Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series A 115 (2008) 121–146
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

On hyperbolic Coxeter polytopes with mutually intersecting facets

Anna Felikson [1], Pavel Tumarkin [2]

Independent University of Moscow, B. Vlassievskii 11, 119002 Moscow, Russia

Received 6 December 2006
Available online 13 June 2007

Abstract

We prove that, apart from some well-known low-dimensional examples, any compact hyperbolic Coxeter polytope has a pair of disjoint facets. This is one of very few known general results concerning combinatorics of compact hyperbolic Coxeter polytopes. We also obtain a similar result for simple non-compact polytopes.

Keywords: Coxeter polytope; Missing face; Gale diagram; Simple polytope

1. Introduction

A Coxeter polytope in the spherical, hyperbolic or Euclidean space is a polytope whose dihedral angles are all integer submultiples of π. These polytopes are very important among acute-angled polytopes since a group generated by reflections with respect to the facets of a Coxeter polytope is discrete. On the other hand, a fundamental chamber of any (finitely generated) discrete reflection group in these spaces is a Coxeter polytope.

Already in 1934, H.S.M. Coxeter [4] proved that any spherical Coxeter polytope (containing no pair of opposite points of the sphere) is a simplex and any compact Euclidean Coxeter polytope is either a simplex or a direct product of simplices.

However, hyperbolic Coxeter polytopes are still far from being classified. It was proved by E. Vinberg [14] that no compact hyperbolic Coxeter polytope exists in dimensions $d \geqslant 30$;

E-mail addresses: felikson@mccme.ru (A. Felikson), pasha@mccme.ru (P. Tumarkin).
[1] Partially supported by grant NS-5666.2006.1.
[2] Partially supported by grants MK-6290.2006.1 and NS-5666.2006.1.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.04.006

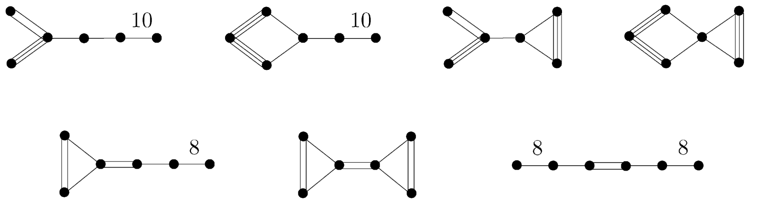

Fig. 1. Coxeter diagrams of Esselmann polytopes (or *Esselmann diagrams* for short).

M. Prokhorov [11] and A. Khovanskij [9] proved that no hyperbolic Coxeter polytope of finite volume exists in dimensions $d \geqslant 996$. These bounds do not look sharp: the examples are known only up to dimension 8 in compact case and up to dimension 21 in the non-compact case.

Besides the restriction on the dimension and some series of examples, there exists a classification of hyperbolic Coxeter polytopes of certain combinatorial types. More precisely, compact simplices were classified by F. Lannér [10], and non-compact simplices were classified by several authors (see e.g. [3] or [13]). Simplicial prisms were listed by I. Kaplinskaya [8]; F. Esselmann [5] obtained the classification of the remaining compact hyperbolic Coxeter d-dimensional polytopes with $d+2$ facets. These consist of seven 4-dimensional polytopes with mutually intersecting facets (we call these polytopes *Esselmann polytopes* and reproduce the list in Fig. 1). P. Tumarkin [12] classified those non-compact hyperbolic Coxeter d-dimensional polytopes with $d+2$ facets that do not have disjoint facets. The only simple polytope from this list is shown in Fig. 2 and is of dimension 4.

This paper is devoted to the proof of the following theorem:

Theorem A. *Let P be a compact hyperbolic Coxeter d-dimensional polytope. If $d > 4$ then P has a pair of disjoint facets.*

If $d \leqslant 4$ and P has no pair of disjoint facets then P is either a simplex or one of the seven Esselmann polytopes.

A d-dimensional polytope is *simple* if any vertex of P is contained in exactly d facets, or equivalently, facets of P at each vertex are in general position. The classification of spherical polytopes implies that any compact hyperbolic Coxeter polytope is simple. While proving Theorem A, we slightly change the proof to obtain a similar result concerning simple non-compact hyperbolic Coxeter polytopes of finite volume.

Theorem B. *Let P be a simple non-compact hyperbolic Coxeter d-dimensional polytope of finite volume. If $d > 9$ then P has a pair of disjoint facets.*

If $d \leqslant 9$ and P has no pair of disjoint facets then P is either a simplex or the 4-dimensional polytope shown in Fig. 2.

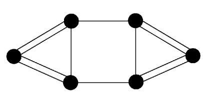

Fig. 2. Coxeter diagram of the unique simple non-compact Coxeter d-polytope that has $d+2$ facets and has no pair of disjoint facets.

The paper is organized as follows. In Section 2 we recall some information about Coxeter polytopes. In Section 3 we introduce some technical tools we use for proving the theorems. Section 4 is devoted to the proof of the theorems.

We prove both Theorems A and B simultaneously. The proof is by induction on the dimension d. The most general case is $d \geqslant 9$. In this case the proof is by examination of the combinatorics of the Coxeter diagram $\Sigma(P)$ while making use of a recent result of D. Allcock (Theorem 2). Some minor technical refinements generalize the proof to $d \geqslant 7$ (see Section 4.5).

The small dimensions are considered in Sections 4.1–4.4. In dimensions $d = 2$ and 3 the argument is purely combinatorial (Lemma 7). In dimensions from 4 to 6 the proof also uses a computational technique developed in Section 3 based on the notion of local determinants.

2. Preliminaries

In this section we list the essential facts about Coxeter diagrams and Gale diagrams. Concerning Coxeter diagrams we follow mainly [15] and [16]. For details about Gale diagrams see [7]. At the end of the section we recall results of R. Borcherds [2] and D. Allcock [1] concerning Coxeter faces of Coxeter polytopes.

In what follows we write *d-polytope* instead of "d-dimensional polytope," *k-face* instead of "k-dimensional face" and *facet* instead of "face of codimension one."

2.1. Coxeter diagrams

2.1.1. An abstract Coxeter diagram Σ is a finite 1-dimensional simplicial complex with weighted edges, where weights w_{ij} are positive, and if $w_{ij} < 1$ then $w_{ij} = \cos\frac{\pi}{k}$ for some integer $k \geqslant 3$. A *subdiagram* of Σ is a subcomplex with the same weights as in Σ. The *order* $|\Sigma|$ is the number of vertices of the diagram Σ.

If Σ_1 and Σ_2 are subdiagrams of an abstract Coxeter diagram Σ, we denote by $\langle \Sigma_1, \Sigma_2 \rangle$ a subdiagram of Σ spanned by the vertices of Σ_1 and Σ_2.

Given an abstract Coxeter diagram Σ with vertices $v_1, \dots, v_n$ and weights w_{ij}, we construct a symmetric $n \times n$ matrix $M(\Sigma) = (c_{ij})$, where $c_{ii} = 1$, $c_{ij} = -w_{ij}$ if v_i and v_j are adjacent, and $c_{ij} = 0$ otherwise. By determinant, rank and signature of Σ we mean the determinant, the rank and the signature of $M(\Sigma)$.

We can suppress the weights but indicate the same information by labeling the edges of a Coxeter diagram in the following way: if the weight w_{ij} equals $\cos\frac{\pi}{m}$, v_i and v_j are joined by an $(m-2)$-fold edge or a simple edge labeled by m; if $w_{ij} = 1$, v_i and v_j are joined by a bold edge; if $w_{ij} > 1$, v_i and v_j are joined by a dotted edge labeled by w_{ij} (or without any label).

We write $[v_i, v_j] = m$ if w_{ij} equals $\cos\frac{\pi}{m}$, and $[v_i, v_j] = \infty$ if $w_{ij} \geqslant 1$.

By a *multiple* edge we mean an edge of weight $\cos\frac{\pi}{m}$ for $m \geqslant 4$. By a *multi-multiple* edge we mean an edge of weight $\cos\frac{\pi}{m}$ for $m \geqslant 6$.

An abstract Coxeter diagram Σ is *elliptic* if $M(\Sigma)$ is positive definite. An $n \times n$ matrix is called *indecomposable* if it cannot be transformed to a block-diagonal one by simultaneous permutations of columns and rows. Clearly, connected components of Σ correspond to indecomposable components of $M(\Sigma)$. A diagram Σ is *parabolic* if any indecomposable component of $M(\Sigma)$ is degenerate and positive semidefinite; a connected diagram Σ is a *Lannér* diagram if Σ is indefinite but any proper subdiagram of Σ is elliptic; a connected diagram Σ is a *quasi-Lannér* diagram if Σ is not a Lannér diagram, Σ is indefinite, but any proper subdiagram of Σ is either elliptic or parabolic; Σ is *superhyperbolic* if its negative inertia index is greater than 1.

The list of connected elliptic and parabolic diagrams with their standard notation is contained in [16, Tables 1, 2]. See also [16, Tables 3, 4] for the lists of Lannér and quasi-Lannér diagrams. We need the following properties of these lists:

- there are finitely many Lannér diagrams of order greater than 3, and the maximal order of a Lannér diagram is 5;
- any Lannér diagram of order 5 contains a subdiagram of the type H_4 or F_4;
- any Lannér diagram of order 4 contains a subdiagram of the type H_3 or B_3;
- any Lannér diagram of order 3 contains a multiple edge;
- Lannér diagrams of order greater than 3 contain no multi-multiple edges;
- any quasi-Lannér diagram of order n contains a connected parabolic subdiagram of order $n-1$.

2.1.2. It is convenient to describe Coxeter polytopes by their Coxeter diagrams. Let P be a Coxeter polytope with facets $f_1, \dots, f_r$. The Coxeter diagram $\Sigma(P)$ of the polytope P is a diagram with vertices $v_1, \dots, v_r$; two edges v_i and v_j are not joined if the hyperplanes spanned by f_i and f_j are orthogonal; v_i and v_j are joined by an edge with weight

$$w_{ij} = \begin{cases} \cos\frac{\pi}{k}, & \text{if } f_i \text{ and } f_j \text{ form a dihedral angle } \frac{\pi}{k}; \\ 1, & \text{if } f_i \text{ is parallel to } f_j; \\ \cosh\rho, & \text{if } f_i \text{ and } f_j \text{ diverge and } \rho \text{ is the distance from } f_i \text{ to } f_j. \end{cases}$$

If $\Sigma = \Sigma(P)$, then $M(\Sigma)$ coincides with the Gram matrix of outer unit normals to the facets of P (referring to the standard model of hyperbolic d-space in $\mathbb{R}^{d,1}$).

It is shown in [15] that a Coxeter diagram $\Sigma(P)$ of a compact d-dimensional hyperbolic polytope P is a connected diagram of signature $(d, 1)$ without parabolic subdiagrams. In particular, $\Sigma(P)$ contains no bold edge, and any indefinite subdiagram contains a Lannér diagram. Moreover, it is shown there that any compact hyperbolic Coxeter d-polytope P is simple, and elliptic subdiagrams of $\Sigma(P)$ are in one-to-one correspondence with faces of P: a k-face F corresponds to an elliptic subdiagram of order $d-k$ whose vertices correspond to the facets of P containing F.

It is also shown in [15] that if $\Sigma(P)$ is a Coxeter diagram of a non-compact hyperbolic d-polytope P, then for any ideal vertex V of P (i.e. V lies at the boundary of the hyperbolic space) the vertices of $\Sigma(P)$ corresponding to facets containing V compose a parabolic diagram of rank $d-1$, and any parabolic subdiagram of $\Sigma(P)$ may be enlarged to some parabolic subdiagram of rank $d-1$. In particular, if P is simple then any parabolic subdiagram S of $\Sigma(P)$ is connected and has rank $d-1$, i.e. S has order d. Clearly, any indefinite subdiagram of $\Sigma(P)$ contains either a Lannér or quasi-Lannér diagram.

As an easy corollary, we have the following statement.

