Journal of Combinatorial Theory, Series A 149 (2017) 115–147

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Minimum vertex degree thresholds for tiling complete 3-partite 3-graphs ☆

Jie Han [a], Chuanyun Zang [b], Yi Zhao [b]

[a] *Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo, Brazil*
[b] *Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States*

ARTICLE INFO

Article history:
Received 30 March 2015
Available online xxxx

Keywords:
Graph packing
Hypergraph
Absorbing method
Regularity lemma

ABSTRACT

Given positive integers $a \le b \le c$, let $K_{a,b,c}$ be the complete 3-partite 3-uniform hypergraph with three parts of sizes a, b, c. Let H be a 3-uniform hypergraph on n vertices where n is divisible by $a + b + c$. We asymptotically determine the minimum vertex degree of H that guarantees a perfect $K_{a,b,c}$-tiling, that is, a spanning subgraph of H consisting of vertex-disjoint copies of $K_{a,b,c}$. This partially answers a question of Mycroft, who proved an analogous result with respect to codegree for r-uniform hypergraphs for all $r \ge 3$. Our proof uses a lattice-based absorbing method, the concept of fractional tiling, and a recent result on shadows for 3-graphs.

© 2017 Elsevier Inc. All rights reserved.

☆ The first author is supported by FAPESP (Proc. 2013/03447-6, 2014/18641-5, 2015/07869-8). The third author is partially supported by NSA grant H98230-12-1-0283 and NSF grant DMS-1400073.

E-mail addresses: jhan@ime.usp.br (J. Han), czang1@student.gsu.edu (C. Zang), yzhao6@gsu.edu (Y. Zhao).

http://dx.doi.org/10.1016/j.jcta.2017.02.003
0097-3165/© 2017 Elsevier Inc. All rights reserved.

1. Introduction

Given $r \geq 2$, an r-uniform hypergraph (in short, *r-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{r}$, that is, every edge is an r-element subset of V. Given an r-graph H with a set S of d vertices, where $1 \leq d \leq r-1$, we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. The minimum 1-degree is also referred as the *minimum vertex degree*.

Given two r-graphs F and H, an *F-tiling* (also known as *F-packing*) of H is a collection of vertex-disjoint copies of F in H. An F-tiling is called a *perfect F-tiling* (or an *F-factor*) of H if it covers all the vertices of H. An obvious necessary condition for H to contain an F-factor is $|V(F)| \mid |V(H)|$. Given an integer n that is divisible by $|V(F)|$, we define the *tiling threshold* $t_d(n, F)$ to be the smallest integer t such that every r-graph H of order n with $\delta_d(H) \geq t$ contains an F-factor.

As a natural extension of the matching problem, tiling has been intensively studied in the past two decades (see survey [21]). Much work has been done on graphs ($r = 2$), see e.g., [10,2,19,22]. In particular, Kühn and Osthus [22] determined $t_1(n, F)$, for any graph F, up to an additive constant. Tiling problems become much harder for hypergraphs ($r \geq 3$). For example, despite efforts from many researchers [1,6,13,17,18,23,29,31,32], we still do not know the vertex degree threshold for a perfect matching in r-graphs for arbitrary r.

Other than the matching problem, only a few tiling thresholds are known (see survey [34]) Let K_4^3 denote the complete 3-graph on four vertices, and let $K_4^3 - e$ denote the (unique) 3-graph on four vertices with three edges. Recently Lo and Markström [25] proved that $t_2(n, K_4^3) = (1 + o(1))3n/4$, and Keevash and Mycroft [16] determined the exact value of $t_2(n, K_4^3)$ for sufficiently large n. In [24], Lo and Markström proved that $t_2(n, K_4^3 - e) = (1 + o(1))n/2$. Let $\mathcal{C}_4^3$ be the unique 3-graph on four vertices with two edges (this 3-graph was denoted by $K_4^3 - 2e$ in [5], and by Y in [14]). Kühn and Osthus [20] showed that $t_2(n, \mathcal{C}_4^3) = (1 + o(1))n/4$, and Czygrinow, DeBiasio and Nagle [5] determined $t_2(n, \mathcal{C}_4^3)$ exactly for large n. Recently Mycroft [27] determined $t_{r-1}(n, F)$ asymptotically for many r-partite r-graphs F including all complete r-partite r-graphs and loose cycles.

There are fewer tiling results on vertex degree conditions. Lo and Markström [25] determined $t_1(n, K_3^3(m))$ and $t_1(n, K_4^4(m))$ asymptotically, where $K_r^r(m)$ denotes the complete r-partite r-graph with m vertices in each part. Recently Han and Zhao [15] and independently Czygrinow [4] determined $t_1(n, \mathcal{C}_4^3)$ exactly for sufficiently large n. In this paper we extend these results by determining $t_1(n, K)$ asymptotically for all complete 3-partite 3-graphs K, and thus partially answer a question of Mycroft [26].

Given $a \le b \le c$, let $d = \gcd(b-a, c-b)$ and define

$$f(a,b,c) := \begin{cases} 1/4, & \text{if } a=1,\ \gcd(a,b,c)=1 \text{ and } d=1; \\ 6-4\sqrt{2} \approx 0.343, & \text{if } a \ge 2,\ \gcd(a,b,c)=1 \text{ and } d=1; \\ 4/9, & \text{if } \gcd(a,b,c)=1 \text{ and } d \ge 3 \text{ is odd}; \\ 1/2, & \text{if } \gcd(a,b,c)>1 \text{ or } a=b=c=1 \text{ or } d \ge 2 \text{ is even}. \end{cases} \tag{1.1}$$

Given positive integers $a \le b \le c$, let $K_{a,b,c}$ be the complete 3-partite 3-graph with three parts of size a, b, and c, respectively.

Theorem 1.1 *(Main result). For integers $1 \le a \le b \le c$,*

$$t_1(n, K_{a,b,c}) = \left(\max\left\{ f(a,b,c), 1 - \left(\frac{b+c}{a+b+c}\right)^2, \left(\frac{a+b}{a+b+c}\right)^2 \right\} + o(1) \right) \binom{n}{2}.$$

Let us compare Theorem 1.1 with the corresponding result in [27], which states that

$$t_2(n, K_{a,b,c}) = \begin{cases} n/2 + o(n) & \text{if } \gcd(a,b,c)>1 \text{ or } a=b=c=1; \\ an/(a+b+c) + o(n) & \text{if } \gcd(a,b,c)=1 \text{ and } d=1; \\ \max\{an/(a+b+c), n/p\} + o(n) & \text{otherwise,} \end{cases}$$

where p is the smallest prime factor of d. Not only is Theorem 1.1 more complicated, but also it contains a case where the coefficient of the threshold is irrational. In fact, as far as we know, all the previously known tiling thresholds had rational coefficients.

The lower bound in Theorem 1.1 follows from six constructions given in Section 2. Three of them are known as *divisibility barriers* and two are known as *space barriers*. Roughly speaking, the divisibility barriers, known as lattice-based constructions, only prevent the existence of a perfect K-tiling; in contrast, the space barriers are 'robust' because they prevent the existence of an almost perfect K-tiling. Our last construction is based on the fact that if a perfect K-tiling exists, then every vertex is covered by a copy of K, so we call it a *covering barrier*. Such a barrier has never appeared before – see concluding remarks in Section 5.

Our proof of the upper bound of Theorem 1.1 consists of two parts: one is on finding an almost perfect K-tiling in H, and the other is on 'finishing up' the perfect K-tiling. Our first lemma says that H contains an almost perfect K-tiling if the minimum vertex degree of H exceeds those of the space barriers.

Lemma 1.2 *(Almost tiling lemma). Fix integers $1 \le a \le b \le c$. For any $\gamma > 0$ and $\alpha > 0$, there exists an integer n_0 such that the following holds. Suppose H is a 3-graph of order $n > n_0$ with $\delta_1(H) \ge (\max\{1-(\frac{b+c}{a+b+c})^2, (\frac{a+b}{a+b+c})^2\} + \gamma)\binom{n}{2}$, then there exists a $K_{a,b,c}$-tiling covering all but at most αn vertices.*

The absorbing method, initiated by Rödl, Ruciński and Szemerédi [28], has been shown to be effective in finding spanning (hyper)graphs. Our absorbing lemma says that H contains a small $K_{a,b,c}$-tiling that can *absorb* any much smaller set of vertices of H if the minimum vertex degree of H exceeds those of the divisibility barriers and the covering barrier.

Lemma 1.3 *(Absorbing lemma). Fix integers $1 \le a \le b \le c$. For any $\gamma > 0$, there exists $\alpha > 0$ such that the following holds for sufficiently large n. Suppose H is a 3-graph on n vertices such that*

$$\delta_1(H) \ge (f(a,b,c)+\gamma)\binom{n}{2}.$$

Then there exists a vertex set W with $|W| \le \frac{1}{4}\gamma n$ such that for any vertex set $U \subset V(H)\setminus W$ with $|U| \le \alpha n$ and $|U| \in (a+b+c)\mathbb{Z}$, both $H[W]$ and $H[W\cup U]$ have $K_{a,b,c}$-factors.

The upper bound of $t_1(n, K_{a,b,c})$ in Theorem 1.1 follows from Lemmas 1.2 and 1.3 easily.

Proof of Theorem 1.1 (upper bound). Let $1 \le a \le b \le c$ be integers and $\gamma > 0$. Let $\alpha > 0$ be the constant returned by Lemma 1.3 and let $n \in (a+b+c)\mathbb{N}$ be sufficiently large. Suppose that H is a 3-graph on n vertices with $\delta_1(H) \ge (\delta+\gamma)\binom{n}{2}$, where

$$\delta = \max\left\{ f(a,b,c), 1-\left(\frac{b+c}{a+b+c}\right)^2, \left(\frac{a+b}{a+b+c}\right)^2 \right\}.$$

We apply Lemma 1.3 to H and get a vertex set W with $|W| \le \frac{1}{4}\gamma n$ and the described absorbing property. In particular, $|W| \in (a+b+c)\mathbb{N}$. Let $H' = H[V(H)\setminus W]$. Then

$$\delta_1(H') \ge \delta_1(H) - |W|(n-1) \ge (\delta+\gamma)\binom{n}{2} - \frac{\gamma}{2}\binom{n}{2} \ge \left(\delta+\frac{\gamma}{2}\right)\binom{|V(H')|}{2}.$$

Next we apply Lemma 1.2 on H' with $\gamma/2$ in place of γ and get a $K_{a,b,c}$-tiling covering T all but a set U of at most $\alpha|V(H')| < \alpha n$ vertices of H'. Since $|V(H)|, |W|, |V(T)| \in (a+b+c)\mathbb{N}$, $|U| = |V(H)| - |W| - |V(T)| \in (a+b+c)\mathbb{N}$. By the absorbing property of W, there exists a $K_{a,b,c}$-factor on $H[W\cup U]$. Thus we get a $K_{a,b,c}$-factor of H. □

Although this proof is a straightforward application of the absorbing method, there are several new ideas in the proofs of Lemmas 1.2 and 1.3. First, in order to show that almost every $(a+b+c)$-set has many absorbing sets, we use lattice-based absorbing arguments developed recently by Han [12]. Second, in order to prove Lemma 1.2, we use the concept of *fractional homomorphic tiling* given by Buß, Hàn and Schacht [3]. Third,

we need a recent result of Füredi and Zhao [9] on the shadows of 3-graphs, which can be viewed as a vertex degree version of the well-known Kruskal–Katona Theorem for 3-graphs.

The rest of the paper is organized as follows. We prove the lower bound in Theorem 1.1 by six constructions in Section 2. We prove Lemma 1.3 in Section 3 and Lemma 1.2 in Section 4, respectively. Finally, we give concluding remarks in Section 5.

Notations. Throughout this paper we let $1 \le a \le b \le c$ be three integers and $k = a+b+c \ge 3$. When it is clear from the context, we write $K_{a,b,c}$ as K for short. By $x \ll y$ we mean that for any $y > 0$ there exists $x_0 > 0$ such that for any $x < x_0$ the following statement holds. When $x \ll y \ll w$ and $x \ll z \ll w$, we simply write $x \ll y, z \ll w$ (and this should not be confused with $x \ll y$ and $z \ll w$). We omit the floor and ceiling functions when they do not affect the proof.

2. Extremal examples

In this section, we prove the lower bound in Theorem 1.1 by six constructions. Following the definition in [27], we say a 3-partite 3-graph $K_{a,b,c}$ is of *type 0* if $\gcd(a,b,c) > 1$ or $a = b = c = 1$. We say $K_{a,b,c}$ is of *type* $d \ge 1$ if $\gcd(a,b,c) = 1$ and $\gcd(b-a, c-b) = d$.

Construction 2.1 *(Space barrier I). Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \frac{a}{k}n - 1$ and $|V_1| + |V_2| = n$. Let G_1 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_1| \ge 1$. Then $\delta_1(G_1) = \binom{n-1}{2} - \binom{(1-\frac{a}{k})n}{2} = (1 - (1-a/k)^2)\binom{n}{2} + o(n^2)$. Since $a \le b \le c$, we have $a \le k/3$ and $0 < 1-(1-a/k)^2 \le 5/9$.*

We claim that G_1 has no perfect $K_{a,b,c}$-tiling. Indeed, consider a copy K' of $K_{a,b,c}$ in G_1. We observe that at least one color class of K' is a subset of V_1 – otherwise V_2 contains at least one vertex from each color class; since K' is complete, there is an edge in V_2, contradicting the definition of G_1. Hence a $K_{a,b,c}$-tiling in G_1 covers at most $\frac{|V_1|}{a}k < n$ vertices, so it cannot be perfect.

Construction 2.2 *(Space barrier II). Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \frac{a+b}{k}n - 1$ and $|V_1| + |V_2| = n$. Let G_2 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_1| \ge 2$. Then $\delta_1(G_2) = \binom{\frac{a+b}{k}n-1}{2} = ((a+b)^2/k^2)\binom{n}{2} + o(n^2)$. Since $a \le b \le c$, we have $a+b \le 2k/3$ and $0 < (a+b)^2/k^2 \le 4/9$.*

We claim that G_2 has no perfect $K_{a,b,c}$-tiling. Similarly as in the previous case, for any copy K' of $K_{a,b,c}$ in G_2, at least two color classes of K' are subsets of V_1. Hence a $K_{a,b,c}$-tiling in G_2 covers at most $\frac{|V_1|}{a+b}k < n$ vertices, so it cannot be perfect.

Construction 2.3 *(Divisibility barrier I). Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \lfloor \frac{n}{2} \rfloor + 1$ and $|V_1| + |V_2| = n$. Let H_1 be the 3-graph on $V_1 \cup V_2$ such that $H_1[V_1]$ and $H_1[V_2]$ are two complete 3-graphs. Then $\delta_1(H_1) \ge \binom{\frac{n}{2}-2}{2} = \frac{1}{4}\binom{n}{2} + o(n^2)$.*

We claim that H_1 has no perfect $K_{a,b,c}$-tiling. Indeed, each copy of $K_{a,b,c}$ must be a subgraph of $H_1[V_1]$ or $H_1[V_2]$. Since $k \geq 3$, due to the choice of V_1 and V_2, we have $|V_1| \neq |V_2| \bmod k$ and therefore k cannot divide both $|V_1|$ and $|V_2|$. Hence H_1 has no perfect $K_{a,b,c}$-tiling.

Construction 2.4 *(Divisibility barrier II). Suppose that $K_{a,b,c}$ is of type d for some even d. Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| + |V_2| = n$ and $|V_2| \in [\frac{n}{2} - 2, \frac{n}{2} + 2]$ is odd, and $\gcd(a,b,c) \nmid |V_2|$ if $\gcd(a,b,c) > 1$. Note that we can pick $|V_2|$ satisfying these conditions because in the interval $[\frac{n}{2} - 2, \frac{n}{2} + 2]$, there are at least two consecutive odd numbers, therefore at least one of them is not divisible by $\gcd(a,b,c)$. Let H_2 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_2|$ is even (0 or 2). Then $\delta_1(H_2) = \min\{\binom{|V_1|-1}{2} + \binom{|V_2|}{2}, |V_1|(|V_2|-1)\} = \frac{1}{2}\binom{n}{2} + o(n^2)$.*

We claim that H_2 has no perfect $K_{a,b,c}$-tiling. Consider a copy K' of $K_{a,b,c}$ in H_2. Since every edge intersects V_2 in an even number of vertices and K' is complete, no color class of K' intersects both V_1 and V_2. Moreover, either 0 or 2 color classes of K' are subsets of V_2. Thus $|V(K') \cap V_2| \in \{0, a+b, a+c, b+c\}$. If $\gcd(a,b,c) > 1$, then $|V(K') \cap V_2|$ is divisible by $\gcd(a,b,c)$. Since $\gcd(a,b,c) \nmid |V_2|$, there is no perfect $K_{a,b,c}$-tiling. Otherwise, either $a = b = c = 1$ or $\gcd(b-a, c-b)$ is even. In either case, all of $a+b, a+c$ and $b+c$ are even and thus $|V(K') \cap V_2|$ is even. Since $|V_2|$ is odd, H_2 has no perfect $K_{a,b,c}$-tiling.

