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An Existence Theorem for Group Divisible Designs of Large Order

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The following result gives a partial answer to a question of R. M. Wilson regarding the existence of group divisible designs of large order. Let k and u be positive integers, $2 \leqslant k \leqslant u$. Then there exists an integer $m_0 = m_0(k, u)$ such that there exists a group divisible design of group type m^u with block size k and index one for all integer $m \geqslant m_0$ if and only if

(i) $u-1 \equiv 0 \bmod (k-1)$,

(ii) $u(u-1) \equiv 0 \bmod k(k-1)$.

This is a generalization of the well-known result of Chowla, Erdős, and Straus on the existence of transversal designs of large order. © 2002 Elsevier Science (USA)

Key Words: group divisible designs; transversal designs; large order.

1. INTRODUCTION

DEFINITION 1.1. Let v be a non-negative integer, λ and t be positive integers and K be a set of positive integers. A group divisible t-design (or t-GDD) of order v, index λ and block sizes from K is a triple $(X, \Gamma, \mathscr{A})$ where,

1. X is a set of v elements (called *points*),

2. $\Gamma = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of X which partition X (called *groups*),

3. $\mathscr{A}$ is a family of subsets of X each of cardinality from K (called *blocks*) such that each block intersects any given group in at most one point,

4. each t-set of points from t distinct groups is in exactly λ blocks.

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Definition 1.1 is a generalization of the concept of group divisible designs (GDDs) for $t \geqslant 2$. Note that a 2-*GDD* is called a *GDD*.

A *list* over a set S is a mapping $l: S \to \mathbf{N}_0$ ($\mathbf{N}_0$ the set of non-negative integers); in other words, by a list over the set S we mean a collection of elements s of S in which each element s occurs with non-negative multiplicity $l(s)$. A list can be informally written as $(s_1, s_2, \ldots)$, where s_i's are not necessarily distinct. By the group type of a t-*GDD* $(X, \Gamma, \mathscr{A})$ we mean the list $(|G| \,|\, G \in \Gamma)$ of group sizes. If a t-*GDD* has n_i groups of size g_i, $1 \leqslant i \leqslant r$, then the list contains each g_i counted n_i times. In this case we denote its group type by $(g_1^{n_1}, g_2^{n_2}, \ldots, g_r^{n_r})$. Group divisible t-designs with equal group sizes are called uniform.

Group divisible designs with block size k and k groups of uniform group size m are called *transversal designs* and denoted by $TD_\lambda(t, k, m)$. If $\lambda = 1$, then we use $TD(t, k, m)$. If $t = 2$ we use $TD_\lambda(k, m)$.

Theorem 1.2 due to Blanchard [4] describes an existence result for $TD(t, k, m)$'s of large order, when $t \geqslant 2$. The $t = 2$ case of Theorem 1.2 is the theorem of Chowla, Erdős, and Straus [1].

THEOREM 1.2. *For any positive integers t and k, $1 \leqslant t \leqslant k$, there is an integer $m^* = m^*(t, k)$ such that for any integer $m \geqslant m^*$ there is a $TD(t, k, m)$.*

R. M. Wilson was interested in the following question: for fixed number of groups u and fixed block size k does a *GDD* of group type m^u with block size k and index one exist for sufficiently large m if the necessary conditions $m(u-1) \equiv 0 \bmod (k-1)$ and $m^2u(u-1) \equiv 0 \bmod k(k-1)$ are satisfied? We are able to prove that for fixed number of groups u and fixed block size k if the group size m is sufficiently large and relatively prime to $k(k-1)$ then a *GDD* of group type m^u with block size k and index one exists if the necessary arithmetic conditions are satisfied. In fact our result given in the theorem below is more general.

THEOREM 1.3. *Let k and u be positive integers, $2 \leqslant k \leqslant u$. Then there exists an integer $m_0 = m_0(k, u)$ such that there exists a GDD of group type m^u block size k and index one for all positive integer m satisfying $m \geqslant m_0$ if and only if*

(i) $u - 1 \equiv 0 \bmod (k-1)$,

(ii) $u(u-1) \equiv 0 \bmod k(k-1)$.

