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Monochromatic solutions to systems of exponential equations

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ABSTRACT

Let $n \in \mathbb{N}$, R be a binary relation on $[n]$, and $C_1(i,j), \ldots, C_n(i,j) \in \mathbb{Z}$, for $i,j \in [n]$. We define the exponential system of equations $\mathcal{E}(R, (C_k(i,j))_{i,j,k})$ to be the system

$$X_i^{Y_1^{C_1(i,j)} \cdots Y_n^{C_n(i,j)}} = X_j, \text{ for } (i,j) \in R,$$

in variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. The aim of this paper is to classify precisely which of these systems admit a monochromatic solution ($X_i, Y_i \neq 1$) in an arbitrary finite colouring of the natural numbers. This result could be viewed as an analogue of Rado's theorem for exponential patterns.

1. Introduction

In 2011, Sisto [16] made the surprising observation that an arbitrary 2-colouring of the natural numbers admits infinitely many integers $a, b > 1$ such that a, b, a^b all receive the same colour. He went on to ask if a similar result holds for k-colourings of the natural numbers with $k > 2$. Brown [3], simplifying and extending the proof of Sisto, gave further examples of exponential, monochromatic patterns that are present in an arbitrary 2-colouring and also proved some weaker results for monochromatic patterns in more

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colours. In a recent paper [15] we answered Sisto's question by showing that any *finite* colouring of the positive integers admits $a, b > 1$ such that $\{a, b, a^b\}$ is monochromatic and went on to develop, in this context, a theory of patterns defined by compositions of the exponential function. In the present paper we turn from the study of patterns arising as compositions of the exponential function, to understand exponential patterns that arise as solutions to systems of equations. There is a vast literature on finding patterns in arbitrary finite partitions of the integers, [1,2,4–9,12,17,13,19] and we refer the reader to the introduction in [15] for a brief discussion of this theory or to [6] for a indepth survey of the most classical elements of the theory.

The motivation for the study of monochromatic solutions to equations lies in the seminal work of Rado [14], who classified the systems of homogeneous linear equations that admit a solution in an arbitrary finite colouring of the natural numbers. More precisely, we say that an $m \times n$ matrix A is *partition regular* if every finite colouring of $\mathbb{N}$ admits monochromatic $x_1, \dots, x_n \in \mathbb{N}$, for which $Ax = 0$, where $x = (x_1, \dots, x_n)$. Rado classified the partition regular matrices by giving a simple criterion on the columns of such matrices. It is in this spirit that the present paper sets out.

It is worth pointing out that, even in the classical, linear theory, there is a distinction between studying patters which solve linear systems and patterns which arise as fixed linear compositions of several free variables. These two types of partition regularity are sometimes termed "kernel partition regular" and "image partition regular", respectively. So, while Rado's theorem gave a complete understanding of what linear systems $Ax = 0$ can be solved in an arbitrary colouring, it was not until the work Hindman and Leader [11] that a classification of "image" partition regular systems was fully understood. We refer the reader to the survey of Hindman [10], for details.

Before going further, let us lay out some basic terminology. Let $k \in \mathbb{N}$ and X be a non-empty set. We call a function $f : \mathbb{N} \to X$ a *finite colouring* if X is finite, and a *k-colouring*, if $|X| \leq k$. As is standard, we refer to the elements of X as *colours*. We say that a collection $\mathcal{A}$, of ordered tuples of integers, is *partition regular* if for every finite colouring $f : \mathbb{N} \to X$ we can find $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{N}$, such that $f(x_1) = \cdots = f(x_n)$ and $(x_1, \dots, x_n) \in \mathcal{A}$. We say that a linear system of equations is *partition regular* if its solution set in $\mathbb{N}$ is partition regular. We say that an exponential system of equations is *partition regular* if its solution set in $\mathbb{N} \setminus \{1\}$ is partition regular. That is, for exponential equations, we only consider solutions where each coordinate at least 2, to remove the trivial cases. It shall also be convenient to define the binary operation $\star$ as $a \star b = a^b$, for $a, b \in \mathbb{N}$.

For $n \in \mathbb{N}$, let R be a binary relation on $[n]$. Given integers $C_1(i, j), \dots, C_n(i, j) \in \mathbb{Z}$, for $i, j \in [n]$, we define the system of equations $\mathcal{E}\left(R, \{C_k(i, j)\}_{i,j,k}\right)$ by

$$X_i^{Y_1^{C_1(i,j)} \cdots Y_n^{C_n(i,j)}} = X_j, \text{ for } (i, j) \in R, \tag{1}$$

where $X_1, \dots, X_n, Y_1, \dots, Y_n$ are variables.