Construction 2.5 *(Divisibility barrier III). Suppose that $K_{a,b,c}$ is of type d for some odd $d \geq 3$, let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| + |V_2| = n$ and $|V_1| \in [\frac{n}{3} - 1, \frac{n}{3} + 1]$ and $d \nmid (|V_1| - \frac{n}{k}a)$. Let H_3 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_1| = 1$. Then $\delta_1(H_3) = \min\{|V_1|(|V_2|-1), \binom{|V_2|}{2}\} = \frac{4}{9}\binom{n}{2} + o(n^2)$.*

We claim that H_3 has no perfect $K_{a,b,c}$-tiling. Consider a copy K' of $K_{a,b,c}$ in H_3. Similarly as in the previous case, exactly one color class of K' is a subset of V_1, which implies $|V_1 \cap V(K')| \in \{a,b,c\}$. Since $\gcd(b-a, c-b) = d$, we have $a \equiv b \equiv c \bmod d$ and thus $|V_1 \cap V(K')| \equiv a \bmod d$. If H_3 contains a perfect $K_{a,b,c}$-tiling $\mathcal{K}$, then $|V_1| - \frac{n}{k}a = |V(\mathcal{K}) \cap V_1| - \frac{n}{k}a \equiv 0 \bmod d$, contradicting our assumption on $|V_1|$. Hence H_3 has no perfect $K_{a,b,c}$-tiling.

Construction 2.6 *(Covering barrier). Let $\alpha = \sqrt{2} - 1$ and suppose that V is partitioned into $\{v\} \cup V_1 \cup V_2 \cup V_3$ such that $|V_1| = |V_2| = \alpha n$ and $|V| = n$. Define a 3-graph F on V whose edge set consists of all triples vxy with $x \in V_1, y \in V_2$ and all triples e in $V_1 \cup V_2 \cup V_3$ such that $e \cap V_1 = \emptyset$ or $e \cap V_2 = \emptyset$. Therefore, $\delta_1(F) = (6 - 4\sqrt{2})\binom{n}{2} + o(n^2) \approx 0.343\binom{n}{2}$.*

It is easy to see that v is not contained in any copy of $K_{2,2,2}$, and hence not contained in any copy of $K_{a,b,c}$ with $a > 1$. Therefore, F has no perfect $K_{a,b,c}$-tiling with $a > 1$.

Proof of Theorem 1.1 (lower bound). Given positive integers $a \le b \le c$ and $n \in k\mathbb{N}$, where $k = a+b+c$, let $t_1 = t_1(n, K_{a,b,c})$ be the tiling threshold. By Constructions 2.1 and 2.2, we have $t_1 \ge (1-(1-a/k)^2)\binom{n}{2} + o(n^2)$ and $t_1 \ge ((a+b)^2/k^2)\binom{n}{2} + o(n^2)$. Furthermore, assume $K_{a,b,c}$ has type d. First, by definition, d is even if and only if $\gcd(a,b,c) > 1$, or $a = b = c = 1$, or $d \ge 2$ is even. By Construction 2.4, we have $t_1 \ge \frac{1}{2}\binom{n}{2} + o(n^2)$ in this case. Second, assume that $d \ge 3$ is odd, then by Construction 2.5, we have $t_1 \ge \frac{4}{9}\binom{n}{2} + o(n^2)$. Finally assume that $d = 1$. If $a = 1$, by Construction 2.3, we have $t_1 \ge \frac{1}{4}\binom{n}{2} + o(n^2)$. If $a \ge 2$, then by Construction 2.6, we have $t_1 \ge (6-4\sqrt{2})\binom{n}{2} + o(n^2)$. □

3. Proof of the absorbing lemma

3.1. Preparation

We need a simple counting result, which, for example, follows from the result of Erdős [7] on supersaturation. Given $l_1, \dots, l_r \in \mathbb{N}$, let $K^{(r)}_{l_1,\dots,l_r}$ denote the complete r-partite r-graph whose jth part has exactly l_j vertices for all $j \in [r]$.

Proposition 3.1. *Given $\mu > 0$, $r, m, l_1, \dots, l_r \in \mathbb{N}$, there exists $\mu' > 0$ such that the following holds for sufficiently large n. Let H be an r-graph on n vertices with a vertex partition $V_1 \cup \cdots \cup V_m$. Suppose $i_1, \dots, i_r \in [m]$ and H contains at least μn^r edges $e = \{v_1, \dots, v_r\}$ such that $v_1 \in V_{i_1}, \dots, v_r \in V_{i_r}$. Then H contains at least $\mu' n^{l_1+\cdots+l_r}$ copies of $K^{(r)}_{l_1,\dots,l_r}$ whose jth part is contained in V_{i_j} for all $j \in [r]$.*

Given a 3-graph H, its *shadow* ∂H is the set of the pairs that are contained in at least one edge of H. We need a recent result of Füredi and Zhao [9] on the shadows of 3-graphs. The union of two (overlapping) complete 3-graphs of order about $\sqrt{d}n$ shows that Lemma 3.2 is (asymptotically) best possible.

Lemma 3.2. *([9]) Given $d \in [\frac{1}{4}, \frac{47-5\sqrt{57}}{24})$, let n be sufficiently large. If H is a 3-graph on n vertices with $\delta_1(H) \ge d\binom{n}{2}$, then $|\partial H| \ge (4\sqrt{d} - 2d - 1)\binom{n}{2}$.*

The next lemma says that for any 3-graph, after a removal of a small portion of edges, any two vertices with a positive codegree in the remaining 3-graph have a linear codegree in H.

Lemma 3.3. *Given $\epsilon > 0$ and an n-vertex 3-graph $H = (V, E)$, there exists a vertex set $V_0' \subseteq V$ and a subhypergraph H' of H such that the following holds*

(i) $|V_0'| \le 3\epsilon n$,
(ii) $\deg_{H'}(v) \ge \deg_H(v) - \epsilon\binom{n}{2}$ *for any* $v \in V \setminus V_0'$,
(iii) $\deg_H(S) > \epsilon^2 n$ *for any pair of vertices* $S \in \partial H'$.

Proof. If an edge $e \in E(H)$ contains a pair $S \in \binom{e}{2}$ with $\deg_H(S) \le \epsilon^2 n$, then it is called *weak*, otherwise called *strong*. Let H' be the subhypergraph of H induced on strong edges. Then (iii) holds. Let

$$V_0' = \left\{ v \in V : v \text{ is contained in at least } \epsilon \binom{n}{2} \text{ weak edges} \right\}.$$

Then (ii) holds. Note that the number of weak edges in H is at most $\binom{n}{2}\epsilon^2 n$. If $|V_0'| > 3\epsilon n$, then there are more than $3\epsilon n \cdot \epsilon\binom{n}{2}/3 = \binom{n}{2}\epsilon^2 n$ weak edges in H, a contradiction. Thus (i) holds. □

We use the notion of reachability introduced by Lo and Markström [24,25]. Given an r-graph F of order f, $\beta > 0$, $i \in \mathbb{N}$, two vertices u, v in an r-graph H on n vertices are (F, β, i)-*reachable (in H)* if and only if there are at least βn^{if-1} $(if-1)$-sets W such that both $H[\{u\} \cup W]$ and $H[\{v\} \cup W]$ contain F-factors. In this case, we call W a *reachable set* for u and v. A vertex set A is (F, β, i)-*closed in H* if every two vertices in A are (F, β, i)-reachable in H. For $x \in V(H)$, let $\tilde{N}_{F,\beta,i}(x)$ be the set of vertices that are (F, β, i)-reachable to x.

We use the following two results from [25].

Proposition 3.4. *([25, Proposition 2.1]) Given $\beta, \epsilon > 0$ and positive integers f and $i_0' > i_0$, there exists $\beta' > 0$ such that the following holds for sufficiently large n. Let F be an r-graph on f vertices. Given an n-vertex r-graph H and a vertex $x \in V(H)$ with $|\tilde{N}_{F,\beta,i_0}(x)| \ge \epsilon n$, then $\tilde{N}_{F,\beta,i_0}(x) \subseteq \tilde{N}_{F,\beta',i_0'}(x)$. In other words, if $x, y \in V(H)$ are (F, β, i_0)-reachable in H and $|\tilde{N}_{F,\beta,i_0}(x)| \ge \epsilon n$, then x, y are (F, β', i_0')-reachable in H.*

The following lemma is essentially [25, Lemma 4.2]. In fact, [25, Lemma 4.2] shows that the density of $K_{c,c,c+1}$'s containing both x and y in the part of size $c+1$ is positive. By averaging, this implies that the density of $K_{a,b,c+1}$'s containing both x and y in the part of size $c+1$ is positive.

Lemma 3.5. *([25]) Let $a \le b \le c$ be positive integers and $K = K_{a,b,c}$. Given $\epsilon > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large n. For any n-vertex 3-graph H, two vertices $x, y \in V(H)$ are $(K, \eta, 1)$-reachable if the number of pairs $S \in N(x) \cap N(y)$ with $\deg(S) \ge \epsilon n$ is at least $\epsilon\binom{n}{2}$.*

3.2. Auxiliary Lemmas

Given positive integers $a \le b \le c$, let $K = K_{a,b,c}$, and $k = a + b + c \ge 3$. We call an m-set A an *absorbing m-set* for a k-set S if $A \cap S = \emptyset$ and both $H[A]$ and $H[A \cup S]$ contain K-factors. Denote by $\mathcal{A}^m(S)$ the set of all absorbing m-sets for S.

Our proof of the Absorbing Lemma is based on the following lemma.

Lemma 3.6. *Given $0 < \eta \le 1/(2k)$, $\beta > 0$, and $i_0 \in \mathbb{N}$, there exists $\alpha > 0$ such that the following holds for all sufficiently large integers n. Suppose $H = (V, E)$ is an n-vertex 3-graph with the following two properties*

($\diamondsuit$) For any $v \in V$, there are at least ηn^{k-1} copies of K containing it.
($\triangle$) There exists $V_0 \subset V$ with $|V_0| \le \eta^2 n$ such that $V(H) \setminus V_0$ is (K, β, i_0)-closed in H.

Then there exists a vertex set W with $V_0 \subseteq W \subseteq V$ and $|W| \le \eta n$ such that for any vertex set $U \subseteq V \setminus W$ with $|U| \le \alpha n$ and $|U| \in k\mathbb{Z}$, both $H[W]$ and $H[U \cup W]$ contain K-factors.

Proof. Let

$$\eta_1 = \frac{\eta}{2}\left(\frac{\beta}{2}\right)^{k-1} \text{ and } \alpha = \frac{\eta_1^2}{32 i_0 k}.$$

There are two steps in our proof. In the first step, we build an absorbing family $\mathcal{F}_1$ that can absorb any small portion of vertices in $V \setminus V_0$. In the second step, we put the vertices in $V_0 \setminus V(\mathcal{F}_1)$ into a family $\mathcal{F}_2$ of copies of K. Then $V(\mathcal{F}_1 \cup \mathcal{F}_2)$ is the desired absorbing set.

Fix a k-set $S = \{v_1, v_2, \dots, v_k\} \subset V \setminus V_0$. Let $m = i_0 k^2 - i_0 k$. We claim that there are at least $\eta_1 n^m$ absorbing m-sets for S, namely, $|\mathcal{A}^m(S)| \ge \eta_1 n^m$. Indeed, we first find a k-set $S' = \{v_1, u_2, \dots, u_k\} \subset V \setminus V_0$ such that $S' \cap S = \{v_1\}$ and S' spans a copy of K. By ($\diamondsuit$), there are at least

$$\eta n^{k-1} - (k-1)n^{k-2} - (\eta^2 n)n^{k-2} \ge \eta n^{k-1}/2$$

choices for S' because there are at most n^{k-2} k-sets containing v_1 and another fixed vertex, and $|V \setminus V_0| \le \eta^2 n$. For each $i = 2, \dots, k$, since $V \setminus V_0$ is (K, β, i_0)-closed, there are at least $\beta n^{i_0 k - 1}$ reachable $(i_0 k - 1)$-sets S_i for u_i and v_i. We greedily choose pairwise disjoint sets $S_2, \dots, S_k$ – when choosing S_i, we need to avoid the vertices in $S \cup S' \cup \bigcup_{j=2}^{i-1} S_j$ so there are at least $\beta n^{i_0 k-1}/2$ choices for S_i. Let $A = (S' \setminus \{v_1\}) \cup \left(\bigcup_{i=2}^{k} S_i\right)$, then $|A| = m$. We claim that both $H[A]$ and $H[A \cup S]$ contain K-factors. Indeed, by the definition of reachability, each $S_i \cup \{u_i\}$ spans i_0 copies of K and thus $H[A]$ contains a K-factor. Furthermore, since S' spans a copy of K and each $S_i \cup \{v_i\}$ spans i_0 copies of K, $H[A \cup S]$ also contains a K-factor. Thus A is an absorbing m-set for S. In total, we get at least

$$\frac{\eta}{2} n^{k-1} \left(\frac{\beta}{2} n^{i_0 k - 1}\right)^{k-1} = \eta_1 n^m$$

such m-sets, thus $|\mathcal{A}^m(S)| \ge \eta_1 n^m$.

Now we build the family $\mathcal{F}_1$ by standard probabilistic arguments. Choose a family $\mathcal{F}$ of m-sets in H by selecting each of the $\binom{n}{m}$ possible m-sets independently with probability $p = \eta_1 n^{1-m}/(8m)$. Then by Chernoff's bound, with probability $1 - o(1)$ as $n \to \infty$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \le 2p\binom{n}{m} \le \frac{\eta_1 n}{4m} \text{ and } |\mathcal{A}^m(S) \cap \mathcal{F}| \ge \frac{p|\mathcal{A}^m(S)|}{2} \ge \frac{\eta_1^2 n}{16m}, \text{ for all } S \in \binom{V \setminus V_0}{k}. \tag{3.1}$$

Furthermore, the expected number of pairs of m-sets in $\mathcal{F}$ that are intersecting is at most

$$\binom{n}{m} \cdot m \cdot \binom{n}{m-1} \cdot p^2 \le \frac{\eta_1^2 n}{64m}.$$

Thus, by using Markov's inequality, we derive that with probability at least $1/2$,

$$\mathcal{F} \text{ contains at most } \frac{\eta_1^2 n}{32m} \text{ intersecting pairs of } m\text{-sets.} \tag{3.2}$$

Hence, there exists a family $\mathcal{F}$ with the properties in (3.1) and (3.2). By deleting one member of each intersecting pair and removing m-sets that are not absorbing sets for any k-set $S \subseteq V \setminus V_0$, we get a subfamily $\mathcal{F}_1$ consisting of pairwise disjoint m-sets. Let $W_1 = V(\mathcal{F}_1)$ and thus $|W_1| = |V(\mathcal{F}_1)| = m|\mathcal{F}_1| \le m|\mathcal{F}| \le \eta_1 n/4$. Since every m-set in $\mathcal{F}_1$ is an absorbing m-set for some k-set S, $H[W_1]$ has a K-factor. For any k-set S, by (3.1) and (3.2) above we have

$$|\mathcal{A}^m(S) \cap \mathcal{F}_1| \ge \frac{\eta_1^2 n}{16m} - \frac{\eta_1^2 n}{32m} = \frac{\eta_1^2 n}{32m}. \tag{3.3}$$

For any set $U \subseteq V \setminus (V_0 \cup W_1)$ of size $|U| \le \alpha n$ and $|U| \in k\mathbb{Z}$, we arbitrarily partition it into at most $\frac{\alpha n}{k}$ k-sets. By the definition of $\mathcal{F}_1$, each such k-set has at least $\frac{\eta_1^2 n}{32m} \ge \frac{\alpha n}{k}$ absorbing sets in $\mathcal{F}_1$ so we can find a distinct absorbing set in $\mathcal{F}_1$ for each of the k-sets. As a result, $H[U \cup W_1]$ contains a K-factor.

