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Refining the bijections among ascent sequences, (2+2)-free posets, integer matrices and pattern-avoiding permutations

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ABSTRACT

The combined work of Bousquet-Mélou, Claesson, Dukes, Jelínek, Kitaev, Kubitzke and Parviainen has resulted in non-trivial bijections among ascent sequences, (2+2)-free posets, upper-triangular integer matrices, and pattern-avoiding permutations. To probe the finer behaviour of these bijections, we study two types of restrictions on ascent sequences. These restrictions are motivated by our results that their images under the bijections are natural and combinatorially significant. In addition, for one restriction, we are able to determine the effect of poset duality on the corresponding ascent sequences, matrices and permutations, thereby answering a question of the first author and Parviainen in this case. The second restriction should appeal to Catalaniacs.

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1. Introduction

In the last decade, an interesting collection of results has emerged from the study of (2+2)-free posets, or interval orders as they are also known, and their connection to permutations avoiding a non-standard permutation pattern of length three. The starting point for this story was the introduction in Bousquet-Mélou et al. [3] of a new type of permutation pattern that the authors termed a *bivincular pattern*. In that paper it was proven that length-n permutations avoiding the bivincular pattern $2|3\overline{1}$ were in one-to-one correspondence with unlabelled (2+2)-free posets on n elements. This was shown by encoding both structures as an integer sequence of length n that has come to be known as an *ascent sequence*. Via ascent sequences, Bousquet-Mélou et al. were able to solve the long-standing open problem of enumerating unlabelled (2+2)-free posets.

In [11], these ascent sequences were shown to uniquely encode another set of objects: all square upper-triangular matrices of non-negative integers whose entries sum to n which have neither rows nor columns consisting of only zeros. Such matrices are known as *Fishburn matrices* since they were introduced by Fishburn [13]; these matrices and the three previously mentioned classes are enumerated by the Fishburn numbers [23, A022493].

These initial two papers linking (bijectively) four different discrete objects led to a series of papers that studied these bijections and built upon the correspondences. From the enumerative viewpoint, Dukes et al. [10] and Kitaev & Remmel [17] considered these objects according to several statistics (such as the number of minimal/maximal elements) and presented multivariate generating functions for these statistics. In Claesson et al. [5], the bijections from the original papers were lifted to achieve bijections between labelled (2+2)-free posets, upper-triangular matrices whose entries partition a set, and a form of coloured ascent sequences. This lift of the correspondences to the labelled setting was used to give a bijection from unlabelled (2+2)-free posets to Fishburn matrices [9], which is equivalent to the definition of the *characteristic matrix* in [13, §2.3].

In another direction, and more recently, it has emerged that refinements of these correspondences have equally compelling stories to tell. Duncan & Steingrímsson [12] studied pattern avoidance in ascent sequences and established bijections between pattern avoiding ascent sequences and other combinatorial objects such as set partitions and objects enumerated by the Catalan and Narayana numbers. Jelínek [24] presented a new method to derive formulas for the generating functions of interval orders. The method generalised the results of [10,17] and also allowed the enumeration of self-dual interval orders with respect to several statistics. Using his newly derived generating function formulas, Jelínek proved a bijective relationship between self-dual interval orders and upper-triangular matrices having no zero rows [24]. Andrews & Jelínek [1] built on Jelínek's work and proved several power series identities involving the refined generating functions for interval orders and self-dual interval orders. Keller and Young [16] considered the difficult question of determining which ascent sequences map to semiorders; also known as unit

interval orders, semiorders are posets that are both (2+2) and (3+1)-free. See also [17] for a consideration of this question.

The present paper adds to this body of work by analyzing two types of restrictions on ascent sequences. One motivation for these restrictions is that their images through the bijections of [3,9,11] are combinatorially significant in the subsets they identify, e.g. series-parallel posets and 231-avoiding permutations. Moreover, the analysis of the images of these ascent sequences allows us to prove results about duals of each of the structures, thus going some way in answering an open problem of Dukes & Parviainen [11].

The first type of restriction we study (in Section 3) begins with a restriction on the types of ascents one may have in an ascent sequence. In particular, when the bijection of [3] recursively builds a (2+2)-free poset from an ascent sequence, there are some ascents that cause complicated and unnatural modifications to the poset, while the bijection treats all other ascents in a very natural way. Our first restriction is to those ascent sequences that contain only these ascents that result in this latter natural behaviour. A motivation for this restriction is that this good behaviour carries through to the general framework of bijections. Indeed, the images of these new *restricted ascent sequences* RAsc through the bijections given in [3,9,11] are proven to be simple restrictions: the subset RMatrices of the matrices from [11] having only positive diagonal entries, the subset RPosets of (2+2)-free posets which have a chain of the maximal possible length, and the set RPerms of permutations avoiding the barred pattern $3\overline{1}52\overline{4}$. This set RPerms was already identified in [3] in the context of modified ascent sequences. See Fig. 1.1 for a diagram outlining our sets and maps of interest.

In Section 4 we give a partial solution to an open problem of Dukes & Parviainen [11] by addressing the topic of structural duality. The dual P^* of a (2+2)-free poset P is also a (2+2)-free poset. This observation prompts the question as to whether one can derive the ascent sequence (resp. permutation) corresponding to P^* from the ascent sequence (resp. permutation) corresponding to P. This question seems intractable in general because of the complicated map between some ascent sequences and posets as mentioned in the previous paragraph. However, consistent with our motivation for restricting to better-behaved sets, we can answer this duality question completely for all posets in RPosets, which we do in Section 4.

In Section 5, we consider the Catalan family CAsc of 101-avoiding ascent sequences studied in [12], and investigate their images under the bijections of [3,9,11]. (The restricted ascent sequences of [17] are also enumerated by Catalan numbers but are different from CAsc.) The results are perhaps even nicer than the R-families and are shown in Fig. 1.1. The posets that arise are the series-parallel interval orders, i.e., those that are both (2+2)-free and N-free. This class of posets appears in [6,7], while series-parallel posets in general are widespread in the literature, partially because their recursive structure permits many polynomial-time algorithms (see, for example, [14] and the references therein). The matrices and permutations which correspond to these ascent sequences are those matrices from RMatrices that are termed *SE-free* in [25], and 231-avoiding permutations, respectively.

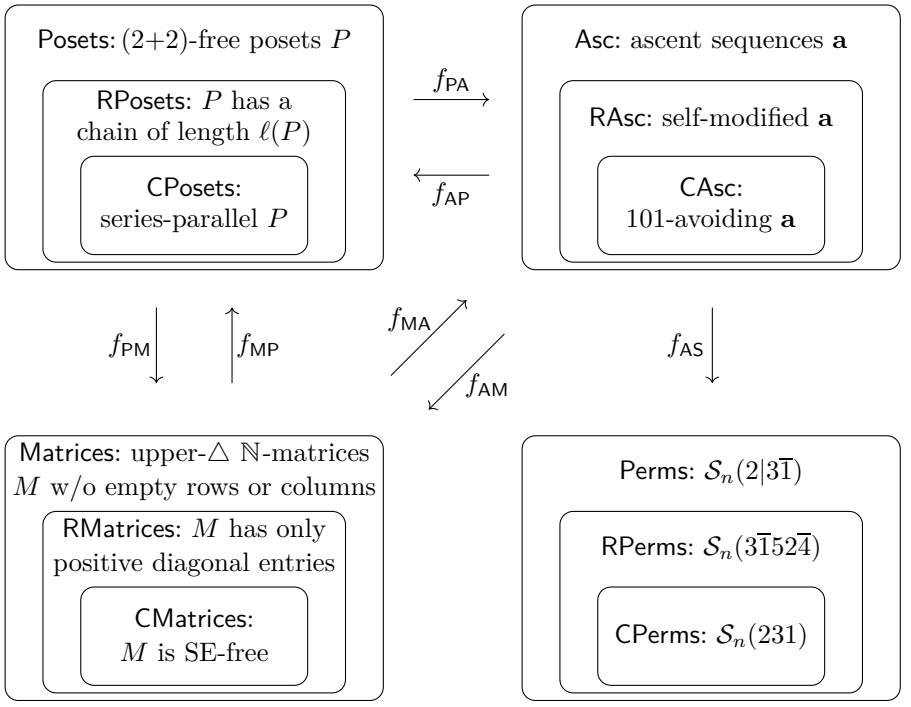

Fig. 1.1. A diagrammatic summary of the sets and bijections of interest.

We conclude in Section 6 with some open questions.

2. Preliminaries

While the intimate connections between four different types of objects are certainly a strength of this area of study, the drawback for our present purposes is there is a considerable amount of background that needs to be introduced, including all four classical sets and many of the bijections among them. Our use of the words "classical" refers to the full sets considered by most of the papers mentioned in the Introduction, and as shown by the largest boxes in Fig. 1.1: ascent sequences, (2+2)-free posets, upper-triangular matrices with non-negative integer entries having neither rows nor columns of all zeros, and permutations avoiding $2|3\overline{1}$.

2.1. The classical sets

An *ascent sequence* is a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers such that $a_1 = 0$, and for all i with $1 < i \leq n$ we have $a_i \leq \mathsf{asc}(a_1, \ldots, a_{i-1}) + 1$, where $\mathsf{asc}(a_1, \ldots, a_k)$ denotes the number of ascents in the sequence $(a_1, \ldots, a_k)$. For example $(0, 1, 0, 1, 3)$ is an ascent sequence whereas $(0, 1, 0, 2, 4)$ is not. Let Asc_n be the set of all

ascent sequences of length n, and let Asc denote the union of these sets over all n, with the same convention applying to all the notation below when the subscript n is dropped.

Let Posets_n be the set of (2+2)-free posets on n elements, meaning posets that have no induced subposet isomorphic to a disjoint union of two 2-element chains. We will be interested in a different defining property of (2+2)-free posets, as we now describe. Let $P = (P, \preceq)$ be a poset with n elements. Given $x \in P$, the set $D(x) = \{y \in P : y \prec_P x\}$ is called the *strict downset* of x. A fact described as "well known" in [2] and which is easy to check is that a poset is (2+2)-free if and only if the set of strict downsets of elements of P can be linearly ordered by inclusion. We let $\ell(P)$ denote the number of distinct nonempty such downsets, so that $D(P) = (D_0, \ldots, D_{\ell(P)})$ is the sequence of downsets of P linearly ordered by inclusion. In other words $\emptyset = D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_{\ell(P)}$. For example, for the poset P in the top left of Fig. 2.1, we have

$$D(P) = (\emptyset, \{p_1, p_2\}, \{p_1, p_2, p_5\}, \{p_1, p_2, p_3, p_5\}).$$

(Note that while this example P is labelled for the purposes of the explanation, the elements of Posets are unlabelled.) We will call D_i *level* i of P, and an element $x \in P$ with $D(x) = D_i$ for some i will be said to *lie at level* i of P. Let $L_i = L_i(P)$ be the set of elements lying at level i of P and set $L(P) = (L_0, \ldots, L_{\ell(P)})$. Again for P from Fig. 2.1,

$$L(P) = (\{p_1, p_2, p_5\}, \{p_6\}, \{p_3\}, \{p_4\}).$$

Let $\mathsf{Matrices}_n$ be the set of all upper-triangular matrices whose entries are all non-negative integers such that there is neither a row nor a column containing only zeros, and whose sum of all entries is n. Observe that the dimension d of an element of $\mathsf{Matrices}_n$ satisfies $1 \leq d \leq n$.