Note that we require these designs to exist for every value of m beyond m_0. Hence, m is not in the necessary conditions because one of the values of m in question is relatively prime to $k(k-1)$. The $u = k$ case of Theorem 1.3

is equivalent to the theorem of Chowla *et al.* on the existence of transversal designs of large order.

2. GROUP DIVISIBLE SUBDESIGNS

DEFINITION 2.1. Let G be a *t-GDD* $(X, \Gamma, \mathscr{A})$ with index λ. A group divisible subdesign H (or sub *t-GDD*) of G is a *t-GDD* $(X', \Gamma', \mathscr{A}')$ such that

1. $X' \subseteq X$,
2. Γ' consists of non-empty groups $G \cap X'$, where G is a group of Γ,
3. $\mathscr{A}'$ consists of the blocks A of $\mathscr{A}$, which are completely contained in X'.

The fundamental construction of Wilson [12] for *GDD*'s is generalized in Theorem 2.2 below for *t-GDD*'s containing sub *t-GDD*'s, where $t \geqslant 2$:

THEOREM 2.2 (The Fundamental Construction). *Let $(X, \Gamma, \mathscr{A})$ be a t-GDD of index one and let s_x be a positive integral weight assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation*

$$S_Y = \bigcup_{x \in Y} S_x$$

for $Y \subseteq X$, put $X^* = S_X$ and $\Gamma^* = \{S_G : G \in \Gamma\}$. Suppose that for each block $A \in \mathscr{A}$, a t-GDD $(S_A, \{S_x : x \in A\}, \mathscr{B}_A)$ of index one exists and denote $\mathscr{A}^* = \bigcup_{A \in \mathscr{A}} \mathscr{B}_A$. Then $(X^*, \Gamma^*, \mathscr{A}^*)$ is a t-GDD of index one which contains a sub t-GDD $(S_A, \{S_x : x \in A\}, \mathscr{B}_A)$ for each $A \in \mathscr{A}$.

Using Wilson's terminology, we refer to a *t-GDD* with a weighting as a recipe and the additional *t-GDD*'s for each block as ingredients.

Proof. The proof that $(X^*, \Gamma^*, \mathscr{A}^*)$ is a *t-GDD* of index one, where $t \geqslant 2$, can be found in Mohácsy and Ray-Chaudhuri's paper, see [9, Theorem 3.2]. Definition 2.1 implies that $(X^*, \Gamma^*, \mathscr{A}^*)$ contains a sub *t-GDD* $(S_A, \{S_x : x \in A\}, \mathscr{B}_A)$ for each $A \in \mathscr{A}$. ∎

3. A COMPOSITION THEOREM

From now on we will assume that $\lambda = 1$. For two positive integers n and m, let $I_n^m = \{m, m+1, \ldots, m+n-1\}$. If $m = 1$, then we write $I_n = \{1, 2, \ldots, n\}$. It

easily follows from Definition 1.1 that a uniform *t-GDD* $(X, \Gamma, \mathscr{A})$ of group type m^u can be constructed on the point set $X = I_u \times I_m$ with groups $\Gamma = \{\{i\} \times I_m : i \in I_u\}$.

Theorem 3.1 is a generalization of Wilson's additive recursive construction for transversal designs; see [1, Chap. X, Theorem 3.1].

