q, q \right) \\ &= (-1)^{\binom{|N|}{2}} \varepsilon^{\binom{|M|}{2}+N(n+M)} \alpha^{(2m-|M|)n+\binom{|N|}{2}} \beta^{n|N|+\binom{|M|+|N|}{2}+\binom{|N|}{2}} q^X \\ & \quad \times \prod_{j=1}^{|M|} \frac{(-\varepsilon \alpha \beta q^{j+n-1-\frac{|M|}{2}-\frac{|N|}{2}}; q)_{|N|}}{(q, -q^{j+1-|M|}/\alpha^2; q)_{j-1} (-\varepsilon q^{j-\frac{|M|}{2}-\frac{|N|}{2}} \beta/\alpha; q)_{|N|}} \\ & \quad \times \prod_{j=1}^{m/2} \frac{(\alpha^2 \beta^2 q^{2j-3-|M|-|N|}; q^2)_{\lceil \frac{|M|+|N|+n+1}{2} \rceil} (\alpha^2 \beta^2 q^{2j-1-|M|-|N|}; q^2)_{\lceil \frac{|M|+|N|+n}{2} \rceil}}{(\alpha^2 \beta^2 q^{2j-3-|M|-|N|}; q^2)_{|M|+|N|+n} (\alpha^2 \beta^2 q^{2j-1-|M|-|N|}; q^2)_n} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{j=1}^m \frac{(\alpha^2 \beta^2 q^{n-1}; q)_{n+j-1} (q, q^{1-m-n}; q)_{j-1} (\alpha^2 q^{M-m+1}; q^2)_{j-1}}{(\alpha^2, \alpha \beta q^{\frac{M+N}{2}}, -\alpha \beta q^{\frac{M-N}{2}}; q)_{n+j-1}} \\
& \times \prod_{j=1}^n \frac{(q^{2[j/2]+1}, q^{-2[(j-1)/2]+1-m+|M|} / \alpha^2, \beta^2 q^{-|N|+2[(j-1)/2]+1}, q^2)_{\frac{m}{2}}}{(q^{j+1-n} / \alpha^2; q)_{|M|} (\beta^2 q^{j-1-|N|}; q)_{|N|}} \\
& \times \prod_{j=1}^{|M|+|N|} \frac{1}{(\alpha^2 \beta^2 q^{j+2n-2-|M|-|N|}; q)_{j-1}} \prod_{j=1}^{|N|} \frac{1}{(q, -q^{j-|N|-1} \beta^2; q)_{j-1}} \\
& \times Q^{|M|, |N|, n}(\alpha q^{-\frac{|M|}{2}}, -\varepsilon \beta q^{-\frac{|N|}{2}}, q^{\frac{m+1}{2}}),
\end{aligned}$$

where

$$\begin{aligned}
X = & 2 \binom{m}{3} + \binom{m}{2} - \frac{1}{4} |N| (|M|^2 + |MN| + 2|N| - 2) \\
& + \frac{1}{2} \{m(m+M-1) + |M|(|M|+1-m-n) - |N|\} n \\
& + \frac{1}{2} \left(\binom{|M|+|N|+n}{3} - \binom{n}{3} \right) \\
& + \frac{m}{2} \left(\left\lceil \frac{(|M|+|N|+n-1)^2}{4} \right\rceil + \left\lceil \frac{(n-1)^2}{4} \right\rceil \right).
\end{aligned}$$

3.6. Final steps

We can now complete the proof of [Theorem 2.1](#). Assume first that m is even. Combining [Corollary 3.2](#), [Proposition 3.4](#) and [Corollary 3.10](#), with the substitutions

$$(\alpha, \beta, n) \mapsto (q^{\frac{1}{2}(1-b-c-m)}, q^{\frac{1}{2}(1+a+c+m)}, b),$$