Proposition 2.1.1. *Let P be a simple hyperbolic Coxeter d-polytope. Then $\Sigma(P)$ contains either a Lannér or quasi-Lannér diagram, and $\Sigma(P)$ does not contain parabolic diagrams of order less than d.*

Lemma 1. *Let $\Sigma(P)$ be a Coxeter diagram of a hyperbolic Coxeter d-polytope P of finite volume. Then no proper subdiagram of $\Sigma(P)$ is a diagram of a hyperbolic Coxeter d-polytope of finite volume.*

Proof. Suppose that a proper subdiagram $\Sigma \subset \Sigma(P)$ is a diagram of a Coxeter d-polytope of finite volume. The vertices of Σ determine a polytope P'. Denote by G_P and $G_{P'}$ the groups generated by reflections with respect to the facets of P and P', respectively. The group $G_{P'}$ is a subgroup of G_P. Since P' is of finite volume, $G_{P'}$ has a finite index in G_P. At the same time, the number of facets of P is more than P' has. This contradicts the main result of [6] which claims that if P and P' are finite volume Coxeter polytopes in $\mathbb{H}^n$ or $\mathbb{E}^n$, G_P and $G_{P'}$ are the groups generated by reflections in the facets of P and P', respectively, and $G_{P'} \subseteq G_P$ is a finite index subgroup, then the number of facets of P does not exceed the number of facets of P'. □

Corollary 1. *If a Coxeter diagram of a simple Coxeter polytope P contains a quasi-Lannér subdiagram then P is a simplex.*

Proof. Any quasi-Lannér diagram of order $d+1$ is a Coxeter diagram of non-compact hyperbolic Coxeter d-dimensional simplex of finite volume (see e.g. [15]). Suppose that P is not a simplex. Lemma 1 implies that if P is a d-polytope of finite volume then $\Sigma(P)$ contains no quasi-Lannér subdiagrams of order $d+1$. Clearly, $\Sigma(P)$ does not contain any quasi-Lannér subdiagram of order greater than $d+1$. Further, since P is simple, any connected parabolic subdiagram of $\Sigma(P)$ should have order d, so $\Sigma(P)$ contains no quasi-Lannér subdiagram of order less than $d+1$, either. □

2.2. Gale diagrams and missing faces

We do not use the content of this section throughout the paper except for the proof of the Theorems A and B for 4-polytopes.

Every combinatorial type of simple d-polytope with $d+k$ facets can be represented by its *Gale diagram* $\mathcal{G}$. This consists of $d+k$ points $a_1,\dots,a_{d+k}$ on $(k-2)$-dimensional unit sphere $\mathbb{S}^{k-2} \subset \mathbb{R}^{k-1}$ centered at the origin. Each point a_i corresponds to a facet f_i of P. The combinatorial type of a convex polytope can be read off from the Gale diagram in the following way: for any $J \subset \{1,\dots,d+k\}$ the intersection of facets $\{f_j \mid j \in J\}$ is a proper (that is, non-empty) face of P if and only if the origin is contained in the interior of $\operatorname{conv}\{a_j \mid j \notin J\}$ (where $\operatorname{conv} X$ is a convex hull of the set X).

The points $a_1,\dots,a_{d+k} \in \mathbb{S}^{k-2}$ compose a Gale diagram of some d-dimensional polytope P with $d+k$ facets if and only if every open half-space H^+ in $\mathbb{R}^{k-1}$ bounded by a hyperplane H through the origin contains at least two of the points $a_1,\dots,a_{d+k}$.

Notice that the definition of Gale diagram introduced above concerns simple polytopes only, and it is "dual" to the standard one (see, for example, [7]): usually Gale diagram is defined in terms of vertices of polytope instead of facets. Notice also that the definition above takes simplices out of consideration: usually one means the origin of $\mathbb{R}^1$ with multiplicity $d+1$ by the Gale diagram of a d-simplex, however we exclude the origin since we consider simple polytopes only, and the origin is not contained in $\mathcal{G}$ for any simple polytope except the simplex.

Let P be a simple polytope. The facets $f_1,\dots,f_m$ of P compose a *missing face* of P if $\bigcap_{i=1}^m f_i = \emptyset$ but any proper subset of $\{f_1,\dots,f_m\}$ has a non-empty intersection.

Lemma 2. *Let P be a simple d-polytope with $d+k$ facets $\{f_i\}$, let $\mathcal{G}=\{a_i\} \subset \mathbb{S}^{k-2}$ be a Gale diagram of P, and let $I \subseteq \{1,\dots,d+k\}$. Then the set $M_I=\{f_i \mid i \in I\}$ is a missing face of P if and only if the following two conditions hold*:

(1) *there exists a hyperplane H through the origin separating the set $\widehat{M}_I = \{a_i \mid i \in I\}$ from the rest points of $\mathcal{G}$;*
(2) *for any proper subset $J \subset I$ no hyperplane through the origin separates the set $\widehat{M}_J = \{a_i \mid i \in J\}$ from the remaining points of $\mathcal{G}$.*

Proof. Suppose first that both conditions hold. Since P is simple, (1) implies that $\operatorname{conv}(G \setminus \widehat{M}_I)$ does not contain the origin, so $\bigcap_{i\in I} f_i = \emptyset$. If $\bigcap_{i\in J} f_i$ is also empty for some $J \subsetneq I$, we obtain that $\operatorname{conv}(\mathcal{G} \setminus \widehat{M}_J)$ does not contain the origin, so there exists a hyperplane H through the origin such that $\mathcal{G} \setminus \widehat{M}_J$ is contained in one of halfspaces H^+ and H^-, say H^+. Then $\mathcal{G} \cap H^-$ is a subset of $\widehat{M}_J$, i.e. some subset of $\widehat{M}_J$ is separated by a hyperplane through the origin, which contradicts (2).

Now suppose that M_I is a missing face. Then there exists a hyperplane H through the origin such that $\mathcal{G} \setminus \widehat{M}_I$ is contained in a halfspace H^+. Since P is simple, we may assume that $\mathcal{G} \cap H = \emptyset$. To prove (1) suppose the contrary, i.e. $a_{i_0} \in H^+$ for some $i_0 \in I$. Then $\mathcal{G} \setminus \widehat{M}_{I\setminus i_0}$ is also contained in H^+, that means that $\bigcap_{i\in I\setminus i_0} f_i$ is empty in contradiction to the definition of missing face. To prove (2) notice that if some hyperplane H_J separates $\widehat{M}_J$ for some $J \subsetneq I$ then $\bigcap_{i\in J} f_i = \emptyset$, which also contradicts the definition of missing face. □

Suppose that P is a simple hyperbolic Coxeter polytope. The definition of missing face implies that for any Lannér or quasi-Lannér subdiagram $L \subset \Sigma(P)$ the facets corresponding to L compose a missing face of P (and any missing face of P corresponds to some Lannér or quasi-Lannér diagram in $\Sigma(P)$).

2.3. Faces of Coxeter polytopes

Let P be a hyperbolic Coxeter d-polytope, and denote by $\Sigma(P)$ its Coxeter diagram. Let S_0 be an elliptic subdiagram of $\Sigma(P)$. By [15, Theorem 3.1], S_0 corresponds to a face of P of dimension $d - |S_0|$. Denote this face by $P(S_0)$. $P(S_0)$ itself is an acute-angled polytope, but it might not be a Coxeter polytope. R. Borcherds proved the following sufficient condition for $P(S_0)$ to be a Coxeter polytope.

Theorem 1. *[2, Ex. 5.6] Suppose P is a Coxeter polytope with diagram $\Sigma(P)$, and $S_0 \subset \Sigma(P)$ is an elliptic subdiagram that has no A_n or D_5 component. Then $P(S_0)$ itself is a Coxeter polytope.*

Facets of $P(S_0)$ correspond to those vertices that together with S_0 comprise an elliptic or positive semidefinite subdiagram of $\Sigma(P)$. The following result of D. Allcock shows how to compute dihedral angles of $P(S_0)$.

Let a and b be the facets of $P(S_0)$ coming from facets A and B of P, i.e. $a = A \cap P(S_0)$ and $b = B \cap P(S_0)$. Denote by v_A and v_B the nodes of $\Sigma(P)$ corresponding to the facets A and B. We say that a node of $\Sigma(P)$ *attaches* to S_0 if it is joined with some nodes of S_0 by edges of any type. Then the angles of $P(S_0)$ can be computed in the following way.

Theorem 2. *[1, Theorem 2.2] Under the hypotheses of Theorem* 1,

(1) *If neither v_A nor v_B attaches to S_0, then $\angle ab = \angle AB$.*
(2) *If just one of v_A and v_B attaches to S_0, say to the component S_0^i, then*

(a) *if* $A \perp B$ *then* $a \perp b$;
(b) *if* v_A *and* v_B *are joined by a simple edge, and adjoining* v_A *and* v_B *to* S_0^i *yields a diagram* B_k (*respectively* D_k, E_8 *or* H_4) *then* $\angle ab = \pi/4$ (*respectively* $\pi/4$, $\pi/6$ *or* $\pi/10$);
(c) *otherwise, a and b do not meet.*

(3) *If* v_A *and* v_B *attach to different components of* S_0, *then*
(a) *if* $A \perp B$ *then* $a \perp b$;
(b) *otherwise, a and b do not meet.*

(4) *If* v_A *and* v_B *attach to the same component of* S_0, *say* S_0^i, *then*
(a) *if A and B are not joined and* $S_0^i \cup \{A, B\}$ *is a diagram* E_6 (*respectively* E_8 *or* F_4) *then* $\angle ab = \pi/3$ (*respectively* $\pi/4$ *or* $\pi/4$);
(b) *otherwise, a and b do not meet.*

Let $w \in \Sigma(P)$ be a *neighbor* of S_0, so that w attaches to S_0 by some edges. We call w a *good neighbor* if $\langle S_0, w\rangle$ is either an elliptic diagram or a positive semidefinite diagram, and *bad* otherwise. We denote by $\overline{S}_0$ the subdiagram of $\Sigma(P)$ consisting of vertices corresponding to facets of $P(S_0)$. The diagram $\overline{S}_0$ is spanned by good neighbors of S_0 and by all vertices not joined to S_0 (in other words, $\overline{S}_0$ is spanned by all vertices of $\Sigma(P) \setminus S_0$ except bad neighbors of S_0). If $P(S_0)$ is a Coxeter polytope, denote its Coxeter diagram by Σ_{S_0}.

Corollary 2. *Suppose that* $P(S_0)$ *is a Coxeter polytope.*

(a) *If* S_0 *has no good neighbors then* $\overline{S}_0 = \Sigma_{S_0}$. *In particular, this always holds for* $S_0 = H_4$ *and* $G_2^{(m)}$ *where* $m \geqslant 7$, *for* $S_0 = H_4$ *if* $d > 4$, *and for* $S_0 = G_2^{(6)}$ *if* $d > 3$.
(b) *If* $S_0 = B_n$, $n \geqslant 2$, *and* Σ_{S_0} *contains a subdiagram S of the type* H_4 *or* F_4, *then S is contained in* $\overline{S}_0$, *too.*

Proof. To prove (a) one should only notice that all neighbors of diagrams listed in item (a) (except for F_4 and $G_2^{(6)}$) are bad. Any good neighbor of F_4 or $G_2^{(6)}$ leads to a parabolic subdiagram of $\Sigma(P)$ of order 4 and 3, respectively, which contradicts Proposition 2.1.1 in case of $d > 4$ and $d > 3$.

Item (b) follows immediately from Theorem 2. □

Notice also that any face of a simple polytope is a simple polytope itself. In particular, if P is simple then for any elliptic subdiagram $S \subset \Sigma$ the polytope $P(S)$ is also simple.

3. Technical tools

3.1. Local determinants

Let Σ be a Coxeter diagram, and let T be a subdiagram of Σ such that $\det(\Sigma \setminus T) \neq 0$. A *local determinant* of Σ on a subdiagram T is

$$\det(\Sigma, T) = \frac{\det \Sigma}{\det(\Sigma \setminus T)}.$$

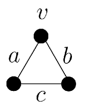

Fig. 3. Diagram $L_{a,b,c}$.

Proposition 3.1.1. *[14, Proposition 12] If a Coxeter diagram Σ consists of two subdiagrams Σ_1 and Σ_2 having a unique vertex v in common, and no vertex of $\Sigma_1 \setminus v$ attaches to $\Sigma_2 \setminus v$, then*

$$\det(\Sigma, v) = \det(\Sigma_1, v) + \det(\Sigma_2, v) - 1.$$

Proposition 3.1.2. *[14, Proposition 13] If a Coxeter diagram Σ is spanned by two disjoint subdiagrams Σ_1 and Σ_2 joined by a unique edge v_1v_2 such that $[v_1, v_2] = m$, then*

$$\det\bigl(\Sigma, \langle v_1, v_2\rangle\bigr) = \det(\Sigma_1, v_1)\det(\Sigma_2, v_2) - \cos^2\frac{\pi}{m}.$$

Denote by L_{abc} a Lannér diagram of order 3 containing subdiagrams of the dihedral groups $G_2^{(a)}$, $G_2^{(b)}$ and $G_2^{(c)}$. Let v be the vertex of $L_{a,b,c}$ that does not belong to $G_2^{(c)}$. Denote by $D(a,b,c)$ the local determinant $\det(L_{a,b,c}, v)$, see Fig. 3.