In the second step, by $(\diamondsuit)$, we greedily build $\mathcal{F}_2$, a collection of copies of K that cover the vertices in $V_0 \setminus W_1$. Indeed, assume that we have built $i < |V_0 \setminus W_1| \le \eta^2 n$ copies of K. Together with the vertices in W_1, at most $ki + \eta_1 n/4 \le k\eta^2 n + \eta_1 n/4$ vertices have already been covered by $\mathcal{F}$. So for any vertex $v \in V_0$ not yet covered, we find the desired copy of K containing v by $(\diamondsuit)$, because $(k\eta^2 n + \eta_1 n/4) \cdot n^{k-2} < \eta n^{k-1}$.

Let $W = V(\mathcal{F}_2) \cup W_1$, we get the desired absorbing set W with $|W| \le k\eta^2 n + \eta_1 n/4 < \eta n$. □

So it remains to show that $(\diamondsuit)$ and $(\triangle)$ hold in the 3-graph H. We first study the property $(\diamondsuit)$. Throughout this subsection, let $d_0 = 6 - 4\sqrt{2} \approx 0.343$. Note that $(4\sqrt{d_0} - 2d_0 - 1) + d_0 = 1$ because $\sqrt{d_0} = 2 - \sqrt{2}$.

Lemma 3.7. *For any $\gamma > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large n. Let H be an n-vertex 3-graph with $\delta_1(H) \geq (d_0 + \gamma)\binom{n}{2}$. Then each vertex $v \in V(H)$ is contained in at least ηn^{k-1} copies of K.*

Proof. Let $\epsilon = \gamma/12$. Let η be returned by Lemma 3.5 when $\gamma\epsilon^2/2$ plays the role of ϵ. Suppose that n is sufficiently large and H is an n-vertex 3-graph with $\delta_1(H) \geq (d_0+\gamma)\binom{n}{2}$. We apply Lemma 3.3 on H and get V_0' and H' satisfying (i)–(iii). Let $H'' = H'[V \setminus V_0']$ and $n' = |V \setminus V_0'|$. By Lemma 3.3 (ii), we have

$$\delta_1(H'') \geq (d_0 + \gamma)\binom{n}{2} - \epsilon\binom{n}{2} - |V_0'|(n-2) > d_0\binom{n'}{2}$$

because $|V_0'| \leq 3\epsilon n$. Since $\frac{1}{4} < d_0 < \frac{47-5\sqrt{57}}{24} \approx 0.385$ and $n' \geq (1-3\epsilon)n$ is sufficiently large, Lemma 3.2 implies that

$$\partial H'' \geq (4\sqrt{d_0} - 2d_0 - 1)\binom{n'}{2} \geq (4\sqrt{d_0} - 2d_0 - 1)(1-6\epsilon)\binom{n}{2}.$$

Since $\delta_1(H) \geq (d_0 + \gamma)\binom{n}{2}$, for every $x \in V(H)$, we have

$$|N_H(x) \cap \partial H''| \geq \left(d_0 + \gamma + (4\sqrt{d_0} - 2d_0 - 1) - 6\epsilon - 1\right)\binom{n}{2} \geq \frac{\gamma}{2}\binom{n}{2},$$

by the definitions of d_0 and ϵ.

Fix $x \in V(H)$ and note that every $S \in N_H(x) \cap \partial H''$ has degree at least $\epsilon^2 n$ in H. Therefore, the number of (S, y) with $S \in N_H(x) \cap \partial H''$ and $y \in N_H(S)$ is at least $\frac{\gamma}{2}\binom{n}{2} \cdot \epsilon^2 n$. By averaging, there exists a vertex y such that

$$|N_H(y) \cap N_H(x) \cap \partial H''| \geq \gamma\epsilon^2\binom{n}{2}/2.$$

This means that x and y have at least $\gamma\epsilon^2\binom{n}{2}/2$ common neighbors with degree at least $\epsilon^2 n$. By Lemma 3.5, x and y are $(K, \eta, 1)$-reachable. Hence, there are at least ηn^{k-1} $(k-1)$-sets W such that $H[\{x\} \cup W]$ forms a copy of K. □

Now we consider the property $(\triangle)$. Following the approach in [12], given a 3-graph H, we first find a partition of $V(H)$ such that all but one part are (K, β, i)-closed in H and then study the reachability between different parts. The following lemma provides such a partition.

Lemma 3.8. *Given $\delta \geq 1/4$ and $\gamma > 0$, there exist constants $0 < \beta \ll \epsilon \ll \gamma$ such that the following holds for sufficiently large n. Let H be an n-vertex 3-graph with $\delta_1(H) \geq (\delta + \gamma)\binom{n}{2}$. Then there is a partition $\mathcal{P}$ of $V(H)$ into $V_0, V_1, \ldots, V_t$ such that*

- $|V_0| \le 4\epsilon n$,
- $t \le \lfloor 1/(\delta+\gamma/2)\rfloor$, *and*
- $|V_i| \ge \epsilon^2 n$ *and* V_i *is* $(K, \beta, 2^{\lfloor 1/(\delta+\gamma/2)\rfloor - 1})$*-closed in* H *for all* $i \in [t]$.

Proof. Let $s = \lfloor 1/(\delta+\gamma/2)\rfloor$. Then we may choose $\epsilon > 0$ such that $\epsilon \ll \min\{(s+1)(\delta+\gamma/2) - 1, 1/k\}$. Let η be the constant returned from applying Lemma 3.5 with $\epsilon^2/16$ in place of ϵ. Note that we may require $\eta \ll \epsilon$ because the conclusion of Lemma 3.5 holds with η replaced by any positive $\eta' < \eta$. Furthermore, let

$$1/n \ll \beta = \beta_{s-1} \ll \cdots \ll \beta_1 \ll \beta_0 \le \eta, \alpha \ll \epsilon.$$

Let $H = (V, E)$ be an n-vertex 3-graph with $\delta_1(H) \ge (\delta+\gamma)\binom{n}{2}$. We apply Lemma 3.3 to H and obtain V_0' and H' satisfying (i)–(iii).

Given $v \in V$ and $0 \le i \le s-1$, let $\tilde{N}_i(v) = \tilde{N}_{K,\beta_i,2^i}(v)$ be the set of vertices in H that are $(K, \beta_i, 2^i)$-reachable to x (note that $\tilde{N}_i(v)$ may contain the vertices of V_0'). Throughout this proof, we say 2^i-reachable (respectively, 2^i-closed) for $(K, \beta_i, 2^i)$-reachable (respectively, $(K, \beta_i, 2^i)$-closed) for short.

Fix $x \in V \setminus V_0'$, we claim that $|\tilde{N}_0(x)| \ge \frac{3}{4}\epsilon^2 n$. To see this, let

$$D = \left\{ v \in V : |N_{H'}(v) \cap N_{H'}(x)| \ge \frac{\epsilon^2}{16}\binom{n}{2} \right\}.$$

Since $\deg_H(p) > \epsilon^2 n$ for any $p \in \partial H'$, Lemma 3.5 implies that two vertices $x, v \in V$ are 1-reachable if $|N_{H'}(v) \cap N_{H'}(x)| \ge \epsilon^2\binom{n}{2}/16$. Therefore $D \subseteq \tilde{N}_0(x)$. Let t be the number of pairs (p, u) where $p \in N_{H'}(x)$ and $u \in N_{H'}(p)$. By Lemma 3.3 (iii), we have $t \ge \deg_{H'}(x) \cdot \epsilon^2 n$. We also know that $|N_{H'}(v) \cap N_{H'}(x)| < \frac{\epsilon^2}{16}\binom{n}{2}$ if $v \notin D$, and $|N_{H'}(v) \cap N_{H'}(x)| \le \deg_{H'}(x)$ otherwise. Consequently,

$$\deg_{H'}(x)\,\epsilon^2 n \le t \le n \cdot \frac{\epsilon^2}{16}\binom{n}{2} + |D| \cdot \deg_{H'}(x).$$

So we get $|D| \ge \epsilon^2 n - \epsilon^2 n\binom{n}{2}/(16\deg_{H'}(x))$. Since $x \in V \setminus V_0'$ and $\delta \ge 1/4$, by Lemma 3.3 (ii),

$$\deg_{H'}(x) \ge (\delta+\gamma-\epsilon)\binom{n}{2} \ge \left(\delta+\frac{\gamma}{2}\right)\binom{n}{2} > \frac{1}{4}\binom{n}{2}. \tag{3.4}$$

Consequently $|D| \ge \frac{3}{4}\epsilon^2 n$ and in turn, $|\tilde{N}_0(x)| \ge \frac{3}{4}\epsilon^2 n$.

Since $|\tilde{N}_0(x)| \ge \frac{3}{4}\epsilon^2 n$, by Proposition 3.4 and the choice of β_i's, we derive $\tilde{N}_i(x) \subseteq \tilde{N}_{i+1}(x)$ for all $0 \le i < s-1$ and all $x \in V \setminus V_0'$. Furthermore, if a set $W \subseteq V \setminus V_0'$ is 2^i-closed in H for some $i \le s-1$, then W is 2^{s-1}-closed in H.

Given any set $S \subseteq V \setminus V_0'$ of $s+1$ vertices, by the Inclusion–Exclusion principle,

$$\sum_{x\in S} \deg_{H'}(x) - \sum_{x,y\in S} |N_{H'}(x) \cap N_{H'}(y)| \le |\bigcup_{x\in S} N_{H'}(x)| \le \binom{n}{2},$$

which implies $\sum_{x,y\in S} |N_{H'}(x) \cap N_{H'}(y)| \ge ((s+1)(\delta+\gamma/2)-1)\binom{n}{2}$ by (3.4). Since $\epsilon \ll (s+1)(\delta+\gamma/2)-1$, there are two vertices $x, y \in S$ such that $|N_{H'}(x) \cap N_{H'}(y)| \ge \frac{\epsilon^2}{16}\binom{n}{2}$, so x, y are 1-reachable to each other. Consequently, if $s = 1$, then $V \setminus V_0'$ is 1-closed and we get the desired partition $\mathcal{P} = \{V_0', V \setminus V_0'\}$.

We may thus assume that $s \ge 2$ and there are two vertices in $V \setminus V_0'$ that are *not* 2^{s-1}-reachable to each other (otherwise we are done). Let t' be the largest integer such that there exist $v_1, \dots, v_{t'} \in V \setminus V_0'$ such that no two of them are $2^{s+1-t'}$-reachable to each other. Earlier arguments show that t' exists and $2 \le t' \le s$. Fix such $v_1, \dots, v_{t'} \in V \setminus V_0'$. By Proposition 3.4, we can assume that any two of them are not $2^{s-t'}$-reachable to each other. Then $\tilde{N}_{s-t'}(v_i)$, $i \in [t']$ satisfy the following properties.

(a) Any $v \in (V \setminus V_0') \setminus \{v_1, \dots, v_{t'}\}$ must be in $\tilde{N}_{s-t'}(v_i)$ for some $i \in [t']$ – otherwise $\{v, v_1, \dots, v_{t'}\}$ contradicts the definition of t'.

(b) $|\tilde{N}_{s-t'}(v_i) \cap \tilde{N}_{s-t'}(v_j)| < \alpha n$ for any $i \ne j$ – otherwise there are at least

$$\frac{\alpha n}{(2^{s+1-t'}k-1)!}\left(\beta_{s-t'} n^{2^{s-t'}k-1} - n^{2^{s-t'}k-2}\right)\left(\beta_{s-t'} n^{2^{s-t'}k-1} - 2^{s-t'}k\, n^{2^{s-t'}k-2}\right)$$

reachable $(2^{s+1-t'}k-1)$-sets for v_i, v_j because there are at least αn vertices $w \in \tilde{N}_{s-t'}(v_i) \cap \tilde{N}_{s-t'}(v_j)$, at least $\beta_{s-t'} n^{2^{s-t'}k-1} - n^{2^{s-t'}k-2}$ $2^{s-t'}$-reachable sets S for v_i and w that do not contain v_j, and at least $\beta_{s-t'} n^{2^{s-t'}k-1} - 2^{s-t'}kn^{2^{s-t'}k-2}$ $2^{s-t'}$-reachable sets for v_j and w that avoid $\{v_i\} \cup S$; finally, we divide by $(2^{s+1-t'}k-1)!$ to eliminate the effect of over-counting. Since $\beta_{s-t'} \gg \beta_{s+1-t'}$, this gives at least $\beta_{s+1-t'} n^{2^{s+1-t'}k-1}$ reachable $(2^{s+1-t'}k-1)$-sets for v_i, v_j, contradicting the assumption that v_i, v_j are not $2^{s+1-t'}$-reachable to each other.

For $i \in [t']$, let $V_i = (\tilde{N}_{s-t'}(v_i) \cup \{v_i\}) \setminus (V_0' \cup \bigcup_{j\in[t']\setminus\{i\}} \tilde{N}_{s-t'}(v_j))$. We observe that V_i is $2^{s-t'}$-closed for all $i \in [t']$. Indeed, if there exist $u_1, u_2 \in V_i$ that are not $2^{s-t'}$-reachable to each other, then $\{u_1, u_2, v_1, \dots, v_{t'}\} \setminus \{v_i\}$ contradicts the definition of t'. Without loss of generality, we may assume $|V_1| \ge \cdots \ge |V_{t'}|$. Let t be the largest integer $i \in [t']$ such that $|V_i| \ge \epsilon^2 n$. Let $V_0 = V \setminus (\bigcup_{1\le i\le t} V_i)$. Clearly $V_0' \subseteq V_0$. By (a) and (b), we have $|V_0| \le |V_0'| + \binom{t'}{2}\alpha n + t'\epsilon^2 n \le 4\epsilon n$. So $\mathcal{P} = \{V_0, V_1, \dots, V_t\}$ is the desired partition. □

We use the following definitions introduced by Keevash and Mycroft [16]. Let $r, t > 0$ be integers and let F be an r-graph of order f. Suppose that H is an r-graph with a partition $\mathcal{P} = \{V_0, V_1, \dots, V_t\}$ of $V(H)$. The *index vector* $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^t$ of a subset $S \subset V(H)$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of

S with $V_1, \dots, V_t$. We call a vector $\mathbf{i} \in \mathbb{Z}^t$ an *s-vector* if all its coordinates are nonnegative and their sum is s. Given $\mu > 0$, an r-vector $\mathbf{v} \in \mathbb{Z}^t$ is called a *μ-robust edge vector* if at least $\mu|V(H)|^r$ edges $e \in E(H)$ satisfy $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{v}$; an f-vector $\mathbf{v} \in \mathbb{Z}^t$ is called a *μ-robust F-vector* if at least $\mu|V(H)|^f$ copies F' of F in H satisfy $\mathbf{i}_{\mathcal{P}}(V(F')) = \mathbf{v}$. Let $I^{\mu}_{\mathcal{P}}(H)$ be the set of all μ-robust edge vectors and let $I^{\mu}_{\mathcal{P},F}(H)$ be the set of all μ-robust F-vectors. Let $L^{\mu}_{\mathcal{P},F}(H)$ be the lattice generated by the vectors of $I^{\mu}_{\mathcal{P},F}(H)$, in other words, $L^{\mu}_{\mathcal{P},F}(H)$ consists of all linear combinations of the vectors of $I^{\mu}_{\mathcal{P},F}(H)$.

When $\mathbf{i}$ is a μ-robust edge vector and F is a complete r-partite r-graph, Proposition 3.1 implies that there exists $\mu' > 0$ such that the edges with index vector $\mathbf{i}$ give rise to at least $\mu' n^f$ copies of F with certain index vectors. For example, when $t = 2$, $F = K_{a,b,c}$ (so $r = 3$) and $(1, 2) \in I^{\mu}_{\mathcal{P}}(H)$, we have $(a, b + c), (b, a + c)$ and $(c, a + b) \in I^{\mu'}_{\mathcal{P},F}(H)$ for some $\mu' > 0$. For $j \in [t]$, let $\mathbf{u}_j \in \mathbb{Z}^t$ be the jth *unit vector*, namely, $\mathbf{u}_j$ has 1 on the jth coordinate and 0 on other coordinates.