yields [\(2.10\)](#), with

$$\begin{aligned}
C = & \frac{(-1)^{\binom{|N|}{2}} \varepsilon^{\binom{|M|}{2} + N(n+M)} q^X}{2^{\frac{1}{2}m(a+b+M+N)+ab}} \prod_{j=1}^b \frac{(q; q)_{a+c+m+j-1} (q; q)_{j-1} (-q^{\frac{1}{2}(-a-b+M-N)+j}; q)_a}{(q; q)_{a+j-1} (q; q)_{c+m+j-1}} \\
& \times \prod_{j=1}^m \frac{(q; q)_{j-1} (q^2; q^2)^{\frac{1}{2}(a+c+N)+j-1}}{(q; q)_{\frac{a+b+M+N}{2}+j-1} (q^2; q^2)^{\frac{b+c-M}{2}+j-1}} \\
& \times \prod_{j=1}^{|M|} \frac{(-\varepsilon q^{j+\frac{a}{2}+\frac{b}{2}-\frac{|M|}{2}-\frac{|N|}{2}}; q)_{|N|}}{(q, -q^{j+b+c-|M|+m}; q)_{j-1} (-\varepsilon q^{j+\frac{a}{2}+\frac{b}{2}+c+m-\frac{|M|}{2}-\frac{|N|}{2}}; q)_{|N|}} \\
& \times \prod_{j=1}^{m/2} \frac{(q^{2j-1+a-b-|M|-|N|}; q^2)^{\lceil \frac{|M|+|N|+b+1}{2} \rceil} (q^{2j+1+a-b-|M|-|N|}; q^2)^{\lceil \frac{|M|+|N|+b}{2} \rceil}}{(q^{2j-1+a-b-|M|-|N|}; q^2)_{|M|+|N|+b} (q^{2j+1+a-b-|M|-|N|}; q^2)_b}
\end{aligned}$$

$$\begin{aligned} & \times \prod_{j=1}^b \frac{(q^{2[j/2]+1}, q^{-2[(j-1)/2]+b+c+|M|}, q^{a+c+m-|N|+2[(j-1)/2]+2}; q^2)_{\frac{m}{2}}}{(q^{j+c+m}; q)_{|M|} (q^{j+a+c+m-|N|}; q)_{|N|}} \\ & \times \prod_{j=1}^{|M|+|N|} \frac{1}{(q^{j+a+b-|M|-|N|}; q)_{j-1}} \prod_{j=1}^{|N|} \frac{1}{(q, -q^{j+a+c+m-|N|}; q)_{j-1}} \end{aligned}$$

for a certain exponent X .

We now express C in terms of the function \tilde{H}_q . The product $\prod_{j=1}^{m/2}$ is equal to

$$\begin{aligned} & \prod_{j=1}^{m/2} \frac{(q; q^2)_{[(a+1)/2]+j-1} (q; q^2)_{[a/2]+j}}{(q; q^2)_{(a+b+|M|+|N|)/2+j-1} (q; q^2)_{(a+b-|M|-|N|)/2+j}} \\ & = \frac{\tilde{H}_{q^2} \left(\left[\frac{a+1}{2} \right] + \frac{m-1}{2}, \left[\frac{a}{2} \right] + \frac{m+1}{2}, \frac{a+b+|M|+|N|-1}{2}, \frac{a+b-|M|-|N|+1}{2} \right)}{\tilde{H}_{q^2} \left(\left[\frac{a+1}{2} \right] - \frac{1}{2}, \left[\frac{a}{2} \right] + \frac{1}{2}, \frac{a+b+|M|+|N|+m-1}{2}, \frac{a+b-|M|-|N|+m+1}{2} \right)}. \quad (3.23) \end{aligned}$$

All other factors can be converted using [Lemma A.4](#) and [Lemma A.5](#). The end result can then be simplified using [Lemma A.3](#). For instance, the factor $\tilde{H}_q(a)$ appearing from $\prod_{j=1}^b (q; q)_{a+j-1}^{-1}$ combines with factors from (3.23) as

$$\frac{\tilde{H}_q(a)}{\tilde{H}_{q^2} \left(\left[\frac{a+1}{2} \right] - \frac{1}{2}, \left[\frac{a}{2} \right] + \frac{1}{2} \right)} = \tilde{H}_{q^2} \left(\left[\frac{a}{2} \right], \left[\frac{a+1}{2} \right] \right).$$