Our main result will tell us that the above system of equations has a monochromatic solution in every finite colouring of $\mathbb{N}$ if and only if an associated system of *linear* equations in the variables $Y_1, \ldots, Y_n$ has a solution. As Rado's theorem gives a nice classification of partition regular linear equations, Theorem 1, when used in tandem with Rado's theorem, yields an effective method for determining if a given exponential system is partition regular.

To define the *associated linear system*, $\mathcal{L}\left(R, \{C_k(i,j)\}_{i,j,k}\right)$, we treat R as a directed graph $D = ([n], R)$ and let $\mathcal{L}\left(R, \{C_k(i,j)\}_{i,j,k}\right)$ be the system of equations, indexed by the (not necessarily directed) cycles C of D,

$$\sum_{e \in C} (-1)^{d(e)} \left(C_1(e)Y_1 + \ldots + C_n(e)Y_n\right) = 0, \tag{2}$$

where, for each cycle C, we fix some orientation and then define $d(e) = 0$ if the edge e is oriented in the same way as the cycle and $d(e) = 1$ if the orientation of the edge and the cycle are different.

For example, the exponential system $X_1^Y = X_2$, $X_2^Y = X_3$, $X_3^Y = X_4$, $X_4 = X_1^{ZW}$, has as its associated linear system the single equation $3y - z - w = 0$. Note that it is also important that we include loops as cycles. For example, the one-equation exponential system $X^{Y_1} = X^{Y_2 Y_3}$ has associated linear system $x - y_1 - y_2 = 0$. We may now state our main theorem.

Theorem 1. *For $n \in \mathbb{N}$, let $R \subseteq [n] \times [n]$, and $C_k(i,j) \in \mathbb{Z}$, for each $i,j,k \in [n]$. The system of exponential equations $\mathcal{E}\left(R, \{C_k(i,j)\}\right)$ is partition regular if and only if $\mathcal{L}\left(R, \{C_k(i,j)\}\right)$ is partition regular.*

So, for example, we deduce that the equation

$$X^{Y_1 \cdot Y_2} = X^{Y_3 \cdot Y_4},$$

is partition regular as the associated linear system $y_1 + y_2 - y_3 - y_4 = 0$ is partition regular. Sisto's original question can also be neatly written in this form: he asked if the equation $X^Y = Z$ is partition regular. This question is immediately answered by this result as its associated linear system is *empty*, as there are no cycles in the corresponding directed graph, and therefore is trivially partition regular.

On the other hand, the exponential equation

$$X^{Y^2} = X^Z$$

is *not* partition regular as the associated linear system consists of the single equation $z - 2y = 0$, which is not partition regular.

For a more complicated example, we see that the exponential system

$$X_1^{Y_1} = X_2,\ X_2^{Y_2} = X_3,\ X_3^{Y_3} = X_1^{Y_4^2},\ X_1^{Y_2} = X_1^{Y_4},$$

is partition regular as its associated linear system is

$$y_1 + y_2 + y_3 - 2y_4 = 0, \quad y_2 - y_4 = 0,$$

which is seen to be partition regular by Rado's theorem (discussed in Section 2). On the other hand, the system

$$X_1^Y = X_2,\ X_2^Y = X_3,\ X_3^Y = X_4,\ X_4 = X_1^{ZW},$$

is not partition regular as its associated linear system $3y - z - w = 0$ is not partition regular, again by Rado's Theorem.

Another illustrative example is perhaps most naturally expressed in a "compositional" form: For every finite colouring of $\mathbb{N}$, one can find $a, b, c > 1$ so that $\{a, b, c, bc, a^b, a^c, a^{bc}\}$ is monochromatic. This follows from the fact that the system $X_1^{Y_1} = X_2$, $X_1^{Y_2} = X_3$, $X_1^{Y_1 Y_2} = X_1^{Y_3}$ is partition regular, which again can be checked easily by applying Theorem 1 and considering its associated linear system.