Given a partition $\mathcal{P} = \{V_0, V_1, \dots, V_t\}$ of $V(H)$, the following lemma shows that $V(H) \setminus V_0$ is closed if $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu'}_{\mathcal{P},F}(H)$ for all $1 \le j < l \le t$. Because of potential applications to other problems, we prove the following lemma for r-graphs with $r \ge 3$.

Lemma 3.9. *Let $i_0, r, t > 0$ be integers and let F be an r-graph of order f. Given constants $\epsilon, \beta, \mu' > 0$, there exists $\beta' > 0$ and an integer $i'_0 > 0$ such that the following holds for sufficiently large n. Let H be an r-graph on n vertices with a partition $\mathcal{P} = \{V_0, V_1, \dots, V_t\}$ such that for each $j \in [t]$, $|V_j| \ge \epsilon^2 n$ and V_j is (F, β, i_0)-closed in H. If $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu'}_{\mathcal{P},F}(H)$ for all $1 \le j < l \le t$, then $V(H) \setminus V_0$ is (F, β', i'_0)-closed in H.*

Proof. We call a set I of f-vectors in $\mathbb{Z}^t$ *a base* if all $\mathbf{u}_j - \mathbf{u}_l$, $j < l$, can be written as linear combinations of the vectors in I, namely, there exist $a^{j,l}_{\mathbf{v}} \in \mathbb{Z}$ such that $\mathbf{u}_j - \mathbf{u}_l = \sum_{\mathbf{v} \in I} a^{j,l}_{\mathbf{v}} \mathbf{v}$. For example, the set of all f-vectors in $\mathbb{Z}^t$ is a base. Since there are $\binom{f+t-1}{t-1}$ f-vectors in $\mathbb{Z}^t$, there are at most $2^{\binom{f+t-1}{t-1}}$ bases. Given a base I, we denote by C_I the largest $|a^{j,l}_{\mathbf{v}}|$ over all $\mathbf{v} \in I$ and $j < l$. Let $C' = \max C_I$ over all bases I.

Given integers r, t, i_0 and constants $\epsilon, \beta, \mu' > 0$, let n be sufficiently large, in particular, $n \gg C'$. Suppose that H is an r-graph satisfying all the assumptions, in particular, $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu'}_{\mathcal{P},F}(H)$ for all $j < l$. We claim that for any $j < l$, any $x_j \in V_j$ and any $x_l \in V_l$ are $(F, \beta_{j,l}, i_{j,l})$-reachable for some $\beta_{j,l} > 0$ and some $i_{j,l} \ge i_0$. Once this is done, since $|\tilde{N}_{F,\beta,i_0}(v)| \ge |V_j| - 1 \ge \epsilon^2 n/2$ for any $j \in [t]$ and $v \in V_j$, we can apply Proposition 3.4 with $\epsilon^2/2$ in place of ϵ and $i'_0 = \max\{i_{j,l}\}$ and derive that any $x_j \in V_j$ and any $x_l \in V_l$ are $(F, \tilde{\beta}, i'_0)$-reachable for some $\tilde{\beta} > 0$. For the same reason, any two vertices in V_j, $j \in [t]$, are (F, β'', i'_0)-reachable for some $\beta'' > 0$. We thus conclude that any two vertices of $V(H) \setminus V_0$ are (F, β', i'_0)-reachable with $\beta' = \min\{\tilde{\beta}, \beta''\}$.

Below we prove this claim for $j = 1$ and $l = 2$. Let $I = I^{\mu'}_{\mathcal{P},F}(H)$. By our assumption, there exist $a_{\mathbf{v}} \in \mathbb{Z}$, $\mathbf{v} \in I$ such that $\mathbf{u}_1 - \mathbf{u}_2 = \sum_{\mathbf{v}} a_{\mathbf{v}} \mathbf{v}$ and $|a_{\mathbf{v}}| \le C'$ for all $\mathbf{v} \in I$. For each $\mathbf{v} \in I$, if $a_{\mathbf{v}} \ge 0$, then let $p_{\mathbf{v}} = a_{\mathbf{v}}$ and $q_{\mathbf{v}} = 0$; otherwise let $p_{\mathbf{v}} = 0$ and $q_{\mathbf{v}} = -a_{\mathbf{v}}$. Hence

$$\mathbf{u}_1 - \mathbf{u}_2 = \sum_{\mathbf{v}\in I}(p_{\mathbf{v}} - q_{\mathbf{v}})\mathbf{v} \quad i.e., \quad \sum_{\mathbf{v}\in I} q_{\mathbf{v}}\mathbf{v} + \mathbf{u}_1 = \sum_{\mathbf{v}\in I} p_{\mathbf{v}}\mathbf{v} + \mathbf{u}_2. \tag{3.5}$$

By comparing the sums of all the coordinates from two sides of either equation in (3.5), we obtain that $\sum_{\mathbf{v}\in I} p_{\mathbf{v}} = \sum_{\mathbf{v}\in I} q_{\mathbf{v}}$, which we denote by C. Then $C \le |I|C' \le \binom{f+t-1}{t-1}C' < \mu' n/(4f)$ because n is sufficiently large. We greedily select $p_{\mathbf{v}} + q_{\mathbf{v}}$ vertex-disjoint copies of F with index vector $\mathbf{v}$ that do not contain x_1 or x_2 for all $\mathbf{v} \in I$. This gives rise to two disjoint families $\mathcal{K}^p$ and $\mathcal{K}^q$, where $\mathcal{K}^p$ consists of $p_{\mathbf{v}}$ copies of F with index vector $\mathbf{v}$ for all $\mathbf{v} \in I$, and $\mathcal{K}^q$ consists of $q_{\mathbf{v}}$ vertex-disjoint copies of F with index vector $\mathbf{v}$ for all $\mathbf{v} \in I$. Note that $|V(\mathcal{K}^p)| = |V(\mathcal{K}^q)| = fC$. When selecting any copy of F, we need to avoid at most $2fC$ vertices, which are incident to at most $2fCn^{f-1} \le \mu' n^f/2$ copies of F. Therefore, there are at least $\mu' n^f/2$ choices for each copy of F in $\mathcal{K}^p$ and $\mathcal{K}^q$ and in turn, at least $(\mu' n^f/2)^{2C}$ choices for $\mathcal{K}^p$ and $\mathcal{K}^q$.

By (3.5), we have $\mathbf{i}_{\mathcal{P}}(V(\mathcal{K}^q)) + \mathbf{u}_1 = \mathbf{i}_{\mathcal{P}}(V(\mathcal{K}^p)) + \mathbf{u}_2$. This implies that we may write $V(\mathcal{K}^p) = \{y_1, \dots, y_{fC}\}$, $V(\mathcal{K}^q) = \{z_1, \dots, z_{fC}\}$ such that $y_1 \in V_1$, $z_1 \in V_2$, and for $i \ge 2$, y_i and z_i are from the same part of $\mathcal{P}$ (and thus are (F, β, i_0)-reachable to each other). We next select a reachable $(i_0 f - 1)$-set S_i for y_i, z_i for $i \ge 2$ such that $S_2, \dots, S_{fC}$ are disjoint and also disjoint from $V(\mathcal{K}^p \cup \mathcal{K}^q) \cup \{x_1, x_2\}$. When selecting each S_i, we need to avoid at most constantly many vertices and thus there are at least $\frac{\beta}{2} n^{i_0 f - 1}$ choices for each S_i. Finally, since x_1 and y_1 and respectively, x_2 and z_1 are (F, β, i_0)-reachable, we can pick two disjoint $(i_0 f - 1)$-sets S_1, S_0 such that S_1 is a reachable set for x_1 and y_1, S_0 is a reachable set for x_2 and z_1, and S_1, S_0 are disjoint from $V(\mathcal{K}^p \cup \mathcal{K}^q) \cup \{x_1, x_2\} \cup \bigcup_{i=2}^{fC} S_i$. Again there are at least $\frac{\beta}{2} n^{i_0 f - 1}$ choices for each of S_1, S_0. We claim that $A := \bigcup_{i=0}^{fC} S_i \cup V(\mathcal{K}^p \cup \mathcal{K}^q)$ is a reachable set for x_1 and x_2. Indeed, $H[A \cup \{x_1\}]$ contains an F-factor because $H[S_i \cup \{z_i\}]$ for $i \ge 2$, $H[S_1 \cup \{x_1\}]$, $H[S_0 \cup \{z_1\}]$ and $\mathcal{K}^p$ all contain F-factors; on the other hand, $H[A \cup \{x_2\}]$ contains an F-factor because $H[S_i \cup \{y_i\}]$ for $i \ge 1$, $H[S_0 \cup \{x_2\}]$ and $\mathcal{K}^q$ all contain F-factors. Let $i_{1,2} = i_0 fC + C + i_0$ and $\beta_{1,2} = \left(\frac{\mu'}{2}\right)^{2C} \left(\frac{\beta}{2}\right)^{fC+1} /(i_{1,2} f - 1)!$. The procedure above provides at least

$$\frac{\left(\frac{\mu'}{2} n^f\right)^{2C} \left(\frac{\beta}{2} n^{i_0 k - 1}\right)^{fC+1}}{(i_{1,2} f - 1)!} = \beta_{1,2} n^{i_{1,2} f - 1}$$

reachable $(i_{1,2} f - 1)$-sets for x_1 and x_2. □

3.3. Proof of Lemma 1.3

The following simple fact will be used later for finding linear combinations of robust K-vectors.

Fact 3.10. Let $a, b, c \in \mathbb{Z}$. If $\gcd(a, b, c) = 1$ and $\gcd(b - a, c - b)$ is odd, then $\gcd(a + b, a + c, b + c) = 1$.

Proof. Let $l = \gcd(a+b, a+c, b+c)$. Then $l \mid (b-a)$ and $l \mid (c-b)$ and consequently $l \mid \gcd(b-a, c-b)$. Thus l is odd. On the other hand, $l \mid 2(a+b+c)$. Since l is odd, it follows that $l \mid (a+b+c)$. Consequently, $l \mid a, l \mid b$ and $l \mid c$, which implies $l \mid \gcd(a,b,c) = 1$, namely, $l = 1$. □

Proof of Lemma 1.3. Fix $\delta \geq f(a,b,c)$ and $\gamma > 0$. Let $\eta = \eta(\gamma)$ be the constant returned by Lemma 3.7. In addition, assume that $\eta \leq \min\{1/(2k), \gamma/4, \mu_1'/2\}$, where μ_1' is the constant returned by Proposition 3.1 with inputs $\mu = 1/8$, $l_1 = b$, and $l_2 = c$. Let $i_0 = 2^{\lfloor 1/(\delta+\gamma/2)\rfloor - 1}$. Let $\beta \ll \epsilon \ll \gamma$ be the constants returned by Lemma 3.8, and assume that $\epsilon \leq \eta^2/4$. We pick $0 < \mu \ll \epsilon$ and let μ' the constant returned by Proposition 3.1 with μ, $l_1 = a$, $l_2 = b$, and $l_3 = c$. We apply Lemma 3.9 with β, i_0 and μ' and get β' and i_0'. Finally, we apply Lemma 3.6 with β', η and i_0', and get $\alpha > 0$.

Let n be sufficiently large and let H be a 3-graph on n vertices such that $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$. It suffices to verify the assumptions ($\triangle$) and ($\diamondsuit$) in Lemma 3.6, which provides the desired vertex set W (here $|W| \leq \eta n \leq \gamma n/4$).

If $\delta_1(H) \geq (6-4\sqrt{2}+\gamma)\binom{n}{2}$, then ($\diamondsuit$) holds by Lemma 3.7; otherwise by the definition of $f(a,b,c)$, we may assume that $a = 1$ and $\delta_1(H) \geq (\frac{1}{4}+\gamma)\binom{n}{2}$. By Proposition 3.1, there are at least $\mu_1'(n-1)^{b+c} \geq \eta n^{k-1}$ copies of $K_{b,c}^{(2)}$ in the link graph[1] of each vertex of H (thus ($\diamondsuit$) holds).

In the rest of the proof we verify ($\triangle$) in cases depending on the type of $K = K_{a,b,c}$. We first apply Lemma 3.8 to H and obtain a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_t\}$ of $V(H)$ such that $|V_0| \leq 4\epsilon n \leq \eta^2 n$, $t \leq \lfloor 1/(\delta+\gamma/2)\rfloor$, $|V_i| \geq \epsilon^2 n$ and V_i is (K, β, i_0)-closed in H for all $i \in [t]$. In particular, $t = 1$ when $d = \gcd(b-a, c-b)$ is even (and $\delta \geq \frac{1}{2}$); $t \leq 2$ if $d \geq 3$ is odd (and $\delta \geq \frac{4}{9}$); $t \leq 3$ if $d = 1$ (and $\delta \geq \frac{1}{4}$).

We are done if $t = 1$. When $t \geq 2$, we consider μ-robust edge vectors in H with respect to the partition $\mathcal{P}$. By Lemma 3.9, it suffices to verify the assumption in Lemma 3.9, that is, $(1,-1) \in L_{\mathcal{P},K}^{\mu'}(H)$ when $t = 2$ and respectively, $(1,-1,0)$, $(1,0,-1)$, $(0,1,-1) \in L_{\mathcal{P},K}^{\mu'}(H)$ when $t = 3$. For convenience, write

$$\mathbf{t}_1 = (a, b+c), \mathbf{t}_2 = (b, a+c), \mathbf{t}_3 = (c, a+b), \mathbf{t}_4 = (a+b+c, 0)$$

and

$$\mathbf{t}_i' = (a+b+c, a+b+c) - \mathbf{t}_i \text{ for } 1 \leq i \leq 4.$$

Claim 3.11. *For any partition $\mathcal{P}' = \{V_0, V', V''\}$ of $V(H)$ with $|V_0| \leq 4\epsilon n$ and $|V''|$, $|V'| \geq \epsilon^2 n$, we have $(1,2)$ or $(2,1) \in I_{\mathcal{P}'}^{3\mu}(H)$.*

Proof. Without loss of generality, assume that $|V'| \leq n/2$. Fix $v \in V'$. We observe that v is contained in at least ϵn^2 crossing edges (those with index vector $(1,2)$ or $(2,1)$)

[1] Given a 3-graph H and a vertex $v \in V(H)$, the link graph is defined as the graph with the vertex set $V(H) \setminus \{v\}$ and the edge set $\{S \setminus \{v\} : v \in S, S \in E(H)\}$.

– otherwise $\delta_1(H) \leq \binom{n/2}{2} + \epsilon n^2 + |V_0|n < (\frac{1}{4}+\gamma)\binom{n}{2}$, contradicting our assumption on $\delta_1(H)$. Hence v is in at least $\epsilon n^2/2$ edges with index vector $(1,2)$ or $\epsilon n^2/2$ edges with index vector $(2,1)$. Without loss of generality, assume that at least half of the vertices in V' are in at least $\epsilon n^2/2$ edges with index vector $(1,2)$. Thus the number of edges with index vector $(1,2)$ is at least $\frac{1}{2}\epsilon^2 n \cdot \epsilon n^2/2 \geq 3\mu n^3$ as $\mu \ll \epsilon$. This means that $(1,2) \in I^{3\mu}_{\mathcal{P}'}(H)$. □

Case 1: K is of type $d \geq 3$ with d odd.