We also observe that

$$\begin{aligned} \prod_{j=1}^{|M|} (-\varepsilon q^{j+\frac{a+b-|M|-|N|}{2}}; q)_{|N|} &= \frac{\tilde{H}_q^{-\varepsilon} \left(\frac{a+b-|M|-|N|}{2}, \frac{a+b+|M|+|N|}{2} \right)}{\tilde{H}_q^{-\varepsilon} \left(\frac{a+b+|M|-|N|}{2}, \frac{a+b-|M|+|N|}{2} \right)} \\ &= \frac{\tilde{H}_{q^2} \left(\frac{a+b-|M|-|N|}{2}, \frac{a+b+|M|+|N|}{2} \right)}{\tilde{H}_q \left(\frac{a+b-M-N}{2}, \frac{a+b+M+N}{2} \right) \tilde{H}_q^{-} \left(\frac{a+b+M-N}{2}, \frac{a+b-M+N}{2} \right)}, \end{aligned}$$

where most factors on the right eventually cancel. Applying [Lemma A.5](#) to

$$\prod_{j=1}^b (-q^{\frac{1}{2}(-a-b+M-N)+j}; q)_a$$

leads, apart from functions \tilde{H}_q^{-} and a power of q , to the factor

$$D = \frac{D_{(a-b+M-N)/2} D_{(-a+b+M-N)/2}}{D_{(-a-b+M-N)/2} D_{(a+b+M-N)/2}},$$

where $D_k = 2^{\min(k,0)}$. Since $-a-b+M-N < 0$ and $a+b+M-N > 0$, it is easy to see that $D = 2^{\frac{a+b}{2} - \frac{1}{2} \max(|a-b|, |M-N|)}$. Finally, we express \tilde{H}_q in terms of H_q . In this

way, we find that (2.10) holds up to some factor q^X , where X is independent of q . Since all functions in (2.10) are invariant up to a sign under replacing q by q^{-1} (here we use (2.9)), we must have $X = 0$. This proves Theorem 2.1 in the case when m is even.

If m is odd we invoke Lemma 3.5. Since a rational function is determined by infinitely many values, it is enough to prove that the right-hand side of (2.10), considered as a function of m , has the same form as the right-hand side of (3.7). Since (2.7) is a Laurent polynomial in α, β, γ , the second factor in (2.10) is rational in $q^{m/2}$. The final factor in (2.11), involving $H_q^{-\varepsilon}$, is rational in q^m by Lemma A.3. Consider now the remaining q -hyperfactorials in (2.11). We rewrite H_q^- as H_{q^2}/H_q and then apply Lemma A.3 to all factors of the form $f(x+m)$, where $f = H_q$ or H_{q^2} . This leads to a product of the form

$$\prod_{j=1}^6 \frac{H_{q^2}(a_j + \frac{m}{2})}{H_{q^2}(b_j + \frac{m}{2})} \prod_{j=1}^{10} \frac{H_{q^2}(c_j + \frac{m-1}{2})}{H_{q^2}(d_j + \frac{m-1}{2})} \prod_{j=1}^{10} \frac{H_{q^4}(e_j + \frac{m}{2})}{H_{q^4}(f_j + \frac{m}{2})} \prod_{j=1}^{10} \frac{H_{q^4}(g_j + \frac{m-1}{2})}{H_{q^4}(h_j + \frac{m-1}{2})},$$

where a_1, \dots, h_{10} are all integers. One may check that

$$\begin{aligned} \sum_{j=1}^6 (b_j - a_j) &= \sum_{j=1}^{10} (d_j - c_j) = \sum_{j=1}^{10} (e_j - f_j) = \sum_{j=1}^{10} (g_j - h_j) \\ &= \frac{a + b + M + N}{2} = a + b - C. \end{aligned}$$

Applying Lemma A.2, it follows that the right-hand side of (2.10) indeed behaves as (3.11) as a function of m . This completes the proof of Theorem 2.1.

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Appendix A. The q -hyperfactorial

In this Appendix, we collect some elementary properties of the q -hyperfactorials H_q and \tilde{H}_q defined in (2.6).

Lemma A.1. *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be non-negative integers, such that $\sum a_j = \sum b_j$ and let H be any one of the functions H_q, H_q^-, \tilde{H}_q or \tilde{H}_q^- . Then, there exists a rational function f such that*

$$\prod_{k=1}^n \frac{H(a_k + \frac{m}{2})}{H(b_k + \frac{m}{2})} = f(q^{m/2}) \quad (\text{A.1})$$

for each non-negative integer m .