To understand where the relationship arises, between the exponential system $\mathcal{E}$ and its associated linear system $\mathcal{L}$, it is most natural to start by considering the "only if" implication in Theorem 1. Let us first apply the function $\nu \circ \nu$ to both sides of the equations in the exponential system $\mathcal{E}$, where ν is an appropriate logarithm-type function; that is, a function satisfying $\nu(a^b) = b\nu(a)$. If we then sum these equations over "cycles", we eliminate the terms of the form $\nu^2(X_i)$ and obtain a system of the general shape of $\mathcal{L}$. From here it is not hard to see that if c is a colouring forbidding a monochromatic solution to $\mathcal{L}$, then $c(\nu(x))$ is a colouring forbidding a solution to the original, exponential equation. Theorem 1 tells us that this restriction is in fact the only restriction to partition regularity of these systems.

To prove the "if" direction of the theorem, we show that if $\mathcal{A} \subseteq \mathbb{N}^n$ is partition regular we can "lift" $\mathcal{A}$ to find an associated exponential pattern that is also partition regular. More precisely, if $n \in \mathbb{N}$, $\mathcal{A} \subset \mathbb{N}^n$, and $W : \mathbb{N}^n \to \mathbb{N}$ is an arbitrary "weight" function, we define the *exponential $\mathcal{A}$-pattern with weight W*, as follows. For each $(x_1, \ldots, x_n) \in \mathcal{A}$, we include into the associated exponential pattern, the following $(W(x_1, \ldots, x_n) + n + 1)$-tuple consisting of the elements

$$a, b^{x_1}, \ldots, b^{x_n},$$

along with

$$a^{b^1}, a^{b^2}, \ldots, a^{b^{W(x_1,\ldots,x_n)}},$$

for each $a, b > 1$. We show the following.

Theorem 2. *Let $n \in \mathbb{N}$, and $W : \mathbb{N}^n \to \mathbb{N}$ be a function. If $\mathcal{A} \subseteq \mathbb{N}^n$ is partition regular, then the associated exponential $\mathcal{A}$-pattern, with weight W, is also partition regular.*

In the next section, we quickly recall some of the theory relevant to this paper and introduce the central definitions. In Section 3, we prove Theorem 2. Finally, in Section 4, we use this result to deduce our classification theorem, Theorem 1.

2. Preliminaries

Although we do not use it directly in our arguments, it is useful to recall Rado's classical theorem on partition regular systems of linear equations [14]. This is both for motivation and to apply to examples, as we have done in the introduction. For $m, n \in \mathbb{N}$, we let A be a $m \times n$ matrix with integer entries. We say that A is *partition regular* if the collection $\{x \in \mathbb{N}^n : Ax = 0\}$ is partition regular. If we let $v_1, \ldots, v_n \in \mathbb{Z}^m$ denote the column vectors of A, we say that A satisfies the *columns property* if one can partition $[n] = S_0 \cup \cdots \cup S_d$, for some $d \in [0, n-1]$, so that $\sum_{j \in S_0} v_j = 0$, while $\sum_{j \in S_i} v_j$, lies in the $\mathbb{Q}$-linear span of the vectors of $S_0 \cup \cdots \cup S_{i-1}$, for each $i \in [n]$. Rado's theorem establishes that these two properties of A are equivalent.

Theorem 3. *For $m, n \in \mathbb{N}$, let A be an $m \times n$ matrix with integer entries. A is partition regular if and only if A has the columns property.*

Although we shall not require them explicitly in the present paper, it is convenient to recall the *Rado colourings*. These colourings were introduced by Rado to demonstrate the non-partition regularity of matrices without the columns property [14] (see [6]). For a prime p and $x \in \mathbb{N}$, we define the colouring $c_p \colon \mathbb{N} \to [p-1]$ by defining $c_p(x)$ as the coefficient of p^k in the base-p expansion of x, where k is the largest integer so that p^k divides x. Rado proved that if A fails to have the columns property then, for sufficiently large primes p, if $x_1, \ldots, x_n$ are integers such that $c_p(x_1) = \cdots = c_p(x_n)$ then $Ax \neq 0$, where $x = (x_1, \ldots, x_n)$.

Turning now to introduce some important notions, let $f \colon \mathbb{N} \to [k]$ be a k-colouring and let $f_n \colon \mathbb{N} \to [k]$ be k-colourings, for $n \in \mathbb{N}$. We say that the sequence $\{f_n\}$ *converges to* f and write $f_n \to f$ if for every $m \in \mathbb{N}$ there exists some $M \in \mathbb{N}$ so that for all $n \geq M$, $f_n(x) = f(x)$, for all $x \in [m]$. As expected, we also say that a sequence *converges* if there exists some f that the sequence converges to. The following basic fact on sequences of colourings $f \colon \mathbb{N} \to [k]$ is often referred to as the *compactness property.*