THEOREM 3.1. *If there is a GDD of group type n^u with block size k, a GDD of group type $(n+w)^u$ with block size k containing a sub GDD of group type w^u, a $TD(u+1, s)$ and a $TD(k, p)$ with $p \leqslant s$, then there exists a GDD of group type $(sn+pw)^u$ with block size k containing a sub GDD of group type $(pw)^u$.*

Proof. A $TD(u, s)$ is called resolvable if the blocks can be partitioned into s parallel classes each class containing s blocks. It is not hard to see that deleting a group of a $TD(u+1, s)$ yields a resolvable $TD(u, s)$. Let $(I_u \times I_s, \{\{i\} \times I_s : i \in I_u\}, \mathscr{A})$ denote the resolvable $TD(u, s)$ obtained by deleting a group from the given $TD(u+1, s)$. Let $C_1 \cup C_2 \cup, \ldots, \cup C_p \cup C_{p+1} \cup, \ldots, \cup C_s$ denote the partition of the blocks of the resolvable $TD(u, s)$ into s parallel classes. Note that $|C_i| = s$ for $i = 1, 2, \ldots, s$.

Now, let us construct the block set of the *GDD* of group type $(sn+pw)^u$ with block size k containing a sub *GDD* of group type $(pw)^u$ on the point set $I_u \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1}))$ with groups $\{\{i\} \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1})) : i \in I_u\}$. Note that $(I_s \times I_n) \cap (I_p \times I_w^{n+1}) = \varnothing$. The sub *GDD* will be constructed on the point set $I_u \times (I_p \times I_w^{n+1})$ with groups $\{\{i\} \times (I_p \times I_w^{n+1}) : i \in I_u\}$.

For each A_1 of $\mathscr{A}$ contained in $C_{p+1} \cup, \ldots, \cup C_s$ we construct a *GDD* of group type n^u with block size k on the point set $A_1 \times I_n$ with groups $\{\{a\} \times I_n : a \in A_1\}$ and block set $\mathscr{K}_{A_1}$. (See Fig. 1. The *GDD* constructed for A_1 is indicated by the striped ovals.)

For each A_2 of $\mathscr{A}$ contained in C_r for some r, $1 \leqslant r \leqslant p$, we construct a *GDD* of group type $(n+w)^u$ with block size k containing a sub *GDD* of group type w^u on the point set $(A_2 \times I_n) \cup (I_u \times \{r\} \times I_w^{n+1})$ with groups $\{(\{(i, j)\} \times I_n) \cup (\{i\} \times \{r\} \times I_w^{n+1}) : i \in I_u$ and $(i, j) \in A_2\}$ and block set

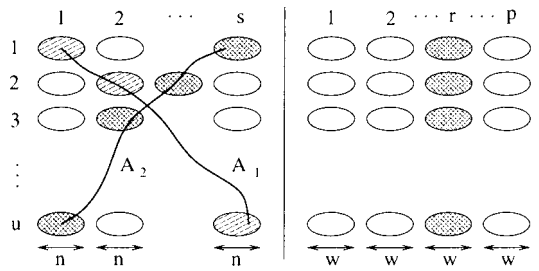


FIG. 1. An additive construction for *GDD*s.

$\mathscr{L}'_{A_2}$. The sub *GDD* of group type w^u is constructed on the point set $I_u \times (\{r\} \times I_w^{n+1})$ with groups $\{\{i\} \times (\{r\} \times I_w^{n+1}) : i \in I_u\}$ and block set $\mathscr{L}''_{A_2}$. (See Figure 1. The *GDD* constructed for A_2 is indicated by the crosshatched ovals.) In fact $\mathscr{L}''_{A_2} = \mathscr{L}'_{A_2} \mid (I_u \times (\{r\} \times I_w^{n+1}))$, that is, $\mathscr{L}''_{A_2}$ is the blocks of $\mathscr{L}'_{A_2}$ contained in $I_u \times (\{r\} \times I_w^{n+1})$. Let us define $\mathscr{L}_{A_2} = \mathscr{L}'_{A_2} \setminus \mathscr{L}''_{A_2}$.

Finally we construct a GDD of group type $(pw)^u$ with block size k on the point set $I_u \times (I_p \times I_w^{n+1})$ with groups $\{\{i\} \times (I_p \times I_w^{n+1}) : i \in I_u\}$ with block set $\mathscr{M}$ using the fundamental construction for GDD's (Theorem 2.2) with the *GDD* of group type w^u with block size k as the recipe and the $TD(k, p)$ as the ingredient.