Proof. It is easy to check that

$$\begin{aligned} \tilde{H}_q\left(a + \frac{m}{2}\right) &= \begin{cases} \tilde{H}_q(m/2)(q; q)_{m/2}^a \prod_{j=1}^a (q^{1+m/2}; q)_{j-1}, & m \text{ even,} \\ \tilde{H}_q((m+1)/2)(q^{1/2}; q)_{(m+1)/2}^a \prod_{j=1}^a (q^{1+m/2}; q)_{j-1}, & m \text{ odd.} \end{cases} \end{aligned} \quad (\text{A.2})$$

It follows that (A.1) holds for $H = \tilde{H}_q$, with

$$f(x) = \prod_{k=1}^n \frac{\prod_{j=1}^{a_k} (qx; q)_{j-1}}{\prod_{j=1}^{b_k} (qx; q)_{j-1}}.$$

Let us now replace \tilde{H}_q by H_q . If m is even, (A.1) is multiplied with $q^{Q(m)/2}$, where Q is the polynomial

$$Q(m) = \sum_{k=1}^n \left(\binom{b_k + \frac{m}{2} + 1}{3} - \binom{a_k + \frac{m}{2} + 1}{3} \right).$$

Note that the cubic and quadratic terms in Q cancel, so $q^{Q(m)/2}$ is a rational function in $q^{m/2}$. Moreover, since $\frac{1}{8} \binom{2m+1}{3} = \binom{m+1}{3} + \frac{m}{8}$, the same multiplier appears when m is odd. This proves the case $H = H_q$. The remaining two cases follow since $H_q^- = H_{q^2}/H_q$ and $\tilde{H}_q^- = \tilde{H}_{q^2}/\tilde{H}_q$. \square

We will also need the following variation of Lemma A.1.

Lemma A.2. Let a_j, b_j, c_j and d_j be non-negative integers such that

$$\sum_{j=1}^k (a_j - b_j) = \sum_{j=1}^l (c_j - d_j) = \lambda.$$

Then there exists a rational function f such that

$$\prod_{j=1}^k \frac{H_q(a_j + \frac{m}{2})}{H_q(b_j + \frac{m}{2})} \prod_{j=1}^l \frac{H_q(c_j + \frac{m-1}{2})}{H_q(d_j + \frac{m-1}{2})} = \left(q^{-\frac{1}{4} \binom{m+1}{2}} (q^{1/2}; q^{1/2})_m \right)^\lambda f(q^{m/2})$$

for each non-negative integer m .

Proof. It follows from (A.2) that

$$\prod_{j=1}^k \frac{H_q(a_j + \frac{m}{2})}{H_q(b_j + \frac{m}{2})} = \begin{cases} \left(q^{-\frac{1}{16}(m+2)m} (q; q)_{m/2} \right)^\lambda f(q^{m/2}), & m \text{ even,} \\ \left(q^{-\frac{1}{16}(m+1)^2} (q^{1/2}; q)_{(m+1)/2} \right)^\lambda f(q^{m/2}), & m \text{ odd,} \end{cases}$$

where f is rational. Replacing m by $m - 1$, it follows that

$$\prod_{j=1}^l \frac{H_q(c_j + \frac{m-1}{2})}{H_q(d_j + \frac{m-1}{2})} = \begin{cases} \left(q^{-\frac{1}{16}m^2}(q^{1/2}; q)_{m/2}\right)^\lambda g(q^{m/2}), & m \text{ even} \\ \left(q^{-\frac{1}{16}(m+1)(m-1)}(q; q)_{(m-1)/2}\right)^\lambda g(q^{m/2}), & m \text{ odd}, \end{cases}$$

again with g rational. Using

$$(q^{1/2}; q^{1/2})_m = \begin{cases} (q^{1/2}; q)_{m/2}(q; q)_{m/2}, & m \text{ even}, \\ (q^{1/2}; q)_{(m+1)/2}(q; q)_{(m-1)/2}, & m \text{ odd} \end{cases}$$

we obtain the desired result. \square

The following two lemmas are straight-forward to verify.