Fact 4. *Given a sequence of colourings $f_n \colon \mathbb{N} \to [k]$ there exists some $f \colon \mathbb{N} \to [k]$ and a strictly increasing sequence $\{N(n)\}_n \subseteq \mathbb{N}$ for which*

$$f_{N(n)} \to f,$$

as $n \to \infty$. $\square$

We also make use of the following consequence of the compactness property. Given a partition regular collection $\mathcal{A} \subseteq \mathbb{N}^n$ and a positive integer k, there exists a minimum integer $P(\mathcal{A}; k)$ such that every k-colouring of $\mathbb{N}$ admits a monochromatic $(x_1, \dots, x_n) \in \mathcal{A}$ with $x_1, \dots, x_n \leq P(\mathcal{A}; k)$.

We make considerable use of van der Waerden's classical theorem [18], which states that for every $k, l \in \mathbb{N}$, there exists a minimal integer $W_k(l)$ such that every k-colouring of an arithmetic progression of length $W_k(l)$ contains a monochromatic sub-progression of length l.

Now, for $r \in \mathbb{N}$, let $f_1, \dots, f_r \colon \mathbb{N} \to [k]$ be k-colourings. We call a sequence of colours $c_1, \dots, c_r \in [k]$ *large for* $f_1, \dots, f_r$ if for every $M \in \mathbb{N}$ we can find an arithmetic progression P_M of length M, such that $f_i(P_M) = c_i$, for each $i \in [r]$. The following two facts now follow easily from van der Waerden's theorem.

Corollary 5. *If f is a finite colouring of $\mathbb{N}$, there exists a colour that is large with respect to f.* $\square$

Lemma 6. *For $r \in \mathbb{N}$, let $f_1, \dots, f_r, f_{r+1} \colon \mathbb{N} \to [k]$ be k-colourings. If $c_1, \dots, c_r \in [k]$ is large with respect to $f_1, \dots, f_r$, then there exists a colour $c_{r+1} \in [k]$ so that $c_1, \dots, c_{r+1}$ is large with respect to $f_1, \dots, f_{r+1}$.*

Proof. Let $M \in \mathbb{N}$ be a parameter. Now use the fact that $c_1, \dots, c_r$ is large for $f_1, \dots, f_r$ to find a progression P_M of length $W_k(M)$ so that $f_i(P_M) = \{c_i\}$ for $i \in [r]$. Applying van der Waerdens's theorem to the colouring f_{r+1}, along the progression P_M, we obtain a monochromatic progression $P'_M \subseteq P_M$ of length M, for which $f_i(P'_M) = c_i$, for $i \in [r]$ and $f_{r+1}(P'_M) = c(M)$, for some $c(M) \in [k]$. To finish, apply the above for all choices of $M \in \mathbb{N}$. There is some value of $[k]$ that is attained infinitely often as a value of $c(M)$. We set c_{r+1} to be this value. $\square$

3. Lifting partition regular patterns

In this section we prove Theorem 2 on "lifting" partition regular patterns. The core of the proof is contained in the following lemma, which works by constructing, at each stage, a huge number of sequences which approximate some "idealized" colourings $\widetilde{f}_1, \dots, \widetilde{f}_r$. Each "idealized" colouring will then act as a mold to help us look for future sequences which, in turn, approximate a new idealized colouring $\widetilde{f}_{r+1}$. Assuming that we don't find the appropriate pattern, we shall observe that the colouring along our new sequences becomes more and more restricted, until we obtain a contradiction.

Our proof here is infinitary however, it is possible to replace our infinitary arguments with finitary ones to obtain explicit bounds on the various quantities. However, this exchange would come at the cost of added clutter and difficulty for the reader.

Before we give the proof of our main lemma, we make two key definitions. For each positive integer d, we define the *d-sequence* to be the sequence of integers $\{2^{d2^x}\}_x$. Given a colouring $f \colon \mathbb{N} \to [k]$, we define *the restriction of f to the d-sequence* to be the colouring $f(d;\ \cdot\ ) \colon \mathbb{N} \to [k]$, with $f(d; x) = f\left(2^{d2^x}\right)$, for $x \in \mathbb{N}$.