A sequence $\mathbf{a} = (a_1, \ldots, a_r)$ of non-negative integers is said to *contain* a sequence $\mathbf{b} = (b_1, \ldots, b_s)$ *as a pattern* if there exists a subsequence of $\mathbf{a}$ of length s whose elements are in the same relative order as those of $\mathbf{b}$. We say $\mathbf{a}$ is $\mathbf{b}$-*avoiding* if it does not contain $\mathbf{b}$. For example, $(0, 2, 1, 3, 1, 0, 2)$, which we write as 0213102 for short, contains the pattern 0101 because of its subsequence 0202, but avoids the pattern 1010.

When considering pattern-avoidance in sequences that are permutations, we allow for a more general notion of pattern: a permutation $\pi = \pi_1 \ldots \pi_n$ is said to *contain* the pattern $2|3\overline{1}$ if there exists an occurrence $\pi_i\pi_j\pi_k$ of 231 in π with the additional conditions that $j = i+1$ and $\pi_i = \pi_k+1$. For example, 32541 contains $2|3\overline{1}$ because of the occurrence 251 of 231, whereas 31452 avoids $2|3\overline{1}$ even though it has three occurrences of the classical pattern 231. The pattern $2|3\overline{1}$ is an example of a *bivincular pattern* as introduced in [3] since it puts conditions on both the entries and positions of an occurrence. As usual, we let $\mathcal{S}_n(2|3\overline{1})$ denote the set of permutations of length n that avoid $2|3\overline{1}$, and this is exactly our set Perms_n.

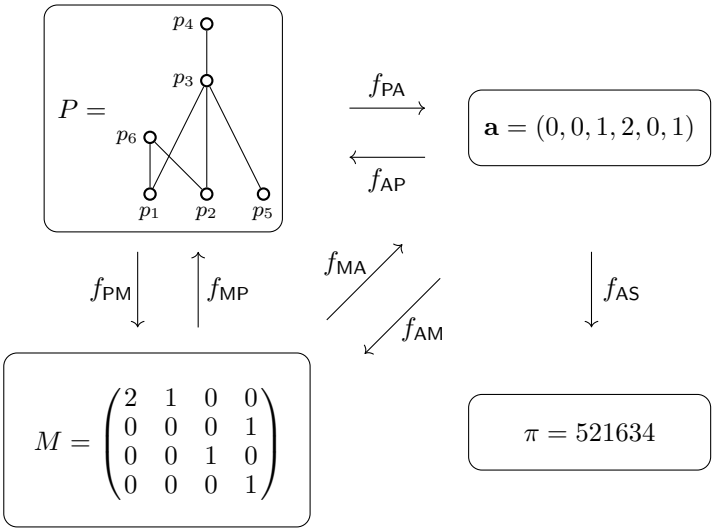

Fig. 2.1. The examples used when defining the bijections in Subsection 2.2.

2.2. The bijections

In this subsection, we gather the classical bijections from the literature [3,9,11] that we need. We refer the reader to these references for the proofs of bijectivity and our claims that particular pairs of maps are inverses. Due to its length, the reader may prefer to skip this subsection and instead refer back to it as a reference.

To denote the bijections from [3,9,11], we will use labels according to their domain and codomain, but rather than use the labels Asc, Posets, $\mathsf{Matrices}$ and Perms, we use the single-letter subscripts A, P, M and S ("S" for "symmetric group"). For example, f_{AP} denotes the bijection of [3] from ascent sequences to (2+2)-free posets.

If $a < b$ are integers, we use the notation $[a, b]$ for the set $\{a, \ldots, b\}$ and $[a, b)$ for the set $\{a, \ldots, b-1\}$, etc.

For each of our bijections, we will refer to the example in Fig. 2.1.

2.2.1. Ascent sequences and matrices

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an ascent sequence and define the truncated sequence $\mathbf{a}^{(k)} = (a_1, \ldots, a_k)$. If M is a $d \times d$ matrix, then we write $\dim(M) = d$. Let $\mathsf{mindex}(M)$ be the lowest index of a row whose rightmost entry is non-zero. For example, $\mathsf{mindex}(M) = 2$ for M from Fig. 2.1. The following map first appears in [11] where it is shown to be a bijection to $\mathsf{Matrices}_n$.

Definition 2.1. Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{Asc}_n$, we define $f_{\mathsf{AM}}(\mathbf{a})$ recursively. First, $f_{\mathsf{AM}}(\mathbf{a}^{(1)}) = (1)$, a 1×1 matrix. Supposing $M^{(k)} = f_{\mathsf{AM}}(\mathbf{a}^{(k)})$ for some $k \in [1, n)$, we have the following three cases for defining $M^{(k+1)} = f_{\mathsf{AM}}(\mathbf{a}^{(k+1)})$.

AM1 If $a_{k+1} \in [0, \mathsf{mindex}(M^{(k)}))$ then let $M^{(k+1)}$ be the matrix $M^{(k)}$ with the entry at position $(a_{k+1}+1, \dim(M^{(k)}))$ increased by one.

AM2 If $a_{k+1} = \dim(M^{(k)})$ then let $M^{(k+1)}$ be the result of appending a new row and column of zeros to the matrix $M^{(k)}$, and inserting a one into the new diagonal entry.

AM3 If $a_{k+1} \in [\mathsf{mindex}(M^{(k)}), \dim(M^{(k)}))$ then let $M^{(k+1)}$ be the outcome of the following: in $M^{(k)}$, insert a new (empty) row between rows a_{k+1} and $1+a_{k+1}$, and insert a new (empty) column between columns a_{k+1} and $1+a_{k+1}$. Let the new row be filled with all zeros except let the rightmost entry be 1. Move all the entries in the rightmost column above where it was prised apart to the left to the new empty spaces, and fill their former positions with zeros. Finally let all other entries in the new column be zero.

Then $f_{\mathsf{AM}}(\mathbf{a}) = f_{\mathsf{AM}}(\mathbf{a}^{(n)}) = M^{(n)}$.

Example 2.2. Let $\mathbf{a} = (0,0,1,2,0,1)$ as in Fig. 2.1.

- $f_{\mathsf{AM}}(\mathbf{a}^{(1)}) = (1) = M^{(1)}$.
- Since $a_2 \in [0, \mathsf{mindex}(M^{(1)}) = 1)$, AM1 applies and $M^{(2)} = (2)$.
- Since $a_3 = \dim(M^{(2)})$, AM2 applies and $M^{(3)} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
- Since $a_4 = \dim(M^{(3)})$, AM2 applies and $M^{(4)} = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.
- Since $a_5 \in [0, \mathsf{mindex}(M^{(4)}) = 3)$, AM1 applies and $M^{(5)} = \left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.
- Since $a_6 \in [\mathsf{mindex}(M^{(5)}) = 1, \dim(M^{(5)}) = 3)$, AM3 applies and we prise the matrix $M^{(5)}$ apart between rows 1 and 2, and columns 1 and 2 to get $\left(\begin{smallmatrix} 2 & & 0 & 1 \\ & & & \\ 0 & & 1 & 0 \\ 0 & & 0 & 1 \end{smallmatrix}\right)$. We fill the new second row with zeros except for a 1 in column 4. We move the single entry 1 in column 4 above the new row into the new second column, preserving this entry's row index, and put zeros in its former position and in the rest of the new column. We get $M^{(6)} = \left(\begin{smallmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) = f_{\mathsf{AM}}(\mathbf{a})$.

As is clear in the above definition, AM3 is by far the most involved, and this will be a feature of AP3 too. When AM3 is used in a recursive step, the results of the bijection quickly become intractable. The point of the R-families is that they are defined exactly so that AM3 and AP3 are never invoked, resulting in more manageable analysis.

We could define f_{MA} as just the inverse of the bijection f_{AM} but it will be useful to write an explicit definition following [11]. Example 2.2 in reverse serves as an example of Definition 2.3. Let $\mathsf{rowsum}_i(M)$ denote the sum of the entries in row i of M.

Definition 2.3. Given $M \in \mathsf{Matrices}_n$, define $\mathsf{reduce}(M) \in \mathsf{Matrices}_{n-1}$ in the following fashion.

MA1 If $\mathsf{rowsum}_{\mathsf{mindex}(M)}(M) > 1$ then let $\mathsf{reduce}(M)$ equal M with the value at position $(\mathsf{mindex}(M), \dim(M))$ reduced by 1.

MA2 If $\mathsf{rowsum}_{\mathsf{mindex}(M)}(M) = 1$ and $\mathsf{mindex}(M) = \dim(M)$ then let $\mathsf{reduce}(M)$ equal M with row $\dim(M)$ and column $\dim(M)$ removed.

MA3 If $\mathsf{rowsum}_{\mathsf{mindex}(M)}(M) = 1$ and $\mathsf{mindex}(M) < \dim(M)$ then perform the following modifications to M to form $\mathsf{reduce}(M)$. For $j \in [1, \mathsf{mindex}(M))$, move the entry in position $(j, \mathsf{mindex}(M))$ to position $(j, \dim(M))$. Then delete row $\mathsf{mindex}(M)$ and (the empty) column $\mathsf{mindex}(M)$.

Now recursively define $M^{(n)} = M$, and $M^{(k)} = \mathsf{reduce}(M^{(k+1)})$ for $k \in [1, n)$. Let $a_k = \mathsf{mindex}(M^{(k)}) - 1$ for $i \in [1, n]$ and define $f_{\mathsf{MA}}(M) = (a_1, \ldots, a_n)$, which is an ascent sequence [11]. Note that in following this recursive procedure, the sequence $(a_1, \ldots, a_n)$ is constructed from right-to-left.

Remark 2.4. A bijection between ascent sequences and Fishburn matrices is given in [4] which differs from ours only in the definition of AM3. Since our results are confined to those ascent sequences where AM3 is never invoked, our results would work equally well using the bijection of Chen, Yan and Zhou.

2.2.2. Ascent sequences and posets

We next introduce the bijections between Asc_n and Posets_n from [3]. Given a (2+2)-free poset P, recall that $\ell(P)$ denotes the highest index of a level or, equivalently, $\ell(P)+1$ is the number of levels in P. Let $\ell^\star(P)$ denote the minimum index of a level that contains a maximal element. Recall the sequence $L(P)$ of level sets defined in Subsection 2.1. The definition of f_{AP} below is very similar in structure to Definition 2.1 above of f_{AM}.

Definition 2.5. Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{Asc}_n$, we define $f_{\mathsf{AP}}(\mathbf{a})$ recursively. First, $f_{\mathsf{AP}}(\mathbf{a}^{(1)})$ is the poset consisting of a single element p_1. Supposing $P^{(k)} = f_{\mathsf{AP}}(\mathbf{a}^{(k)})$ for some $k \in [1, n)$, we have the following three cases for defining $P^{(k+1)} = f_{\mathsf{AP}}(\mathbf{a}^{(k+1)})$.

AP1 If $a_{k+1} \in [0, \ell^\star(P^{(k)})]$ then let $P^{(k+1)}$ be the result of adding to $P^{(k)}$ a new maximal element p_{k+1} that covers the same elements as do the elements in $L_{a_{k+1}}(P^{(k)})$.

