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NOTE

A Simple Proof of a Conjecture of Simion

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Simion had a unimodality conjecture concerning the number of lattice paths in a rectangular grid with the Ferrers diagram of a partition removed. Hildebrand recently showed the stronger result that these numbers are log concave. Here we present a simple proof of Hildebrand's result. © 2002 Elsevier Science (USA)
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1. INTRODUCTION

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$ and λ' the conjugate of λ. Let $R(m, n)$ denote the rectangular grid with m rows and n columns where $m \geqslant \lambda_1'$ and $n \geqslant \lambda_1$. Consider the grid with the Ferrers diagram of λ removed from the upper left corner of $R(m, n)$. Let $N(m, n, \lambda)$ denote the number of paths in $R(m, n)$ such that the path starts at the lower left corner, the path ends at the upper right-hand corner, and at each step the path goes up one unit or to the right one unit but never inside the removed Ferrers diagram of λ. It is well known that there would be $\binom{m+n}{n}$ such paths if there were no Ferrers diagram removed. Simion [5] proposed a unimodality conjecture for $N(m, n, \lambda)$. This conjecture is also described in [2, 4]. The description in here is based on that in [4].

Conjecture 1 (Simion). For each integer ℓ and each partition λ, the sequence

$$N(\lambda_1', \lambda_1 + \ell, \lambda), N(\lambda_1' + 1, \lambda_1 + \ell - 1, \lambda), \ldots, N(\lambda_1' + \ell, \lambda_1, \lambda)$$

is unimodal.

A sequence of positive numbers $x_0, x_1, \ldots, x_\ell$ is unimodal if $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_k \geqslant \cdots \geqslant x_\ell$ for some k and is log concave in i if $x_{i-1}x_{i+1} \leqslant x_i^2$ for $0 < i < \ell$.

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It is well known that a log-concave sequence is also unimodal. Very recently, Hildebrand [3] showed the following stronger result.

THEOREM 1 (Hildebrand). *The sequence in Simion's conjecture is log concave.*

The key idea behind Hildebrand's proof is to show

$$N(m,n+1,\lambda)N(m+1,n,\lambda)\leqslant N(m,n,\lambda)N(m+1,n+1,\lambda) \tag{1}$$

and

$$N(m-1,n+1,\lambda)N(m+1,n+1,\lambda)\leqslant N^2(m,n+1,\lambda). \tag{2}$$

Note that (1) and (2) yield

$$N(m-1,n+1,\lambda)N(m+1,n,\lambda)\leqslant N(m,n,\lambda)N(m,n+1,\lambda). \tag{3}$$

By symmetry, this implies

$$N(m+1,n-1,\lambda)N(m,n+1,\lambda)\leqslant N(m,n,\lambda)N(m+1,n,\lambda). \tag{4}$$

Further, (3) and (4) yield

$$N(m+1,n-1,\lambda)N(m-1,n+1,\lambda)\leqslant N^2(m,n,\lambda),$$

the desired result. So, to show Theorem 1, it suffices to show (1) and (2).

2. PROOF OF (1) AND (2)

A matrix A is said to be totally positive of order 2 (or a TP_2 matrix, for short) if all the minors of order 2 of A have nonnegative determinants. A sequence of positive numbers $x_0, x_1, x_2, \ldots, x_\ell$ is log concave if and only if the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_\ell \\ 0 & x_0 & x_1 & \cdots & x_{\ell-1} \end{pmatrix}$$

is TP_2 (see, e.g., [1, Proposition 2.5.1]). The following lemma is a special case of [1, Theorem 2.2.1].

LEMMA 1. *The product of two finite TP_2 matrices is also TP_2.*

COROLLARY 1. *Let* $a_0, a_1, \ldots, a_\ell$ *be nonnegative and* $x_0, x_1, \ldots, x_\ell$ *positive. Denote* $A_m = \sum_{i=0}^{m} a_i$ *and* $X_m = \sum_{i=0}^{m} x_i$ *for* $m = 0, 1, \ldots, \ell$.

(i) *Assume* $a_i x_{i+1} \leqslant a_{i+1} x_i$ *for all* i. *Then* $A_m X_{m+1} \leqslant A_{m+1} X_m$ *for all* m.

(ii) *If the sequence* $x_0, x_1, \ldots, x_\ell$ *is log concave, then so is the sequence* $X_0, X_1, \ldots, X_\ell$.

Proof. Note that

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_\ell \\ a_0 & a_1 & a_2 & \cdots & a_\ell \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_\ell \\ A_0 & A_1 & A_2 & \cdots & A_\ell \end{pmatrix}$$

and

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_\ell \\ 0 & x_0 & x_1 & \cdots & x_{\ell-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_\ell \\ 0 & X_0 & X_1 & \cdots & X_{\ell-1} \end{pmatrix}.$$

The statement follows immediately from Lemma 1. ∎

We now prove (1) and (2) by induction on λ_1, the largest part of λ. If $\lambda_1 = 0$, i.e., $\lambda = \emptyset$, then both (1) and (2) are easily verified since $N(m, n, \lambda) = \binom{m+n}{n}$, so we proceed to the induction step. Let $\lambda_1 \geqslant 1$ and $r = \lambda_1'$. Denote by μ the partition $(\lambda_1 - 1, \ldots, \lambda_r - 1)$. Then

$$N(m, n, \lambda) = \sum_{k=\lambda_1'}^{m} N(k, n-1, \mu).$$

However, the sequence $N(k, n-1, \mu)$ is log concave in k by the induction hypothesis. Hence $N(m, n, \lambda)$ is log concave in m by Corollary 1(ii). This proves (2). On the other hand, we have by the induction hypothesis

$$N(k, n, \mu) N(k+1, n-1, \mu) \leqslant N(k+1, n, \mu) N(k, n-1, \mu).$$

Thus by Corollary 1(i),

$$\sum_{k=\lambda_1'}^{m} N(k,n,\mu) \sum_{k=\lambda_1'}^{m+1} N(k,n-1,\mu) \leqslant \sum_{k=\lambda_1'}^{m} N(k,n-1,\mu) \sum_{k=\lambda_1'}^{m+1} N(k,n,\mu).$$

This gives (1).

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