In this case, $\delta_1(H) \geq (\frac{4}{9}+\gamma)\binom{n}{2}$. Thus $t = 2$ and $\mathcal{P} = \{V_0, V_1, V_2\}$. By Claim 3.11, without loss of generality, assume that $(1,2) \in I^{\mu}_{\mathcal{P}}(H)$. If $I^{\mu}_{\mathcal{P}}(H) = \{(1,2)\}$, then assume that $|V_2| = pn$ for some $0 < p < 1$. The number of edges with index vector $(1,2)$ is at most

$$|V_1|\binom{|V_2|}{2} < (1-p)p^2n^3/2 \leq \frac{4}{9}\cdot\frac{n^3}{6},$$

where the second inequality becomes an equality when $p = 2/3$. Thus, $e(H) \leq \frac{4}{9}\frac{n^3}{6} + 3\mu n^3 + |V_0|n^2 < \frac{4}{9}\binom{n}{3} + 5\epsilon n^3$ (where $3\mu n^3$ bounds the number of edges with other index vectors), contradicting our assumption on $\delta_1(H)$. Therefore, $|I^{\mu}_{\mathcal{P}}(H)| \geq 2$ and there are 3 possibilities: $I^{\mu}_{\mathcal{P}}(H) \supseteq \{(1,2),(3,0)\}$, $I^{\mu}_{\mathcal{P}}(H) \supseteq \{(1,2),(0,3)\}$ and $I^{\mu}_{\mathcal{P}}(H) \supseteq \{(1,2),(2,1)\}$. By Proposition 3.1,

$$I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4\} \text{ or } I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_4\} \text{ or}$$
$$I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_1,\mathbf{t}'_2,\mathbf{t}'_3\},$$

respectively. If $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$,

$$\mathbf{t}_4-\mathbf{t}_1 = (b+c,-(b+c)),\ \mathbf{t}_4-\mathbf{t}_2 = (a+c,-(a+c)),$$
$$\mathbf{t}_4-\mathbf{t}_3 = (a+b,-(a+b)) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since K is of type $d \geq 3$ and d is odd, Fact 3.10 implies that $\gcd(b+c,a+c,a+b) = 1$ and hence $(1,-1) = x(\mathbf{t}_4-\mathbf{t}_1)+y(\mathbf{t}_4-\mathbf{t}_2)+z(\mathbf{t}_4-\mathbf{t}_3) \in L^{\mu'}_{\mathcal{P},K}(H)$ for some integers x,y,z. Otherwise $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_4\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$ or $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_1,\mathbf{t}'_2,\mathbf{t}'_3\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$, it is easy to see that in either case

$$(a,-a),(b,-b),(c,-c) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since $\gcd(a,b,c) = 1$, we conclude that $(1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

Case 2: K is of type 1 and $t = 2$.

By Claim 3.11, without loss of generality, assume that $(1,2) \in I^{\mu}_{\mathcal{P}}(H)$. By Proposition 3.1, we have

$$\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \in I^{\mu'}_{\mathcal{P},K}(H),$$

and thus

$$\mathbf{t}_2 - \mathbf{t}_1 = (b-a, a-b), \mathbf{t}_3 - \mathbf{t}_2 = (c-b, b-c) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since $K_{a,b,c}$ is of type 1, namely, $\gcd(b-a, c-b) = 1$, we conclude that $(1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

Case 3: K is of type 1 and $t = 3$.

If $(1,2,0) \in I^{\mu}_{\mathcal{P}}(H)$, then the arguments in Case 2 show that $(1,-1,0) \in L^{\mu'}_{\mathcal{P},K}(H)$. If we also have $(0,1,2) \in I^{\mu}_{\mathcal{P}}(H)$, then $(0,1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$. Consequently $(1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$, and we are done. In general, let T be the set of all vectors with three coordinates $0,1,2$ (in any order). If

$$I^{\mu}_{\mathcal{P}}(H) \text{ contains two members of } T \text{ whose 0's are on different coordinates,} \tag{3.6}$$

then the arguments above show that $(1,-1,0), (0,1,-1), (1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

We claim that (3.6) holds if $(1,1,1) \notin I^{\mu}_{\mathcal{P}}(H)$. In this case, we prove a stronger statement than (3.6): *for each $i \in [3]$, $I^{\mu}_{\mathcal{P}}(H)$ contains a member of T whose ith coordinate is positive.* Fix $i \in [3]$. By applying Claim 3.11 to $\mathcal{P}' = \{V_0, V_i, V_{i+1} \cup V_{i+2}\}$ (the addition is modulo 3), we may assume that at least $3\mu n^3$ edges have index vector $(1,2)$ with respect to $\mathcal{P}'$. Since $(1,1,1) \notin I^{\mu}_{\mathcal{P}}(H)$, at most μn^3 of these edges have index vector $(1,1,1)$ with respect to $\mathcal{P}$. Thus, there exists $j \neq i$ such that at least μn^3 of these edges intersect V_j with two vertices. This proves the desired statement.

What remains is the case when $(1,1,1) \in I^{\mu}_{\mathcal{P}}(H)$. In this case, by Proposition 3.1,

$$(a,b,c), (b,a,c), (a,c,b), (b,c,a), (c,a,b), (c,b,a) \in I^{\mu'}_{\mathcal{P},K}(H).$$

This implies $(y,-y,0), (0,y,-y), (y,0,-y) \in L^{\mu'}_{\mathcal{P},K}(H)$ for all $y \in \{b-a, c-b\}$. Since $\gcd(b-a, c-b) = 1$, we derive that $(1,-1,0), (0,1,-1), (1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$. □

4. Proof of the almost tiling lemma

4.1. The weak regularity and cluster hypergraphs

Let $H = (V, E)$ be a 3-graph and let V_1, V_2, V_3 be mutually disjoint non-empty subsets of V. We denote the number of edges with one vertex in each V_i, $i \in [3]$ by $e(V_1, V_2, V_3)$, and the density of H with respect to (V_1, V_2, V_3) by

$$d(V_1, V_2, V_3) = \frac{e(V_1, V_2, V_3)}{|V_1||V_2||V_3|}.$$

The triple (V_1, V_2, V_3) of mutually disjoint subsets $V_1, V_2, V_3 \subseteq V$ is called (ϵ, d)-*regular* for $\epsilon > 0$ and $d \geq 0$ if

$$|d(A_1, A_2, A_3) - d| \leq \epsilon$$

for all triples of subsets $A_i \subseteq V_i$, $i \in [3]$, satisfying $|A_i| \geq \epsilon |V_i|$. The triple (V_1, V_2, V_3) is called ϵ-*regular* if it is (ϵ, d)-regular for some $d \geq 0$. By definition, if $A_i \subseteq V_i$, $i \in [3]$, has size $|A_i| \geq p|V_i|$ for some $p \geq \epsilon$, then (A_1, A_2, A_3) is $(\epsilon/p, d)$-regular.

Let $H = (V, E)$ be an n-vertex 3-graph, a partition of V into $V_0, V_1, \ldots, V_t$ is called an (ϵ, t)-*regular partition* if

(i) $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| \leq \epsilon n$,
(ii) for all but at most $\epsilon \binom{t}{3}$ sets $\{i, j, l\} \in \binom{[t]}{3}$, the triple (V_i, V_j, V_l) is ϵ-regular.

We call $V_1, \ldots, V_t$ *clusters*. Given an (ϵ, t)-regular partition $\mathcal{P} = \{V_0, V_1, V_2, \ldots, V_t\}$ and $d > 0$, the *cluster hypergraph* $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ is defined as the 3-graph whose vertices are $V_1, \ldots, V_t$ and $\{V_i, V_j, V_l\}$ forms an edge of $\mathcal{R}$ if and only if (V_i, V_j, V_l) is ϵ-regular and $d(V_i, V_j, V_l) \geq d$.

We need a simple corollary of the Weak Regularity Lemma, which is a straightforward extension of Szemerédi's regularity lemma for graphs [30]. The following proposition shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Since its proof is the same as that of [3, Proposition 15], we omit the proof.

Proposition 4.1. *([3]) For $0 < \epsilon < d \ll \delta$ and $t_0 \geq 0$ there exist T and n_2 such that the following holds. Suppose H is a 3-graph on $n > n_2$ vertices with $\delta_1(H) \geq \delta \binom{n}{2}$. Then there exists an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ such that the cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ satisfies $\delta_1(\mathcal{R}) \geq (\delta - \epsilon - d)\binom{t}{2}$.*

Next we show that every regular triple can be almost perfectly tiled by copies of $K_{a,b,c}$ provided the sizes of its three parts is somewhat balanced.

Proposition 4.2. *Let $a \leq b \leq c$ be integers, $0 < 2\epsilon \leq d$, and m be sufficiently large. Suppose (V_1, V_2, V_3) is (ϵ, d)-regular, $|V_1| \leq |V_2| \leq |V_3| = m$, and*

$$\frac{|V_1|}{a} \geq \frac{|V_2|}{b} \geq \frac{|V_3|}{c}. \tag{4.1}$$

Then there is a $K_{a,b,c}$-tiling on $V_1 \cup V_2 \cup V_3$ covering all but at most $\frac{c}{a}\epsilon(|V_1| + |V_2| + |V_3|)$ vertices.

Proof. We will greedily pick vertex disjoint $K_1, K_2, \ldots, K_s$ until $|V_i \setminus \bigcup_{\ell=1}^{s} V(K_\ell)| < \epsilon m$ for some $i \in [3]$, where each K_ℓ is a copy of $K_{a,b,c}$ or $K_{k,k,k}$ by the algorithm described below. This gives rise to a $K_{a,b,c}$-tiling because each copy of $K_{k,k,k}$ consists of three

vertex disjoint copies of $K_{a,b,c}$. Our algorithm is as follows. For $i \in [3]$, let $U_i^0 = V_i$. For $j \in [s]$, let

$$\left\{U_1^j, U_2^j, U_3^j\right\} = \left\{V_i \setminus \bigcup_{\ell=1}^{j} V(K_\ell) : i \in [3]\right\} \text{ such that } |U_1^j| \le |U_2^j| \le |U_3^j|,$$

and $U^j = U_1^j \cup U_2^j \cup U_3^j$. In other words, U_1^j, U_2^j, U_3^j are the subsets of V_1, V_2, V_3 obtained from removing the vertices of $K_1, \dots, K_j$ and arranged in the ascending order of size. Suppose that we have already found $K_1, \dots, K_j$ and $|U_1^j| \ge \epsilon m$. We let K_{j+1} be a copy of $K_{k,k,k}$ from U^j if

$$|U_3^j| - |U_1^j| \le c - a; \tag{4.2}$$

otherwise we let K_{j+1} be a copy of $K_{a,b,c}$ with a vertices from U_1^j, b vertices from U_2^j, and c vertices from U_3^j. In either case this is possible because $|U_i^j| \ge \epsilon m$ for $i \in [3]$; by the regularity, we have $d(U_1^j, U_2^j, U_3^j) \ge d - \epsilon \ge \epsilon$ and

$$e(U_1^j, U_2^j, U_3^j) \ge \epsilon |U_1^j||U_2^j||U_3^j| \ge \epsilon^4 m^3.$$

By Proposition 3.1, we can find a copy of $K_{k,k,k}$ or $K_{a,b,c}$ from U^j. The algorithm terminates when $|U_1^s| < \epsilon m$. We need to show that $|U^s| \le \frac{c}{a}\epsilon(|V_1|+|V_2|+|V_3|)$. By (4.1), $|V_1|+|V_2|+|V_3| \ge \frac{k}{c}m$ and thus $\frac{c}{a}\epsilon(|V_1|+|V_2|+|V_3|) \ge \frac{k}{a}\epsilon m$. So it suffices to show that $|U^s| \le \frac{k}{a}\epsilon m$.

First, assume that (4.2) holds for some $0 \le j < s$. In this case $K_{j+1} \cong K_{k,k,k}$ and

$$|U_3^{j+1}| - |U_1^{j+1}| = |U_3^j| - |U_1^j| \le c - a.$$

Therefore $K_\ell \cong K_{k,k,k}$ for all $\ell > j$ and consequently $|U_3^s| - |U_1^s| \le c - a$. Since $|U_1^s| < \epsilon m$, it follows that $|U^s| < 3\epsilon m + 2(c-a)$. If $a = c$, then $|U^s| \le 3\epsilon m = \frac{k}{a}\epsilon m$ and we are done. Otherwise $\frac{k}{a} \ge 3 + \frac{1}{a}$. Since m is large enough, it follows that $|U^s| < (3 + \frac{1}{a})\epsilon m \le \frac{k}{a}\epsilon m$, as desired.

Second, assume that (4.2) fails for all $0 \le j < s$. We claim that for all $0 \le j \le s$,

$$\frac{|U_1^j|}{a} \ge \frac{|U_2^j|}{b} \ge \frac{|U_3^j|}{c}. \tag{4.3}$$

This suffices because $|U_1^s| < \epsilon m$ and (4.3) with $j = s$ together imply that $|U^s| \le (1 + \frac{b}{a} + \frac{c}{a})|U_1^s| < \frac{k}{a}\epsilon m$.

Let us prove (4.3) by induction. The $j = 0$ case follows from (4.1) and the assumption $|V_1| \le |V_2| \le |V_3|$. Suppose that (4.3) holds for some $j \ge 0$. By our algorithm, K_{j+1} is a copy of $K_{a,b,c}$ with a vertices from U_1^j, b vertices from U_2^j, and c vertices from U_3^j. Let $\tilde{U}_i^j = U_i^j \setminus V(K_{j+1})$ for $i \in [3]$ and thus $|\tilde{U}_1^j|/a = |U_1^j|/a - 1$, $|\tilde{U}_2^j|/b = |U_2^j|/b - 1$ and $|\tilde{U}_3^j|/c = |U_3^j|/c - 1$. By the inductive hypothesis,

$$\frac{|\tilde{U}_1^j|}{a} \geq \frac{|\tilde{U}_2^j|}{b} \geq \frac{|\tilde{U}_3^j|}{c}. \tag{4.4}$$

Since $|U_i^j| \geq \epsilon m \geq b + c$ for all $i \in [3]$,

$$b^2 - a^2 \leq (b-a)|U_1^j| \leq b|U_2^j| - a|U_1^j|$$

and

$$c^2 - b^2 \leq (c-b)|U_2^j| \leq c|U_3^j| - b|U_2^j|$$

which implies that

$$\frac{|\tilde{U}_2^j|}{a} \geq \frac{|\tilde{U}_1^j|}{b}, \quad \text{and} \quad \frac{|\tilde{U}_3^j|}{b} \geq \frac{|\tilde{U}_2^j|}{c}. \tag{4.5}$$

Now we separate cases according to the order of $|\tilde{U}_1^j|$, $|\tilde{U}_2^j|$ and $|\tilde{U}_3^j|$. Since $|\tilde{U}_3^j| - |\tilde{U}_1^j| = |U_3^j| - |U_1^j| - (c-a) > 0$, we only have three cases.

Case 1. $|\tilde{U}_1^j| \leq |\tilde{U}_2^j| \leq |\tilde{U}_3^j|$. Then (4.3) for $j+1$ follows from (4.4) immediately.
Case 2. $|\tilde{U}_2^j| \leq |\tilde{U}_1^j| \leq |\tilde{U}_3^j|$. Together with (4.4) and (4.5), we derive that

$$\frac{|\tilde{U}_2^j|}{a} \geq \frac{|\tilde{U}_1^j|}{b} \geq \frac{|\tilde{U}_2^j|}{b} \geq \frac{|\tilde{U}_3^j|}{c}.$$

Case 3. $|\tilde{U}_1^j| \leq |\tilde{U}_3^j| \leq |\tilde{U}_2^j|$. Together with (4.4) and (4.5), we derive that

$$\frac{|\tilde{U}_1^j|}{a} \geq \frac{|\tilde{U}_2^j|}{b} \geq \frac{|\tilde{U}_3^j|}{b} \geq \frac{|\tilde{U}_2^j|}{c}.$$

This implies that (4.3) holds for $j+1$ and we are done. □

When $a = b = c$, the proof of Lemma 1.2 is a standard application of the regularity method. This was given implicitly in [17] and stated as [25, Lemma 4.4] without a proof. For completeness, we include the proof here.