AP2 If $a_{k+1} = 1 + \ell(P^{(k)})$ then let $P^{(k+1)}$ be the result of adding to $P^{(k)}$ a new element p_{k+1} covering all maximal elements of $P^{(k)}$.

AP3 If $a_{k+1} \in (\ell^\star(P^{(k)}), \ell(P^{(k)})]$ then let $P^{(k+1)}$ be the outcome of the following: to $P^{(k)}$, add a new element p_{k+1} covering the same elements as the elements in $L_{a_{k+1}}(P^{(k)})$. Let $\mathcal{M}$ be the set of maximal elements of $P^{(k)}$ lying at any level less than a_{k+1}. Add all relations $x \preceq y$ where $x \in \mathcal{M}$ and y is any element of $L_{a_{k+1}}(P^{(k)}) \cup \cdots \cup$

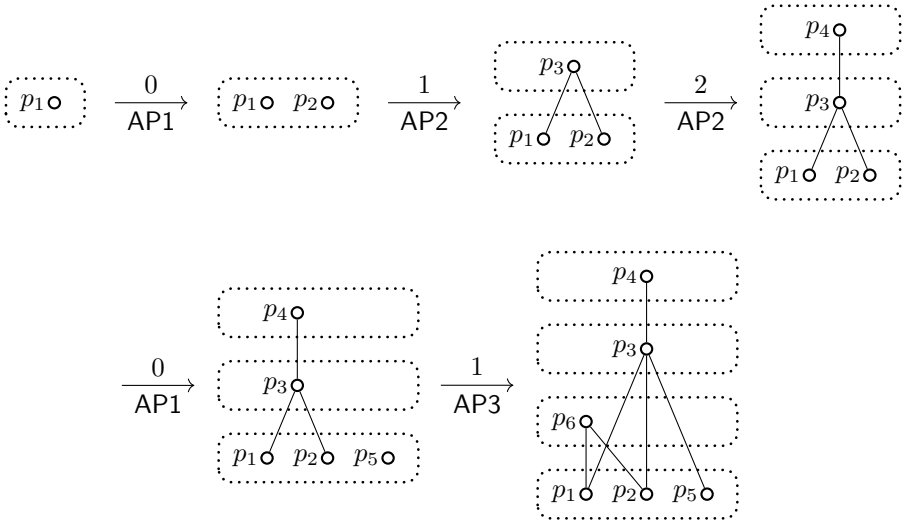

Fig. 2.2. The recursive construction of Example 2.6.

$L_{\ell(P^{(k)})}(P^{(k)})$; here we do not consider the new element p_{k+1} to be an element of $L_{a_{k+1}}(P^{(k)})$.

Then $f_{\mathsf{AP}}(\mathbf{a}) = f_{\mathsf{AP}}(\mathbf{a}^{(n)}) = P^{(n)}$.

Example 2.6. Let $\mathbf{a} = (0, 0, 1, 2, 0, 1)$ as in Fig. 2.1. Certainly, $P^{(1)}$ is the one-element poset. The recursive construction of $P = f_{\mathsf{AP}}(\mathbf{a})$ appears in Fig. 2.2, where the dotted shapes depict the different levels. The element a_{k+1} appears above each arrow and the case name appears below each arrow. The labels p_i are just for expository purposes and are not part of $f_{\mathsf{AP}}(\mathbf{a})$. In the final step, $\mathcal{M} = \{p_5\}$. Note that the new element p_6 in the final step ends up on its own level, and this is true in general for applications of AP3.

To define the inverse f_{PA}, a key observation from Definition 2.5 is that p_{k+1} is always a maximal element of $P^{(k+1)}$ and, furthermore, a_{k+1} will be exactly $\ell^\star(P^{(k+1)})$. Therefore, f_{PA} will be a recursive map that obtains $P^{(k)}$ from $P^{(k+1)}$ by removing a maximal element at level $\ell^\star(P^{(k+1)})$ and records this level value as a_{k+1}. Since all maximal elements at the same level in a (2+2)-free poset are order equivalent, it does not matter which element we remove. Of course, care has to be taken to make sure we invert AP3 correctly: see PA3 below.

Recall that $D_i(P)$ denotes the strict downset of the elements at level i. Again, the definition below parallels the corresponding one from Subsection 2.2.1, i.e., Definition 2.3. Obviously, Fig. 2.2 in reverse serves as an example of Definition 2.7.

Definition 2.7. Given $P \in \mathsf{Posets}_n$, let $i = \ell^\star(P)$ and define $\mathsf{remove}(P) \in \mathsf{Posets}_{n-1}$ in the following fashion.

PA1 If $|L_i| > 1$ then let $\mathsf{remove}(P)$ equal P with a maximal element at level i removed.

PA2 If $|L_i| = 1$ and $i = \ell(P)$ then obtain $\mathsf{remove}(P)$ by deleting the unique element at level i of P.

PA3 If $|L_i| = 1$ and $i < \ell(P)$ then perform the following modifications to P to form $\mathsf{remove}(P)$. Let $\mathcal{M} = D_{i+1}(P) \setminus D_i(P)$. Make each element of $\mathcal{M}$ a maximal element by deleting all relations $x \prec y$ with $x \in \mathcal{M}$. Then remove the unique element that was at level i of P.

Now recursively define $P^{(n)} = P$, and $P^{(k)} = \mathsf{remove}(P^{(k+1)})$ for $k \in [1, n)$. Let $a_k = \ell^\star(P^{(k)})$ for $i \in [1, n]$ and define $f_{\mathsf{PA}}(P) = (a_1, \ldots, a_n)$, which is an ascent sequence [3]. Note that in following this recursive procedure, the sequence $(a_1, \ldots, a_n)$ is constructed from right-to-left.

2.2.3. Matrices and posets

The bijections between matrices and posets are actually simpler than those involving ascent sequences. To provide definitions for f_{MP} and f_{PM}, it is easier to work with labelled posets and remove the labels after the construction, as is done in [9]. We begin with f_{PM}.

Definition 2.8. Let P be a labelled (2+2)-free poset on the set $\{p_1, \ldots, p_n\}$. The poset is uniquely specified by the sets $D(P)$ and $L(P)$ defined in Subsection 2.1 since every element of L_i has the set D_i as its strict downset.

For $j \in [0, \ell(P)]$, let $K_j(P) = D_{j+1}(P) \setminus D_j(P)$ where $D_{\ell(P)+1} \coloneqq P$. Let M' be the matrix with entries $M'_{ij} = L_{i-1}(P) \cap K_{j-1}(P)$ for all $i, j \in [1, \ell(P) + 1]$, and define $f_{\mathsf{PM}}(P)$ to be the matrix whose (i, j) entry is $|M'_{ij}|$.

Example 2.9. Let P be the (2+2)-free poset from the top-left of Fig. 2.1 where

$$D(P) = (\emptyset, \{p_1, p_2\}, \{p_1, p_2, p_5\}, \{p_1, p_2, p_3, p_5\})$$
$$L(P) = (\{p_1, p_2, p_5\}, \{p_6\}, \{p_3\}, \{p_4\}).$$

We have $\ell(P) = 3$ and

$$K(P) = (K_0(P), K_1(P), K_2(P), K_3(P)) = (\{p_1, p_2\}, \{p_5\}, \{p_3\}, \{p_4, p_6\}).$$

From this

$$M' = \begin{pmatrix} \{p_1, p_2\} & \{p_5\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \{p_6\} \\ \emptyset & \emptyset & \{p_3\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{p_4\} \end{pmatrix} \tag{2.1}$$

and so $f_{\mathsf{PM}}(P)$ is the matrix M in the bottom-left of Fig. 2.1.

To define f_{MP} by inverting f_{PM}, the first step is to construct the set-valued matrix as in (2.1), but it is not clear which of the entries $\{p_1, \ldots, p_n\}$ should go where. The key insight is that it doesn't really matter because $f_{\mathsf{MP}}(M)$ will ultimately be an *unlabelled* poset.

Definition 2.10. Let $M \in \mathsf{Matrices}_n$ and let $P' = \{p_1, \ldots, p_n\}$. Replace each of the zero entries in M with the empty set, and replace any non-zero integer a in M with a subset of P' of size a while ensuring there are no duplicated elements in the matrix. Given $p_i \in P'$, let $c(p_i)$ and $r(p_i)$ be the column and row indices of p_i . Let $P = (P', \preceq)$ be the poset whereby $p_i \prec p_j$ if and only if $c(p_i) < r(p_j)$. Finally let $f_{\mathsf{MP}}(M)$ be the unlabelled version of the poset P. □

Example 2.11. Again using M from Fig. 2.1, suppose we assign $\{p_1, \ldots, p_6\}$ as follows:

$$M' = \begin{pmatrix} \{p_1, p_2\} & \{p_3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \{p_4\} \\ \emptyset & \emptyset & \{p_5\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{p_6\} \end{pmatrix}.$$

This gives the strict order relations $p_1, p_2 \prec p_4, p_5, p_6$ and $p_3 \prec p_5, p_6$ and $p_5 \prec p_6$. This is the same poset as in the top left of Fig. 2.1 once all labels are removed.

2.2.4. Ascent sequences to permutations

For an ascent sequence $\mathbf{a}$, we follow [3], which used a technique systemised in [26], to recursively construct a permutation $f_{\mathsf{AS}}(\mathbf{a}) \in \mathcal{S}_n(2|3\overline{1})$.

Given a $2|3\overline{1}$-avoiding permutation $\pi \in \mathcal{S}_{n-1}$, we wish to add the entry n to π and get another $2|3\overline{1}$-avoiding permutation. With this in mind, we say that the site between positions i and $i+1$ in π is *active* if $\pi_i = 1$ or $\pi_i - 1$ is to the left of π_i. The sites immediately before position 1 and immediately after position $n-1$ are always active. We leave it as an exercise to check that the active sites are exactly those where inserting n into π will result in an element of $\mathcal{S}_n(2|3\overline{1})$. We label the active sites as subscripts from left to right beginning with 0. For example, the permutation 521634 becomes ${}_0521_16_23_34_4$.

Recall that for an ascent sequence $\mathbf{a} = (a_1, \ldots, a_n)$, the prefix $(a_1, \ldots, a_k)$ is denoted $\mathbf{a}^{(k)}$.

Definition 2.12. Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{Asc}_n$, we define $f_{\mathsf{AS}}(\mathbf{a})$ recursively. First, $f_{\mathsf{AS}}(\mathbf{a}^{(1)}) = 1$, a permutation of one element. For $k \in (1, n]$, we let $f_{\mathsf{AS}}(\mathbf{a}^{(k)})$ be $f_{\mathsf{AS}}(\mathbf{a}^{(k-1)})$ with k inserted at the active site labelled a_k. Then $f_{\mathsf{AS}}(\mathbf{a})$ is defined to be $f_{\mathsf{AS}}(\mathbf{a}^{(n)})$.

We leave it as an exercise to check that there will indeed exist an active site labelled a_k.