Let $\mathscr{K} = \bigcup_{A \in C_{p+1} \cup, \ldots, \cup C_s} \mathscr{K}_A$ and $\mathscr{L} = \bigcup_{A \in C_1 \cup, \ldots, \cup C_p} \mathscr{L}_A$. We define the block set $\mathscr{N} = \mathscr{K} \cup \mathscr{L} \cup \mathscr{M}$. We claim that $(I_u \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1})), \{\{i\} \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1})) : i \in I_u\}, \mathscr{N})$ is a *GDD* of group type $(sn + pw)^u$ with block size k containing a sub *GDD* $(I_u \times (I_p \times I_w^{n+1}), \{\{i\} \times (I_p \times I_w^{n+1}) : i \in I_u\}, \mathscr{N}')$ of group type $(pw)^u$, where $\mathscr{N}' = \mathscr{N} \mid (I_u \times (I_p \times I_w^{n+1}))$.

First note that we have the right number of blocks.

It is easy to see that

$$|\mathscr{K}_A| = \frac{u(u-1)}{k(k-1)} n^2 \qquad \text{for each} \quad A \in C_{p+1} \cup, \ldots, \cup C_s.$$

Thus,

$$|\mathscr{K}| = (s-p)\, s \frac{u(u-1)}{k(k-1)} n^2. \tag{1}$$

From a similar calculation as above, we obtain

$$|\mathscr{L}_A| = \frac{u(u-1)}{k(k-1)} (n+w)^2 - \frac{u(u-1)}{k(k-1)} w^2$$

and hence

$$|\mathscr{L}| = ps \left[\frac{u(u-1)}{k(k-1)} (n+w)^2 - \frac{u(u-1)}{k(k-1)} w^2 \right] \tag{2}$$

Also

$$|\mathscr{M}| = \frac{u(u-1)}{k(k-1)} (pw)^2. \tag{3}$$

From (1), (2), and (3)

$$|\mathcal{N}| = \frac{u(u-1)}{k(k-1)}\left[(s-p)\,sn^2 + ps(n^2+2nw) + p^2w^2\right]$$
$$= \frac{u(u-1)}{k(k-1)}(sn+pw)^2$$

which equals the number of blocks of a *GDD* of group type $(sn+pw)^u$ with block size k.

Let $(u_1, (a_1, b_1))$ and $(u_2, (a_2, b_2))$ be two distinct points that belong to two distinct groups, where u_i is in I_u and (a_i, b_i) is in $(I_s \times I_n) \cup (I_p \times I_w^{n+1})$ for $i = 1, 2$.

Case 1. Assume that $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\} \subset I_u \times (I_s \times I_n)$. Then there is a unique block A of the resolvable $TD(u, s)$ that contains the 2-set $\{(u_1, a_1), (u_2, a_2)\}$. If A belongs to $C_{p+1} \cup, \ldots, \cup C_s$ then the pair $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\}$ is contained in a unique block of the *GDD* $(A \times I_n, \{\{i\} \times I_n : i \in A\}, \mathcal{H}_A)$ and no other block of $\mathcal{N}$ contains the given two points. If A belongs to C_r for some r, $1 \leqslant r \leqslant p$, then $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\}$ is contained in a unique block of the *GDD* $((A \times I_n) \cup (I_u \times \{r\} \times I_w^{n+1}), \{(\{(i, j)\} \times I_n) \cup (\{i\} \times \{r\} \times I_w^{n+1}) : i \in I_u \text{ and } (i, j) \in A\}, \mathcal{L}'_A)$ containing a sub *GDD* $(I_u \times (\{r\} \times I_w^{n+1})), \{\{i\} \times (\{r\} \times I_w^{n+1}) : i \in I_u\}, \mathcal{L}''_A)$. In fact this block belongs to $\mathcal{L}_A = \mathcal{L}'_A \setminus \mathcal{L}''_A$ and no other blocks can contain the given two points.