Proof of Lemma 1.2 when $a = b = c$**.** Let $0 < 4\epsilon = d \ll \min\{\gamma, \alpha\}$ and $t_0 = 1/\epsilon$. Suppose T and n_2 are the parameters returned by Proposition 4.1 with $\delta = 5/9 + \gamma$. Let H be a 3-graph on n vertices with $\delta_1(H) \geq (\frac{5}{9} + \gamma)\binom{n}{2}$ for some sufficiently large $n \geq n_2$. We apply Proposition 4.1 and obtain an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ and a cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ satisfying $\delta_1(\mathcal{R}) \geq (\frac{5}{9} + \gamma - \epsilon - d)\binom{t}{2}$. Note that each cluster is of size $n/t \geq n/T$. Suppose that $t \equiv r \bmod 3$ for some $r \in \{0, 1, 2\}$. Let $\mathcal{R}'$ be the induced subgraph of $\mathcal{R}$ on clusters $V_{r+1}, \ldots, V_t$. Then $\delta_1(\mathcal{R}') \geq \delta_1(\mathcal{R}) - 2t \geq \left(\frac{5}{9} + \frac{\gamma}{2}\right)\binom{t}{2}$. We apply [11, Theorem 6][2] to $\mathcal{R}'$ and get a perfect

[2] We may alternatively use the exact result in [17,23].

matching M. For each edge $e = \{V_i, V_j, V_l\} \in M$, Proposition 4.2 provides a $K_{a,b,c}$-tiling that covers all but at most $\epsilon(|V_i| + |V_j| + |V_l|)$ vertices of $V_i \cup V_j \cup V_l$. The union of these $K_{a,b,c}$-tilings covers all but at most

$$|V_0| + \epsilon(|V_1| + \cdots + |V_t|) + |V_1| + |V_2| \leq 2\epsilon n + 2n/t \leq 4\epsilon n \leq \alpha n$$

vertices of $V(H)$, as desired. □

We assume that $a < c$ in the next two subsections.

4.2. Fractional homomorphic tilings

To obtain a large $K_{a,b,c}$-tiling in H when $a < c$, we follow the idea of Buß, Hàn and Schacht [3] considering a fractional homomorphism from $K_{a,b,c}$ to the cluster hypergraph $\mathcal{R}$. Let us first define a fractional hom$(K_{a,b,c})$-tiling. As in previous sections, we simply write $K_{a,b,c}$ as K.

Definition 4.3. Given a 3-graph $H = (V, E)$, a function $h : V \times E \to [0, 1]$ is called a fractional hom(K)-tiling of H if

(1) $h(v, e) = 0$ if $v \notin e$,
(2) $h(v) = \sum_{e \in E} h(v, e) \leq 1$,
(3) for every $e \in E$ there exists a labeling $e = uvw$ such that $h(u, e) \leq h(v, e) \leq h(w, e)$ and

$$\frac{h(u, e)}{a} \geq \frac{h(v, e)}{b} \geq \frac{h(w, e)}{c}.$$

Given $e = uvw \in E$, we simply write $h(u, v, w) = (h(u, e), h(v, e), h(w, e))$. We denote by $h_{\min}$ the smallest non-zero value of $h(v, e)$ and by $w(h)$ the (total) weight of h:

$$w(h) = \sum_{(v,e) \in V \times E} h(v, e).$$

For example, suppose that the vertex classes of K are X, Y, Z with $|X| = a$, $|Y| = b$ and $|Z| = c$. We obtain a fractional hom(K)-tiling h by letting $h(x, y, z) = \frac{1}{abc}(a, b, c) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$ for every $xyz \in E(K)$ with $x \in X, y \in Y, z \in Z$.[3] Then $w(h) = k$ (the largest possible) and $h_{\min} = \frac{1}{bc}$. We later refer to $(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$ as the *standard weight* of an edge of K and refer to the function mentioned above as the *standard weight function* on K.

The following proposition shows that a fractional hom(K)-tiling in the cluster hypergraph can be "converted" to an integer K-tiling in the original hypergraph.

[3] In general, we write $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$.

Proposition 4.4. *Let $1 \le a \le b \le c$ be integers. Given $\epsilon, \phi > 0$, $d \ge 2\epsilon/\phi$, and integer $T > 0$, there exists $n_3 \in \mathbb{Z}$ such that the following holds for all $0 < t \le T$ and $n \ge n_3$. Let H be a 3-graph on n vertices with an (ϵ, t)-regular partition $\mathcal{P}$ and a cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$. Suppose that there is a fractional hom(K)-tiling h of $\mathcal{R}$ with $h_{\min} \ge \phi$. Then there exists a K-tiling of H that covers at least $(1 - 2c\epsilon/\phi)\, w(h)n/t$ vertices.*

Proof. Let $\mathcal{R}'$ be the subhypergraph of $\mathcal{R}$ consisting of the hyperedges $e = uvw \in E(\mathcal{R}')$ with $h(u,e)$, $h(v,e)$, $h(w,e) \ge h_{\min} \ge \phi$. For each $u \in V(\mathcal{R}')$, let V_u be the corresponding cluster of H. Since $\mathcal{P}$ is an (ϵ, t)-regular partition, all the clusters have size ℓ for some $\ell \ge (1-\epsilon)n/t$. In each V_u we find disjoint subsets V_u^e of size $h(u,e)\ell$ for all $e \in E(\mathcal{R}')$ with $u \in e$ – this is possible because $\sum_{e \in E(\mathcal{R}')} h(u,e) \le 1$. Note that every edge $e = uvw \in E(\mathcal{R}')$ corresponds to an (ϵ, d')-regular triple (V_u, V_v, V_w) for some $d' \ge d$. Hence for every $e = uvw \in E(\mathcal{R}')$, (V_u^e, V_v^e, V_w^e) is $(\epsilon/\phi, d')$-regular with at least $\phi\ell \ge (1-\epsilon)\phi n/t \ge (1-\epsilon)\phi n_3/T$ vertices in each part. Because of Definition 4.3 (3), the assumptions that $d \ge 2\epsilon/\phi$ and $(1-\epsilon)\phi n_3/T$ is sufficiently large, we can apply Proposition 4.2 and obtain a K-tiling covering at least

$$\left(1 - \frac{c}{a} \cdot \frac{\epsilon}{\phi}\right) h(e)\ell \ge (1 - c\epsilon/\phi)h(e)(1-\epsilon)\frac{n}{t} \ge (1 - 2c\epsilon/\phi)\, h(e)\frac{n}{t}$$

vertices of $V_u \cup V_v \cup V_w$, where $h(e) = h(u,e) + h(v,e) + h(w,e)$. Repeating this to all hyperedges of $\mathcal{R}'$, we obtain a K-tiling that covers at least

$$\sum_{uvw \in E(\mathcal{R}')} (1 - 2c\epsilon/\phi)\, h(e)\frac{n}{t} = (1 - 2c\epsilon/\phi)\, w(h)\frac{n}{t}$$

vertices of H. □

The key of the proof of Lemma 1.2 is that if a maximum K-tiling in $\mathcal{R}$ is not large enough, then we use the minimum degree condition to find a large fractional hom(K)-tiling of $\mathcal{R}$, which gives a large K-tiling in H by Proposition 4.4. The following two propositions show that we can find a fractional hom(K)-tiling h on one or two copies of K together with one or two vertices outside such that $w(h)$ is larger than the standard weight on these copies of K alone.

Given a copy K_1 of K and two vertices $u, u' \notin V(K_1)$, let $\mathcal{L}_1(K_1, u, u')$ denote the family of all 3-graphs on $\{u, u'\} \cup V(K_1)$ whose edge set contains $E(K_1)$ and at least $a+1$ triples $uu'v$ with $v \in V(K_1)$.

Proposition 4.5. *Let $1 \le a \le b \le c$ be integers with $a < c$ and $k = a+b+c$. Let K_1 be a copy of $K_{a,b,c}$ and let $u, u' \notin V(K_1)$ be two vertices. For any 3-graph $L \in \mathcal{L}_1(K_1, u, u')$, there is a fractional hom(K)-tiling h of L with $w(h) \ge k + \frac{1}{abc}$ and $h_{\min} \ge \frac{1}{bc^2}$.*

Proof. Suppose the vertex classes of K_1 are X, Y, Z with $|X| = a$, $|Y| = b$, and $|Z| = c$. Since $\deg(uu') \ge a + 1 = |X| + 1$, we have $N(uu', Y \cup Z) \ne \emptyset$.

If there exists $z \in N(uu', Z)$, then we pick $x \in X$ and $y \in Y$ and assign weights $h(z,u,u') = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$,

$$h(x,y,z) = \left(\frac{1}{bc} - \frac{a}{bc^2}, \frac{1}{ac} - \frac{1}{c^2}, \frac{1}{ab} - \frac{1}{bc}\right) = \frac{c-a}{abc^2}(a,b,c),$$

and assign the standard weight to all other edges of K_1. Then h is a fractional $\hom(K)$-tiling of L with $w(h) = k + \frac{1}{ab} + \frac{1}{ac} - \frac{a}{bc^2} - \frac{1}{c^2} \geq k + \frac{1}{abc}$ and $h_{\min} = \frac{c-a}{bc^2} \geq \frac{1}{bc^2}$.

Otherwise $N(uu', Z) = \emptyset$, then there exists $y \in N(uu', Y)$. First assume $a < b$. We assign $h(y,u,u') = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$,

$$h(x,y,z) = \left(\frac{1}{bc} - \frac{a}{b^2c}, \frac{1}{ac} - \frac{1}{bc}, \frac{1}{ab} - \frac{1}{b^2}\right) = \frac{b-a}{ab^2c}(a,b,c)$$

for some $x \in X$ and $z \in Z$, and the standard weight to all other edges. Then h is a fractional $\hom(K)$-tiling with $w(h) = k + \frac{1}{ab} + \frac{1}{ac} - \frac{a}{b^2c} - \frac{1}{b^2} \geq k + \frac{1}{abc}$ and $h_{\min} = \frac{b-a}{b^2c} \geq \frac{1}{bc^2}$. Second, we assume $a = b$. By the degree condition, we have $N(uu', X) \neq \emptyset$. Pick $x \in N(uu', X)$ and $z \in Z$. By assigning $h(x,u,u') = h(y,u,u') = h(x,y,z) = (\frac{1}{2ac}, \frac{1}{2ac}, \frac{1}{2a^2})$ and the standard weight to all other edges, we get a fractional $\hom(K)$-tiling with $w(h) = k + \frac{1}{a^2} + \frac{1}{ac} - \frac{1}{2a^2} \geq k + \frac{1}{abc}$ and $h_{\min} = \frac{1}{2ac} \geq \frac{1}{bc^2}$ as $c \geq 2$. □

Given two vertex disjoint copies K_1, K_2 of K and a vertex $u \notin V(K_1) \cup V(K_2)$, let $\mathcal{L}_2(K_1, K_2, u)$ denote the family of all 3-graphs on $\{u\} \cup V(K_1) \cup V(K_2)$ whose edge set contains $E(K_1) \cup E(K_2)$ and at least $\max\{a^2 + 2a(b+c), (a+b)^2\} + 1$ triples uvw with $v \in V(K_1)$ and $w \in V(K_2)$.

The following proposition shows that any 3-graph $L \in \mathcal{L}_2(K_1, K_2, u)$ has a $\hom(K)$-tiling with weight greater than $2k$. In its proof we assign weights to an edge as follows. Suppose $0 < \lambda \leq 1$, then $(\frac{a}{c}\lambda, \frac{b}{c}\lambda, \lambda)$ satisfies (3) in Definition 4.3. Furthermore, given $\mu_1, \mu_2 \geq 0$ such that $\frac{a}{c}\lambda + \mu_1 \leq \frac{b}{c}\lambda \leq \lambda - \mu_2$, then $(\frac{a}{c}\lambda + \mu_1, \frac{b}{c}\lambda, \lambda - \mu_2)$ satisfies (3) in Definition 4.3 as well.

Proposition 4.6. *Let $1 \leq a \leq b \leq c$ be integers with $a < c$ and $k = a + b + c$. Let K_1, K_2 be two vertex disjoint copies of $K_{a,b,c}$ and let $u \notin V(K_1)$ be a vertex. For any 3-graph $L \in \mathcal{L}_2(K_1, K_2, u)$, there exists a fractional $\hom(K)$-tiling of L with $w(h) \geq 2k + \frac{1}{abc^2}$ and $h_{\min} \geq \frac{1}{bc^2}$.*

Proof. For $i = 1, 2$, denote the vertex classes of K_i by X_i, Y_i, Z_i with $|X_i| = a$, $|Y_i| = b$, and $|Z_i| = c$. Let L_u be the bipartite graph on $V(K_1) \cup V(K_2)$ such that two vertices $v \in V(K_1)$ and $w \in V(K_2)$ are adjacent if and only if uvw is an edge of L. Then L_u satisfies the following properties.

(i) Since $\deg_L(u) \geq a^2 + 2a(b+c) + 1$, L_u must have an edge not incident to $X_1 \cup X_2$.
(ii) Since $\deg_L(u) \geq (a+b)^2 + 1$, L_u must have an edge incident to $Z_1 \cup Z_2$.

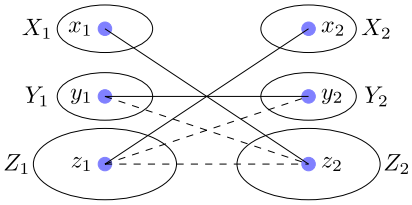


Fig. 1. L_u in the last subcase of Case 1.2.

Let $\lambda = \frac{1}{abc}$. Our proof is now divided into cases based on the values of a, b and c.

Case 1. $b < c$.

First we assume that there is $z_1z_2 \in L_u$ for $z_1 \in Z_1$ and $z_2 \in Z_2$. Let $x_i \in X_i$, $y_i \in Y_i$ for $i = 1, 2$. In this case let $h(u, z_1, z_2) = (\lambda, \lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab} - \lambda)$. Other edges of K_1 or K_2 receive the standard weight $(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$. *In the rest of the proof, any edge of K_1 or K_2 not specified receives the standard weight.* Therefore we get a fractional $\hom(K)$-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \lambda$. We thus assume $L_u[Z_1, Z_2] = \emptyset$ and proceed in two subcases.

Case 1.1. $a < b < c$. We first assume that there exists $z_1y_2 \in L_u$ for $z_1 \in Z_1$ and $y_2 \in Y_2$. Let $x_i \in X_i$ for $i = 1, 2$, $y_1 \in Y_1$ and $z_2 \in Z_2$. In this case we let $h(y_2, z_1, u) = (\frac{a}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_1, y_1, z_1) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab} - \frac{b}{c}\lambda)$, and $h(x_2, y_2, z_2) = (\frac{1}{bc}, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ab} - \frac{a}{b}\lambda)$. So we get a fractional $\hom(K)$-tiling of L with $w(h) = 2k + (1 - \frac{a}{b})\lambda \geq 2k + \frac{1}{b}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

We thus assume that $L_u[Z_1, Y_2] = \emptyset$ and by symmetry, $L_u[Y_1, Z_2] = \emptyset$. By (i), it follows that $L_u[Y_1, Y_2] \neq \emptyset$. By (ii), without loss of generality, assume that $L_u[Z_1, X_2] \neq \emptyset$. Suppose $y_1y_2, z_1x_2 \in L_u$ with $y_1 \in Y_1$, $y_2 \in Y_2$, $z_1 \in Z_1$ and $x_2 \in X_2$. Let $x_1 \in x_1$ and $z_2 \in Z_2$. We let $h(y_1, y_2, u) = (\frac{b}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_2, u, z_1) = (\frac{a}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_1, y_1, z_1) = \big(\frac{1}{bc}, \frac{1}{ac} - \frac{b}{c}\lambda, \frac{1}{ab} - \lambda\big)$, and $h(x_2, y_2, z_2) = \big(\frac{1}{bc} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{b}{c}\lambda, \frac{1}{ab} - \lambda\big)$. So we get a fractional $\hom(K)$-tiling of L with $w(h) = 2k + \frac{b}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

Case 1.2. $a = b < c$. We first assume that both $L_u[Z_1, X_2] \neq \emptyset$ and $L_u[Z_1, Y_2] \neq \emptyset$. Suppose $z_1x_2, z_1'y_2 \in L_u$ with $z_1, z_1' \in Z_1$, $x_2 \in X_2$ and $y_2 \in Y_2$ (we may have $z_1 = z_1'$). We assign the weights $h(z_1, x_2, u) = h(z_1', y_2, u) = (\frac{a}{c}\lambda, \frac{a}{c}\lambda, \lambda)$. If $a \geq 2$, then pick $x_1, x_1' \in X_1$, $y_1, y_1' \in Y_1$ and $z_2 \in Z_2$, and assign $h(x_1, y_1, z_1) = h(x_1', y_1', z_1') = \big(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{a}{c}\lambda\big)$ and $h(x_2, y_2, z_2) = \big(\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda\big)$. If $a = 1$, then pick $x_1 \in X_1$, $y_1 \in Y_1$ and $z_2 \in Z_2$, and assign $h(x_2, y_2, z_2) = \big(\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda\big)$ and

$$h(x_1, y_1, z_1) = h(x_1, y_1, z_1') = \begin{cases} \big(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{2a}{c}\lambda\big) & \text{if } z_1 = z_1', \\ \big(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{a}{c}\lambda\big) & \text{if } z_1 \neq z_1'. \end{cases}$$

In all cases we get a fractional $\hom(K)$-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

We may thus assume that at least one of $L_u[Z_1, X_2]$ and $L_u[Z_1, Y_2]$ is empty, and by symmetry, at least one of $L_u[X_1, Z_2]$ and $L_u[Y_1, Z_2]$ is empty. Since $a = b$, X_i and

Y_i $(i = 1, 2)$ play the same role. Without loss of generality, assume that $L_u[Z_1, Y_2] = L_u[Y_1, Z_2] = \emptyset$. Furthermore, we observe that $L_u[Z_1, X_2] \neq \emptyset$ and $L_u[X_1, Z_2] \neq \emptyset$ – otherwise, as $L_u[Z_1, Z_2] = \emptyset$, it follows that $\deg_L(u) \leq 4a^2 + ac < a^2 + 2a(b + c)$, a contradiction.