It is easy to check (see e.g. [14]) that

$$D(a,b,c) = 1 - \frac{\cos^2(\pi/a) + \cos^2(\pi/b) + 2\cos(\pi/a)\cos(\pi/b)\cos(\pi/c)}{\sin^2(\pi/c)}.$$

Notice that $|D(a,b,c)|$ is an increasing function on each of a, b, c tending to infinity while c tends to infinity.

3.2. Lists $L_\alpha(S_0, d)$, $L_\beta(S_0, d)$ and $L'(\Sigma, C, d)$

Lemma 3. *Let P be a simple Coxeter d-polytope with mutually intersecting facets, and assume that P is not a simplex. Let S_0 be a connected elliptic subdiagram of $\Sigma(P)$ such that*

(I) *$|S_0| < d$ and $S_0 \neq A_n$, D_5.*
(II) *S_0 has no good neighbors in $\Sigma(P)$.*
(III) *If $|S_0| \neq 2$, then $\Sigma(P)$ contains no multi-multiple edges.*
If $|S_0| = 2$, then the edge of S_0 has the maximum multiplicity amongst all edges in $\Sigma(P)$.

Suppose that Theorems A *and* B *hold for any d_1-polytope satisfying $d_1 < d$. Then there exists a subdiagram $S_1 \subset \Sigma(P)$ and two vertices $y_0, y_1 \in \Sigma(P)$ such that the subdiagram $\langle S_0, y_1, y_0, S_1\rangle$ satisfies the following conditions*:

(1) *S_0 and S_1 are connected elliptic diagrams, $S_0, S_1 \neq A_n, D_5$;*
(2) *No vertex of S_1 attaches to S_0 and $|S_0| + |S_1| = d$;*
(3) *$\langle y_0, S_1\rangle$ is either a Lannér diagram or one of the four diagrams shown in Fig.* 5 (*in the latter case y_0 is the marked vertex of the diagram*);
(4) *$\langle S_0, y_1\rangle$ is an indefinite subdiagram, and one of the following holds*:
   (4α) *y_1 is not joined to S_1, and $\langle S_0, y_1\rangle$ is either a Lannér diagram or one of the four diagrams shown in Fig.* 5 (*in the latter case y_0 is the marked vertex of the diagram*);

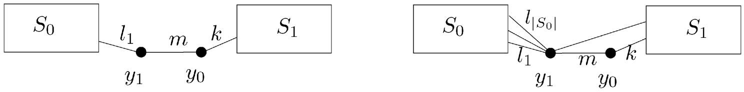

Fig. 4. Diagrams $\langle S_0, y_1, y_0, S_1\rangle$ satisfying the conditions (1)–(6): the left one satisfies condition (4α), and the right one satisfies condition (4β).

(4β) *y_1 is a good neighbor of S_1, and the diagram $\langle y_0, S_1\rangle$ contains no multi-multiple edges*;

(5) *if $|S_0| \neq 2$, then $\langle S_0, y_1, y_0, S_1\rangle$ contains no multi-multiple edges*;
if $|S_0| = 2$, then the edge of S_0 has the maximum possible multiplicity in $\langle S_0, y_1, y_0, S_1\rangle$;

(6) *if $|S_1| = 4$ then S_1 is a diagram of type F_4 or H_4*;
if $|S_1| = 3$ then S_1 is a diagram of type B_3 or H_3;
if $|S_1| = 2$, then the edge of S_1 has the maximum possible multiplicity in $\langle y_0, S_1\rangle$.

Conditions (1)–(6) of the lemma are illustrated in Fig. 4.

Proof. We construct the required diagram in several steps.

(1) *Analyzing the data.* Since S_0 has no good neighbors, $\overline{S}_0 = \Sigma_{S_0}$ (see Corollary 2). Denote by $\dim = d - |S_0|$ the dimension of $P(S_0)$. As a subdiagram of $\Sigma(P)$, the diagram Σ_{S_0} contains no dotted edges. Clearly, $\dim < d$. By the assumption, Theorems A and B hold for polytopes of dimension less than d. By Proposition 2.1.1, $P(S_0)$ is either a compact simplex (and $2 \leqslant \dim \leqslant 4$) or one of the Esselmann polytopes (and $\dim = 4$).

(2) *Choosing S_1.* We take a subdiagram $S_1 \subset \overline{S}_0 = \Sigma_{S_0}$ as follows:

If $\dim = 4$ then $\overline{S}_0$ contains a subdiagram S_1 of type F_4 or H_4.

If $\dim = 3$ then $\overline{S}_0$ contains a subdiagram S_1 of type B_3 or H_3.

If $\dim = 2$ then $\overline{S}_0$ contains a subdiagram of type $G_2^{(k)}$, $k \geqslant 4$, i.e. a multiple edge. We choose S_1 as a diagram $G_2^{(k)} \subset \overline{S}_0$, where k is maximal in $\overline{S}_0$.

Clearly, in all cases conditions (1) and (2) are satisfied. Notice also, that if $\overline{S}_0$ contains a multi-multiple edge, then the diagram S_1 has no good neighbors in $\Sigma(P)$.

(3) *Choosing y_0.* If $P(S_0)$ is a simplex, then S_1 contains all but one vertex of $\overline{S}_0$. Let $y_0 = \overline{S}_0 \setminus S_1$.

If $P(S_0)$ is an Esselmann polytope, then it is always possible to choose $y_0 \in \overline{S}_0 \setminus S_1$ such that the diagram $\langle y_0, S_1\rangle$ coincides with one of the four diagrams shown in Fig. 5 (for y_0 we take the vertex marked by y).

Thus, condition (3) holds.

(4) *Choosing y_1.* We consider two cases.

(α) Suppose that S_1 has no good neighbors in $\Sigma(P)$. Then $\overline{S}_1 = \Sigma_{S_1}$ and $P(S_1)$ is either a compact simplex or an Esselmann polytope. Clearly, $S_0 \subset \overline{S}_1$. If $P(S_1)$ is a simplex, define $y_1 = \overline{S}_1 \setminus S_0$. If $P(S_1)$ is an Esselmann polytope, we define $y_1 \in \overline{S}_1$ such that $\langle S_0, y_1\rangle$ is

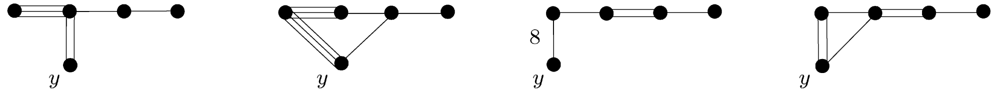

Fig. 5. Subdiagrams of Esselmann diagrams.

one of the four diagrams shown in Fig. 5 (for y_1 we take the vertex marked by y). Hence, $\langle S_0, y_1\rangle$ satisfies condition (4α).

(β) Suppose that S_1 has a good neighbor in $\Sigma(P)$. We choose y_1 as one of good neighbors of S_1. The vertex y_1 is connected to S_1 by exactly one edge, and this edge is simple. The vertex y_1 might also be connected to any vertex of S_0 and to y_0.
Since S_1 has a good neighbor and $|S_1| \leqslant d-2$, Corollary 2 implies that $S_1 \neq F_4, H_4, G_2^{(k)}$ for $k \geqslant 6$ (in particular, S_1 contains no multi-multiple edge). Therefore, $\overline{S}_0$ is neither an Esselmann diagram nor a Lannér diagram of order 5, so $\overline{S}_0$ is a Lannér diagram of order 3 or 4. In the latter case $\overline{S}_0$ contains no multi-multiple edge. In the first case, recall that the diagram S_1 is chosen as a subdiagram of $\overline{S}_0$ containing the edge of maximal multiplicity. Since S_1 contains no multi-multiple edges, $\overline{S}_0 = \langle y_0, S_1\rangle$ does not contain them, either. Thus, condition (4β) holds.

Condition (5) is satisfied by assumption (III) of the lemma, and condition (6) is satisfied by the choice of S_1, which completes the proof. □

Lemma 4. *The number of diagrams $\langle S_0, y_1, y_0, S_1\rangle$ of signature $(d,1)$, $4 \leqslant d \leqslant 8$, satisfying conditions* (1)–(6) *of Lemma* 3*, is finite.*

Proof. Suppose that $S_0 \neq G_2^{(k)}$ for $k \geqslant 6$. Then by condition (3) the diagram $\langle S_0, y_1, y_0, S_1\rangle$ contains no multi-multiple edges. Since $|S_0| + |S_1| = d \leqslant 8$, we obtain that $|\langle S_0, y_1, y_0, S_1\rangle| \leqslant 10$, and we have finitely many possibilities for the diagram.

Now suppose that $S_0 = G_2^{(k)}$, $k \geqslant 6$. Since $|\langle S_0, y_1, y_0, S_1\rangle| = d+2$ and $\operatorname{sign}\langle S_0, y_1, y_0, S_1\rangle = (d,1)$, we have $\det\langle S_0, y_1, y_0, S_1\rangle = 0$. We consider two cases: either the diagram S_1 has a good neighbor in $\langle S_0, y_1, y_0, S_1\rangle$ or not.

Case (α). S_1 has no good neighbors in $\langle S_0, y_1, y_0, S_1\rangle$. In this case the subdiagrams $\langle S_0, y_1\rangle$ and $\langle S_1, y_0\rangle$ are either Lannér diagrams or diagrams shown in Fig. 5. The only edge connecting these diagrams is y_0y_1; we let $m = [y_0, y_1]$ (see Fig. 4). By Proposition 3.1.2, we have

$$\det\bigl(\langle S_0, y_1, y_0, S_1\rangle, \langle y_1, y_0\rangle\bigr) = \det\bigl(\langle S_0, y_1\rangle, y_1\bigr)\cdot\det\bigl(\langle y_0, S_1\rangle, y_0\bigr) - \cos^2\frac{\pi}{m}.$$

Since $\det\langle S_0, y_1, y_0, S_1\rangle = 0$, we obtain

$$\det\bigl(\langle S_0, y_1\rangle, y_1\bigr)\cdot\det\bigl(\langle y_0, S_1\rangle, y_0\bigr) = \cos^2\frac{\pi}{m}.$$

In particular,

$$\bigl|\det\bigl(\langle S_0, y_1\rangle, y_1\bigr)\cdot\det\bigl(\langle y_0, S_1\rangle, y_0\bigr)\bigr| < 1$$

($m=2$ is impossible, since the two indefinite subdiagrams $\langle S_0, y_1\rangle$ and $\langle y_0, S_1\rangle$ should be joined in $\Sigma(P)$). Hence, at least one of $|\det(\langle S_0, y_1\rangle, y_1)|$ and $|\det(\langle y_0, S_1\rangle, y_0)|$ is less than 1.

Suppose that $|\det(\langle S_0, y_1\rangle, y_1)| < 1$. Recall that $S_0 = G_2^{(k)}$, $k \geqslant 6$, and we have $\det(\langle S_0, y_1\rangle, y_1) = D(i,j,k)$, where $i, j \leqslant k$ by assumption. Since $|D(i,j,k)|$ is an increasing function on i, j, k, it is easy to see that if $k \geqslant 6$, $k \geqslant i, j$ and $|D(i,j,k)| < 1$, then (i,j,k) is either $(2,3,7)$ or $(2,3,8)$. So, $\langle S_0, y_1\rangle$ is either $L_{2,3,7}$ or $L_{2,3,8}$, and $k \leqslant 8$. Therefore, the diagram $\langle S_0, y_1, y_0, S_1\rangle$ contains no subdiagram $G_2^{(l)}$ for $l > 8$ and we are left with finitely many diagrams.