Lemma 7. *For $n \in \mathbb{N}$, let $W : \mathbb{N}^n \to \mathbb{N}$ be a function and let $\mathcal{A} \subseteq \mathbb{N}^n$ be partition regular. Let $r, k \in \mathbb{N}$ and let $f : \mathbb{N} \to [k]$ be a colouring that admits no monochromatic, exponential $\mathcal{A}$-system with weight W. Then we can find colours $c_1, \dots, c_r \in [k]$ and corresponding colourings $\widetilde{f}_1, \dots, \widetilde{f}_r : \mathbb{N} \to [k]$ so that the following hold.*

1. *For each $i \in [r]$ there exists a sequence of integers $\{d_i(n)\}_n$ so that*

$$f(d_i(n);\ \cdot\ ) \to \widetilde{f}_i \quad as \quad n \to \infty,$$

where, for each $n \in \mathbb{N}$, $f(d_i(n);\ \cdot\ )$ is the restriction of f to the $d_i(n)$-sequence;
2. *The sequence of colours $c_1, \dots, c_r$ is large with respect to $\widetilde{f}_1, \dots, \widetilde{f}_r$;*
3. *$c_1, \dots, c_r$ are distinct.*

Proof. We apply induction on r. For $r = 1$, choose $d_1(n) = 1$ for all $n \in \mathbb{N}$. Thus $f(d_1(n); \cdot)$ trivially converges to a k-colouring $\tilde{f}_1$. Now, by the corollary to van der Waerden's theorem above (Corollary 5), there exists $c_1 \in [k]$ that is large with respect to $\widetilde{f}_1$. This proves the base case of the induction.

For the inductive step, suppose that we have found colourings $\widetilde{f}_1, \dots, \widetilde{f}_{r-1}$ with associated colours $c_1, \dots, c_{r-1}$ that satisfy the statement of the lemma. In what follows, we let $M \in \mathbb{N}$ be a parameter. Now since $c_1, \dots, c_{r-1}$ is large for $\widetilde{f}_1, \dots, \widetilde{f}_{r-1}$, we may find a progression

$$P_M = \{d(M)x + a(M) : x \in [M]\}$$

of length M with $\tilde{f}_i(P_M) = \{c_i\}$, for each $i \in [r-1]$. Now let $M' \in \mathbb{N}$ be a (new) parameter and define

$$h(M') = \max\left\{W(x_1, \dots, x_n) : x_1, \dots, x_n \in [P(\mathcal{A}; k^{M'})]\right\},$$

while recalling that $P(\mathcal{A}; k)$ is the smallest integer so that every k-colouring of $\mathbb{N}$ admits a monochromatic tuple $x_1, \dots, x_n$ such that $(x_1, \dots, x_n) \in \mathcal{A}$ and $x_1, \dots, x_n \leq P(\mathcal{A}; k)$. For each appropriate M, M', we consider numbers of the form $2^{d(M)x2^y}$, where $x \leq P(\mathcal{A}; k^{M'})$ and $y \leq M'$. In particular, we define a colouring $F = F_{M,M'} : [P(\mathcal{A}; k^{M'})] \to [k]^{M'}$ by

$$F(x) = \left(f\left(2^{d(M)x2^1}\right), f\left(2^{d(M)x2^2}\right), \dots, f\left(2^{d(M)x2^{M'}}\right)\right)$$

and observe that F defines a $k^{M'}$-colouring of $[P(\mathcal{A}; k^{M'})]$ and therefore we can find a set of positive integers $x_1 = x_1(M, M'), \dots, x_n = x_n(M, M')$, with $(x_1, \dots, x_n) \in \mathcal{A}$, which is monochromatic with respect to the colouring F. We now show that the colouring f along the sequence

$$2^{d(M)x_1 2^1}, \dots, 2^{d(M)x_1 2^{M'}}$$

is rather constrained, provided M is sufficiently large compared to M'.

Claim 8. *If $M \geq 2^{M'} h(M')$, then the colours*

$$f\left(2^{d(M)x_1 2^1}\right), \dots, f\left(2^{d(M)x_1 2^{M'}}\right)$$

are distinct from the colours $c_1, \dots, c_{r-1}$.

Proof. We show that if just one of these elements is coloured by a colour of $\{c_1, \dots, c_{r-1}\}$, we can find a monochromatic exponential $\mathcal{A}$-system, thus obtaining a contradiction. So, assume that there is an element $2^{d(M)x_1 2^y}$, with $y \in [M']$, that receives colour c_p, with $p \in [r-1]$. By the definition of the $x_1, \dots, x_n$, it follows that all of the elements

$$2^{d(M)x_1 2^y}, 2^{d(M)x_2 2^y}, \dots, 2^{d(M)x_n 2^y}$$

receive colour c_p.