Example 2.13. Let $\mathbf{a} = (0, 0, 1, 2, 0, 1)$ as in Fig. 2.1. We get the following sequence of permutations, labelled according to their active sites, with the arrows labelled by a_k:

$$ {}_01_1 \xrightarrow{0} {}_021_1 \xrightarrow{1} {}_021_13_2 \xrightarrow{2} {}_021_13_24_3 \xrightarrow{0} {}_0521_13_24_3 \xrightarrow{1} {}_0521_16_23_34_4 \,, $$

consistent with Fig. 2.1.

2.2.5. Relationships among the bijections

Since the bijections between matrices and posets are very different from those involving ascent sequences, it is not at all clear that Fig. 1.1 is a commutative diagram at the level of Posets, Asc and Matrices. Since our maps are all bijections, this is equivalent to showing that $f_{\mathsf{PM}} = f_{\mathsf{AM}} \circ f_{\mathsf{PA}}$, the truth of which is given as a remark in [9]. The equivalent statement $f_{\mathsf{PM}} \circ f_{\mathsf{AP}} = f_{\mathsf{AM}}$ can be checked using a careful induction argument. The crux of the argument is that the ways in which AP1, AP2 and AP3 from Definition 2.5 change the matrix $f_{\mathsf{PM}}(P^{(k)})$ to $f_{\mathsf{PM}}(P^{(k+1)})$ match exactly with the effects of AM1, AM2 and AM3 respectively from Definition 2.1. In Theorem 4.3, we will use the equivalent statement that $f_{\mathsf{PA}} = f_{\mathsf{MA}} \circ f_{\mathsf{PM}}$.

We will also need a bijection f_{PS} (see Subsection 4.2), and we will define it by $f_{\mathsf{PS}} = f_{\mathsf{AS}} \circ f_{\mathsf{PA}}$. A direct definition of f_{PS} that bypasses ascent sequences appears in [3, Sub. 4.2] but we choose not to include that background here since we do not make heavy use of f_{PS}.

3. Restricted sets

As we saw in the previous section, the bijection f_{AM} (resp. f_{AP}) becomes more complicated when the case AM3 (resp. AP3) arises. In this section, we introduce the subset of Asc for which these cases never arise, denoted RAsc, where the letter R stands for "restricted." There are two reasons for restricting our attention. First, as we will show, the images of this subset under our various bijections have nice definitions, and some of these images are combinatorially significant. Secondly, the bijections' behaviour becomes more tractable, allowing us to obtain results for these restricted sets, as we will see in particular in Section 4.

We first must determine the conditions on an ascent sequence $\mathbf{a}$ that cause these two troublesome cases to arise. Referring to Definition 2.1, we need to avoid the situation when

$$ a_{k+1} \in [\mathsf{mindex}(M^{(k)}), \dim(M^{(k)})). \tag{3.1} $$

Let us translate this expression into properties of $\mathbf{a}$. Looking at all three cases in this definition, we see that $\mathsf{mindex}(M^{(k+1)})$ is nothing more than $a_{k+1}+1$, so $\mathsf{mindex}(M^{(k)}) = a_k+1$. In addition, this observation implies that AM1 corresponds to the case when a_k is not an ascent in $\mathbf{a}$. Moreover, observe that when AM1 applies, the dimension of the matrix does not change, but it increases by 1 for each application of AM2 or AM3. Therefore, $\dim(M^{(k)})$ is exactly $\mathsf{asc}(a_1, \ldots, a_k)+1$. Thus the condition (3.1) that invokes AM3 when constructing $M^{(k+1)}$ is equivalent to

$$a_{k+1} \in (a_k, \mathsf{asc}(a_1, \ldots, a_k)]. \tag{3.2}$$

A similar analysis of Definition 2.5 shows that (3.2) is also exactly the condition that causes AP3 to be used. Thus we are led to the following definition of a subset of Asc, which was introduced in [3] under the name "self-modified" ascent sequences.

Definition 3.1. Let RAsc_n be the subset of Asc consisting of those ascent sequences $\mathbf{a} = (a_1, \ldots, a_n)$ such that

$$a_k \in [0, a_{k-1}] \cup \{1 + \mathsf{asc}(a_1, \ldots, a_{k-1})\} \quad \text{for all } k > 1. \tag{3.3}$$

In other words, if a_k is larger than a_{k-1}, then a_k must be the largest it can be under the conditions on an ascent sequence. For example, $(0, 1, 0, 2)$ is in RAsc whereas $(0, 1, 0, 1)$ is not. Taking these ideas a step further, we get the following equivalent definition of RAsc, which will be useful later.

Lemma 3.2. RAsc_n *is the subset of* Asc *consisting of those ascent sequences* $\mathbf{a} = (a_1, \ldots, a_n)$ *such that*

$$a_k \in [0, a_{k-1}] \cup \{1 + \mathsf{max}(a_1, \ldots, a_{k-1})\} \quad \textit{for all } k > 1. \tag{3.4}$$

Proof. We argue by induction on the number of ascents. If the first ascent is at position i_1 so that $0 = a_{i_1} < a_{i_1+1}$ then we have

$$a_{i_1+1} = 1 + \mathsf{asc}(a_1, \ldots, a_{i_1}) = 1 + \mathsf{max}(a_1, \ldots, a_{i_1}) = 1.$$

With the jth ascent at position i_j, we assume

$$a_{i_j+1} = 1 + \mathsf{asc}(a_1, \ldots, a_{i_j}) = 1 + \mathsf{max}(a_1, \ldots, a_{i_j}).$$

Thus at the $(j+1)$st ascent, we have (note the subtle difference below between the indices i_{j+1} and $i_j + 1$)

$$\begin{aligned} a_{i_{j+1}+1} &= 1 + \mathsf{asc}(a_1, \ldots, a_{i_{j+1}}) \\ &= 1 + \mathsf{asc}(a_1, \ldots, a_{i_j+1}) \\ &= 1 + \mathsf{asc}(a_1, \ldots, a_{i_j}) + 1 \\ &= 1 + \mathsf{max}(a_1, \ldots, a_{i_j}) + 1 \\ &= \mathsf{max}(a_1, \ldots, a_{i_j+1}) + 1 \\ &= \mathsf{max}(a_1, \ldots, a_{i_{j+1}}) + 1. \quad \square \end{aligned}$$

The rest of this section will be working towards proving Corollary 3.8, which states that, under the classical bijections defined in the previous section, RAsc_n maps to the sets we now define.

Definition 3.3.

- Let $\mathsf{RMatrices}_n$ be the set of matrices in $\mathsf{Matrices}_n$ all of whose diagonal entries are positive.
- Let $\mathsf{RPosets}_n$ be the set of those posets in Posets_n that have a chain of length $\ell(P)$.
- Let $\mathsf{RPerms}_n = \mathcal{S}_n(3\overline{1}52\overline{4})$ (defined next).

The set $\mathcal{S}_n(3\overline{1}52\overline{4})$ is enumerated in [19,20] (see also [23, A098569]) and appears in [3] as the image $f_{\mathsf{AS}}(\mathsf{RAsc}_n)$. A permutation π is said to avoid the barred permutation $3\overline{1}52\overline{4}$ if every occurrence of the pattern 231 in π plays the role of 352 in an occurrence of the pattern 31524 in π. In other words, if we have $i < j < k$ with $\pi_k < \pi_i < \pi_j$, there must also exist ℓ and m such that $i < \ell < j < k < m$ and $\pi_i\pi_\ell\pi_j\pi_k\pi_m$ is an occurrence of 31524.

Since a poset P in Posets has $\ell(P)+1$ levels, the maximal possible length of a chain is $\ell(P)$, so $\mathsf{RPosets}_n$ consists of those posets that have a chain that contains an element from every level.

The following observation will be crucial for several results stated in this paper. It not only gives the image of an element of $\mathsf{RMatrices}$ under f_{MA} but, combined with Corollary 3.8(a), shows that all elements of RAsc take a particular form. We use i^j to denote a sequence of j copies of i. From this point on, we will denote the entry in row i and column j of the matrix M by m_{ij} or $m_{i,j}$

Lemma 3.4. *Suppose that* $M \in \mathsf{RMatrices}$ *with* $\dim(M) = d$. *Then*

$$f_{\mathsf{MA}}(M) = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1\,d}}, \ldots, 0^{m_{1d}}). \tag{3.5}$$

Proof. This follows from the definition of f_{MA} by considering how $\mathbf{a} = f_{\mathsf{MA}}(M)$ is constructed from right-to-left in Definition 2.3. Since the diagonal entries of M are all positive, we never have to apply MA3. □

Example 3.5. The matrix $M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in bijection with the ascent sequence $\mathbf{a} = (0, 0, 1, 2, 0)$.

The next lemma shows the essential condition for the desired bijection between RAsc_n and $\mathsf{RMatrices}_n$.

Proposition 3.6. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{Asc}_n$ *and* $M = f_{\mathsf{AM}}(\mathbf{a})$ *with* $\dim(M) = d$. *There will be a zero on the diagonal of* M *if and only if there exists* $i \in [1, n-1]$ *such that* $a_i < a_{i+1} \leq \mathsf{asc}(a_1, \ldots, a_i)$.

Proof. To show this we will consider Definition 2.1 and how the matrix entries change during construction with respect to the rules AM1, AM2 and AM3.

Suppose that there exists $i \in [1, n-1]$ with $a_i < a_{i+1} \leq \mathsf{asc}(a_1, \ldots, a_i)$. Recall from the second paragraph of this section that $a_i = \mathsf{mindex}(M^{(i)}) - 1$ and $\mathsf{asc}(a_1, \ldots, a_i) = \dim(M^{(i)}) - 1$. So when constructing $M^{(i+1)}$ from $M^{(i)}$, AM3 applies, and $M^{(i+1)}_{1+a_{i+1}, 1+a_{i+1}} = 0$. This entry is not in the rightmost column of the matrix and cannot therefore be increased by any subsequent applications of AM1 or AM2. Depending on the subsequent values in the ascent sequence, if AM3 is used then the 0 at position $(1 + a_{i+1}, 1 + a_{i+1})$ may be permuted amongst the diagonal entries but will never again be in the rightmost column, and therefore never again accessible to change.

For the converse, suppose that for all $i \in [1, n-1]$,

$$a_{i+1} \in [0, a_i] \cup \{1 + \mathsf{asc}(a_1, \ldots, a_i)\}. \tag{3.6}$$

The procedure for constructing M will begin with $M^{(1)} = (1)$, a 1-by-1 matrix. Because of (3.6), only AM1 and AM2 will be needed to construct the subsequent $M^{(i)}$. Since AM1 and AM2 both preserve the property of all diagonal entries being strictly positive, M will also have that desired property. □

We next provide a similarly essential ingredient for the desired bijection between RPosets and RMatrices.

Proposition 3.7. *Let $P \in \mathsf{Posets}_n$ and let $M = f_{\mathsf{PM}}(P)$ with $dim(M) = \ell(P) + 1$. All the entries on the diagonal of M will be non-zero if and only if P has a chain of length $\ell(P)$.*

Proof. Consider the construction of M from P as given in Definition 2.8, especially the intermediate matrix M' given by

$$M'_{ij} = L_{i-1} \cap (D_j \backslash D_{j-1})$$

for all $i, j \in [1, \ell(P) + 1]$. Suppose $m_{ii} \neq 0$ for all $i \in [1, \ell(P) + 1]$. We have that $m_{ii} \neq 0$ if and only if $M'_{ii} \neq \emptyset$, which is equivalent to

$$L_{i-1} \cap (D_i \backslash D_{i-1}) \neq \emptyset. \tag{3.7}$$

In other words, there exists at least one element $w_i \in L_{i-1} \cap (D_i \backslash D_{i-1})$ for all $1 \leq i \leq \ell(P) + 1$. Therefore,

$$w_1 \prec_P w_2 \prec_P \cdots \prec_P w_{\ell(P)}, \tag{3.8}$$

and so P has a chain of length $\ell(P)$.

