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Positroids and Schubert matroids

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A B S T R A C T

Postnikov gave a combinatorial description of the cells in a totally nonnegative Grassmannian. These cells correspond to a special class of matroids called positroids. We prove his conjecture that a positroid is exactly an intersection of cyclically shifted Schubert matroids. This leads to a combinatorial description of positroids that is easily computable.

1. Introduction

A ***positroid*** is a matroid that can be represented by a $k \times n$-matrix with nonnegative maximal minors. The classical theory of total positivity concerns matrices in which all minors are nonnegative, and this subject was extended by Lusztig [4].

Lusztig introduced the totally nonnegative variety $G_{\geqslant 0}$ in an arbitrary reductive group G and the totally nonnegative part $(G/P)_{\geqslant 0}$ of a real flag variety (G/P). He also conjectured that $(G/P)_{\geqslant 0}$ is made up of cells, and this was proved by Rietsch [6].

In this paper, we will restrict our attention to $(\mathrm{Gr}_{kn})_{\geqslant 0}$, the ***totally nonnegative Grassmannian***. Then there is a more refined decomposition using matroid strata. Postnikov obtained a relationship between $(\mathrm{Gr}_{kn})_{\geqslant 0}$ and certain planar bicolored graphs, producing a combinatorially explicit cell decomposition of $(\mathrm{Gr}_{kn})_{\geqslant 0}$ [5]. The cells correspond to positroids.

One of the results of [5] is that each cell is an intersection of $(\mathrm{Gr}_{kn})_{\geqslant 0}$ and Schubert cells corresponding to a combinatorial object called the Grassmann necklace. This result implies that each positroid is included in an intersection of cyclically shifted Schubert matroids. We extend this result: each positroid is exactly an intersection of certain cyclically shifted Schubert matroids.

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A more detailed formulation of the main result follows. Let $[n] := \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ be the collection of all k-element subsets of $[n]$. Fix some $t \in [n]$. We define the ordering $<_t$ on $[n]$ by the total order $t <_t t+1 <_t \cdots <_t n <_t 1 \cdots <_t t-1$. For $I, J \in \binom{[n]}{k}$, where

$$I = \{i_1, \ldots, i_k\}, \quad i_1 <_t i_2 \cdots <_t i_k$$

and

$$J = \{j_1, \ldots, j_k\}, \quad j_1 <_t j_2 \cdots <_t j_k,$$

we set

$$I \leqslant_t J \quad \text{if and only if} \quad i_1 \leqslant_t j_1, \ldots, i_k \leqslant_t j_k.$$

For each $I \in \binom{[n]}{k}$ and $w \in S_n$, we define the ***cyclically shifted Schubert matroid*** as

$$\mathrm{SM}_I^t := \left\{ J \in \binom{[n]}{k} \;\middle|\; I \leqslant_t J \right\}.$$

We will show that a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ is a positroid if and only if it can be written as $\mathrm{SM}_{I_1}^1 \cap \mathrm{SM}_{I_2}^2 \cap \cdots \cap \mathrm{SM}_{I_n}^n$ for a Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$, $I_1, \ldots, I_n \in \binom{[n]}{k}$. Our proof is purely combinatorial.

The paper is organized as follows. In Section 2, we go over the basics of matroids and the totally nonnegative Grassmannian. In Section 3, we review ⅃-diagrams and Γ-graphs. In Section 4, we give the proof of our main result. In Section 5, we introduce the upper Grassmann necklace. In Section 6, we view lattice path matroids as special cases of positroids.

Remark 1. After this paper was finished, further developments on this subject using geometry of flag manifolds, or Poisson geometry were obtained in [3] and [8].

2. Preliminaries and the main result

We would like to guide the readers unfamiliar with basics in this section to [2] and [5].

A ***matroid*** of rank k on the set $[n]$ is a nonempty collection $\mathcal{M} \subseteq \binom{[n]}{k}$ of k-element subsets of $[n]$, called ***bases*** of $\mathcal{M}$, that satisfies the exchange axiom:

For any $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $I \setminus \{i\} \cup \{j\} \in \mathcal{M}$.

An element in the Grassmannian Gr_{kn} can be understood as a collection of n vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ spanning the space $\mathbb{R}^k$ modulo the simultaneous action of GL_k on the vectors. The vectors v_i are the columns of a $k \times n$-matrix A that represents the element of the Grassmannian. Then an element $V \in \mathrm{Gr}_{kn}$ represented by A gives the matroid $\mathcal{M}_V$ whose bases are the k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$. Here, $\Delta_I(A)$ denotes the determinant of A_I, the k-by-k submatrix of A with the column set I.

Then Gr_{kn} has a subdivision into ***matroid strata*** $S_\mathcal{M}$ labeled by some matroids $\mathcal{M}$:

$$S_\mathcal{M} := \{V \in \mathrm{Gr}_{kn} \mid \mathcal{M}_V = \mathcal{M}\}.$$

The elements of the stratum $S_\mathcal{M}$ are represented by matrices A such that $\Delta_I(A) \neq 0$ if and only if $I \in \mathcal{M}$.