Case 2. Assume that $(u_1, (a_1, b_1))$ is in $I_u \times (I_s \times I_n)$ and $(u_2, (a_2, b_2))$ is in $I_u \times (I_p \times I_w^{n+1})$. Then there is a unique block A of the parallel class C_{a_2} of the resolvable $TD(u, s)$, that covers the point (u_1, a_1). Thus, similar to case 1, $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\}$ is contained in some block of the *GDD* $((A \times I_n) \cup (I_u \times \{a_2\} \times I_w^{n+1}), \{(\{(i, j)\} \times I_n) \cup (\{i\} \times \{a_2\} \times I_w^{n+1}) : i \in I_u \text{ and } (i, j) \in A\}, \mathcal{L}'_A)$ containing a sub *GDD* $(I_u \times (\{a_2\} \times I_w^{n+1}), \{\{i\} \times (\{a_2\} \times I_w^{n+1}) : i \in I_u\}, \mathcal{L}''_A)$. Again, this block belongs to $\mathcal{L}_A = \mathcal{L}'_A \setminus \mathcal{L}''_A$. This is the only block that contains the given two points, since there is only one block of the resolvable $TD(u, s)$ that contains the point (u_1, a_1) and belongs to the parallel class C_{a_2}.

Case 3. If $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\} \subset I_u \times (I_p \times I_w^{n+1})$, then $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\}$ is contained in a unique block of the *GDD* $(I_u \times (I_p \times I_w^{n+1}), \{\{i\} \times (I_p \times I_w^{n+1}) : i \in I_u\}, \mathcal{M})$. No other blocks of $\mathcal{H} \cup \mathcal{L}$ contain $\{(u_1, (a_1, b_1)), (u_2, (a_2, b_2))\}$, since any block of $\mathcal{H} \cup \mathcal{L}$ intersects $I_u \times (I_p \times I_w^{n+1})$ in at most one point.

The argument of case 3 also proves that the GDD $(I_u \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1}))$, $\{\{i\} \times ((I_s \times I_n) \cup (I_p \times I_w^{n+1})) : i \in I_u\}, \mathcal{N})$ of group type $(sn+pw)^u$ with block size k obtained by the above construction contains a sub GDD $(I_u \times (I_p \times I_w^{n+1})$, $\{\{i\} \times (I_p \times I_w^{n+1}) : i \in I_u\}, \mathcal{M})$ of group type $(pw)^u$. ∎

4. A CONSTRUCTION FOR GROUP DIVISIBLE DESIGNS

Theorem 4.1 gives a construction for GDD's, which we need in the proof of our main theorem. The greatest common divisor d of two integers a and b is denoted by $(a, b) = d$ and the least common multiple m of two integers a and b is denoted by $[a, b] = m$.

THEOREM 4.1. *Let k and u be positive integers, $2 \leqslant k \leqslant u$. Let $m^*(2, u+1)$ be as in Theorem 1.2. Then there exist positive integers $n = n(k, u)$ and $w = w(k, u)$ such that $(n, w) = 1$, $n \geqslant m^*(2, u+1)$, and there exist a GDD of group type n^u with block size k and a GDD of group type $(n+w)^u$ with block size k containing a sub GDD of group type w^u if and only if*

(i) $u-1 \equiv 0 \mod(k-1)$,

(ii) $u(u-1) \equiv 0 \mod k(k-1)$.

Theorem 4.2 given by Mohácsy and Ray-Chaudhuri [10] is used to prove Theorem 4.1.