To show that if P has a chain of length $\ell(P)$ then M has only non-zero diagonal entries, it suffices to show that the condition in (3.8) implies that in (3.7) for all i. So

consider the construction of P from M, as given in Definition 2.10. A chain in P of length $\ell(P)$ must arise from a sequence of $\ell(P)+1$ nonempty entries in M', each strictly southeast of the previous one. But since $\dim(M') = \ell(P)+1$ by definition of f_{PM}, these nonempty entires in M' must all be along the diagonal, yielding (3.7) for all i. □

Combining the previous two propositions gives the main result of this section, which shows that RAsc, RMatrices, RPosets and RPerms are all in bijection, and the bijections we need are exactly the restricted versions of the classical ones. Part (c) of the theorem was already proved as [3, Prop. 10].

Corollary 3.8. *The classical bijections among* Asc_n, $\mathsf{Matrices}_n$, Posets_n *and* Perms_n *restrict to bijections among* RAsc_n, $\mathsf{RMatrices}_n$, $\mathsf{RPosets}_n$ *and* RPerms_n, *specifically,*

(a) f_{AM} *maps* RAsc_n *bijectively to* $\mathsf{RMatrices}_n$;
(b) f_{PM} *maps* $\mathsf{RPosets}_n$ *bijectively to* $\mathsf{RMatrices}_n$;
(c) *[3]* f_{AS} *maps* RAsc_n *bijectively to* RPerms_n.

Proof. Part (a) follows from Proposition 3.6, the definitions of RAsc_n and $\mathsf{RMatrices}_n$, and the fact that $f_{\mathsf{AM}} : \mathsf{Asc}_n \to \mathsf{Matrices}_n$ is a bijection.

Similarly, (b) follows from Proposition 3.7, the definitions of $\mathsf{RPosets}_n$ and $\mathsf{RMatrices}_n$, and the fact that $f_{\mathsf{PM}} : \mathsf{Posets}_n \to \mathsf{Matrices}_n$ is a bijection. □

4. Poset duality under the bijections

If a poset P is (2+2)-free, then it is clear that the dual poset P^* obtained by reversing all its inequalities is also (2+2)-free. An open question in [11] asks how $\mathbf{a}$ is related to $\mathbf{a}^*$, where $\mathbf{a}$ and $\mathbf{a}^*$ are the ascent sequences corresponding to P and P^* respectively. While this question appears intractable for general (2+2)-free posets, in this section we answer it for RPosets. In addition, we extend the answer to give the corresponding notion of duality for RPerms. Combined with the duality result for RMatrices given by [9, Theorem 10], we get a complete understanding of how poset duality acts on our four R-families according to our bijections. In fact, one major motivation for our restriction to the R-families is their amenability to adopting an analogue of poset duality.

In view of Corollary 3.8, we can abuse notation by using the same f notation for our bijections even though our domains will now be R-families as opposed to the domains of Asc, Matrices, Posets and Perms that we had before.

Definition 4.1. Let $f : \mathsf{RPosets} \to \mathsf{Struct}$ be a bijection where Struct is a collection of objects. Given $P \in \mathsf{RPosets}$ with $f(P) = s$, we write s^* for the unique object $f(P^*)$, and we call s^* the *dual* of s according to f.

Example 4.2. As a first example, we consider the dual of an element $M = (m_{ij})$ of RMatrices according to the bijection f_{PM}. Define $\mathsf{flip}(M)$ to be the reflection of M through

its antidiagonal, i.e., if $\dim(M) = d$, then $\mathsf{flip}(M)_{ij} = m_{d+1-j,\,d+1-i}$. Observe that $M \in \mathsf{RMatrices}_n$ if and only if $\mathsf{flip}(M) \in \mathsf{RMatrices}_n$. Theorem 10 from [9] states that $M^* = \mathsf{flip}(M)$.

4.1. Duality for ascent sequences

We will use Example 4.2 as a basis for determining the dual of an element of RAsc according to f_{PA}.

Theorem 4.3. *Let* $\mathbf{a} \in \mathsf{RAsc}_n$. *By Lemma 3.4, we have*

$$\mathbf{a} = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1\,d}}, \ldots, 0^{m_{1d}}) \tag{4.1}$$

where $M = f_{\mathsf{AM}}(\mathbf{a})$. *The dual ascent sequence* $\mathbf{a}^*$ *according to* f_{PA} *is given by*

$$\begin{aligned}\mathbf{a}^* = (&0^{m_{dd}}, 1^{m_{d-1\,d-1}}, 0^{m_{d-1\,d}}, 2^{m_{d-2\,d-2}}, 1^{m_{d-2\,d-1}}, 0^{m_{d-2\,d}},\\ &\ldots, (d-1)^{m_{11}}, (d-2)^{m_{12}}, \ldots, 0^{m_{1d}}).\end{aligned} \tag{4.2}$$

Proof. Let $P = f_{\mathsf{AP}}(\mathbf{a})$. We first observe that, since $f_{\mathsf{PA}} = f_{\mathsf{MA}} \circ f_{\mathsf{PM}}$, we get $\mathbf{a}^* = f_{\mathsf{PA}}(P^*) = f_{\mathsf{MA}}(\mathsf{flip}(M))$. Considering the definition of $\mathsf{flip}(M)$ and applying Lemma 3.4 yields the result. □

Example 4.4. If $\mathbf{a} = (0,0,1,2,0)$, then $M = \begin{pmatrix} 2&0&1\\ 0&1&0\\ 0&0&1 \end{pmatrix}$. Thus $\mathsf{flip}(M) = \begin{pmatrix} 1&0&1\\ 0&1&0\\ 0&0&2 \end{pmatrix}$, and so $\mathbf{a}^* = (0,1,2,2,0)$.

There is an alternative way to construct $\mathbf{a}^*$ that avoids the need to write it out in the form (4.1). This alternative uses a map pan that takes *any* sequence of numbers as its input and returns an ascent sequence $\mathsf{pan}(\mathbf{a})$, which we call the *panorama* of $\mathbf{a}$. First, we need a preliminary definition.

Definition 4.5. Given a sequence $(a_1, a_2, \ldots, a_k)$ of real numbers, the *view* v_i of the element a_i is defined in the following recursive fashion:

(a) if j is the minimum index greater than i such that $a_j > a_i$, then define $v_i = v_j + 1$;
(b) if no such j exists then $v_i := 0$.

We can think of the view v_i as counting left-to-right maxima starting at the entry a_i. For example, the sequence $(0,1,2,0,4,3,1,2)$ has $(3,2,1,1,0,0,1,0)$ as its sequence of views.

Definition 4.6. Let $\mathbf{a}$ be a sequence of real numbers. We construct the *panorama sequence* $\mathsf{pan}(\mathbf{a})$ of $\mathbf{a}$ in the following manner. Suppose the maximum value of $\mathbf{a}$ occurs in positions

$i_1, i_2, \ldots, i_j$. Begin $\mathsf{pan}(\mathbf{a})$ with the sequence of views $(v_{i_1}, v_{i_2}, \ldots, v_{i_j})$. Now continue this process with the next highest value in $\mathbf{a}$, concatenating the corresponding view values onto the right end of $\mathsf{pan}(\mathbf{a})$. Repeat this process until the view of every element of $\mathbf{a}$ has been added to $\mathsf{pan}(\mathbf{a})$.

Example 4.7. For $\mathbf{a} = (0, 1, 2, 0, 4, 3, 1, 2)$ as above, $\mathsf{pan}(\mathbf{a}) = (0, 0, 1, 0, 2, 1, 3, 1)$.

As promised, we get the following alternative construction of $\mathbf{a}^*$.

Proposition 4.8. *Let* $\mathbf{a} \in \mathsf{RAsc}$ *and let* $\mathbf{a}^*$ *be its dual according to* f_{PA}. *Then* $\mathbf{a}^* = \mathsf{pan}(\mathbf{a})$.

Proof. Consider $\mathsf{pan}(\mathbf{a})$ with $\mathbf{a}$ as given in (3.5). By Corollary 3.8(a) and by definition of $\mathsf{RMatrices}_n$, we know that the diagonal entries m_{jj} are all positive. With this observation in mind, we see that the left-to-right maxima that determine the view of the element a_i come from those entries in $\mathbf{a}$ of the form $j^{m_{j+1\,j+1}}$ with $j > i$ that appear to the right of a_i. It follows that the panorama $\mathsf{pan}(\mathbf{a})$ is exactly the sequence given in (4.2), as required. □

Observe that $\mathsf{pan}(\mathbf{a})$ in Example 4.7 is an ascent sequence, while Proposition 4.8 implies the same is true for all $\mathbf{a} \in \mathsf{RAsc}$; this is no coincidence. In fact, one can show that $\mathsf{pan}(\mathbf{a})$ is an ascent sequence for *any* sequence of real numbers $\mathbf{a}$. One method of proof is to show $\mathsf{pan}(\mathbf{a})$ has a stronger property, originally defined in [15], which we recall now. Another reason for introducing this natural stronger property is that it will be useful in the next section.

Definition 4.9. A *restricted growth function (RGF)* is a sequence $\mathbf{a}$ of non-negative integers such that each $j > 0$ that appears in $\mathbf{a}$ is preceded by an appearance of $j - 1$. Equivalently, for all $j > 0$ that appear in $\mathbf{a}$, the first appearance of j is preceded by an appearance of every i satisfying $0 \leq i < j$.

For example, $(0, 1, 0, 2, 1, 3)$ is an RGF whereas $(0, 1, 0, 1, 3, 2)$ is not.

Lemma 4.10. $\mathsf{RAsc} \subseteq \mathsf{RGF} \subseteq \mathsf{Asc}$ *and both containments are strict.*

Proof. Both containments are trivial for sequences of length $n = 1$, so assume $n > 1$ and let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{RAsc}_n$. By (3.4), the first time $j > 0$ appears, it must take the form $1 + \mathsf{max}(a_1, \ldots, a_{i-1})$ for some i. Thus $j - 1$ appears before j, and $\mathbf{a}$ is an RGF.

Next let $\mathbf{a} = (a_1, \ldots, a_n)$ be an RGF, and let $l(j)$ denote the position in $\mathbf{a}$ of the leftmost appearance of the number j, when such a position exists. Note that to prove that $\mathbf{a} \in \mathsf{Asc}$, it suffices to show that $a_{l(j)} \leq 1 + \mathsf{asc}(a_1, \ldots, a_{l(j)-1})$ for each j. By definition of RGFs, $l(0) < l(1) < \cdots < l(m)$, where m is the maximum value in $\mathbf{a}$. Hence position $l(i) - 1$ is always an ascent for $1 \leq i \leq m$. Thus $\mathsf{asc}(a_1, \ldots, a_{l(j)-1}) \geq j - 1$, from which we conclude $a_{l(j)} = j \leq 1 + \mathsf{asc}(a_1, \ldots, a_{l(j)-1})$, as required.