Suppose $z_1x_2, x_1z_2 \in L_u$, where $z_1 \in Z_1$, $x_2 \in X_2$, $x_1 \in X_1$, $z_2 \in Z_2$. By (i), there exists $y_1y_2 \in L_u$, where $y_1 \in Y_1$, $y_2 \in Y_2$ (see Fig. 1). We assign the weights $h(u, x_2, z_1) = h(u, x_1, z_2) = h(y_1, y_2, u) = (\frac{a}{c}\lambda, \frac{a}{c}\lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = (\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda)$. This gives a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda + \frac{2a}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$. Note that $h(u) = \frac{2a}{c}\lambda + \lambda = \frac{2a+c}{a^2c^2} \leq 1$ because $a \geq 1$ and $c \geq 2$. Thus this weight assignment is possible.

Case 2. $a < b = c$.

Since $b = c$, Y_i and Z_i $(i = 1, 2)$ play the same role. Thus by (i), without loss of generality, assume that there exists $z_1z_2 \in L_u$ for $z_1 \in Z_1$ and $z_2 \in Z_2$. Furthermore, generalizing (ii), we know that there must be an edge incident to $Y_1 \cup Y_2$ and without loss of generality, say that edge is incident to Y_1. We now proceed with three cases.

Case 2.1. There exists $y_1x_2 \in L_u$ where $y_1 \in Y_1$ and $x_2 \in X_2$. Pick $x_1 \in X_1$ and $y_2 \in Y_2$. We assign $h(u, z_1, z_2) = (\lambda, \lambda, \lambda)$, $h(x_2, y_1, u) = (\frac{a}{c}\lambda, \lambda, \lambda)$, $h(x_1, y_1, z_1) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$, and $h(x_2, y_2, z_2) = \left(\frac{1}{c^2} - \frac{a}{c}\lambda, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$. Thus, we get a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

Case 2.2. There exists $y_1y_2 \in L_u$ where $y_1 \in Y_1$ and $y_2 \in Y_2$. Pick $x_i \in X_i$ for $i = 1, 2$. We assign the weights $h(u, z_1, z_2) = h(u, y_1, y_2) = (\lambda, \lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$ and get a fractional hom(K)-tiling of L with $w(h) = 2k + 2\lambda$ and $h_{\min} = \lambda$.

Case 2.3. There exists $y_1z_2' \in L_u$ where $y_1 \in Y_1$ and $z_2' \in Z_2$ (it is possible to have $z_2 = z_2'$). We assign the weights $h(z_2, u, z_1) = h(z_2', u, y_1) = (\frac{a}{c}\lambda, \lambda, \lambda)$. Pick $x_1 \in X_1$, $x_2 \in X_2$ and distinct $y_2, y_2' \in Y_2$, which is possible as $b > a \geq 1$. Let $h(x_1, y_1, z_1) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$ and $h(x_2, y_2, z_2) = h(x_2, y_2', z_2') = \left(\frac{1}{c^2}, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda\right)$. Thus, we get a fractional hom(K)-tiling of L with $w(h) = 2k + 2\lambda - \frac{2a}{c}\lambda \geq 2k + \frac{2}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

In all cases we obtain a fractional hom(K)-tiling with $w(h) \geq 2k + \frac{\lambda}{c} = 2k + \frac{1}{abc^2}$ and $h_{\min} \geq \frac{a}{c}\lambda = \frac{1}{bc^2}$. □

4.3. Proof of Lemma 1.2 when $a < c$

Let H be a 3-graph on n vertices. Given $0 \leq \beta \leq 1$, a K-tiling of H is called *β-deficient* if it covers all but at most βn vertices of $V(H)$.

Proposition 4.7. *Given $0 < d \leq 3/5$ and $\beta, \rho > 0$, there exists an n_0 such that the following holds. If every 3-graph H on $n > n_0$ vertices with $\delta_1(H) \geq d\binom{n}{2}$ has a β-deficient*

K-tiling, then every 3-graph H' on $n' > \max\{n_0, 5\}$ vertices with $\delta_1(H') \geq (d-\rho)\binom{n'}{2}$ has a $(\beta + 2k\rho)$-deficient K-tiling.

Proof. Let H' be a 3-graph on n' vertices with $\delta_1(H') \geq (d-\rho)\binom{n'}{2}$. By adding a set A of $2\rho n'$ new vertices and all the triples of $V(H') \cup A$ that intersect A as edges, we obtain a 3-graph H on $n = n' + 2\rho n'$ vertices. Thus

$$\delta_1(H) = \delta_1(H') + 2\rho n'(n'-1) + \binom{2\rho n'}{2} \geq (d-\rho)\binom{n'}{2} + 4\rho\binom{n'}{2} + \binom{2\rho n'}{2}.$$

Note that $3\rho\binom{n'}{2} \geq 2d\rho n'^2$ because $d \leq 3/5$ and $n' \geq 5$. Thus, $\delta_1(H) \geq d\binom{n'}{2} + 2d\rho n'^2 + d\binom{2\rho n'}{2} = d\binom{n}{2}$. By assumption, H has a β-deficient K-tiling. After removing at most $2\rho n'$ copies of K that intersect A, we obtain a $(\beta + 2k\rho)$-deficient K-tiling of H'. □

Proof of Lemma 1.2 when $a < c$. Since $a < c$, we have $k \geq 4$. Let $\delta = \max\{1 - (\frac{b+c}{k})^2, (\frac{a+b}{k})^2\}$. Since $a \leq b \leq c$, it follows that $\delta \leq \max\{5/9, 4/9\} = 5/9$. Without loss of generality, assume that $0 < \gamma \leq \min\{3/5 - \delta, 2\delta, a/(3k)\}$. Assume for a contradiction that there is an α such that for all n_0 there is some 3-graph H on $n > n_0$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$ but which does not contain an α-deficient K-tiling. Let α_0 be the supremum of all such α.

Let $\epsilon \ll \gamma\alpha_0$. By the definition of α_0, there is an integer n_0 such that

$$\text{all 3-graphs } H \text{ on } n > n_0 \text{ vertices with } \delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$$
$$\text{have an } (\alpha_0+\epsilon)\text{-deficient } K\text{-tiling.} \tag{4.6}$$

We may also assume that n_0 is sufficiently large so that we can apply Proposition 3.1 with $r = 3$, $m = 1$, $l_1 = a$, $l_2 = b$, $l_3 = c$ on 3-graphs of order at least $\alpha_0 n_0/2$. Our goal is to show that there exists an n_1 such that all 3-graphs H on $n > n_1$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$ have an $(\alpha_0-\epsilon)$-deficient K-tiling, thus contradicting the definition of α_0.

Let n_2 and T be the integers returned from Proposition 4.1 with inputs ϵ, $d = 2bc^2\epsilon$, $t_0 = \max\{n_0, k/\epsilon\}$. Let n_3 be the integer returned from Proposition 4.4 with inputs ϵ, $\phi = 1/(bc^2)$, d and T. Let $n_1 = \max\{n_0, n_2, n_3\}$ and let H be a 3-graph on $n > n_1$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$. We assume that H does *not* contain an $(\alpha_0-\epsilon)$-deficient K-tiling – otherwise we are done. After applying Proposition 4.1 to H with the constants chosen above, we get an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ and a cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ on $t > t_0$ vertices with $\delta_1(\mathcal{R}) \geq (\delta+\gamma-(2bc^2+1)\epsilon)\binom{t}{2}$. By (4.6) and assumption $\delta+\gamma \leq 3/5$, we can apply Proposition 4.7 and obtain an $(\alpha_0+\epsilon+2k(2bc^2+1)\epsilon)$-deficient K-tiling of $\mathcal{R}$. Let $\mathcal{M} = \{K_1, K_2, \ldots, K_m\}$ be a largest K-tiling in $\mathcal{R}$ and let U be the set of uncovered vertices.

Claim 4.8. *Let h be a fractional hom(K)-tiling of $\mathcal{R}$ with $h_{\min} \geq \frac{1}{bc^2}$. Then $w(h) < (1-\alpha_0+\sqrt{\epsilon}/2)t \leq mk+\sqrt{\epsilon}t$.*

Proof. We know that $|U| \leq (\alpha_0+\epsilon+2k(2bc^2+1)\epsilon)t \leq (\alpha_0+5kbc^2\epsilon)t$. As $\epsilon \ll 1$, it follows that

$$mk+\sqrt{\epsilon}t \geq (1-\alpha_0-5kbc^2\epsilon)t+\sqrt{\epsilon}t \geq (1-\alpha_0+\sqrt{\epsilon}/2)t.$$

So it suffices to show that $w(h) < (1-\alpha_0+\sqrt{\epsilon}/2)t$. Suppose this is not the case. By Proposition 4.4, there is a K-tiling of H that covers at least

$$\left(1-2bc^3\epsilon\right)w(h)\frac{n}{t} \geq \left(1-2bc^3\epsilon\right)(1-\alpha_0+\sqrt{\epsilon}/2)t\frac{n}{t} \geq (1-\alpha_0+\epsilon)\,n$$

vertices (as $\epsilon \ll 1$). Therefore it is an $(\alpha_0-\epsilon)$-deficient K-tiling, contradicting our assumption on H. □

In the rest of the proof we will derive a contradiction to Claim 4.8. Immediately Claim 4.8 implies that

$$|U| \geq \frac{\alpha_0}{2}t \tag{4.7}$$

otherwise $\mathcal{M}$ gives a fractional hom(K)-tiling h with $w(h) = mk \geq (1-\alpha_0/2)t \geq (1-\alpha_0+\sqrt{\epsilon}/2)t$, as $\epsilon \ll \alpha_0$.

Let $E_3 = \{e \in E(\mathcal{R}) : e \subseteq U\}$ and $E_2 = \{e \in E(\mathcal{R}) : |e \cap U| = 2\}$.

Claim 4.9. *$|E_3| \leq \gamma\binom{|U|}{3}/2$ and $|E_2| \leq \delta\binom{|U|}{2}mk$.*

Proof. By (4.7) and Proposition 3.1, we have $|E_3| \leq \gamma\binom{|U|}{3}/2$, as otherwise there exists a copy of K in U, contradicting the maximality of $\mathcal{M}$.

Suppose, to the contrary, that $|E_2| > \delta\binom{|U|}{2}mk$. Let $\mathcal{A}$ be the set of all triples iuu', $i \in [m]$, $u \neq u' \in U$ such that uu' is adjacent to at least $a+1$ vertices in K_i. By the definition of $\mathcal{L}_1$, $iuu' \in \mathcal{A}$ if and only if $\mathcal{R}[V(K_i) \cup \{u,u'\}] \in \mathcal{L}_1(K_i,u,u')$. Let $\mathcal{A}_0$ be a largest matching in $\mathcal{A}$. By the maximality of $\mathcal{A}_0$, for any $i \in [m] \setminus V(\mathcal{A}_0)$ and any $u \neq u' \in U \setminus V(\mathcal{A}_0)$, at least $k-a$ vertices of K_i are not adjacent to uu'. Counting the number of non-edges $e \notin E(\mathcal{R})$ with $|e \cap U| = 2$, we have

$$(k-a)(m-|\mathcal{A}_0|)\binom{|U|-2|\mathcal{A}_0|}{2} \leq \binom{|U|}{2}mk-|E_2| < (1-\delta)\binom{|U|}{2}mk.$$

Since $(1-\delta)k \leq (\frac{b+c}{k})^2k = \frac{(k-a)^2}{k}$, it follows that

$$(m-|\mathcal{A}_0|)\binom{|U|-2|\mathcal{A}_0|}{2} \leq \frac{k-a}{k}m\binom{|U|}{2}. \tag{4.8}$$

We claim that $|\mathcal{A}_0| \geq \gamma\alpha_0 m$. Indeed, (4.7) implies that $|U| \geq \alpha_0 t/2 \geq \alpha_0 mk/2 \geq 2\alpha_0 m$ (as $k \geq 4$). If $|\mathcal{A}_0| < \gamma\alpha_0 m$, then $m - |\mathcal{A}_0| \geq (1-\gamma\alpha_0)m$ and $|U| - 2|\mathcal{A}_0| \geq |U| - 2\gamma\alpha_0 m \geq (1-\gamma)|U|$. Thus (4.8) implies that

$$\begin{aligned}\frac{k-a}{k} m \binom{|U|}{2} &\geq (1-\gamma\alpha_0)m\binom{(1-\gamma)|U|}{2} \\ &\geq (1-\gamma\alpha_0)(1-2\gamma)m\binom{|U|}{2} > (1-3\gamma)m\binom{|U|}{2}\end{aligned}$$

contradicting $\gamma \leq \frac{a}{3k}$. Now let $\mathcal{A}' \subseteq \mathcal{A}_0$ be of size $\gamma\alpha_0 m$. By Proposition 4.5, for each member of $\mathcal{A}'$, there is a fractional $\hom(K)$-tiling h' of $\mathcal{R}[V(K_i) \cup \{u, u'\}]$ with $w(h') \geq k + \frac{1}{abc}$ and $h'_{\min} \geq \frac{1}{bc^2}$. This gives rise to a fractional $\hom(K)$-tiling h of $\mathcal{R}$ with $h_{\min} \geq \frac{1}{bc^2}$ and $w(h) \geq mk + \gamma\alpha_0 m/(abc)$.

To complete the proof, we need a lower bound for m. Recall that $\delta_1(\mathcal{R}) \geq (1 - \left(\frac{b+c}{k}\right)^2 + \gamma - (2bc^2+1)\epsilon)\binom{t}{2}$. Thus if $|U| > \frac{b+c}{k}t$, then $\binom{|U|}{2} \geq (\frac{b+c}{k})^2\binom{t}{2} - t$ and

$$\delta_1(\mathcal{R}[U]) \geq \delta_1(\mathcal{R}) - \binom{t}{2} + \binom{|U|}{2} > (\gamma - (2bc^2+1)\epsilon)\binom{t}{2} - t \geq \frac{\gamma}{2}\binom{t}{2},$$

where the last inequality holds because $t \geq t_0 \geq 1/\epsilon$. This implies that $|E_3| > \frac{1}{3}|U|\gamma\binom{t}{2}/2 > \gamma\binom{|U|}{3}/2$, contradicting the first part of Claim 4.9. Therefore $|U| \leq \frac{b+c}{k}t$ and $|V(\mathcal{M})| = mk \geq \frac{a}{k}t$, which gives $m \geq \frac{a}{k^2}t$. The fractional $\hom(K)$-tiling h of $\mathcal{R}$ thus satisfies

$$w(h) \geq mk + \frac{\gamma\alpha_0 m}{abc} \geq mk + \frac{\gamma\alpha_0 t}{k^2 bc} > mk + \sqrt{\epsilon}t,$$

as $\epsilon \ll \gamma\alpha_0$, contradicting Claim 4.8. □

Let $\mathcal{T}$ be the set of all triples uij, $u \in U$, $i \neq j \in [m]$ such that there are at least $\delta k^2 + 1$ edges uvw of $\mathcal{R}$ with $v \in V(K_i)$ and $w \in V(K_j)$. Since $\delta k^2 + 1 = \max\{a^2 + 2a(b+c), (a+b)^2\} + 1$, by the definition of $\mathcal{L}_2$, $uij \in \mathcal{T}$ if and only if $\mathcal{R}[V(K_i) \cup V(K_j) \cup \{u\}] \in \mathcal{L}_2(K_i, K_j, u)$. Let $\mathcal{T}_0$ be a largest matching in $\mathcal{T}$.