Lemma A.3. For m a non-negative integer,

$$\tilde{H}_q(m) = \tilde{H}_{q^2} \left(\frac{m-1}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m+1}{2} \right).$$

The same identity holds if \tilde{H}_q is replaced by H_q .

Lemma A.4. For k, l and m non-negative integers,

$$\begin{aligned} \prod_{j=1}^k (q; q)_{m+j-1} &= \frac{\tilde{H}_q(k+m)}{\tilde{H}_q(m)}, \\ \prod_{j=1}^k (q^{l+j}; q)_m &= \frac{\tilde{H}_q(l, k+l+m)}{\tilde{H}_q(k+l, l+m)}, \\ \prod_{j=1}^k (q^{l+j}; q)_{j-1} &= \frac{\tilde{H}_{q^2}(\frac{l}{2}, \frac{l+1}{2}, \frac{l-1}{2} + k, \frac{l}{2} + k)}{\tilde{H}_q(l+k)}. \end{aligned}$$

Moreover, for $k, 2l+1$ and m non-negative integers,

$$\begin{aligned} \prod_{j=1}^k (q^{l+1+(j-1)/2}; q)_m &= \frac{\tilde{H}_q(l, l, l+m + [k/2], l+m + [(k+1)/2])}{\tilde{H}_q(l+m, l+m, l + [k/2], l + [(k+1)/2])}, \\ \prod_{j=1}^k (q^{l+1+[j/2]}; q)_m &= \frac{\tilde{H}_q(l, l+1, l+m + [(k+1)/2], l+m + [(k+2)/2])}{\tilde{H}_q(l+m, l+m+1, l + [(k+1)/2], l + [(k+2)/2])}, \\ \prod_{j=1}^k (q^{l+\frac{k}{2}-[\frac{j-1}{2}]}; q)_m &= \frac{\tilde{H}_q(l-\varepsilon, l+\varepsilon, l+m+k/2, l+m+k/2)}{\tilde{H}_q(l+m-\varepsilon, l+m+\varepsilon, l+k/2, l+k/2)}, \end{aligned}$$

where $\varepsilon = 0$ for k even and $\varepsilon = 1/2$ for k odd.

We also need the following variation of the second identity in [Lemma A.4](#).

Lemma A.5. For k and m non-negative integers and l an arbitrary integer,

$$\prod_{j=1}^k (-q^{l+j}; q)_m = \frac{C_{k+l} C_{l+m}}{C_l C_{k+l+m}} \frac{\tilde{H}_q^-(|l|, |k+l+m|)}{\tilde{H}_q^-(|k+l|, |l+m|)},$$

where $C_n = 1$ for $n \geq 0$ and $C_n = 2^n q^{(n-n^3)/6}$ for $n < 0$.

Proof. By induction on k , the result is reduced to

$$(-q^{k+l+1}; q)_m = \frac{C_{k+l+1} C_{k+l+m}}{C_{k+l} C_{k+l+m+1}} \frac{\tilde{H}_q^-(|k+l+m+1|, |k+l|)}{\tilde{H}_q^-(|k+l+m|, |k+l+1|)}.$$

By induction on m , this is in turn reduced to

$$1 + q^{k+l+m+1} = \frac{C_{k+l+m+1}^2}{C_{k+l+m} C_{k+l+m+2}} \frac{\tilde{H}_q^-(|k+l+m|, |k+l+m+2|)}{\tilde{H}_q^-(|k+l+m+1|)^2}.$$

It is easy to check that

$$\frac{\tilde{H}_q^-(|n+1|, |n-1|)}{\tilde{H}_q^-(|n|)^2} = \begin{cases} 1 + q^n, & n \geq 1, \\ 1, & n = 0, \\ 1 + q^{-n}, & n \leq -1. \end{cases}$$

Thus, the result holds for any solution to the recursion

$$\frac{C_n^2}{C_{n-1} C_{n+1}} = \begin{cases} 1, & n \geq 1, \\ 2, & n = 0, \\ q^n, & n \leq -1. \end{cases}$$

The given solution corresponds to the initial values $C_0 = C_1 = 1$. \square

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