Suppose that $|\det(\langle y_0, S_1\rangle, y_0)| < 1$. Since the diagram $\langle y_0, S_1\rangle$ is either a Lannér diagram or one of the diagrams shown in Fig. 5, it is easy to check that if $|S_1| > 2$ then $\det(\langle y_0, S_1\rangle, y_0) > 1$. Therefore, $|S_1| = 2$, $d = 4$. Again, it is easy to see that there are only 5 triples (i, j, k) for which $i, j \leqslant k$ and $|D(i, j, k)| < 1$: $(i, j, k) = (2, 3, 7), (2, 4, 5), (2, 3, 8), (3, 3, 4)$ and $(2, 5, 5)$. For each of these triples there exist finitely many triples (i', j', k') satisfying the condition $|D(i', j', k') \cdot D(i, j, k)| < 1$. So, in the case when S_1 has no good neighbors in $\langle S_0, y_1, y_0, S_1\rangle$ the lemma is proved.

Case (β). y_1 is a good neighbor of S_1 in $\langle S_0, y_1, y_0, S_1\rangle$. Note that any edge of $\langle S_0, y_1, y_0, S_1\rangle$ belongs to either $\langle S_0, y_1\rangle$ or $\langle y_1, y_0, S_1\rangle$. By Lemma 3.1.1, we have

$$\det\big(\langle S_0, y_1, y_0, S_1\rangle, y_1\big) = \det\big(\langle S_0, y_1\rangle, y_1\big) + \det\big(\langle y_1, y_0, S_1\rangle, y_1\big) - 1.$$

On the other hand,

$$\det\big(\langle S_0, y_1, y_0, S_1\rangle, y_1\big) = \frac{\det(\langle S_0, y_1, y_0, S_1\rangle)}{\det(\langle S_0, y_0, S_1\rangle)} = 0.$$

Therefore,

$$\det\big(\langle S_0, y_1\rangle, y_1\big) + \det\big(\langle y_1, y_0, S_1\rangle, y_1\big) = 1.$$

Since $\langle y_0, S_1\rangle$ and $\langle y_1, y_0, S_1\rangle$ are indefinite diagrams, we obtain that $\det(\langle y_1, y_0, S_1\rangle, y_1) > 0$, so $\det(\langle S_0, y_1\rangle, y_1) < 0$. Furthermore, $|\langle y_1, y_0, S_1\rangle| = d$, which implies

$$\big|\det\langle y_1, y_0, S_1\rangle\big| < d! \tag{*}$$

(since the absolute value of each of the summands in the standard expansion of the determinant does not exceed 1). At the same time, by condition (4β) the diagram $\langle y_0, S_1\rangle$ contains no multi-multiple edges, and we have finitely many possibilities for $\det\langle y_0, S_1\rangle$. Therefore, there exists a positive constant M such that

$$M < \big|\det\langle y_0, S_1\rangle\big|. \tag{**}$$

Combining $(*)$ and $(**)$, we obtain

$$0 < \det\big(\langle y_1, y_0, S_1\rangle, y_1\big) < \frac{d!}{M},$$

hence,

$$1 - \frac{d!}{M} < \det\big(\langle S_0, y_1\rangle, y_1\big) < 0. \tag{***}$$

Recall that $S_0 = G_2^{(k)}$ and that the diagram $\langle S_0, y_1\rangle$ contains no $G_2^{(l)}$ for $l > k$. In particular, $\det(\langle S_0, y_1\rangle, y_1) = D(i, j, k)$ for some $i, j \leqslant k$. By $(***)$, we have finitely many possibilities for k. By the assumption, $\langle S_0, y_1, y_0, S_1\rangle$ contains no subdiagram of the type $G_2^{(l)}$ for $l > k$, so we have finitely many possibilities for the whole diagram $\langle S_0, y_1, y_0, S_1\rangle$. □

According to Lemma 4, for each $S_0 = G_2^{(k)}, B_3, B_4, H_3, H_4, F_4$ we can write down the complete list

$$L(S_0, d)$$

of diagrams $\langle S_0, y_1, y_0, S_1\rangle$ of signature $(d,1)$, $4 \leqslant d \leqslant 8$, satisfying conditions (1)–(6) of Lemma 3 and containing no parabolic diagrams of order less than d. Define also a list

$$L(d) = \bigcup_{k=6}^{\infty} L\big(G_2^{(k)}, d\big).$$

By Lemma 4, the list $L(d)$ is also finite. In view of condition (4) of Lemma 3, the list $L(S_0, d)$ naturally splits into two disjoint parts

$$L(S_0, d) = L_\alpha(S_0, d) \cup L_\beta(S_0, d),$$

where the list $L_\alpha(S_0, d)$ consists of diagrams satisfying condition (4α), and the list $L_\beta(S_0, d)$ consists of diagrams satisfying condition (4β). Similarly, the list $L(d)$ splits into two parts

$$L_\alpha(d) = \bigcup_{k=6}^{\infty} L_\alpha\big(G_2^{(k)}, d\big) \quad \text{and} \quad L_\beta(d) = \bigcup_{k=6}^{\infty} L_\beta\big(G_2^{(k)}, d\big).$$

These lists were obtained by a computer. Usually they are not very short. In what follows we reproduce some parts of the lists as far as we need.

Remark. It is easy to see that the bounds obtained in the proof of Lemma 4 are not optimal. In real computations we usually analyze concrete data to reduce calculations.

The following lemma is obvious:

Lemma 5. *For any diagram Σ and any constant C the number of diagrams $\langle \Sigma, x\rangle$ (spanned by Σ and a single vertex x) containing no subdiagrams $G_2^{(k)}$ for $k > C$ is finite.*

Hence, for any diagram Σ, a constant C and dimension d, it is possible to write down a complete list

$$L'(\Sigma, C, d)$$

of diagrams $\langle \Sigma, x\rangle$ of signature $(d,1)$ containing no subdiagrams $G_2^{(k)}$ for $k > C$.

Given Σ, C and d, the list $L'(\Sigma, C, d)$ can be obtained by a computer. We reproduce some of these lists as far as we need. To shorten the computations we use the following:

(1) Suppose that $\langle \Sigma, x\rangle \in L'(\Sigma, C, d)$, and $|\Sigma| \geqslant d+1$. Then $|\langle \Sigma, x\rangle| \geqslant d+2$, and $\det\langle \Sigma, x\rangle = 0$. To check the determinant is faster than to find the signature. So, first we compute the determinant and in the rare cases when it vanishes we compute the signature.
(2) Suppose that $\Sigma \subset \Sigma(P)$, where P is a simple hyperbolic d-polytope without a pair of disjoint facets. Suppose that Σ contains a connected elliptic subdiagram $S \neq A_k, D_5$. Suppose also that $\overline{S} \not\subset \Sigma$ (since $|\overline{S}| + |S| > d$, this always holds if $|\Sigma| \leqslant d + |B|$, where B is the set of bad neighbors of S in Σ). In this case there exists $x \in \Sigma(P) \setminus \Sigma$ which is either a good neighbor of S or is not joined to S. Denote by $L'(\Sigma, C, d, S^{(g,n)})$, $L'(\Sigma, C, d, S^{(g)})$ and $L'(\Sigma, C, d, S^{(n)})$ the sublists of $L'(\Sigma, C, d)$ which consist of diagrams $\langle \Sigma, x\rangle$ satisfying the following conditions (g,n), (g) and (n), respectively:
   (g,n) either x is a *good* neighbor of S or x is *not* a neighbor of S;
   (g) x is a *good* neighbor of S;
   (n) x is *not* a neighbor of x.

Now we may assume (in the assumptions above) that $\Sigma(P)$ contains a diagram $\langle \Sigma, x\rangle$ from one of the lists $L'(\Sigma, C, d, S^{(g,n)})$, $L'(\Sigma, C, d, S^{(g)})$ and $L'(\Sigma, C, d, S^{(n)})$. This hugely reduces the computations.

4. Proof of Theorems A and B

The plan of the proof is as follows. We assume that there exists a simple hyperbolic Coxeter d-polytope P with mutually intersecting facets, and P is not a simplex. Then, using Theorem 1, Corollary 1 and the classification of Lannér diagrams, we find a Coxeter face of P of sufficiently small codimension. In view of Theorem 2, this face often has no pair of disjoint facets either. This enables us to carry out an induction in large dimensions ($d \geqslant 7$). In small dimensions (up to 6) the existence of simplices and Esselmann polytopes forces us to involve also a computer case-by-case check based on computations of local determinants.

For a part of the proof (in dimensions 4–6) we also need the following lemma.

Lemma 6. *Let P be a simple Coxeter hyperbolic d-polytope without a pair of disjoint facets. If P is neither a simplex nor an Esselmann polytope nor the polytope shown in Fig.* 2*, then P has at least $d+3$ facets.*

Proof. The lemma follows immediately from the classification of hyperbolic Coxeter d-polytopes with $d+1$ and $d+2$ facets. The polytopes with $d+1$ facets are simplices, compact polytopes with $d+2$ facets are either Esselmann polytopes or simplicial prisms, and the latter have disjoint facets; any simple non-compact polytope with $d+2$ facets is either a simplicial prism or the polytope shown in Fig. 2. □

4.1. Dimensions 2 *and* 3

The following lemma does not involve hyperbolic geometry.

Lemma 7. *Let P be a simple d-polytope and $d = 2$ or* 3*. If P has no pair of disjoint facets then P is a simplex.*

Proof. For $d = 2$ the statement is evident.

To prove it for $d = 3$ note that any simple 3-polytope different from simplex has at least one 2-face which is not a triangle. Denote such a face by f. Let a and b be non-adjacent edges of f. Denote by f_a and f_b the faces of P such that $a = f_a \cap f$ and $b = f_b \cap f$. By assumption of the lemma $f_a \cap f_b \neq \emptyset$. Since P is simple, $f_a \cap f_b$ is an edge. Therefore, the set $\partial P \setminus \{f \cap f_a \cap f_b\}$ has two connected components M_1 and M_2 (here ∂P is the boundary of P). Each of these components M_i contains at least one face m_i of P, hence m_1 and m_2 are two disjoint facets of P. □

4.2. Dimension 4

Lemma 7 does not hold for 4-polytopes. Moreover, for any $k \geqslant 6$ there exists a simple 4-polytope with k facets having no pair of disjoint facets. More precisely, the duals of the cyclic polytopes $C(k, 4)$ are simple, have k facets, and any two of its facets intersect in a 2-face (i.e.

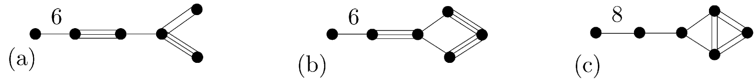

Fig. 6. Intermediate results for $d = 4$.

these polytopes are 2-neighborly); see [7] for definitions and details. Furthermore, there are already known seven Esselmann compact Coxeter hyperbolic 4-polytopes with 6 facets containing no pair of disjoint facets (see Fig. 1), and one non-compact 4-polytope which is combinatorially equivalent to a product of two simplices (see Fig. 2).

Proposition 4.2.1. *Let P be a simple hyperbolic Coxeter 4-polytope having no pair of disjoint facets. If P is not an Esselmann polytope then $\Sigma(P)$ contains no multi-multiple edge.*

Proof. Suppose that $\Sigma(P)$ contains a multi-multiple edge. Choose $S_0 = G_2^{(k)}$, $k \geqslant 6$, as an edge of maximal multiplicity in $\Sigma(P)$. Clearly, S_0 has no good neighbors, so by Lemma 3, $\Sigma(P)$ contains a subdiagram $\langle S_0, y_1, y_0, S_1\rangle$ from the list $L(4)$.