Next, choose an integer $N = N(M)$ to be large enough so that the colouring $f(d_p(N); \cdot)$ agrees with $\widetilde{f}_p$ on every element of $[\max P_M]$. Such a choice of N exists, as we are granted $f(d_p(n); \cdot) \to \widetilde{f}_p$ as $n \to \infty$, by the induction hypothesis. As a result, we have that P_M is coloured by $f(d_p(N); \cdot)$ exactly as it is coloured by $\widetilde{f}_p$.

We claim that $(x_1, \dots, x_n)$, $a = 2^{d_p(N)2^{a(M)}}$ and $b = 2^{d(M)2^y}$ define an exponential $\mathcal{A}$-system that is monochromatic in the colour c_p. We already know that $f(b^{x_1}) = \cdots = f(b^{x_n}) = c_p$, so it only remains to check the colour of $a \star (b^l)$, for each $l \in [0, W(x_1, \dots, x_n)]$. So fix $l \in [0, W(x_1, \dots, x_n)]$ and write

$$f\left(a^{b^l}\right) = f\left(2 \star \left(d_p(N)2^{a(M)+d(M)l2^y}\right)\right) = f\left(d_p(N); a(M) + d(M)l2^y\right).$$

Now since $l2^y \leq W(x_1, \dots, x_n)2^{M'} \leq M$, our choice of N allows us to conclude that the above is equal to

$$\tilde{f}_p\left(a(M) + d(M)l2^y\right) = c_p,$$

where this last inequality holds as $P_M = \{a(M) + d(M)x : x \in [M]\}$ is a progression with the property that $\tilde{f}_p(P_M) = c_p$, as we assumed above.

Hence we have found an exponential $\mathcal{A}$-system, monochromatic in c_p. This contradicts the assumption on f and completes the proof of the claim. □

So for each $M' \in \mathbb{N}$, we set $d'(M') = d\left(2^{M'}h(M')\right) \cdot x_1\left(2^{M'}h(M'), M'\right)$ and apply the compactness property (i.e. Fact 4) to find a subsequence of the $\{d'(M')\}$ for which the sequence of colourings $\left(f(d'(M'); \cdot)\right)_{M' \in \mathbb{N}}$ converges. We take $\{d_r(M)\}_M$ to be this subsequence and $\tilde{f}_r$ to be the corresponding limiting colouring, to satisfy Conclusion 1.

Now note that we have $c_1, \dots, c_{r-1} \notin \tilde{f}_r(\mathbb{N})$, for if $\tilde{f}_r(x_0) = c_i$ for some $x_0 \in \mathbb{N}$ and $i \in [r-1]$, it would follow, for sufficiently large M', that the integer

$$2 \star (d'(M')2^{x_0}) = 2 \star \left(d\left(2^{M'}h(M')\right) \cdot x_1\left(2^{M'}h(M'), M'\right) \cdot 2^{x_0} \right),$$

would receive the colour c_i, which is in contradiction with Claim 8.

Finally, we choose the colour c_r. This is easily done; since $c_1, \dots, c_{r-1}$ is large for $\widetilde{f}_1, \dots, \widetilde{f}_{r-1}$, by Lemma 6, we may find a colour $c_r \in [k]$ so that $c_1, \dots, c_r$ is large with respect to $\tilde{f}_1, \dots, \tilde{f}_r$. This satisfies Conclusion 2. To see that we have satisfied Conclusion 3, it is enough to note that c_r must be distinct from $c_1, \dots, c_{r-1}$ as $c_1, \dots, c_{r-1} \notin \tilde{f}_r(\mathbb{N})$.

So we have constructed a sequence of integers $\{d_r(M)\}$ and a colour $c_r \in [k] \setminus \{c_1, \dots, c_{r-1}\}$ so that $f(d_r(M); \cdot)$ converges to a colouring $\tilde{f}_r$ with the property that $c_1, \dots, c_r$ is large for $\tilde{f}_1, \cdots, \tilde{f}_r$. With this we have satisfied Conclusions 1, 2 and 3 and thus we conclude the induction step. Thus we complete the proof of Lemma 7, by induction. □

We may now deduce Theorem 2 from Lemma 7.