To see the strict containments, the shortest examples are $(0,1,0,1) \in \mathsf{RGF} \setminus \mathsf{RAsc}$ while $(0,1,0,1,3) \in \mathsf{Asc} \setminus \mathsf{RGF}$. □

Remark 4.11. Clearly, generalised ballot sequences (also known as Yamanouchi words) are RGFs, and so we have (slightly) generalised the first author's result [8] that generalised ballot sequences are ascent sequences.

Remark 4.12. We leave it as an exercise for the reader to justify our earlier assertion that $\mathsf{pan}(\mathbf{a})$ is an RGF and hence an ascent sequence for any sequence of real numbers $\mathbf{a}$. (In fact, one can constructively prove the converse that every RGF is a panorama sequence.) A harder exercise is to show there is another way in which $\mathsf{pan}(\mathbf{a})$ is "nicer" than $\mathbf{a}$: if $\mathbf{a}$ is an RGF, then $\mathsf{pan}(\mathbf{a}) \in \mathsf{RAsc}$. Consequently, $\mathsf{pan}^2(\mathbf{a}) \in \mathsf{RAsc}$ for any sequence $\mathbf{a}$ of real numbers. Moreover, this latter fact combined with Proposition 4.8 tells us that $\mathsf{pan}^2(\mathbf{a}) = \mathbf{a}$ if and only if $\mathbf{a} \in \mathsf{RAsc}$. This last observation gives an alternative definition of RAsc.

As we mentioned, we have not been able to answer in full the open question from [11] by determining $\mathbf{a}^*$ for all $\mathbf{a} \in \mathsf{Asc}$. However, we assert that RAsc is in some sense the largest subset that behaves nicely with respect to poset duality. Indeed, since $(\mathbf{a}^*)^* = \mathbf{a}$ by definition, the fact that $\mathsf{pan}^2(\mathbf{a}) = \mathbf{a}$ if and only if $\mathbf{a} \in \mathsf{RAsc}$ tells us that $\mathbf{a}^*$ can be as simple as $\mathsf{pan}(\mathbf{a})$ if and only if $\mathbf{a} \in \mathsf{RAsc}$.

4.2. Duality for permutations

For our one remaining notion of duality, we use the definition of the dual of an ascent sequence to determine the dual π^* of an element π of $\mathsf{RPerms} = \mathcal{S}_n(3\overline{1}52\overline{4})$ according to $f_{\mathsf{PS}} := f_{\mathsf{AS}} \circ f_{\mathsf{PA}}$.

Our definition of π^* requires the use of what are perhaps the three best known involutions on a permutation $\pi = (\pi_1, \ldots, \pi_n)$: its inverse π^{-1}, its *reverse* $\mathrm{rev}(\pi) := (\pi_n, \ldots, \pi_1)$, and its *complement* $\mathrm{comp}(\pi) := (n+1-\pi_1, \ldots, n+1-\pi_n)$. Together, these involutions allow us to state the result with the most technical proof of this paper.

Theorem 4.13. *Let $\pi \in \mathcal{S}_n(3\overline{1}52\overline{4}) = \mathsf{RPerms}$. The dual permutation according to f_{PS} is given by $\pi^* = (\mathrm{comp}(\mathrm{rev}(\pi)))^{-1}$.*

Proof. Let $\mathbf{a} \in \mathsf{RAsc}_n$ and let $\pi = f_{\mathsf{AS}}(\mathbf{a}) \in \mathcal{S}_n(3\overline{1}52\overline{4})$. See Example 4.14 below for an example pertinent to the key elements of this proof; we will use the symbol ✓ in this proof to denote paragraphs or single statements that are demonstrated in the example. Theorem 4.3 tells us how to construct $\mathbf{a}^*$ from $\mathbf{a}$: if

$$\mathbf{a} = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1\,d}}, \ldots, 0^{m_{1d}}) \quad (4.3)$$

then $\mathbf{a}^*$ is given by

$$\mathbf{a}^* = (0^{m_{dd}}, 1^{m_{d-1\,d-1}}, 0^{m_{d-1\,d}}, 2^{m_{d-2\,d-2}}, 1^{m_{d-2\,d-1}}, 0^{m_{d-2\,d}}, \ldots, (d-1)^{m_{11}}, (d-2)^{m_{12}}, \ldots, 0^{m_{1d}}).$$

Here, $\mathbf{a} = f_{\mathsf{MA}}(M)$ and $\mathbf{a}^* = f_{\mathsf{MA}}(\mathsf{flip}(M))$ as shown in the proof of Theorem 4.3. ✓.

In Definition 2.12, we saw a recursive definition of $f_{\mathsf{AS}}(\mathbf{a})$. However, as shown in [3, Cor. 9], when $\mathbf{a} \in \mathsf{RAsc}$, we have the following equivalent but simpler definition. Let $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathsf{RAsc}_n$ and let k be the largest value in this sequence. Let $W_i(\mathbf{a})$ be the list of all positions $j \in [1, n]$ such that $\mathbf{a}_j = i$ written in decreasing order. Define

$$f_{\mathsf{AS}}(\mathbf{a}) = W_0(\mathbf{a})W_1(\mathbf{a})\ldots W_k(\mathbf{a}) = \bigoplus_{i=0}^{k} W_i(\mathbf{a}) \tag{4.4}$$

to be the concatenation of these lists ✓.

The proof will rely on several different ways to sum the m_{ab} values which we now identify. The sums use three different ways to traverse the upper-triangular entries of (m_{ij}).

T traversal $(1,1) \to (2,2) \to (1,2) \to (3,3) \to (2,3) \to (1,3) \to \ldots \to (2,d) \to (1,d)$.
R traversal $(1,d) \to (2,d) \to \ldots \to (d,d) \to (1,d-1) \to \ldots \to (d-1,d-1) \to \ldots \to (1,1)$.
S traversal $(1,d) \to (1,d-1) \to \ldots \to (1,1) \to (2,d) \to \ldots (2,2) \to \ldots \to (d-1,d) \to (d-1,d-1) \to (d,d)$.

Let T_{ij} be the sum of m_{ab} using T traversal until we reach the pair $(a,b) = (i,j)$. Let R_{ij} be the sum of m_{ab} using R traversal until we reach the pair $(a,b) = (i,j)$. Let S_{ij} be the sum of m_{ab} using S traversal until we reach the pair $(a,b) = (i,j)$. The quantities T'_{ij}, R'_{ij}, and S'_{ij} are those where one sums m'_{ab}'s in place of m_{ab}'s, where the two are related via $m'_{ij} = m_{d+1-j,\,d+1-i}$. In other words, the primed versions of the sums are obtained by traversing in the prescribed orders but over the entries of $\mathsf{flip}(M)$ instead of M ✓. Notice that we have the following identities:

$$T_{ij} + R_{ij} = n + m_{ij} \tag{4.5}$$

$$T'_{ij} + R'_{ij} = n + m'_{ij} \tag{4.6}$$

$$S'_{ij} = R_{d+1-j,\,d+1-i}\ . \tag{4.7}$$

Let us use the notation $[x]_a$ for the list $(x, x-1, \ldots, x-a+1)$, and define $[x]_0$ to be the empty list. Applying $f_{\mathsf{AS}}(\mathbf{a})$ we find that $W_{i-1}(\mathbf{a}) = [T_{id}]_{m_{id}} \cdots [T_{ii}]_{m_{ii}}$. This is due to the T traversal matching the order of the powers m_{ij} appearing in (4.3). Thus

$$\pi = f_{\mathsf{AS}}(\mathbf{a}) = \bigoplus_{i=1}^{d} \bigoplus_{j=d}^{i} [T_{ij}]_{m_{ij}} \tag{4.8}$$

where $\oplus$ denotes left-to-right concatenation of sequences, and where the index j in the inner concatenation runs from d down to i ✓. Similarly, working with $\mathsf{flip}(M)$ instead of M, we have

$$\pi^* = f_{\mathsf{AS}}(\mathbf{a}^*) = \bigoplus_{i=1}^{d} \bigoplus_{j=d}^{i} [T'_{ij}]_{m'_{ij}} \ \checkmark \tag{4.9}$$

We now have bona fide expressions for both π and π^*. It remains to show that they satisfy the equation stated in the theorem.

The value $\pi(a)$ is obtained in the following way:

$$\pi(a) = T_{ij} + S_{ij} - m_{ij} + 1 - a \tag{4.10}$$

where (i, j) is the unique pair such that $a \in (S_{ij} - m_{ij}, S_{ij}]$ ✓. Indeed, to determine $\pi(a)$, we first have to determine which sequence $[T_{ij}]_{m_{ij}}$ from (4.8) will include the ath entry of π. The order from left-to-right in which the $[T_{ij}]_{m_{ij}}$ appear matches the S traversal, which is why we pick (i, j) in the stated way. Next we need the $\pi(a)$ values for $a \in (S_{ij} - m_{ij}, S_{ij}]$ to give the sequence $[T_{ij}]_{m_{ij}}$. Indeed, the smallest such a is $S_{ij} - m_{ij} + 1$, which gives $\pi(a) = T_{ij}$ in (4.10). As a increases all the way to S_{ij}, we get all the $\pi(a)$ values decreasing all the way to $T_{ij} - m_{ij} + 1$, as required. This argument also makes it clear that the pair (i, j) is also the unique pair such that $b = \pi(a) \in (T_{ij} - m_{ij}, T_{ij}]$ ✓.

The statement of the theorem is equivalent to showing $\pi^*(a) = b$ if and only if $\pi(n+1-b) = n+1-a$. Suppose that $\pi^*(a) = b$. Then we have $\pi^*(a) = T'_{ij} + S'_{ij} - m'_{ij} + 1 - a$ where (i, j) is the unique pair such that $a \in (S'_{ij} - m'_{ij}, S'_{ij}]$. Equivalently (i, j) is the unique pair such that $b = \pi^*(a) \in (T'_{ij} - m'_{ij}, T'_{ij}]$. The rest of this proof will start with the statement that

$$T'_{ij} + S'_{ij} - m'_{ij} + 1 - a = b, \tag{4.11}$$

where (i, j) is the unique pair such that $b \in (T'_{ij} - m'_{ij}, T'_{ij}]$ and consist of manipulations to remove all primed terms with a view to recovering a statement that is equivalent to $\pi(n + 1 - b) = n + 1 - a$.