The list $L_\alpha(4)$ contains two Esselmann diagrams only. The list $L_\beta(4)$ contains two Esselmann diagrams and the diagrams shown in Fig. 6(a)–(c). By Lemma 1, $\Sigma(P)$ contains no Esselmann diagrams. Hence, we are left with three diagrams shown in Fig. 6(a)–(c). In these cases $\Sigma(P)$ contains no subdiagrams $G_2^{(l)}$ for $l > 6$, 6, and 8, respectively. Since none of these diagrams is a diagram of a 4-dimensional Coxeter polytope (see Lemma 6), $\Sigma(P)$ should contain some diagram from the list $L'(\Sigma, C, 4)$, where Σ ranges over the diagrams from Fig. 6(a)–(c), and $C = 6$, 6, and 8, respectively. However, these lists are empty: a straightforward computer check shows that taking any diagram $\langle \Sigma, x\rangle$, where x attaches to Σ by edges of multiplicity at most $C - 2$, we obtain either a superhyperbolic diagram or a diagram with positive inertia index $\geqslant 5$ (in fact, we compute the signature only for those diagrams whose determinant vanishes, see the remark below the definition of the list $L'(\Sigma, C, d)$). Thus, we come to a contradiction with the assumption of the proposition. □

To prove the main result of this section, i.e. Lemma 9, we need the following lemma:

Lemma 8. *Let P be a simple hyperbolic Coxeter 4-polytope having no pair of disjoint facets. Suppose that P is not a simplex.*

(a) *Let $v_1, \dots, v_6$ be any six vertices of $\Sigma(P)$. Then the subdiagram spanned by $v_1, \dots, v_6$ contains two disjoint Lannér diagrams of order 3 each.*
(b) *The order of any Lannér subdiagram of $\Sigma(P)$ equals 3.*

Remark. The lemma involves combinatorics only. For any simple polytope P we may consider a "diagram of missing faces" instead of Coxeter diagram and missing faces instead of Lannér diagrams.

Proof. Consider a Gale diagram $\mathcal{G}$ of the 4-polytope P. Denote by $f_1, \dots, f_n$ the facets of P. Then $\mathcal{G}$ is a set of n points at $(n - 6)$-dimensional sphere $\mathbb{S}^{n-6}$. Denote by $b_1, \dots, b_n$ the points corresponding to the facets $f_1, \dots, f_n$, respectively. Since P is simple, we may assume that $b_i \neq b_j$ for $i \neq j$. Denote by v_i the vertex of $\Sigma(P)$ corresponding to a facet f_i, $i = 1, \dots, n$.

Consider an $(n-6)$-dimensional plane Π spanned by $b_7, \dots, b_n$ and the origin. Again, we may assume that Π does not contain points b_i for $i \leqslant 6$. The hyperplane Π separates $\mathbb{S}^{n-6}$ into two hemispheres. Since P has no disjoint facets, each of the hemispheres contains at least 3 points from $\{b_1, \dots, b_6\}$ (see Lemma 2). Hence, three points (say b_1, b_2, b_3) belong to one halfspace, the rest belong to another, which means that $\langle v_1, v_2, v_3 \rangle$ and $\langle v_4, v_5, v_6 \rangle$ are Lannér diagrams (again, see Lemma 2), and (a) is proved.

To prove (b) suppose that $\langle v_k, v_{k+1}, v_{k+2}, v_{k+3} \rangle$ is a Lannér diagram. Consider the corresponding points $b_k, b_{k+1}, b_{k+2}, b_{k+3}$ in the Gale diagram. By Lemma 2, there exists an $(n-6)$-plane Π through the origin separating these four points. We can rotate the hyperplane Π around the origin until it meets one of the points $b_1, \dots, b_n$. It cannot meet first any of $b_k, b_{k+1}, b_{k+2}, b_{k+3}$ (if Π passes through one of these points then the other three are separated by a plane, so the four points do not correspond to a Lannér diagram). Hence, Π will meet first some point $x_1 \in \{b_1, \dots, b_n\}$ distinct from $b_k, b_{k+1}, b_{k+2}, b_{k+3}$. Now, we can rotate Π around x_1 and the origin until Π meets some $x_2 \in \{b_1, \dots, b_n\}$, $x_2 \neq b_k, b_{k+1}, b_{k+2}, b_{k+3}$, and so on. We have freedom to rotate Π until it passes through $(n-6)$ points $x_1, \dots, x_{n-6}$ (where $x_i \in \{b_1, \dots, b_n\}$, $x_i \neq b_k, b_{k+1}, b_{k+2}, b_{k+3}$). Now Π separates $\mathbb{S}^{n-6}$ into two hemispheres: one contains 4 points and another contains $n-(n-6)-4=2$ points. This contradicts the assumption that P have no pair of disjoint faces. Therefore, no Lannér subdiagram of $\Sigma(P)$ is of order 4. Similarly, it cannot be of order greater than 4. Since no Lannér subdiagram is of order 2, we obtain that the order of any Lannér diagram equals 3. $\square$

Lemma 9. *Let P be a simple hyperbolic Coxeter 4-polytope. If P has no pair of disjoint facets, then P is either a simplex or one of seven Esselmann polytopes or the polytope shown in Fig. 2.*

Proof. By Proposition 4.2.1, the diagram $\Sigma(P)$ contains no multi-multiple edges. Let $\Sigma \subset \Sigma(P)$ be a subdiagram of order 6 (by Lemma 6, such a subdiagram does exist). By Lemma 8, we can assume that $\Sigma = \langle S_1, S_2 \rangle$, where S_1 and S_2 are Lannér diagrams. There are only 11 Lannér diagrams of order 3 containing no edges of multiplicity greater than 5. We check all possible pairs of S_1 and S_2 (66 possibilities) and connect the vertices of S_1 with the vertices of S_2 by edges of all possible multiplicities (2, 3, 4, 5 for each of 6 edges). In all but 39 cases we obtain that $\det\langle S_1, S_2 \rangle \neq 0$. Further, 3 of these 39 cases correspond to Esselmann diagrams; one diagram is the diagram of the polytope shown in Fig. 2; 4 diagrams contain parabolic subdiagrams of order less than 4; 11 of these 39 diagrams contain Lannér subdiagrams of order 4, so they cannot be subdiagrams of $\Sigma(P)$ by Lemma 8(b). We are left with 20 diagrams none of which is a diagram of Coxeter 4-polytope: any of them has order 6, but none of them is an Esselmann diagram or a diagram of a 4-prism (see [5] and [8]). Therefore, $\Sigma(P)$ contains a subdiagram appearing in one of the lists $L'(\Sigma, 5, 4)$, where Σ ranges over the 20 diagrams mentioned above. However, these lists are empty, and the lemma is proved. $\square$

4.3. Dimension 5

In this section we suppose that P is a simple hyperbolic Coxeter 5-polytope having no pair of disjoint facets. We also assume that P is not a simplex.

Proposition 4.3.1. *$\Sigma(P)$ contains neither a subdiagram of the type F_4 nor a subdiagram of the type H_4.*

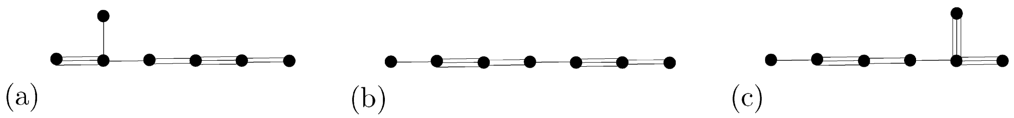

Fig. 7. Intermediate results for $d = 5$, $S_0 = H_3$.

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = F_4$ or H_4. Then by Corollary 2, $\overline{S}_0 = \Sigma_{S_0}$, and Σ_{S_0} contains no dotted edges. On the other hand, $P(S_0)$ is a 1-dimensional polytope, i.e. a segment, so Σ_{S_0} should consist of a dotted edge. □

Proposition 4.3.2. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $\Sigma(P)$ contains a multi-multiple edge. Choose $S_0 = G_2^{(k)}$, $k \geqslant 6$, as an edge of maximal multiplicity in $\Sigma(P)$. Clearly, S_0 has no good neighbors, and $\Sigma(P)$ contains a subdiagram appearing in the list $L(5)$. But all diagrams from the list $L(5)$ contain a subdiagram of the type H_4, which contradicts Proposition 4.3.1. □

Proposition 4.3.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = H_3$. It follows from Proposition 4.3.1 that S_0 has no good neighbors. So, $\Sigma(P)$ contains a subdiagram appearing in the list $L(H_3, 5)$. The only diagram from the list $L_\alpha(H_3, 5)$ containing neither a multi-multiple edge nor a subdiagram of the type H_4 is shown in Fig. 7(a). In the list $L_\beta(H_3, 5)$ there are two diagrams containing neither a multi-multiple edge nor a subdiagram of the type H_4; these diagrams are shown in Fig. 7(b) and (c).

Consider the diagram Σ shown in Fig. 7(a). By Lemma 6, Σ is not a diagram of a 5-polytope. Thus, if $\Sigma(P)$ contains Σ, then $\Sigma(P)$ also contains some diagram from the list $L'(\Sigma, 5, 5)$. Further, denote by S the subdiagram of Σ of the type B_4. Then Σ_S is the diagram of a Coxeter 1-polytope, i.e. Σ_S contains two vertices. Therefore, $\Sigma(P)$ should contain a diagram from the list $L'(\Sigma, 5, 5, S^{(g,n)})$ which happens to be empty.

Now, consider the diagrams shown in Fig. 7(b) and (c). By Lemma 6, none of them is a diagram of a 5-polytope. Thus, if $\Sigma(P)$ contains one of these two diagrams (denote it by Σ), then $\Sigma(P)$ also contains some diagram from the list $L'(\Sigma, 5, 5)$. Furthermore, denote by $S \subset \Sigma$ a diagram of the type H_3 having 2 neighbors in Σ. By Proposition 4.3.2, S has no good neighbors in $\Sigma(P)$. Hence, the diagram $\Sigma(P)$ should contain a diagram from the list $L'(\Sigma, 5, 5, S^{(n)})$. This list turns out to be empty in both cases.

The contradiction shows that the diagrams shown in Fig. 7(b) and (c) cannot be subdiagrams of $\Sigma(P)$, which finishes the proof. □

Proposition 4.3.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $\Sigma(P) \supset S_0 = G_2^{(5)}$. It follows from Proposition 4.3.3 that S_0 has no good neighbors, so $\Sigma(P)$ contains a subdiagram appearing in the list $L(S_0, 5)$. However, in the list $L(S_0, 5)$ there is no diagram containing neither a multi-multiple edge nor a subdiagram of the types H_3 and F_4. □

It follows from Propositions 4.3.2 and 4.3.4 that any multiple edge in $\Sigma(P)$ is a double edge.

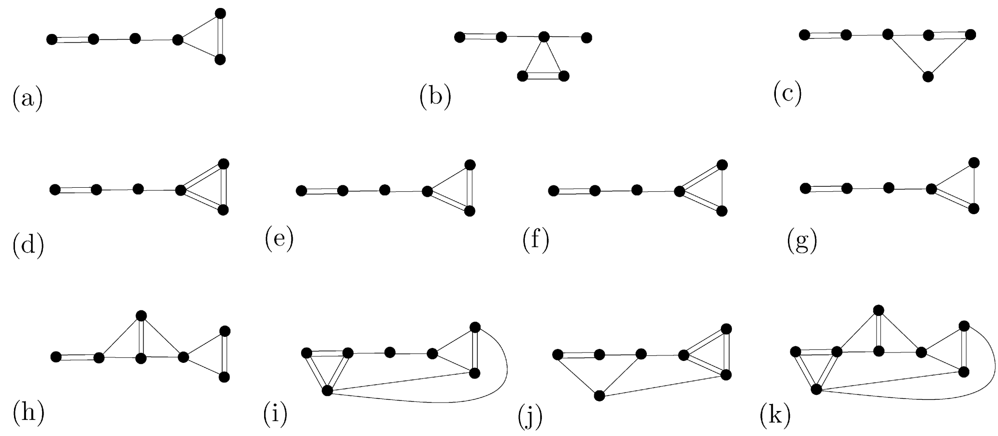

Fig. 8. Intermediate results for $d = 5$, $S_0 = B_4$.

Proposition 4.3.5. *$\Sigma(P)$ contains no subdiagram of the type B_4.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_4$. Then $\overline{S}_0 \neq \Sigma_{S_0}$, since Σ_{S_0} is a dotted edge and $\overline{S}_0$ is not. Let u and v be the vertices of $\overline{S}_0$. By Theorem 2, at least one of u and v is a good neighbor of S_0 (we assume that u is a good neighbor, so $\langle S_0, u\rangle$ is either B_5, or $\widetilde{B}_4$, or $\widetilde{C}_4$). Suppose that v is not a neighbor of S_0. Then by Theorem 2, v attaches to u. If $[u, v] = 3$, then $\langle S_0, u, v\rangle$ is either of the type B_6, or contains a subdiagram of the type F_4, or is a quasi-Lannér diagram, respectively. If $[u, v] = 4$, then $\langle S_0, u, v\rangle$ is either of the type $\widetilde{C}_5$ or contains a subdiagram of the type $\widetilde{C}_2$ or, again, is a quasi-Lannér diagram. Now recall that $\Sigma(P)$ contains neither quasi-Lannér diagrams nor elliptic diagrams of order greater than 5, and any connected parabolic subdiagram of $\Sigma(P)$ should be of order 5. Therefore, v is also a good neighbor of S_0, the diagram $\langle S_0, v\rangle$ is either B_5 or $\widetilde{B}_4$ or $\widetilde{C}_4$, and the diagram $\Sigma = \langle S_0, u, v\rangle$ coincides with one of the diagrams shown in Fig. 8(a)–(g).