Let us define the totally nonnegative Grassmannian and its cells.

Definition 2. (See [5, Definition 3.1].) The ***totally nonnegative Grassmannian*** $\mathrm{Gr}_{kn}^{\mathrm{tnn}} \subset \mathrm{Gr}_{kn}$ is the quotient $\mathrm{Gr}_{kn}^{\mathrm{tnn}} = \mathrm{GL}_k^+ \setminus \mathrm{Mat}_{kn}^{\mathrm{tnn}}$, where $\mathrm{Mat}_{kn}^{\mathrm{tnn}}$ is the set of real $k \times n$-matrices A of rank k with nonnegative maximal minors $\Delta_I(A) \geqslant 0$ and GL_k^+ is the group of $k \times k$-matrices with positive determinant.

Definition 3. (See [5, Definition 3.2].) ***Totally nonnegative Grassmann cells*** $S_\mathcal{M}^{\mathrm{tnn}}$ in $\mathrm{Gr}_{kn}^{\mathrm{tnn}}$ are defined as $S_\mathcal{M}^{\mathrm{tnn}} := S_\mathcal{M} \cap \mathrm{Gr}_{kn}^{\mathrm{tnn}}$. $\mathcal{M}$ is called a ***positroid*** if the cell $S_\mathcal{M}^{\mathrm{tnn}}$ is nonempty.

Note that from the above definitions, we get

$$S_{\mathcal{M}}^{\mathrm{tnn}} = \left\{ \mathrm{GL}_k^+ \bullet A \in \mathrm{Gr}_{kn}^{\mathrm{tnn}} \;\middle|\; \Delta_I(A) > 0 \text{ for } I \in \mathcal{M},\ \Delta_I(A) = 0 \text{ for } I \notin \mathcal{M} \right\}.$$

In [5], Postnikov constructed a bijection between the cells and combinatorial objects called Grassmann necklaces.

Definition 4. (See [5, Definition 16.1].) A ***Grassmann necklace*** is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of subsets $I_r \subseteq [n]$ such that:

- if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$,
- if $i \notin I_i$ then $I_{i+1} = I_i$.

The indices are taken modulo n. In particular, we have $|I_1| = \cdots = |I_n|$.

An example of a Grassmann necklace would be $I_1 = \{1, 2, 4\}$, $I_2 = \{2, 4, 5\}$, $I_3 = \{3, 4, 5\}$, $I_4 = \{4, 5, 2\}$, $I_5 = \{5, 1, 2\}$. Two of the results in [5] are the following:

Lemma 5. *(See [5, Lemma 16.3].) For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ of rank k on the set $[n]$, let $\mathcal{I}_{\mathcal{M}} = (I_1, \ldots, I_n)$ be the sequence of subsets such that I_i is the minimal member of $\mathcal{M}$ with respect to $\leqslant_i$. Then $\mathcal{I}_{\mathcal{M}}$ is a Grassmann necklace.*

Theorem 6. *(See [5, Theorem 17.2].) Let $S_{\mathcal{M}}^{\mathrm{tnn}}$ be a nonnegative Grassmann cell, and let $\mathcal{I}_{\mathcal{M}} = (I_1, \ldots, I_n)$ be the Grassmann necklace corresponding to $\mathcal{M}$. Then*

$$S_{\mathcal{M}}^{\mathrm{tnn}} = \bigcap_{i=1}^{n} \Omega_{I_i}^{i} \cap \mathrm{Gr}_{kn}^{\mathrm{tnn}},$$

where $\Omega_{I_i}^{i}$ is the cyclically shifted Schubert cell, which is the set of elements $V \in \mathrm{Gr}_{kn}$ such that I_i is the lexicographically minimal base of $\mathcal{M}_V$ with respect to ordering $<_i$ on $[n]$.

These results imply the following:

Corollary 7. *Let $\mathcal{M}$ be a positroid and let $\mathcal{I}_{\mathcal{M}} = (I_1, \ldots, I_n)$ be the associated Grassmann necklace. Then*

$$\mathcal{M} \subseteq \bigcap_{i=1}^{n} \mathrm{SM}_{I_i}^{i}.$$

So the bases of a positroid are included in an intersection of cyclically shifted Schubert matroids. But we do not yet know if they are actually equal. Postnikov therefore conjectured that each positroid is exactly the intersection of cyclically shifted Schubert matroids. This is what we are going to prove in our paper:

Theorem 8. *$\mathcal{M}$ is a positroid if and only if for some Grassmann necklace $(I_1, \ldots, I_n)$,*

$$\mathcal{M} = \bigcap_{i=1}^{n} \mathrm{SM}_{I_i}^{i}.$$

In other words, $\mathcal{M}$ is a positroid if and only if the following holds: $H \in \mathcal{M}$ if and only if $H \geqslant_t I_t$ for all $t \in [n]$.