THEOREM 4.2 (Mohácsy and Ray-Chaudhuri [10]). *Let t, k, and u be positive integers, $t \leqslant k \leqslant u$. Then there exists an integer $d_1 = d_1(t, k, u)$ such that for any positive integers b, $a \geqslant 2$ and $d \geqslant d_1$ there exists a t-GDD of group type $(ba^d)^u$ with block size k and index one if and only if*

$$\binom{u-s}{t-s} \equiv 0 \quad mod \binom{k-s}{t-s} \quad for \quad s = 0, 1, \ldots, t-1.$$

Proof of Theorem 4.1. First we prove the necessity of the conditions. If a GDD of group type n^u with block size k and a GDD of group type w^u with block size k exist then

$$\begin{aligned} &\text{(i)} \quad n(u-1) \equiv 0 \quad \mod(k-1), \\ &\text{(ii)} \quad n^2u(u-1) \equiv 0 \quad \mod k(k-1), \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\text{(i)} \quad w(u-1) \equiv 0 \qquad \bmod (k-1), \\ &\text{(ii)} \quad w^2u(u-1) \equiv 0 \qquad \bmod k(k-1). \end{aligned} \tag{5}$$

Equivalently,

$$\begin{aligned} &\text{(i)} \quad (u-1) \equiv 0 \qquad \bmod \frac{k-1}{(k-1, n)}, \\ &\text{(ii)} \quad u(u-1) \equiv 0 \qquad \bmod \frac{k(k-1)}{(k(k-1), n^2)}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\text{(i)} \quad (u-1) \equiv 0 \qquad \bmod \frac{k-1}{(k-1, w)}, \\ &\text{(ii)} \quad u(u-1) \equiv 0 \qquad \bmod \frac{k(k-1)}{(k(k-1), w^2)}. \end{aligned} \tag{7}$$

It follows from (6) and (7),

$$\begin{aligned} &\text{(i)} \quad (u-1) \equiv 0 \qquad \bmod \left[\frac{k-1}{(k-1, n)}, \frac{k-1}{(k-1, w)} \right], \\ &\text{(ii)} \quad u(u-1) \equiv 0 \qquad \bmod \left[\frac{k(k-1)}{(k(k-1), n^2)}, \frac{k(k-1)}{(k(k-1), w^2)} \right]. \end{aligned} \tag{8}$$

Since $(n, w) = 1$, $[k-1/(k-1, n), k-1/(k-1, w)] = k-1$ and $[k(k-1)/(k(k-1), n^2), k(k-1)/(k(k-1), w^2)] = k(k-1)$. Thus, the conditions of the theorem follow.

Now we prove the sufficiency of the conditions. It is easy to check that the conditions of Theorem 4.2 for $t = 2$ are satisfied. Thus, from Theorem 4.2 there exists an integer $d_1 = d_1(2, k, u)$ such that if $d \geqslant d_1$ then there exists a *GDD* of group type $(2^d)^u$ with block size k. By the Chowla, Erdős, Straus theorem (t=2 case of Theorem 1.2), there is an integer $m^*(2, u+1)$, such that a transversal design $TD(u+1, m)$ exists for all integers $m \geqslant m^*(2, u+1)$. Then, obviously there is a $TD(u, 2^d)$ for all integers $d \geqslant m^*(2, u+1)$. Let $d_0 = d_0(k, u) = max(d_1(2, k, u), m^*(2, u+1))$.

We can construct a *GDD* of group type $(2^{2d})^u$ with block size k containing a sub *GDD* of group type $(2^d)^u$ for all $d \geqslant d_0$ applying the fundamental construction theorem, Theorem 2.2, with the $TD(u, 2^d)$ as the recipe and

the *GDD*'s of group type $(2^d)^u$ with block size k as ingredients. It follows from Theorem 4.2 that there exists a *GDD* of group type $(2^d(2^d-1))^u$ with block size k for all integers $d \geqslant d_0$.

Let d_2 be a fixed integer such that $d_2 \geqslant d_0$.Choose an integer t satisfying $(5, t) = 1$ and $5^{d_2} \leqslant t$. From the Chowla, Erdős, Straus theorem, there exist $TD(u+1, t)$ and $TD(k, 5^{d_2})$. Note that a $TD(k, 5^{d_2})$ exists, since a $TD(u+1, 5^{d_2})$ exists and $u \geqslant k$.