Since $\Sigma(P)$ contains no multiple edges except for double edges, $\Sigma(P)$ contains also some diagram from the list $L'(\Sigma, 4, 5)$. For the diagram from Fig. 8(a) this list consists of the two diagrams shown in Fig. 8(h) and (i) (denote these diagrams by Σ_1 and Σ_2). As Σ ranges over the diagrams shown in Fig. 8(b)–(g), the only diagram from a list $L'(\Sigma, 4, 5)$ which contains neither a subdiagram of type F_4 nor a parabolic subdiagram of order less than 5, is that shown in Fig. 8(j) (denote it by Σ_3).

Similarly, $\Sigma(P)$ contains some diagram appearing in either $L'(\Sigma_1, 4, 5)$ or $L'(\Sigma_2, 4, 5)$ or $L'(\Sigma_3, 4, 5)$. The latter list is empty, and the former two coincide and consist of a unique diagram shown in Fig. 8(k). The latter diagram contains a subdiagram of the type F_4, and we come to a contradiction. □

Proposition 4.3.6. *$\Sigma(P)$ contains no subdiagram of the type B_3.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_3$. By Proposition 4.3.5, S_0 has no good neighbors and $\Sigma(P)$ contains some subdiagram from the list $L(S_0, 5)$. In the list $L(S_0, 5)$ there is no diagram containing neither subdiagram of the type $G_2^{(k)}$, $k \geqslant 5$, nor subdiagram of the types B_4 and F_4. □

Proposition 4.3.7. *$\Sigma(P)$ contains no subdiagram of the type B_2.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_2$. By Proposition 4.3.6, S_0 has no good neighbors, so by Lemma 7, $\overline{S}_0$ is a Lannér diagram of order 4. Hence, $\overline{S}_0$ contains a subdiagram of the type either H_3 or B_3, which is impossible by Propositions 4.3.3 and 4.3.6. □

Lemma 10. *Let P be a simple hyperbolic Coxeter 5-polytope. Then either P has a pair of disjoint facets or P is a non-compact simplex.*

Proof. Suppose that P is not a simplex and P has no pair of disjoint facets. By Propositions 4.3.2, 4.3.4 and 4.3.7, $\Sigma(P)$ contains no multiple edges. At the same time, any Lannér diagram of order greater than 2 contains a multiple edge. Hence, $\Sigma(P)$ contains no Lannér diagram of order greater than 2. By Corollary 1, $\Sigma(P)$ contains no quasi-Lannér diagram as well. This means that $\Sigma(P)$ contains a Lannér diagram of order 2, i.e. a dotted edge. □

4.4. Dimension 6

In this section we suppose that P is a simple Coxeter 6-polytope having no pair of disjoint facets, and P is not a simplex.

Proposition 4.4.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $\Sigma(P)$ contains a multi-multiple edge. Choose $S_0 = G_2^{(k)}$, $k \geqslant 6$, as an edge of maximal multiplicity in $\Sigma(P)$. Clearly, S_0 has no good neighbors and $\Sigma(P)$ contains some diagram appearing in the list $L(S_0, 6)$. This list turns out to be empty. □

Proposition 4.4.2. *$\Sigma(P)$ contains neither subdiagram of the type F_4 nor subdiagram of the type H_4.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram S_0 of the type either F_4 or H_4. Then S_0 has no good neighbors and $\Sigma(P)$ contains a subdiagram from the list $L(S_0, 6)$.

The list $L_\alpha(S_0, 6)$ contains a unique diagram Σ without multi-multiple edges. This diagram is shown in Fig. 9(a). Suppose that $\Sigma \subset \Sigma(P)$. By Lemma 6, Σ is not a diagram of a 6-polytope, hence, $\Sigma(P)$ contains some diagram from the list $L'(\Sigma, 5, 6)$. Further, denote by S the subdiagram of Σ of the type B_5. Then $\Sigma(P)$ contains also some diagram from the list $L'(\Sigma, 5, 6, S^{(g,n)})$. But this list is empty, so $\Sigma(P)$ contains no subdiagram of the type shown in Fig. 9(a).

The list $L_\beta(S_0, 6)$ contains five diagrams without multi-multiple edges. These diagrams are shown in Fig. 9(b)–(f). The diagram shown in Fig. 9(b) contains parabolic subdiagrams $\widetilde{C}_3$

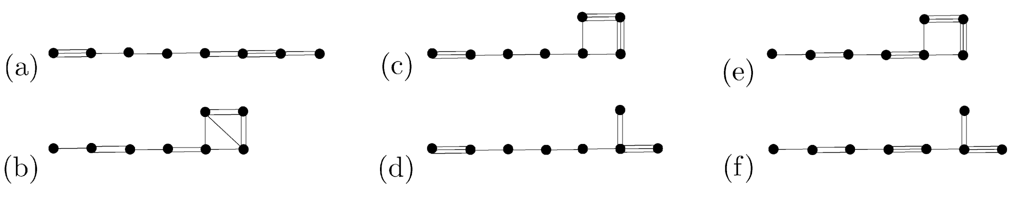

Fig. 9. Intermediate results for $d = 6$, $S_0 = H_4$ and F_4.

Fig. 10. Intermediate results for $d = 6$, $S_0 = H_3$.

and $\widetilde{A}_2$, which is impossible. Suppose that $\Sigma(P)$ contains a subdiagram Σ which is one of the four diagrams shown in Fig. 9(c)–(f). By Lemma 6, Σ is not a diagram of a 6-polytope, hence, $\Sigma(P)$ contains some diagram from the list $L'(\Sigma, 5, 6)$. In the cases Fig. 9(c) and (d) denote by S a subdiagram of Σ of the type H_4 having two neighbors in Σ. In the cases Fig. 9(e) and (f) denote by S a subdiagram of Σ of the type H_3 such that S is disjoint from the subdiagram of the type F_4. Then $\Sigma(P)$ contains some diagram from the list $L'(\Sigma, 5, 6, S^{(g,n)})$. However, this list is empty in each of the four cases. $\square$

Proposition 4.4.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

Proof. Suppose that $\Sigma(P) \supset S_0 = H_3$. By Proposition 4.4.2, S_0 has no good neighbors and $\Sigma(P)$ contains a diagram from the list $L(S_0, 6)$.

In the list $L_\alpha(S_0, 6)$ there is a unique diagram Σ containing neither multi-multiple edges nor subdiagram of the types H_4 and F_4. This diagram is shown in Fig. 10(a). By Lemma 6, Σ is not a diagram of a 6-polytope, so $\Sigma(P)$ contains a subdiagram appearing in the list $L'(\Sigma, 5, 6)$. Denote by S a subdiagram of Σ of the type B_3. Then $\Sigma(P)$ contains a subdiagram from the list $L'(\Sigma, 5, 6, S^{(g,n)})$. This list consists of a unique diagram Σ' shown in Fig. 10(b). The diagram Σ' contains a subdiagram of the type H_4, which is impossible by Proposition 4.4.2.

In the list $L_\beta(S_0, 6)$ there is no diagram containing neither a multi-multiple edge nor a subdiagram of the types H_4 and F_4. $\square$

Proposition 4.4.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $\Sigma(P) \supset S_0 = G_2^{(5)}$. By Proposition 4.4.3, S_0 has no good neighbors, so $\overline{S}_0 = \Sigma_{S_0}$, and $P(S_0)$ is a simple Coxeter 4-polytope without disjoint facets. By Lemma 9, Σ_{S_0} contains either a parabolic subdiagram of the type $\widetilde{C}_3$, or a subdiagram of the type H_4, or a subdiagram of the type F_4, which is impossible by Proposition 4.4.2. $\square$

By Propositions 4.4.1 and 4.4.4, any multiple edge in $\Sigma(P)$ is a double edge.

Proposition 4.4.5. *$\Sigma(P)$ contains no subdiagram of the type B_5.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_5$. The same argument as in Proposition 4.3.5 shows that $\Sigma(P)$ contains a subdiagram Σ which coincides with one of the diagrams shown in Fig. 11(a)–(g). By Lemma 6, none of these diagrams is a diagram of a 6-polytope, so $\Sigma(P)$ contains a diagram from the list $L'(\Sigma, 4, 6)$. The union of these lists contains more than 50 diagrams, but only one of these diagrams contains neither a subdiagram of the type F_4 nor a parabolic subdiagram of rank less than 5. This diagram Σ' is shown in Fig. 11(h). By Lemma 6, this diagram is not a diagram of a 6-polytope, so $\Sigma(P)$ contains a diagram from the list $L'(\Sigma', 4, 6)$. The list $L'(\Sigma', 4, 6)$ consists of a unique diagram Σ'' shown in Fig. 11(i). However, the diagram Σ'' contains a subdiagram of the type F_4, which is impossible by Proposition 4.4.2. $\square$

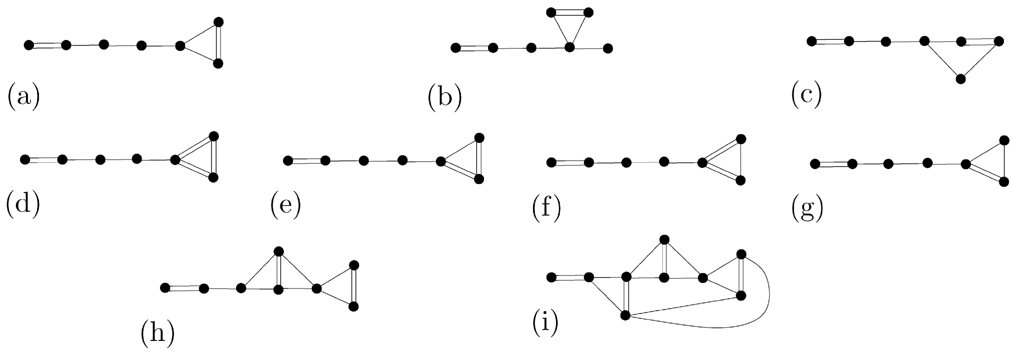

Fig. 11. Intermediate results for $d = 6$, $S_0 = B_5$.

Proposition 4.4.6. *$\Sigma(P)$ contains no subdiagram of the type B_4.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_4$. By Proposition 4.4.5, S_0 has no good neighbors and $\Sigma(P)$ contains a subdiagram from the list $L(S_0, 6)$.

In the list $L_\alpha(S_0, 6)$ there is no diagram containing neither subdiagram $G_2^{(k)}$, $k \geqslant 5$, nor subdiagram of the types B_5 and F_4. In the list $L_\beta(S_0, 6)$ there are two diagrams containing neither a subdiagram $G_2^{(k)}$, $k \geqslant 5$, nor a subdiagram of the types B_5 and F_4. These two diagrams are shown in Fig. 12(a) and (b). Both of these diagrams contain parabolic subdiagrams of order 3, which is impossible. □

Proposition 4.4.7. *$\Sigma(P)$ contains no subdiagram of the type B_3.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = B_3$. By Proposition 4.4.6 S_0 has no good neighbors and $\Sigma(P)$ contains a subdiagram from the list $L(S_0, 6)$. However, in the list $L(S_0, 6)$ there is no diagram containing neither a subdiagram $G_2^{(k)}$, $k \geqslant 5$, nor a subdiagram of the types B_4 and F_4. □

Proposition 4.4.8. *$\Sigma(P)$ contains no subdiagram of the type B_2.*

Proof. Suppose that $\Sigma(P) \supset S_0 = B_2$. By Proposition 4.4.7, S_0 has no good neighbors, and the proof follows the proof of Proposition 4.4.4. □

Lemma 11. *Let P be a simple hyperbolic Coxeter 6-polytope. Then either P has a pair of disjoint facets or P is a non-compact simplex.*

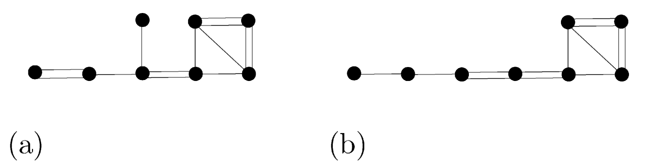

Fig. 12. Intermediate results for $d = 6$, $S_0 = B_4$.