Proof of Theorem 2. Let $n \in \mathbb{N}$, $W : \mathbb{N}^n \to \mathbb{N}$ be a function and let $\mathcal{A} \subseteq \mathbb{N}^n$ be a partition regular pattern. For a contradiction, suppose that f is a k-colouring of $\mathbb{N}$ for which there is no monochromatic exponential $\mathcal{A}$-pattern, with weight W. Now apply Lemma 7 with the choice of $r = k+1$ to find $k+1$ distinct colours $c_1, \dots, c_{k+1} \in [k]$. This is a contradiction. □

4. Proof of Theorem 1

We are now in a position to prove our classification of partition regular, exponential systems. Recall that a binary relation R comes implicitly in the definition of the systems $\mathcal{L}$ and $\mathcal{R}$. In what follows, we regard this relation as a directed graph in the obvious way, thus allowing us to borrow from the terminology of directed graphs. Indeed, call a digraph *weakly connected* if the underlying, undirected graph is connected, and say that a subgraph of G is a *weak component* of G if this subgraph is a component in the underlying, undirected graph.

Proof of Theorem 1. For $n \in \mathbb{N}$, $R \subseteq [n]^2$, and $C_k(i,j) \in \mathbb{Z}$, $i, j, k \in [n]$, we consider the system of exponential equations $\mathcal{E} = \mathcal{E}(R, \{C_k(i,j)\})$, along with the associated linear system of equations $\mathcal{L} = \mathcal{L}(R, \{C_k(i,j)\})$.

Let us first assume that $\mathcal{L}$ is partition regular. We let $\mathcal{A} \subseteq \mathbb{N}^n$ be the collection of positive-integer solutions to the system of equations $\mathcal{L}$ and we define the weight function $W : \mathbb{N}^n \to \mathbb{N}$, by $W(x_1, \dots, x_n) = \left(\sum_{i,j,k} |C_k(i,j)| \right) \sum_i^n x_i$, for all $x_1, \dots, x_n \in \mathbb{N}$.

Now, given a finite colouring f of the integers, apply Theorem 2 to find integers $a, b > 1$ and $(z_1, \dots, z_n) \in \mathcal{A}$ so that all of the integers

$$a, b^{z_1}, \dots, b^{z_n}, a^{b^1}, \dots, a^{b^{W(z_1,\dots,z_n)}}$$

are given the same colour by f.

We define numbers $x_1, \dots, x_n, y_1, \dots, y_n$ that will constitute a monochromatic solution to $\mathcal{E}$. We start by selecting $y_i = b^{z_i}$, for $i \in [n]$.

Now, if $x, y \in [n]$ are in the same weak component of R, and $P_{x,y}$ is a path between x and y we define

$$\omega(P_{x,y}) = \sum_{e \in E(P_{x,y})} (-1)^{d(e)} \left(C_1(e)z_1 + \cdots + C_n(e)z_n\right).$$

We note that this definition does not depend on the choice of P. For if P' is another path from x to y we have

$$\omega(P_{x,y}) - \omega(P'_{x,y}) = \sum_{e \in E(C)} (-1)^{d(e)} \left(C_1(e)z_1 + \cdots + C_n(e)z_n\right),$$

where C is the closed walk formed by first traversing $P_{x,y}$ and then traversing $P'_{x,y}$ backwards. We may then partition the edges of the closed walk C as a union of cycles $C^1, \dots, C^l$, $l \in \mathbb{N}$, and thus decompose the above sum as

$$= \sum_{i=1}^{l} \sum_{e \in E(C_i)} (-1)^{d(e)} \left(C_1(e)z_1 + \cdots + C_n(e)z_n\right),$$

which is clearly 0, as $z_1, \dots, z_n$ is a solution to $\mathcal{L}$.

Hence it makes sense to define $\omega(x, y) = \omega(P_{x,y})$, where $P_{x,y}$ is some path between x, y. Now define R' to be a binary relation on $[n]$, defined by $(x, y) \in R'$ if x, y are in the same weak component of R and $\omega(x, y) \geq 0$. Note that this relation is transitive: if $\omega(x, y) \geq 0$ and $\omega(y, z) \geq 0$ then there are paths $P_{x,y}, P_{y,z}$ from x to y and y to z with $\omega(P_{x,y}), \omega(P'_{x,y}) \geq 0$. Therefore the path $P_{x,z}$, formed by first traversing $P_{x,y}$ and then $P_{y,z}$, satisfies $\omega(P_{x,z}) = \omega(P_{x,y}) + \omega(P_{y,z}) \geq 0$. Thus, $\omega(x, z) \geq 0$.