Now, applying Theorem 3.1 with $s = t$, $n = 2^d(2^d-1)$, $p = 5^{d_2}$ and $w = 2^d$ we construct a *GDD* of group type $(t2^d(2^d-1)+5^{d_2}2^d)^u$ with block size k containing a sub *GDD* of group type $(5^{d_2}2^d)^u$ for all $d \geqslant d_0$. We construct the sub *GDD* of group type $(5^{d_2}2^d)^u$ to contain a sub *GDD* of group type $(5^{d_2})^u$. This can be achieved by applying the fundamental construction theorem, Theorem 2.2. We take the $TD(u, 2^d)$ as the recipe and *GDD*'s of group type $(5^{d_2})^u$ with block size k as ingredients. The existence of a *GDD* of group type $(5^{d_2})^u$ with block size k follows from Theorem 4.2.

We have constructed a *GDD* of group type

$$(\underbrace{t2^d(2^d-1)+5^{d_2}(2^d-1)}_{n} + \underbrace{5^{d_2}}_{w})^u$$

with block size k containing a sub *GDD* of group type $(5^{d_2})^u$ for all integers $d \geqslant d_0$. We choose $d = \phi(3^\alpha) = 2 \cdot 3^{\alpha-1}$, where ϕ is Euler's ϕ function and α is a fixed integer such that $\alpha \geqslant d_0$.

Let $n = t2^d(2^d-1)+5^{d_2}(2^d-1)$ and $w = 5^{d_2}$. We claim that $(n, w) = 1$, $n \geqslant m^*(2, u+1)$ and there exists a *GDD* of group type n^u with block size k.

Since $(t, 5) = 1$ and $d = 2 \cdot 3^{\alpha-1}$, $(n, w) = 1$.

Obviously $n \geqslant m^*(2, u+1)$, since $d \geqslant d_0$.

$n \equiv 2^d - 1 \equiv 0 \bmod 3^\alpha$, thus from Theorem 4.2 there exists a *GDD* of group type n^u with block size k. The proof is complete.

5. A REPRESENTATION THEOREM OF LARGE INTEGERS

THEOREM 5.1. *Let n and w be positive integers such that $(n, w) = 1$. Then there exists an integer $m_1 = m_1(n, w)$ such that any integer $m \geqslant m_1$ can be represented as $m = sn + pw$ where s and p are non-negative integers and $n < p \leqslant s$.*

Proof. Let m be a positive integer. There exist integers s and p such that $m = sn + pw$ if and only if $s = mn^{(\phi(w)-1)} - iw$ and $p = m(1-n^{\phi(w)})/w + in$, where i is an arbitrary integer.

We claim that if i satisfies the inequality

$$\frac{m(n^{\phi(w)}-1)}{wn}+1<i\leqslant\frac{m(wn^{\phi(w)-1}+n^{\phi(w)}-1)}{w(w+n)} \tag{9}$$

then $n<p\leqslant s$.

First we show that it is possible to choose an integer i that satisfies (9), if m is sufficiently large. A simple calculation shows that

$$\frac{m(wn^{\phi(w)-1}+n^{\phi(w)}-1)}{w(w+n)}-\left(\frac{m(n^{\phi(w)}-1)}{wn}+1\right)=\frac{m}{n(w+n)}-1.$$

Thus, if $m\geqslant 2n(w+n)$, then we can choose an integer i satisfying inequality (9). Hence, we set $m_1=m_1(n,w)=2n(w+n)$

Now, we prove that if i satisfies (9), then $n<p\leqslant s$. From (9)

$$i\leqslant\frac{m(wn^{\phi(w)-1}+n^{\phi(w)}-1)}{w(w+n)},$$

$$i(w+n)\leqslant m\left(n^{\phi(w)-1}+\frac{n^{\phi(w)}-1}{w}\right),$$

$$p=\frac{m(1-n^{\phi(w)})}{w}+in\leqslant mn^{\phi(w)-1}-iw=s.$$

Again, from (9)

$$p=\frac{m(1-n^{\phi(w)})}{w}+in>\frac{m(1-n^{\phi(w)})}{w}+\left(\frac{m(n^{\phi(w)}-1)}{wn}+1\right)n=n$$

and the theorem follows.

6. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.3. To prove the necessity of the conditions, first note that if a *GDD* of group type m^u with block size k and index one exists then

(i) $m(u-1)\equiv 0 \bmod (k-1)$,

(ii) $m^2u(u-1)\equiv 0 \bmod k(k-1)$.

Since we require that a *GDD* of group type m^u exist for any integer $m \geqslant m_0$, taking m relative prime to $k(k-1)$ we get the conditions of Theorem 1.3.

Now we prove that the conditions are sufficient for the existence of such *GDD*'s. From Theorem 4.1, there exist positive integers $n = n(k, u)$ and $w = w(k, u)$ such that $(n, w) = 1$, $n \geqslant m^*(2, u+1)$, where $m^*(2, u+1)$ is as in Theorem 1.2, and there exists a *GDD* of group type n^u with block size k and a *GDD* of group type $(n+w)^u$ with block size k containing a sub *GDD* of group type w^u.

Let us set $m_0 = m_0(k, u) = m_1(n(k, u), w(k, u))$, where m_1 is from Theorem 5.1. By Theorem 5.1, if $m \geqslant m_0$ then there exist positive integers s and p such that $m = sn + pw$ and $n < p \leqslant s$.

Since $m^*(2, u+1) \leqslant n \leqslant p \leqslant s$, by Theorem 1.2 there exist a $TD(u+1, s)$ and a $TD(k, p)$. Hence, Theorem 3.1 guarantees the existence of a *GDD* of group type $(sn+pw)^u = m^u$ with block size k. This proves Theorem 1.3.

From Theorem 1.3 we get the following corollary which partially answers R. M. Wilson's question.

COROLLARY 6.1. *Let k and u be positive integers, $2 \leqslant k \leqslant u$. Then there exists an integer $m_0 = m_0(k, u)$ such that for any positive integer m satisfying $m \geqslant m_0$ and $(m, k(k-1)) = 1$ there exists a GDD of group type m^u block size k and index one if and only if*

(i) $m(u-1) \equiv 0 \bmod (k-1)$,

(ii) $m^2 u(u-1) \equiv 0 \bmod k(k-1)$.

REFERENCES

1. T. Beth, D. Jungnickel, and H. Lenz, "Design Theory," Bibliographisches Institut, Mannheim, 1993.
2. J. L. Blanchard, A construction for orthogonal arrays with strength $t \geqslant 3$, *Discrete Math.* **137**, No. 1–3 (1995), 35–44.
3. J. L. Blanchard, The existence of orthogonal arrays of any strength three with large order, unpublished manuscript, California Institute of Technology, Pasadena, 1994.
4. J. L. Blanchard, The existence of orthogonal arrays of any strength with large order, unpublished manuscript, California Institute of Technology, Pasadena, 1994.
5. K. A. Bush, Orthogonal arrays of index unity, *Ann. Math. Statist.* **23** (1952), 426–434.
6. K. I. Chang, "An Existence Theory for Group Divisible Designs," Ph.D. thesis, The Ohio State University, 1976.
7. C. J. Colbourn and J. H. Dinitz, "The CRC Handbook of Combinatorial Designs," CRC Press, Boca Raton, FL, 1996.
8. J. H. Dinitz and D. R. Stinson, "Contemporary Design Theory, a Collection of Surveys," Wiley, New York, 1992.
9. H. Mohácsy and D. K. Ray-Chaudhuri, Candelabra systems and designs, *J. Statist. Plann. Inference*, in press.

10. H. Mohácsy and D. K. Ray-Chaudhuri, A construction for group divisible t-design with strength $t \geqslant 2$ and index unity, *J. Statist. Plann. Inference*, in press.
11. H. Mohácsy, "Candelabra Systems and Designs," Ph.D. thesis, The Ohio State University, December 1998.
12. R. M. Wilson, An existence theory for pairwise balanced designs. I. Composition theorems and morphisms, *J. Combin. Theory Ser. A* **13** (1972), 220–245.