Proof. Suppose that P is not a simplex. By Propositions 4.4.1, 4.4.4 and 4.4.8, $\Sigma(P)$ contains no multiple edges. Now the proof follows the proof of Lemma 10. □

4.5. Large dimensions

In this section we assume that P is a simple hyperbolic Coxeter d-polytope ($d \geqslant 7$) containing no pair of disjoint facets, and P is not a simplex. We also assume that P is such a polytope of minimal possible dimension. We recall that $\Sigma(P)$ contains no quasi-Lannér diagrams (see Corollary 1), so if $S_0 \subset \Sigma(P)$ is an elliptic diagram, $\overline{S}_0 = \Sigma_{S_0}$, and Σ_{S_0} does not contain dotted edges, then the dimension of $P(S_0)$ is at most 4.

Proposition 4.5.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = G_2^{(k)}$ for some $k > 5$. Then S_0 has no good neighbors. Therefore, $P(S_0)$ is a Coxeter $(d-2)$-polytope without a pair of disjoint facets, and we contradict our assumptions. □

Proposition 4.5.2. *$\Sigma(P)$ contains neither subdiagram of the type H_4 nor subdiagram of the type F_4.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = H_4$ or F_4.

For $d = 7$ we check the lists $L(S_0, d)$. The union of these lists for $S_0 = H_4$ and F_4 consists of four diagrams $\Sigma_1, \ldots, \Sigma_4$ shown in Fig. 13(a)–(d). Denote by S a subdiagram of Σ_i of type H_4 having either two ($i = 1, 2$) or three ($i = 3, 4$) bad neighbors. Since any neighbor of S is bad and none of Σ_i is a diagram of a 7-polytope, $\Sigma(P)$ contains a subdiagram from the list $L'(\Sigma_i, 5, 7, S^{(n)})$ for some $i \leqslant 4$. The lists $L'(\Sigma_i, 5, 7, S^{(n)})$ for $i = 1, 2, 3$ are empty, and the list $L'(\Sigma_4, 5, 7, S^{(n)})$ consists of a unique diagram Σ'_4 shown in Fig. 13(e). Again, $\Sigma(P)$ should contain a subdiagram from the list $L'(\Sigma'_1, 5, 7, S^{(n)})$ for the same S. However, this list is empty.

For $d = 8$ we check the lists $L(S_0, d)$ which turn out to be empty.

For $d > 8$ consider the $(d-4)$-polytope $P(S_0)$. By Corollary 2, $\Sigma_{S_0} = \overline{S}_0$. It follows that Σ_{S_0} contains no dotted edges and $P(S_0)$ is a Coxeter $(d-4)$-polytope without pair of disjoint facets. If $d = 9$ or $d = 10$, this contradicts Lemmas 10 and 11, respectively. If $d > 10$, this contradicts the assumption that d is the minimal possible dimension of such a polytope. □

Proposition 4.5.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

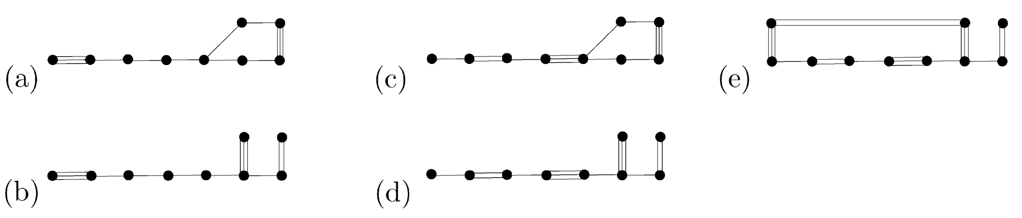

Fig. 13. Intermediate results for $d = 7$, $S_0 = H_4$ and F_4.

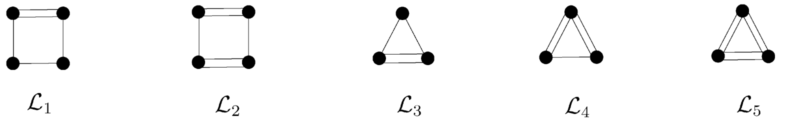

Fig. 14. Notation for some Lannér diagrams.

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = H_3$. By Proposition 4.5.2, S_0 has no good neighbors. Thus, it follows from Corollary 2 that $P(S_0)$ is a Coxeter $(d-3)$-polytope without a pair of disjoint facets. If $d > 7$ then as in Proposition 4.5.2 we have a contradiction.

Suppose that $d = 7$. Then $\overline{S}_0$ is either a Lannér diagram of order 5 or one of the Esselmann diagrams. In any case, $\overline{S}_0$ contains either a subdiagram of the type H_4 or a subdiagram of the type F_4, which is impossible by Proposition 4.5.2. □

Proposition 4.5.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $\Sigma(P)$ contains a subdiagram $S_0 = G_2^{(5)}$. Since $\Sigma(P)$ contains no subdiagram of the type H_3 (Proposition 4.5.3), S_0 has no good neighbors. Thus, $P(S_0)$ is a Coxeter $(d-2)$-polytope without a pair of disjoint facets, and we come to a contradiction. □

As a corollary of Proposition 4.5.4, we may assume that all multiple edges in $\Sigma(P)$ are double edges.

Proposition 4.5.5. *Any Lannér subdiagram of $\Sigma(P)$ is one of the five diagrams shown in Fig.* 14.

Proof. By the assumption $\Sigma(P)$ contains no Lannér diagrams of order 2. Thus, the statement follows from the classification of Lannér diagrams and Proposition 4.5.4. □

Proposition 4.5.6. *If $\Sigma(P)$ contains a subdiagram $S = B_3$ or B_2 then S has at least* 2 *good neighbors. In addition, for any good neighbor u of S the diagram $\langle S, u\rangle$ is of the type B_4 or B_3, respectively.*

Proof. Consider the Coxeter polytope $P(S)$. Suppose that $P(S)$ has no pair of disjoint facets. Then, by assumption, the dimension of $P(S)$ is at most 4, which means that $d = 7$ and $S = B_3$ (see Lemmas 4.3 and 4.4). As in the proof of Proposition 4.5.3, Σ_S contains a subdiagram $\Sigma = H_4$ or F_4. By Corollary 2, $\overline{S}$ also contains Σ, which contradicts Proposition 4.5.2.

Now we may assume that $P(S)$ has a pair of disjoint facets. Let v and u be the vertices of Σ_S joined by a dotted edge. Denote by $\bar{v}$ and $\bar{u}$ the corresponding vertices of $\Sigma(P)$. In view of Theorem 2, we may assume that one of v and u, say v, is a good neighbor of S (otherwise $[v,u] = [\bar{v},\bar{u}] \neq \infty$). Suppose that u is not a neighbor of S. By Proposition 4.5.4, $[\bar{v},\bar{u}] \leqslant 4$. If $[\bar{v},\bar{u}] = 4$ then $\langle S,\bar{u},\bar{v}\rangle = \widetilde{C}_4$ or $\widetilde{C}_3$ which are parabolic of small order. Thus, $[\bar{v},\bar{u}] = 2$ or 3. By item (2*b*) of Theorem 2, we have $[v,u] = 2$ or 4, respectively, in contradiction to the assumption that $[v,u] = \infty$. Therefore, u is also a good neighbor of S.

By Proposition 2.1.1, $\langle S,u\rangle$ is not parabolic, and Proposition 4.5.2 implies that $\langle S,u\rangle \neq F_4$, which finishes the proof. □

Proposition 4.5.7. *$\Sigma(P)$ contains no subdiagram of the type $\mathcal{L}_1$ (see Fig.* 14).

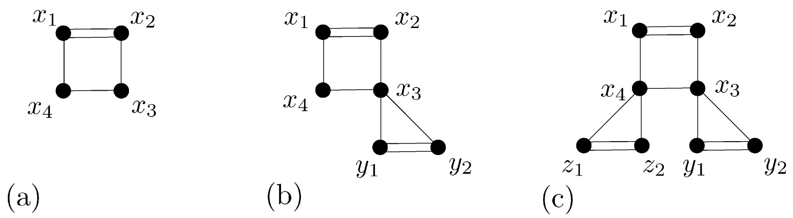

Fig. 15. Notation for the proof of Proposition 4.5.7.

Proof. Suppose the contrary. Denote the vertices of the subdiagram as shown in Fig. 15(a). By Proposition 4.5.6, the subdiagram $B_3 = \langle x_1, x_2, x_3\rangle$ has at least 2 good neighbors y_1 and y_2. By Proposition 4.5.6, $\langle x_1, x_2, x_3, y_i\rangle = B_4$ for $i = 1, 2$. Clearly, $[y_1, y_2] = 4$, otherwise we have either a parabolic subdiagram $\langle x_1, x_2, x_3, y_1, y_2\rangle = \widetilde{B}_4$ or a parabolic subdiagram $\langle x_3, y_1, y_2\rangle = \widetilde{A}_2$. Further, $[x_4, y_i] \neq 3$ (otherwise $\langle x_4, x_3, y_i\rangle = \widetilde{A}_2$), and $[x_4, y_i] \neq 4$ (otherwise $\langle x_2, x_1, x_4, y_i\rangle = \widetilde{C}_3$). Hence, by Proposition 4.5.4 we have $[x_4, y_i] = 2$ and $\langle x_1, x_2, x_3, x_4, y_1, y_2\rangle$ is the diagram shown in Fig. 15(b).

Consider now a pair of good neighbors of the subdiagram $B_3 = \langle x_4, x_3, x_2\rangle$ denoting these neighbors by z_1 and z_2. Then $[z_i, y_j] \neq 3$ for any $i, j \in \{1, 2\}$ (otherwise $\langle z_i, x_3, x_4, y_j\rangle = \widetilde{A}_3$). We also have $[z_i, y_j] \neq 4$ for any $i, j \in \{1, 2\}$ (otherwise $\langle x_1, x_2, x_3, z_i, y_j\rangle = \widetilde{C}_4$ in contradiction to Proposition 4.5.2). Thus, $\langle x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2\rangle$ is the diagram shown in Fig. 15(c). An explicit calculation shows that the subdiagram $\langle z_1, z_2, x_4, x_3, y_1, y_2\rangle$ is superhyperbolic. □

Proposition 4.5.8. *$\Sigma(P)$ contains no subdiagram of the type $\mathcal{L}_2$.*

Proof. Suppose the contrary. Denote the vertices of the subdiagram as shown in Fig. 16(a). By Proposition 4.5.6, the subdiagram $B_3 = \langle x_1, x_2, x_3\rangle$ has at least 2 good neighbors y_1 and y_2. By Proposition 4.5.6, $\langle x_1, x_2, x_3, y_i\rangle = B_4$. Clearly, $[y_1, y_2] = 4$ (see the proof of Proposition 4.5.7). Further, $[y_i, x_4] \neq 2$ and $[y_i, x_4] \neq 4$ (otherwise we have a parabolic subdiagram $\langle x_2, x_3, x_4, y_i\rangle = \widetilde{B}_3$ or $\langle y_i, x_4, x_1, x_2\rangle = \widetilde{C}_3$, respectively). Thus, $[y_i, x_4] = 3$ and $\langle x_1, x_2, x_3, x_4, y_1, y_2\rangle$ is the diagram shown in Fig. 16(b).

Consider a pair of good neighbors of the subdiagram $B_3 = \langle x_4, x_3, x_2\rangle$ denoting them by z_1 and z_2. Then $[z_i, y_j] \neq 3$ or 4 for any $i, j \in \{1, 2\}$ (otherwise, we have, respectively, $\langle z_i, x_1, x_4, y_j\rangle = \widetilde{A}_3$ or $\mathcal{L}_1$). Thus, $[z_i, y_j] = 2$ and $\langle x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2\rangle$ is the diagram

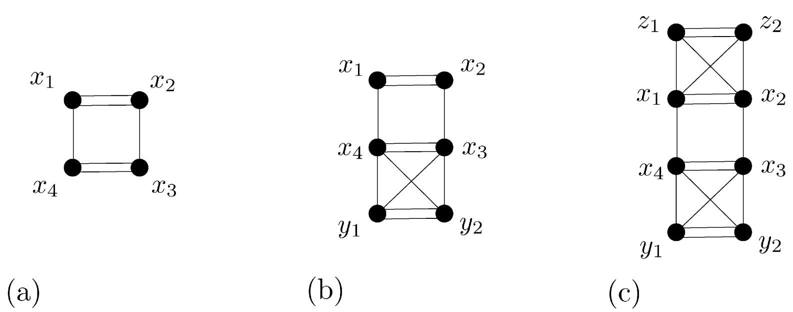

Fig. 16. Notation for the proof of Proposition 4.5.8.