Claim 4.10. $|\mathcal{T}_0| \geq \frac{\gamma\alpha_0}{6k}t$.

Proof. We first derive a lower bound for $|\mathcal{T}|$ by considering $\sum_{u\in U} \deg_{\mathcal{R}}(u)$. First partition the edges of $\mathcal{R}$ intersecting U based on whether they contain one, two or three vertices of U. Next we partition the edges uxy of $\mathcal{R}$ with $u \in U$ and $x \in V(\mathcal{K}_i), y \in V(\mathcal{K}_j)$ (i.e., the edges of $\mathcal{R}$ with exactly one vertex in U) into three classes: (1) those with $i = j$, there are at most $\binom{k}{2}m|U|$ such edges; (2) those with $i \neq j$ and $uij \notin \mathcal{T}$, there are at most $\delta k^2|U|\binom{m}{2}$ such edges; (3) those with $i \neq j$ and $uij \in \mathcal{T}$, there are at most $k^2|\mathcal{T}|$ such edges. Consequently,

$$|U|\delta_1(\mathcal{R}) \le \sum_{u\in U} \deg_{\mathcal{R}}(u) \le 3|E_3| + 2|E_2| + \binom{k}{2} m|U| + \delta k^2 |U| \binom{m}{2} + k^2|\mathcal{T}|.$$

By Claim 4.9, it follows that

$$\begin{aligned} |U|\delta_1(\mathcal{R}) &\le |U| \left(\frac{\gamma}{2} \binom{|U|}{2} + \delta |U| mk + \binom{k}{2} m + \delta k^2 \binom{m}{2} \right) + k^2|\mathcal{T}| \\ &\le |U| \left(\delta \binom{|U|}{2} + \delta |U| mk + \delta k^2 \binom{m}{2} + \binom{k}{2} m \right) + k^2|\mathcal{T}| \quad \text{as } \gamma \le 2\delta \\ &\le |U| \left(\delta \binom{t}{2} + \frac{kt}{2} \right) + k^2|\mathcal{T}|. \end{aligned}$$

On the other hand, $\delta_1(\mathcal{R}) \ge (\delta + \gamma - (2bc^2+1)\epsilon)\binom{t}{2}$. Using $t \ge k/\epsilon$ and $\epsilon \ll \gamma$, we derive that $k^2|\mathcal{T}| \ge |U| \cdot \frac{\gamma}{2}\binom{t}{2}$ or $|\mathcal{T}| \ge \frac{\gamma}{2k^2}\binom{t}{2}|U|$.

By the maximality of $\mathcal{T}_0$, all triples of $\mathcal{T}$ are covered by $V(\mathcal{T}_0)$. Since at most $\binom{m}{2}|\mathcal{T}_0|$ triples of $\mathcal{T}$ are covered by $V(\mathcal{T}_0) \cap U$ and at most $2|\mathcal{T}_0|(m-1)|U|$ triples of $\mathcal{T}$ are covered by $V(\mathcal{T}_0) \setminus U$, it follows that $|\mathcal{T}| \le 2|\mathcal{T}_0|(m-1)|U| + \binom{m}{2}|\mathcal{T}_0|$. Since $mk - k + |U| = t - k$, we have $(mk-k)|U| \le (t-k)^2/4 \le \binom{t}{2}/2$. Consequently,

$$|\mathcal{T}| \le \frac{|\mathcal{T}_0|}{k}\binom{t}{2} + \frac{|\mathcal{T}_0|}{k^2}\binom{t}{2}.$$

Together with $|\mathcal{T}| \ge \frac{\gamma}{2k^2}\binom{t}{2}|U|$, we derive that $|\mathcal{T}_0| \ge \frac{\gamma|U|}{2k+2} \ge \frac{\gamma\alpha_0 t}{6k}$ using (4.7). □

For every $uij \in \mathcal{T}_0$, Proposition 4.6 provides a fractional $\hom(K)$-tiling h of $\mathcal{R}[\{u\} \cup V(K_i) \cup V(K_j)]$ with $w(h) \ge 2k + \frac{1}{abc^2}$ and $h_{\min} \ge \frac{1}{bc^2}$. Furthermore, for every $K_i \in \mathcal{M}$ with $i \notin V(\mathcal{T}_0)$, we assign the standard weight on K_i. Hence, the union of all these fractional $\hom(K)$-tilings gives a fractional $\hom(K)$-tiling of $\mathcal{R}$ with $h_{\min} \ge \frac{1}{bc^2}$ and

$$w(h) \ge \left(2k + \frac{1}{abc^2}\right)|\mathcal{T}_0| + k(m - 2|\mathcal{T}_0|) = mk + \frac{1}{abc^2}|\mathcal{T}_0| \ge mk + \sqrt{\epsilon}t,$$

as $\epsilon \ll \gamma\alpha_0$, contradicting Claim 4.8. This completes the proof of Lemma 1.2. □

5. Concluding remarks

In this paper, we investigate the minimum vertex degree conditions for tiling complete 3-partite 3-graphs K. Our result is best possible, up to the error term γn^2. We remark that in some cases (e.g., $K = K_{1,1,t}$ for $t \ge 2$) it seems possible to remove the error term and obtain exact results – this was done for $K_{1,1,2}$ in [4,15]. In general, in order to obtain an exact result, we need to have a stability version of the almost tiling lemma and a stability version of the absorbing lemma, together with an analysis of the 3-graphs that look like extremal examples. In many cases, when analyzing extremal examples, we

need to know $\mathrm{ex}_1(n,K)$, the vertex-degree Turán number for K, which is a challenging question in general. (The *generalized Turán number* $\mathrm{ex}_d(n,F)$ of an r-graph F is the smallest integer t such that every r-graph H of order n with $\delta_d(H)\geq t+1$ contains a copy of F.)

When proving the lower bound of Theorem 1.1, we introduced the covering barrier. In general, given an r-graph F, let $c_d(n,F)$ denote the minimum integer c such that every r-graph H of order n with $\delta_d(H)\geq c$ has the property that every vertex of H is covered by some copy of F. When F is a graph, it is not hard to see that $c_1(n,F)=(1-1/(\chi(F)-1)+o(1))n$: the lower bound follows from the $(\chi(F)-1)$-partite Turán graph, and the upper bound can be derived after applying the Regularity Lemma to $V(H)\setminus\{v\}$ for an arbitrary vertex v (see [33] for details). Given an r-graph F, trivially

$$\mathrm{ex}_d(n,F)<c_d(n,F)\leq t_d(n,F). \tag{5.1}$$

We know that $c_1(n,F)=\mathrm{ex}_1(n,F)+o(n)$ for all 2-graphs F. Construction 2.6 and Lemma 3.7 together show that $c_1(n,K_{a,b,c})=(6-4\sqrt{2}+o(1))\binom{n}{2}$ if $2\leq a\leq b\leq c$, while Theorem 1.1 shows that $t_1(n,K_{a,b,c})=(6-4\sqrt{2}+o(1))\binom{n}{2}$ for certain a,b,c (for example, $K_{2,3,6}$). This shows that the upper bound for $c_d(n,F)$ in (5.1) could be asymptotically tight. For small 3-graphs F, determining $c_2(n,F)$ seems easier than determining $\mathrm{ex}_2(n,F)$ or $t_2(n,F)$ (known as two difficult problems) – see [8] for recent progress.

Let us give the following constructions of space barriers for complete r-partite r-graph tilings for arbitrary r.

Construction 5.1. *Fix positive integers $i<r$ and $a_1\leq\cdots\leq a_r$. Let $s=a_1+\cdots+a_r$ and H_i be an n-vertex r-graph with $V(H_i)=A_i\cup B_i$ and $|A_i|=(a_1+\cdots+a_i)n/s-1$ such that $E(H_i)$ consists of all r-tuples containing at least i vertices of A_i.*

To see why H_i does not contain a $K_{a_1,\dots,a_r}$-factor, we observe that for each copy of $K_{a_1,\dots,a_r}$, at least i color classes of it are subsets of A_i, and thus at least $a_1+\cdots+a_i$ vertices of it are in A_i. Since $|A_i|<(a_1+\cdots+a_i)n/s$, there is no $K_{a_1,\dots,a_r}$-factor of H_i. Thus the minimum d-degree threshold for a $K_{a_1,\dots,a_r}$-factor is greater than $\max_{i\in[r-1]}\delta_d(H_i)$. Note that $\delta_d(H_{r-d+1})=0$ since any d-set in B_{r-d+1} has degree zero. Thus, $\max_{i\in[r-1]}\delta_d(H_i)=\max_{i\in[r-d]}\delta_d(H_i)$. This means that there are $r-d$ space barriers, e.g., there is only one construction for the $(r-1)$-degree case, and there are two constructions for the vertex degree threshold in 3-graphs.

Since our main idea of proving Lemma 1.2 (see also [14]) is to analyze the bipartite link graph of any uncovered vertex on two existing copies of K in the partial tiling, new ideas are needed to attack the general vertex degree tiling problem. On the other hand, this also suggests that it seems possible to generalize Lemma 1.2 to the one of tiling r-partite r-graphs under minimum $(r-2)$-degree.

Another direction to extend the result of this paper is to study the minimum vertex degree conditions for non-complete 3-partite 3-graphs. Clearly if F is a spanning subgraph of $K_{a,b,c}$ then $t_1(n,F)\leq t_1(n,K_{a,b,c})$. Note that there may be more than one

choice of $K_{a,b,c}$ that contains F as a spanning subgraph. One of the referees pointed out the following example, which shows that

$$t_1(n,F) < \min t_1(n,K_{a,b,c}) \tag{5.2}$$

for some F, where the minimum is taken over all $K_{a,b,c}$ that contain F as a spanning subgraph. Indeed, take a copy of $K_{1,2,3}$ and denote u as the vertex in the vertex class of size one. Add three new vertices x, y, z and new edges uxy and uxz, and denote the resulting graph by F. Then $K_{1,4,4}$ and $K_{1,3,5}$ are the only choices of $K_{a,b,c}$ that contain F as a spanning subgraph. By Theorem 1.1, $t_1(n,K_{1,4,4}) = (\frac{4}{9}+o(1))\binom{n}{2}$ and $t_1(n,K_{1,3,5}) = (\frac{1}{2}+o(1))\binom{n}{2}$. On the other hand, $K_{2,7,9}$ has a perfect F-tiling. By Theorem 1.1, we have $t_1(n,F) \le t_1(n,K_{2,7,9}) = (6-4\sqrt{2}+o(1))\binom{n}{2}$, which implies (5.2).

Acknowledgments

We thank Richard Mycroft for valuable discussion. We also thank two referees for their careful reading and detailed comments that improve the presentation of the paper. In particular, we are grateful to one referee for showing us the example that implies (5.2).

References

[1] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, B. Sudakov, Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels, J. Combin. Theory Ser. A 119 (6) (2012) 1200–1215.
[2] N. Alon, R. Yuster, H-factors in dense graphs, J. Combin. Theory Ser. B 66 (2) (1996) 269–282.
[3] E. Buß, H. Hàn, M. Schacht, Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs, J. Combin. Theory Ser. B 103 (6) (2013) 658–678.
[4] A. Czygrinow, Minimum degree condition for C_4-tiling in 3-uniform hypergraphs, unpublished manuscript.
[5] A. Czygrinow, L. DeBiasio, B. Nagle, Tiling 3-uniform hypergraphs with $K_4^3 - 2e$, J. Graph Theory 75 (2) (2014) 124–136.
[6] A. Czygrinow, V. Kamat, Tight co-degree condition for perfect matchings in 4-graphs, Electron. J. Combin. 19 (2) (2012) 20.
[7] P. Erdős, On extremal problems of graphs and generalized graphs, Israel J. Math. 2 (3) (1964) 183–190.
[8] V. Falgas-Ravry, Y. Zhao, Codegree thresholds for covering 3-uniform hypergraphs, SIAM J. Discrete Math. 30 (4) (2016) 1899–1917.
[9] Z. Füredi, Y. Zhao, On the size of shadows under minimum degree condition, preprint.
[10] A. Hajnal, E. Szemerédi, Proof of a conjecture of P. Erdős, in: Combinatorial Theory and Its Applications, II, Proc. Colloq., Balatonfüred, 1969, North-Holland, Amsterdam, 1970, pp. 601–623.
[11] H. Hàn, Y. Person, M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, SIAM J. Discrete Math. 23 (2009) 732–748.
[12] J. Han, Decision problem for perfect matchings in dense k-uniform hypergraphs, Trans. Amer. Math. Soc. (2017), http://dx.doi.org/10.1090/tran/6999, accepted.
[13] J. Han, Perfect matchings in hypergraphs and the erdős matching conjecture, SIAM J. Discrete Math. 30 (2016) 1351–1357.
[14] J. Han, Y. Zhao, Minimum vertex degree threshold for loose Hamilton cycles in 3-uniform hypergraphs, J. Combin. Theory Ser. B 114 (2015) 70–96.
[15] J. Han, Y. Zhao, Minimum vertex degree threshold for $\mathcal{C}_4^3$-tiling, J. Graph Theory 79 (4) (2015) 300–317.

[16] P. Keevash, R. Mycroft, A geometric theory for hypergraph matching, Mem. Amer. Math. Soc. 233 (2015) 1098.
[17] I. Khan, Perfect matchings in 3-uniform hypergraphs with large vertex degree, SIAM J. Discrete Math. 27 (2) (2013) 1021–1039.
[18] I. Khan, Perfect matchings in 4-uniform hypergraphs, J. Combin. Theory Ser. B 116 (2016) 333–366.
[19] J. Komlós, G. Sárközy, E. Szemerédi, Proof of the Alon–Yuster conjecture, Discrete Math. 235 (1–3) (2001) 255–269.
[20] D. Kühn, D. Osthus, Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree, in: Combinatorics, Prague, 1998, J. Combin. Theory Ser. B 96 (6) (2006) 767–821.
[21] D. Kühn, D. Osthus, Embedding large subgraphs into dense graphs, in: Surveys in Combinatorics 2009, in: London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, pp. 137–167.
[22] D. Kühn, D. Osthus, The minimum degree threshold for perfect graph packings, Combinatorica 29 (1) (2009) 65–107.
[23] D. Kühn, D. Osthus, A. Treglown, Matchings in 3-uniform hypergraphs, J. Combin. Theory Ser. B 103 (2) (2013) 291–305.
[24] A. Lo, K. Markström, Minimum codegree threshold for $(K_4^3 - e)$-factors, J. Combin. Theory Ser. A 120 (3) (2013) 708–721.
[25] A. Lo, K. Markström, F-factors in hypergraphs via absorption, Graphs Combin. 31 (3) (2015) 679–712.
[26] R. Mycroft, Packing in hypergraphs, in: Presentation at the Extremal Combinatorics Workshop of ICMS, Edinburgh, UK, 2014.
[27] R. Mycroft, Packing k-partite k-uniform hypergraphs, J. Combin. Theory Ser. A 138 (2016) 60–132.
[28] V. Rödl, A. Ruciński, E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, Combin. Probab. Comput. 15 (1–2) (2006) 229–251.
[29] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, J. Combin. Theory Ser. A 116 (3) (2009) 613–636.
[30] E. Szemerédi, Regular partitions of graphs, in: Problèmes Combinatoires et Théorie des Graphes, Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976, in: Colloq. Int. Cent. Natl. Rech. Sci., vol. 260, CNRS, Paris, 1978, pp. 399–401.
[31] A. Treglown, Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, J. Combin. Theory Ser. A 120 (7) (2013) 1463–1482.
[32] A. Treglown, Y. Zhao, A note on perfect matchings in uniform hypergraphs, Electron. J. Combin. 23 (1) (2016) 1.16.
[33] C. Zang, Matchings and tilings in hypergraphs, PhD thesis, Georgia State University, 2016.
[34] Y. Zhao, Recent advances on Dirac-type problems for hypergraphs, in: Recent Trends in Combinatorics, in: IMA Vol. Math. Appl., vol. 159, Springer, Cham, 2016, pp. 145–165.