From (4.5)–(4.7), we have $T'_{ij} - m'_{ij} = n - S_{d+1-j,\, d+1-i}$, and

$$S'_{ij} = R_{d+1-j,\, d+1-i} = n + m_{d+1-j,\, d+1-i} - T_{d+1-j,\, d+1-i} \ .$$

The above equality (4.11) is therefore equivalent to

$$n - S_{d+1-j,\, d+1-i} + (n + m_{d+1-j,\, d+1-i} - T_{d+1-j,\, d+1-i}) + 1 - b = a.$$

Subtract both sides of this equation from $n + 1$ to yield

$$T_{d+1-j,\, d+1-i} + S_{d+1-j,\, d+1-i} - m_{d+1-j,\, d+1-i} + 1 - (n + 1 - b) = n + 1 - a$$

where, again, (i,j) is the unique pair such that $b \in (T'_{ij} - m'_{ij}, T'_{ij}]$. Notice that

$$\begin{aligned}
& b \in (T'_{ij} - m'_{ij}, T'_{ij}] \\
\iff\ & T'_{ij} - m'_{ij} < b \leq T'_{ij} \\
\iff\ & n+1-T'_{ij} + m'_{ij} > n+1-b \geq n+1-T'_{ij} \\
\iff\ & n-T'_{ij} + m'_{ij} \geq n+1-b > n-T'_{ij} \\
\iff\ & R'_{ij} \geq n+1-b > R'_{ij} - m'_{ij} \\
\iff\ & S_{d+1-j,\,d+1-i} \geq n+1-b > S_{d+1-j,\,d+1-i} - m_{d+1-j,\,d+1-i} \\
\iff\ & n+1-b \in (S_{d+1-j,\,d+1-i} - m_{d+1-j,\,d+1-i}, S_{d+1-j,\,d+1-i}].
\end{aligned}$$

Thus we now have that

$$T_{d+1-j,\,d+1-i} + S_{d+1-j,\,d+1-i} - m_{d+1-j,\,d+1-i} + 1 - (n+1-b) = n+1-a$$

where (i,j) is the unique pair such that

$$n+1-b \in (S_{d+1-j,\,d+1-i} - m_{d+1-j,\,d+1-i}, S_{d+1-j,\,d+1-i}].$$

Changing the variables to $I := d+1-j$ and $J := d+1-i$, we get

$$T_{I,J} + S_{I,J} - m_{I,J} + 1 - (n+1-b) = n+1-a$$

where (I,J) is the unique pair such that $n+1-b \in (S_{I,J} - m_{I,J}, S_{I,J}]$. By comparing with (4.10), we see that we have arrived at a statement that is equivalent to $\pi(n+1-b) = n+1-a$, as required. □

Example 4.14. Let

$$\mathbf{a} = (0,0,0,1,1,0,2,2,2,2,0,3,1,1)$$

and so

$$\pi = f_{\mathsf{AS}}(\mathbf{a}) = (11,6,3,2,1\,|\,14,13,5,4\,|\,10,9,8,7\,|\,12)$$

where the vertical bars separate each $W_i(\mathbf{a})$ from $W_{i+1}(\mathbf{a})$. We have $\mathbf{a} = f_{\mathsf{MA}}(M)$ where

$$M = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

by (4.3). Thus

$$\mathsf{flip}(M) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

from which we can read off

$$\mathbf{a}^* = (0, 1, 1, 1, 1, 2, 2, 0, 0, 3, 3, 3, 2, 1),$$

yielding

$$\pi^* = (9, 8, 1 \,|\, 14, 5, 4, 3, 2 \,|\, 13, 7, 6 \,|\, 12, 11, 10)$$

which does indeed equal $(\mathrm{comp}(\mathrm{rev}(\pi)))^{-1}$.

From here on, we give examples of other items appearing in the proof. First, we have $T_{33} = 3 + 2 + 1 + 4 = 10$, $R_{33} = 2 + 1 + 1 + 4 = 8$, $S_{33} = 1 + 1 + 3 + 2 + 2 + 4 = 13$, $T'_{33} = 1 + 4 + 2 = 7$, $R'_{33} = 1 + 1 + 3 + 2 + 2 = 9$, and $S'_{33} = 2 + 1 + 1 + 4 + 1 + 2 = 11$. Next, consistent with (4.8) and (4.9), we have

$$\pi = [14]_0[11]_1[6]_1[3]_3 \mid [14]_2[10]_0[5]_2 \mid [12]_0[10]_4 \mid [12]_1,$$

and

$$\pi^* = [14]_0[9]_2[5]_0[1]_1 \mid [14]_1[7]_0[5]_4 \mid [13]_1[7]_2 \mid [12]_3.$$

Finally, let us give some examples of (4.10), starting with $a = 10$. To determine $\pi(10)$ we find that $10 \in (S_{33} - m_{33}, S_{33}] = (9, 13]$. We thus obtain $\pi(10) = T_{33} + S_{33} - m_{33} + 1 - 10 = 10 + 13 - 4 + 1 - 10 = 10$. As a increases to $11, 12, 13$, the only term that changes on the right-hand side of (4.10) is a, and we get $9, 8, 7$ for the respectively values of $\pi(a)$. Notice that the four values we obtained here for $\pi(a)$ are exactly the elements of $[T_{33}]_{m_{33}}$.

5. A Catalan restriction and series-parallel posets

In this section, we consider subsets of the R-families whose cardinalities are given by the Catalan numbers; we thus use the C prefix for naming these subsets. The study of these subsets is also motivated by the results of applying our bijections to these subsets, which allow us to draw connections between some natural families: pattern-avoiding ascent sequences, pattern-avoiding permutations, and series-parallel (2+2)-free posets. Pattern avoidance in ascent sequences is studied from an enumerative perspective in [12].

A sequence is said to be *abab*-avoiding if it is both 0101-avoiding and 1010-avoiding.

Proposition 5.1. *Let* $\mathbf{a}$ *be a sequence of non-negative integers. The following are equivalent:*

(a) **a** *is a* 101*-avoiding ascent sequence;*
(b) **a** *is an abab-avoiding ascent sequence;*
(c) **a** *is a* 0101*-avoiding ascent sequence;*
(d) **a** *is an abab-avoiding RGF.*

Consequently, the number of sequences of length n satisfying these conditions is the Catalan number C_n.

Proof. That (a) implies (b) is immediate, as is the implication from (b) to (c). That (a) and (c) are equivalent is [12, Theorem 2.5]. Lemma 4.10 gives that (d) implies (b), while [12, Lemma 2.4] gives that both (a) or (c) imply (d).

The last assertion is shown in [12, Theorem 2.5] and also follows from the fact that the number of sequences **a** of length n satisfying (d) is shown to be C_n in [21, Exer. 88]. □

Definition 5.2. Let CAsc_n denote the set of those sequences of length n consisting of non-negative integers that satisfy the conditions of Proposition 5.1.

As an example, of the 15 ascent sequences of length 4, the only one containing *abab* is 0101. Thus $14 = C_4 = |\mathsf{CAsc}_4|$.

It is certainly not the case that $\mathsf{CAsc}_n = \mathsf{RAsc}_n$ since $01021 \in \mathsf{RAsc}_n \setminus \mathsf{CAsc}_n$. However, we do have the following relationship.

Proposition 5.3. $\mathsf{CAsc}_n \subseteq \mathsf{RAsc}_n$.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an ascent sequence that is not in RAsc_n. We will show that **a** violates one of the characterizations of CAsc_n from Proposition 5.1. By definition of RAsc_n, there exists i such that $a_{i-1} < a_i < 1 + \mathsf{asc}(a_1, \ldots, a_{i-1})$. If $a_i = a_j$ for some $j < i - 1$, then $a_j a_{i-1} a_i$ is an occurrence of 101 in **a**, violating Proposition 5.1(a). If $a_j > a_i$ for some $j < i - 1$, then the RGF property implied by Proposition 5.1(d) would again imply the appearance of the subsequence $a_i a_{i-1} a_i$ in **a**. Thus we can assume that $a_j < a_i$ for all j with $1 \le j \le i - 1$.

If $\mathbf{a} \in \mathsf{CAsc}_n$, we know **a** is an RGF, so the first appearance a_j of any non-zero element of $(a_1, \ldots, a_{i-1})$ results in position $j - 1$ contributing $+1$ to $\mathsf{asc}(a_1, \ldots, a_{i-1})$. Because $a_j < a_i$, the combined contribution of these first appearances to $\mathsf{asc}(a_1, \ldots, a_{i-1})$ is at most $a_i - 1$. Since $a_i - 1 < \mathsf{asc}(a_1, \ldots, a_{i-1})$, there must then be a value a_j that results in at least two ascents, i.e., there must exist j and k such that $1 < j < k \le i - 1$ with $a_j = a_k$ such that both $j - 1$ and $k - 1$ are ascents. Consequently, $a_j a_{k-1} a_k$ is an occurrence of 101 in **a**, again violating Proposition 5.1(a). □

We next turn to determining the image of CAsc_n under our bijections. We begin with posets since this case is perhaps the most interesting. Recall from Fig. 1.1 that $\mathsf{CPosets}_n$ denotes the series-parallel posets in Posets_n. In other words, $\mathsf{CPosets}_n$ consists of those posets with n elements that are both $(2 + 2)$-free and N-free.

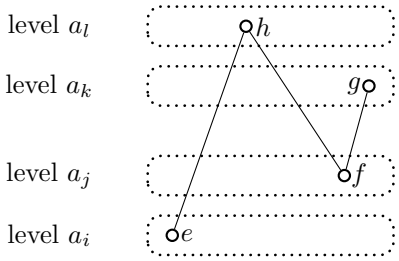

Fig. 5.1. For ease of reference of notation, we give one possible configuration of the N-subposet of P from the proof of Proposition 5.4. This is just one of two possible configurations, since there is no mathematical reason why level a_i should appear below level a_j.

Proposition 5.4. *$f_{\mathsf{AP}} : \mathsf{CAsc}_n \to \mathsf{CPosets}_n$, and f_{AP} is a bijection.*

Proof. Since $f_{\mathsf{AP}} : \mathsf{Asc}_n \to \mathsf{Posets}_n$ is a bijection and since CAsc_n and $\mathsf{CPosets}_n$ both have size C_n by Proposition 5.1 and [21, Exer. 182], it suffices to prove that $f_{\mathsf{AP}}(\mathbf{a}) \in \mathsf{CPosets}_n$ for any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{CAsc}_n$. So suppose $f_{\mathsf{AP}}(\mathbf{a}) = P$ contains an N. Since $\mathbf{a} \in \mathsf{RAsc}_n$ by Proposition 5.3, we know that only rules AP1 and AP2 from Definition 2.5 are used in constructing P. Let i, j, k, l be distinct positions in $\mathbf{a}$ that result (under f_{AP}) in the elements e, f, g, h, respectively, of P which form an N as, for example, in Fig. 5.1. Since $\mathbf{a} \in \mathsf{CAsc}_n \subseteq \mathsf{RAsc}_n$, applying [3, Lem. 7] tells us that e, f, g, h appear at levels a_i, a_j, a_k, a_l respectively of P. Comparing strict downsets, we see that $a_l > a_k > a_j, a_i$. There are two cases to consider depending on the relative values of the positions k and l.

We first consider the case $l > k$. Since $\mathbf{a}$ is an RGF by Proposition 5.1, in constructing P, the first time an element was added at level a_l it must have been added as a top element using the rule AP2. But since $h \not> _P g$, there must be some element h' at level a_l that was added before g (and so also before h). Thus we have $a_l a_k a_l$ appearing in $\mathbf{a}$, contradicting Proposition 5.1(a).

It remains to consider the case $l < k$. With the same reasoning as the previous paragraph, there exists an element g' at level a_k that was added as a top element. Since $e \notin D_g$ and g and g' are at the same level, we know $e \notin D_{g'}$. Thus g' was added to P before e. On the other hand, since $h >_P e$, we know h was added to P after e, and hence so was g. Thus we have $a_k a_i a_k$ appearing in $\mathbf{a}$, again contradicting Proposition 5.1(a). □

We follow Jelínek [25] to define the family $\mathsf{CMatrices}_n$.