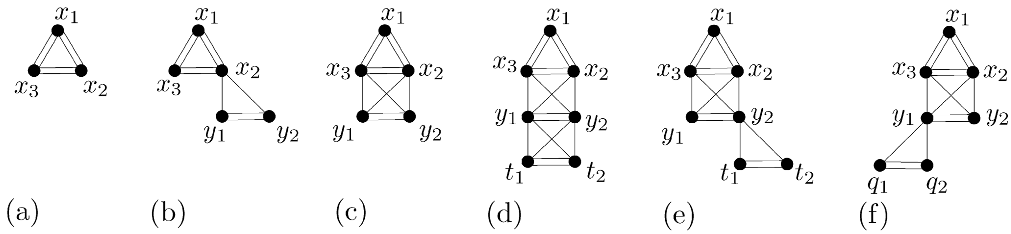

Fig. 17. Notation for the proof of Proposition 4.5.9.

shown in Fig. 16(c). An explicit check shows that the subdiagram $\langle z_1, z_2, x_2, x_3, y_1, y_2\rangle$ is superhyperbolic. □

Proposition 4.5.9. *$\Sigma(P)$ contains no subdiagram of the type $\mathcal{L}_5$.*

Proof. Suppose the contrary. Denote the vertices of the subdiagram as shown in Fig. 17(a). By Proposition 4.5.6, the subdiagram $B_2 = \langle x_1, x_2\rangle$ has at least 2 good neighbors y_1 and y_2. We may assume that $[y_1, x_1] = 2$ and $[y_1, x_2] = 3$. Then we have similar conditions for y_2: $[y_2, x_1] = 2$ and $[y_2, x_2] = 3$ (otherwise we have $[y_2, x_1] = 3$ and $[y_2, x_2] = 2$, so the diagram $\langle x_1, x_2, y_1, y_2\rangle$ is either F_4 (forbidden by Proposition 4.5.2) or a cyclic Lannér diagram forbidden by Propositions 4.5.7 and 4.5.8). Clearly, $[y_1, y_2] = 4$, otherwise either $\langle x_1, x_2, y_1, y_2\rangle = \widetilde{B}_3$ or $\langle x_2, y_1, y_2\rangle = \widetilde{A}_2$. Furthermore, $[y_i, x_3] \neq 4$ (otherwise $\langle y_i, x_3, x_1\rangle = \widetilde{C}_2$), and $[y_1, x_3] = 3$ if and only if $[y_2, x_3] = 3$ (otherwise $\langle y_2, y_1, x_3, x_1\rangle = \widetilde{C}_3$). Thus, $\langle x_1, x_2, x_3, y_1, y_2\rangle$ is one of two diagrams shown in Fig. 17(b) and (c).

Suppose that $\Sigma(P)$ contains no subdiagram of the type shown in Fig. 17(c), i.e. $\langle x_1, x_2, x_3, y_1, y_2\rangle$ is the diagram shown in Fig. 17(b). Consider good neighbors z_1 and z_2 of the subdiagram $B_2 = \langle x_1, x_3\rangle$. Without loss of generality, we may assume that z_1 and z_2 are neighbors of x_3. By the assumption, z_1 and z_2 are not neighbors of x_1 and x_2 (otherwise $\langle x_1, x_2, x_3, z_1, z_2\rangle$ is a subdiagram of the type shown in Fig. 17(c)). It follows that the subdiagram $\langle z_1, x_3, x_2, y_1\rangle$ is either F_4 (forbidden by Proposition 4.5.2) or a cyclic Lannér diagram (forbidden by Propositions 4.5.7 and 4.5.8). We come to a contradiction which shows that $\Sigma(P)$ contains a subdiagram of the type shown in Fig. 17(c). We now assume that $\langle x_1, x_2, x_3, y_1, y_2\rangle$ is this subdiagram.

Consider two good neighbors t_1 and t_2 of the subdiagram $B_3 = \langle x_1, x_2, y_2\rangle$. We have

(1) $[t_i, x_3] = 2$ (otherwise either $\langle t_i, y_2, x_3\rangle = \widetilde{A}_2$ or $\langle x_1, x_3, t_i\rangle = \widetilde{C}_2$);
(2) $[t_1, t_2] = 4$ (otherwise either $\langle t_1, t_2, y_2\rangle = \widetilde{A}_2$ or $\langle x_1, x_2, y_2, t_1, t_2\rangle = \widetilde{B}_4$);
(3) $[t_i, y_1] \neq 4$ (otherwise $\langle x_1, x_3, y_1, t_i\rangle = \widetilde{C}_3$);
(4) either $[t_1, y_1] = [t_2, y_2] = 3$ or $[t_1, y_1] = [t_2, y_2] = 2$ (otherwise $\langle x_1, x_3, y_1, t_1, t_2\rangle = \widetilde{C}_4$).

Therefore, $\langle x_1, x_2, x_3, y_1, y_2, t_1, t_2\rangle$ is one of two diagrams shown in Fig. 17(d) and (e). The diagram shown in Fig. 17(d) is superhyperbolic. Thus, $\langle x_1, x_2, x_3, y_1, y_2, t_1, t_2\rangle$ is the diagram shown in Fig. 17(e).

Consider two good neighbors q_1 and q_2 of the subdiagram $B_3 = \langle x_1, x_3, y_1\rangle$. Reasoning as above shows that the subdiagram $\langle x_1, x_2, x_3, y_1, y_2, q_1, q_2\rangle$ looks like the diagram shown in Fig. 17(f). Then the subdiagram $\langle q_1, y_1, y_2, t_1\rangle$ is either F_4 (forbidden by Proposition 4.5.2) or a cyclic Lannér diagram (forbidden by Propositions 4.5.7 and 4.5.8). The contradiction proves the statement. □

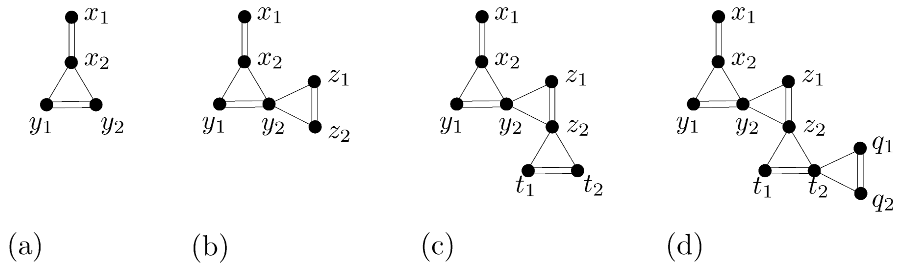

Fig. 18. Notation for the proof of Proposition 4.5.10.

Proposition 4.5.10. *$\Sigma(P)$ contains no subdiagram of the type B_2.*

Proof. Suppose the contrary. Let $\langle x_1, x_2\rangle$ be the vertices of B_2. Let y_1 and y_2 be two good neighbors of $\langle x_1, x_2\rangle$. Clearly, $\langle x_1, x_2, y_1, y_2\rangle$ is the diagram shown in Fig. 18(a).

Let z_1 and z_2 be two good neighbors of $B_2 = \langle y_1, y_2\rangle$. We have $[z_i, x_2] \neq 3$ (otherwise $\langle x_2, y_2, z_i\rangle = \widetilde{A}_2$). If $[z_1, x_2] = 4$, then $[z_2, x_2] = 4$ (otherwise $\langle z_1, z_2, x_2\rangle = \widetilde{C}_2$), and $\langle z_1, z_2, x_2\rangle = \mathcal{L}_5$, which contradicts Proposition 4.5.9. So, $[z_i, x_2] = 2$. Furthermore, $[z_i, x_1] = 2$, otherwise the cycle $\langle x_1, x_2, y_2, z_i\rangle = \mathcal{L}_1$ or $\mathcal{L}_2$, which contradicts Propositions 4.5.7 and 4.5.8. Thus, $\langle x_1, x_2, y_1, y_2, z_1, z_2\rangle$ is the diagram shown in Fig. 18(b).

Let t_1 and t_2 be two good neighbors of $B_2 = \langle z_1, z_2\rangle$. Repeating the argument above we obtain that t_i is not connected to z_1, y_2 and y_1. Moreover, $[t_i, x_2] = 2$ (see $\langle x_2, y_2, z_2, t_i\rangle$), and $[t_i, x_1] = 2$ (otherwise either $\langle t_2, x_1, x_2\rangle = \widetilde{C}_2$ or $\langle t_i, x_1, x_2, y_1\rangle = F_4$). Thus, $\langle x_1, x_2, y_1, y_2, z_1, z_2, t_1, t_2\rangle$ is the diagram shown in Fig. 18(c). This diagram is superhyperbolic, and the proof is complete. $\square$

Remark. If we consider two good neighbors q_1, q_2 of $B_2 = \langle t_1, t_2\rangle$, we obtain a diagram shown in Fig. 18(d), which is evidently superhyperbolic.

Now we are able to finish the proof of the theorems.

Lemma 12. *Let P be a simple hyperbolic Coxeter d-polytope. If $d > 9$ then P has a pair of disjoint facets. If $6 < d \leqslant 9$ then either P has a pair of disjoint facets or P is a non-compact simplex.*

Proof. Suppose that P is not a simplex. It follows from Propositions 4.5.5 and 4.5.10 that $\Sigma(P)$ contains no Lannér subdiagrams of order greater than 2. Therefore, it contains a dotted edge, and the lemma is proved. $\square$

Acknowledgments

The authors would like to thank the referee for many helpful comments and suggestions. We are grateful to D. Panov for pointing out to us the result of D. Allcock, and to D. Allcock who introduced us to his paper [1], which was not published yet.

References

[1] D. Allcock, Infinitely many hyperbolic Coxeter groups through dimension 19, Geom. Topol. 10 (2006) 737–758.

[2] R. Borcherds, Coxeter groups, Lorentzian lattices, and K3 surfaces, Int. Math. Res. Not. 19 (1998) 1011–1031.
[3] N. Bourbaki, Groupes et algèbres de Lie, Hermann, Paris, 1968 (Ch. IV–VI).
[4] H.S.M. Coxeter, Discrete groups generated by reflections, Ann. of Math. 35 (1934) 588–621.
[5] F. Esselmann, The classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets, Comment. Math. Helv. 71 (1996) 229–242.
[6] A. Felikson, P. Tumarkin, Reflection subgroups of reflection groups, Funct. Anal. Appl. 38 (2004) 313–314.
[7] B. Grünbaum, Convex Polytopes, John Wiley & Sons, 1967.
[8] I.M. Kaplinskaya, Discrete groups generated by reflections in the faces of simplicial prisms in Lobachevskian spaces, Math. Notes 15 (1974) 88–91.
[9] A.G. Khovanskij, Hyperplane sections of polyhedra, toric varieties, and discrete groups in Lobachevskij spaces, Funct. Anal. Appl. 20 (1986) 41–50.
[10] F. Lannér, On complexes with transitive groups of automorphisms, Comm. Sem. Math. Univ. Lund. 11 (1950) 1–71.
[11] M.N. Prokhorov, The absence of discrete reflection groups with non-compact fundamental polyhedron of finite volume in Lobachevsky space of large dimension, Math. USSR Izv. 28 (1987) 401–411.
[12] P. Tumarkin, Hyperbolic Coxeter n-polytopes with $n+2$ facets, Math. Notes 75 (2004) 848–854.
[13] E.B. Vinberg, Discrete groups generated by reflections in Lobachevskii spaces, Math. USSR Sb. 1 (1967) 429–444.
[14] E.B. Vinberg, The absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension, Trans. Moscow Math. Soc. 47 (1985) 75–112.
[15] E.B. Vinberg, Hyperbolic reflection groups, Russian Math. Surveys 40 (1985) 31–75.
[16] E.B. Vinberg (Ed.), Geometry II, Encyclopaedia Math. Sci., vol. 29, Springer-Verlag, Berlin, 1993.