From the above, we see that the relation R' is transitive and compares every two elements of $[n]$ that are in the same weak component of R. We may assume, without loss, that R has $m \in [n]$ weak components and that $1, \dots, m$ are vertices from each of the m components (resp.) so that if $x \in [m]$, we have that $\omega(x, y) \geq 0$, for each y which is in the same weak component as x. We define the auxiliary integers k_i, $i \in [n]$, by first putting $k_1 = \cdots = k_m = 0$ and then, for $i > m$, we set

$$k_i = \omega(j, i) \geq 0,$$

where $j \in [m]$ is a representative vertex from the weak component containing i.

We now define $x_i = a^{b^{k_i}}$, for $i \in [m]$. Note that since $k_i \geq 0$, x_i is an integer and certainly $x_i > 1$. We also have that $k_i \leq W(x_1, \dots, x_n)$, for each $i \in [n]$ and therefore all of our choices of the $x_1, \dots, x_n, y_1, \dots, y_n$ receive the same colour from f. It only remains to check that they satisfy the system of equations $\mathcal{E}$. To this end, we note that for $(i, j) \in R$, we have

$$k_j - k_i = C_1((i,j))z_1 + \cdots + C_n((i,j))z_n,$$

which follows from the "independence of path" argument above. So, finally, if $e = (i, j) \in R$ we have

$$\begin{aligned} x_i \star \left(y_1^{C_1(e)} \cdots y_n^{C_n(e)}\right) &= a \star (b \star (k_i + C_1(e)z_1 + \cdots + C_n(e)z_n)) \\ &= a \star (b \star k_j) = x_j, \end{aligned}$$

as desired. This proves that the $x_1, \dots, x_n, y_1, \dots, y_n$ indeed form a solution to $\mathcal{E}$ and thus we have shown that $\mathcal{E}$ is partition regular.

We now show that if $\mathcal{L}$ is *not* partition regular, then we may produce a colouring demonstrating that $\mathcal{E}$ is not partition regular. For $x \in \mathbb{N}$, write x in its prime factorization $x = p_1^{e_1} \cdots p_k^{e_k}$ and define the function ν by setting $\nu(1) = 0$ and letting $\nu(x) = e_1 + \cdots + e_k$, for $x > 1$. This function has three simple properties that shall be useful for us.

1. We have $\nu(x) = 0$ if and only if $x = 1$;
2. $\nu(xy) = \nu(x) + \nu(y)$ and, in particular, $\nu(a^b) = b\nu(a)$;
3. $\nu(x)$ takes integer values.

Now assume that $\mathcal{L}$ is not partition regular and let c be a finite colouring that forbids monochromatic solutions to $\mathcal{L}$. We define a colouring f by

$$f(x) = c(\nu(x)).$$

To see that this colouring has the required property, we would like to apply the function $\nu \circ \nu$ to both sides of each of our equations in $\mathcal{E}$. As we cannot do this directly, we rewrite our equations in the form

$$X_i^{Y_1^{C'(i,j)} \cdots Y_n^{C'(i,j)}} = X_j^{Y_1^{C''(i,j)} \cdots Y_n^{C''(i,j)}}, \tag{3}$$

where $C'(i,j), C''(i,j)$ are non-negative integers, for all $(i, j) \in R$. We now apply the function $\nu^2 = \nu \circ \nu$ to both sizes of every equation at (3). Note that this is possible, as we are assuming that $X_i, Y_i > 1$ and therefore we have that each side of the equation is a positive integer greater than 1 and thus $\nu \circ \nu$ is defined. After rearranging, we obtain

$$C_1(e)\nu(Y_1) + \cdots + C_n(e)\nu(Y_n) = \nu^2(X_j) - \nu^2(X_i),$$

for each $e = (i, j) \in R$.

Now, given a cycle C of R, we may eliminate the ν^2 terms by summing over the cycle C, multiplying by ± 1 according to the orientation of each $e \in C$. We thus obtain the equation

$$\sum_{e \in C} (-1)^{d(e)} \left(C_1(e)\nu(Y_1) + \cdots + C_n(e)\nu(Y_n) \right) = 0,$$

for each cycle C in R. Now, if $y_1, \ldots, y_n$ form a monochromatic solution to this rewritten system of equations, we have that $f(y_1) = \cdots = f(y_n)$ and therefore $c(\nu(y_1)) = \cdots = c(\nu(y_n))$. So if we put $u_1 = \nu(y_1), \ldots, u_n = \nu(y_n)$, we see that $u_1, \ldots, u_n$ satisfy the equation $\mathcal{L}$ and $c(u_1) = \cdots = c(u_n)$. However, our choice of c forbids this situation. Therefore $\mathcal{E}$ is not partition regular. This completes the proof of Theorem 1. $\square$

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