Definition 5.5. An SE-pair of a matrix $M \in \mathsf{Matrices}$ is a pair of non-zero entries m_{ij} and $m_{i'j'}$ such that $i < i'$, $j < j'$ and $i' \leq j$. We say that M is *SE-free* if it contains no SE-pair.

In [25, Lem. 1.2], Jelínek shows that $P \in \mathsf{Posets}$ is series-parallel if and only if $f_{\mathsf{PM}}(P)$ is SE-free.

Definition 5.6. Define $\mathsf{CMatrices}_n$ to be the subset of $\mathsf{Matrices}_n$ consisting of those elements that are SE-free.

Finally, we determine the C-family for permutations. The answer is quite appealing, namely $\mathsf{CPerms} = \mathcal{S}_n(231)$.

Remark 5.7. At this point, it is worth clarifying the containment relations of Fig. 1.1. We already showed in Proposition 5.3 that if $\mathbf{a} \in \mathsf{Asc}$, then $\mathbf{a}$ being 101-avoiding automatically implies that $\mathbf{a}$ is self-modified. That the analogous implications hold for the other three C-families follows from Theorem 5.9, but can also be checked directly as follows:

- If $\pi \in \mathcal{S}_n(231)$ then $\pi \in \mathcal{S}_n(3\overline{1}52\overline{4})$ by definition of $\mathcal{S}_n(3\overline{1}52\overline{4})$.
- If $M \in \mathsf{Matrices}$ is a k-by-k matrix that is SE-free, then it must have only positive diagonal entries. To see this, suppose $m_{ii} = 0$ for some i. By definition of $\mathsf{Matrices}$, we have $2 \leq i \leq k-1$, and there must exist an SE-pair formed by m_{ai} and m_{ib} where $1 \leq a < i$ and $i < b \leq k$.
- We leave it as an exercise to show that if $P \in \mathsf{Posets}$ is series-parallel, then P has a chain of length $\ell(P)$.

Proposition 5.8. *f_{AS} maps CAsc_n bijectively to CPerms.*

Proof. The definition of CAsc_n and the fact that f_{AS} is a bijection tells us that $|f_{\mathsf{AS}}(\mathsf{CAsc}_n)| = C_n$. As is well known, C_n is also the cardinality of CPerms. Thus it suffices to show that $\mathsf{CPerms} \subseteq f_{\mathsf{AS}}(\mathsf{CAsc}_n)$. Crucial to our proof is that, from Propositions 5.1(a) and 5.3, we have $\mathsf{CAsc}_n = \mathsf{RAsc}_n(101)$.

Corollary 3.8(c) and the definition of RPerms give $f_{\mathsf{AS}}(\mathsf{RAsc}_n) = \mathsf{RPerms} \supseteq \mathsf{CPerms}$. Thus if $f_{\mathsf{AS}}(\mathbf{a}) \in \mathsf{CPerms}$, then $\mathbf{a} \in \mathsf{RAsc}_n$. We wish to show the more refined fact that $\mathbf{a} \in \mathsf{CAsc}_n$, i.e., that $\mathbf{a}$ avoids 101. So suppose that $\mathbf{a}$ contains a 101 pattern as $(a_i, a_j, a_k) = (d, c, d)$. Then in $f_{\mathsf{AS}}(\mathbf{a})$, using the definition of $f_{\mathsf{AS}}(\mathbf{a})$ given in (4.4), j will be in $W_c(\mathbf{a})$ which is to the left of $W_d(\mathbf{a})$. In $W_d(\mathbf{a})$, k will be to the left of i since $k > i$. Thus (j, k, i) will be a 231 pattern in $f_{\mathsf{AS}}(\mathbf{a})$, which is the desired contradiction. □

We now have all the necessary ingredients to compile the following "Catalan" refinement of Corollary 3.8

Corollary 5.9. *The classical bijections among Asc_n, $\mathsf{Matrices}_n$, Posets_n and Perms restrict to bijections among CAsc_n, $\mathsf{CMatrices}_n$, $\mathsf{CPosets}_n$ and CPerms. Specifically,*

(a) *f_{AP} maps CAsc_n bijectively to $\mathsf{CPosets}_n$;*
(b) [25] *f_{PM} maps $\mathsf{CPosets}_n$ bijectively to $\mathsf{CMatrices}_n$;*
(c) *f_{AS} maps CAsc_n bijectively to CPerms.*

6. Open problems

Comparing our bijections, the definition of those between Matrices and Posets are perhaps the simplest, which helps with our understanding of that correspondence. In particular, although we determined in Section 4 the effect of poset duality on just the R-families, the result from [9] that $M^* = \mathsf{flip}(M)$ given in Example 4.2 holds for all $M \in \mathsf{Matrices}$. At the other extreme, the bijections f_{AM}, f_{AP}, and their inverses are the most complicated, especially because of the cases AM3 and AP3. It is not surprising, therefore, that our open problems all involve ascent sequences.

As we have already mentioned, we have not answered the following question of [11] for $\mathbf{a} \in \mathsf{Asc} \setminus \mathsf{RAsc}$.

Problem 6.1. Given $P \in \mathsf{Posets}$ with $f_{\mathsf{PA}}(P) = \mathbf{a}$, how do we obtain $f_{\mathsf{PA}}(P^*)$ from $\mathbf{a}$?

The following problem has already been considered in [16,17].

Problem 6.2. Under f_{AP}, which ascent sequences give rise to posets that are both (2+2)-free and (3+1)-free?

Posets that are both (2+2)- and (3+1)-free are known as *semiorders* or *unit interval orders*, and it is well known that they are counted by the Catalan numbers. Analogous to Problem 6.2, a question already answered by [9, Prop. 16] is to determine those $M \in \mathsf{Matrices}$ that map under f_{MP} to semiorders. The answer is those M such that there does not exist $i > i'$ and $j < j'$ with $m_{ij}m_{i'j'} \neq 0$. For example, in Fig. 2.1, $m_{33}m_{24} \neq 0$ while $\{p_3, p_4, p_5, p_6\}$ form a (3+1) in P.

Obtaining sets in bijection with semiorders can be motivated by a famous problem in symmetric functions: a conjecture of Stanley and Stembridge [22] states that the chromatic symmetric functions of incomparability graphs of (3+1)-free posets are e-positive. Guay-Paquet [18] has reduced this conjecture to the case of semiorders. So perhaps an alternative characterization of semiorders would give insight into the conjecture. For example, we have a characterization of semiorders in terms of matrices M in the previous paragraph, and [9, Lem. 15] tells us an easy way to identify which elements of $f_{\mathsf{MP}}(M)$ are incomparable.

Finally, Proposition 5.4 suggests the following modification of Problem 6.2.

Problem 6.3. Under f_{AP}, which ascent sequences give rise to posets that are (2+2)-free, (3+1)-free and N-free?

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References

[1] George E. Andrews, Vít Jelínek, On q-Series identities related to interval orders, European J. Combin. 39 (2014) 178–187.
[2] Kenneth P. Bogart, An obvious proof of Fishburn's interval order theorem, Discrete Math. 118 (1–3) (1993) 239–242.
[3] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, Sergey Kitaev, $(\mathbf{2}+\mathbf{2})$-free posets, ascent sequences and pattern avoiding permutations, J. Combin. Theory Ser. A 117 (7) (2010) 884–909.
[4] Dandan Chen, Sherry H.F. Yan, Robin D.P. Zhou, Equidistributed statistics on Fishburn matrices and permutations, Electron. J. Combin. 26 (1) (2019), P1.11, 14 pp.
[5] Anders Claesson, Mark Dukes, Martina Kubitzke, Partition and composition matrices, J. Combin. Theory Ser. A 118 (5) (2011) 1624–1637.
[6] Filippo Disanto, Luca Ferrari, Renzo Pinzani, Simone Rinaldi, Catalan pairs: a relational-theoretic approach to Catalan numbers, Adv. in Appl. Math. 45 (2010) 505–517.
[7] Filippo Disanto, Luca Ferrari, Renzo Pinzani, Simone Rinaldi, Catalan lattices on series parallel interval orders, in: Associahedra, Tamari Lattices and Related Structures, Birkhäuser/Springer, Basel, 2012, pp. 323–338.
[8] Mark Dukes, Generalized ballot sequences are ascent sequences, Australas. J. Combin. 64 (2016) 61–63.
[9] Mark Dukes, Vít Jelínek, Martina Kubitzke, Composition matrices, (2+2)-free posets and their specializations, Electron. J. Combin. 18 (1) (2011), P44, 9 pp.
[10] Mark Dukes, Sergey Kitaev, Jeffrey Remmel, Einar Steingrímsson, Enumerating (2+2)-free posets by indistinguishable elements, J. Comb. 2 (1) (2011) 139–163.
[11] Mark Dukes, Robert Parviainen, Ascent sequences and upper triangular matrices containing non-negative integers, Electron. J. Combin. 17 (1) (2010), R53, 16 pp.
[12] Paul Duncan, Einar Steingrímsson, Pattern avoidance in ascent sequences, Electron. J. Combin. 18 (1) (2011), Paper 226, 17 pp.
[13] Peter C. Fishburn, Interval Orders and Interval Graphs. A Study of Partially Ordered Sets, John Wiley & Sons, Ltd., Chichester, 1985.
[14] Gary Gordon, Series-parallel posets and the Tutte polynomial, Discrete Math. 158 (1–3) (1996) 63–75.
[15] George Hutchinson, Partitioning algorithms for finite sets, Commun. ACM 6 (10) (1963) 613–614.
[16] Mitchel T. Keller, Stephen J. Young, Hereditary semiorders and enumeration of semiorders by dimension, Preprint, arXiv:1801.00501.
[17] Sergey Kitaev, Jeffrey Remmel, Enumerating (2+2)-free posets by the number of minimal elements and other statistics, Discrete Appl. Math. 159 (17) (2011) 2098–2108.
[18] Guay-Paquet Mathieu, A modular relation for the chromatic symmetric functions of (3+1)-free posets, Preprint, arXiv:1306.2400.
[19] Lara Pudwell, Enumeration Schemes for Pattern-Avoiding Words and Permutations, Ph.D. thesis, Rutgers University, 2008, http://faculty.valpo.edu/lpudwell/papers/pudwell_thesis.pdf.
[20] Lara Pudwell, Enumeration schemes for permutations avoiding barred patterns, Electron. J. Combin. 17 (1) (2010), Paper 29, 27 pp.
[21] Richard P. Stanley, Catalan Numbers, Cambridge University Press, New York, 2015.
[22] Richard P. Stanley, John R. Stembridge, On immanants of Jacobi–Trudi matrices and permutations with restricted position, J. Combin. Theory Ser. A 62 (2) (1993) 261–279.
[23] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 7th July 2018.
[24] Jelínek Vít, Counting general and self-dual interval orders, J. Combin. Theory Ser. A 119 (3) (2012) 599–614.
[25] Jelínek Vít, Catalan pairs and Fishburn triples, Adv. in Appl. Math. 70 (2015) 1–31.
[26] Julian West, Generating trees and the Catalan and Schörder numbers, Discrete Math. 146 (1–3) (1995) 247–262.