3. Le-diagrams and Le-graphs

In [5], Postnikov showed a bijection between positroids and combinatorial objects called Ⅎ-diagrams.

Definition 9. Fix a partition λ that fits inside the rectangle $(n-k)^k$. The boundary of the Young diagram of λ gives the lattice path of length n from the upper right corner to the lower left corner of the rectangle $(n-k)^k$. Let us denote this path as the ***boundary path***. Label each edge in the path by $1, \dots, n$ as we go downwards and to the left. Define $I(\lambda)$ as the set of labels of k vertical steps in the path.

Each column and row corresponds to exactly one labeled edge. Let us index the columns and rows with those labels. We will say that a box is at (i, j) if it is on row i and column j. A ***filling*** of λ is a diagram of λ where each box is either empty or filled with a dot.

Definition 10. (See [5, Definition 6.1].) For a partition λ, let us define a ***⅃-diagram*** L of shape λ as a filling of boxes of the Young diagram of shape λ such that, for any three boxes indexed $(i, j), (i', j), (i, j')$, where $i' < i$ and $j' > j$, if boxes on position (i', j) and (i, j') are filled, then the box on (i, j) is also filled. This property is called the ⅃-property. We will say that a ⅃-diagram is ***full*** if every box is filled.

Fix a ⅃-diagram L of shape λ. For each box at (i, j) of L, we define the ***NW-region*** of it as the collection $\{(i', j') \mid i' < i, j' > j\}$. There is a unique dot (i', j') that minimizes $i - i'$ and $j' - j$ at the same time, due to the ⅃-property. We will say that (i', j') covers (i, j) and write this as $(i', j') \lhd (i, j)$.

Definition 11. (See [5, Definition 6.3].) A ***Γ-graph*** is obtained from a ⅃-diagram in the following way. Place a vertex at the middle of each step in the boundary path of the diagram and mark these vertices by $1, 2, \dots, n$. We will call these vertices the ***boundary vertices***. Now for each dot inside the ⅃-diagram, draw a horizontal line to its right, and vertical line to its bottom until it reaches the boundary of the diagram. Then orient all vertical edges downward and horizontal edges to the left.

Γ-graphs were also used to study TP-basis of positroids in [7]. The source set of the Γ-graph is given by $I(\lambda)$ and the sink set is given by $[n] \setminus I(\lambda)$.

Definition 12. A ***path*** in a Γ-graph is a directed path that starts at some boundary vertex and ends at some boundary vertex. Given a path p, we denote its starting point and end point by p^s and p^e. A ***VD-family*** is a family of paths where no pair of paths share a vertex.

A dot at (i, j) is an ***NW-corner*** of a path p, if p changes direction at (i, j). For each dot at (i, j), there is a path that starts at a boundary vertex i, ends at a boundary vertex j and has the dot at (i, j) as an NW-corner. We call such a path a ***hook path of*** $(\boldsymbol{i}, \boldsymbol{j})$.

Given a VD-family of paths $\{p_1, \dots, p_t\}$, we say that this family represents $J = I(\lambda) \setminus \{p_1^s, \dots, p_t^s\} \cup \{p_1^e, \dots, p_t^e\}$. We consider the empty family to be a VD-family. The following proposition follows as a corollary from [5, Theorem 6.5].

Proposition 13. *(See [5, Theorem 6.5].) Given a ⅃-diagram L, let $\mathcal{M}_L$ be the set that consists of J's such that J is represented by a VD-family in the Γ-graph of L. Then $\mathcal{M}_L$ is a positroid, and this correspondence gives a bijection between ⅃-diagrams and positroids.*

Each ⅃-diagram corresponds to a positroid, and hence a Grassmann necklace. Given a ⅃-diagram L, let us try to find out its corresponding Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ directly from the diagram. It is obvious that $I_1 = I(\lambda)$.

For each box (x, y), we can get a maximal chain $(x_t, y_t) \lhd \cdots \lhd (x_1, y_1)$ such that (x_1, y_1) is the unique dot in $\{(i, j) \mid i \leqslant x, j \geqslant y\}$ that minimizes $x - i$ and $j - y$ at the same time. We will call this the ***chain rooted at*** $(\boldsymbol{x}, \boldsymbol{y})$. Then the collection of hook paths at (x_r, y_r) for $1 \leqslant r \leqslant t$ is a VD-family. So we get $J_{(x,y)} := I_\lambda \setminus \{x_1, \dots, x_t\} \cup \{y_1, \dots, y_t\} \in \mathcal{M}_L$. In Fig. 1, chain rooted at $(5, 9)$ is given by $(1, 10) \lhd (5, 9)$. A chain rooted at $(3, 9)$ is given by $(1, 10)$.

We define the ***boundary strip*** of a Young diagram to be the collection of boxes that touches the boundary path of the diagram. Look at the box that is uppermost and rightmost among the boxes in

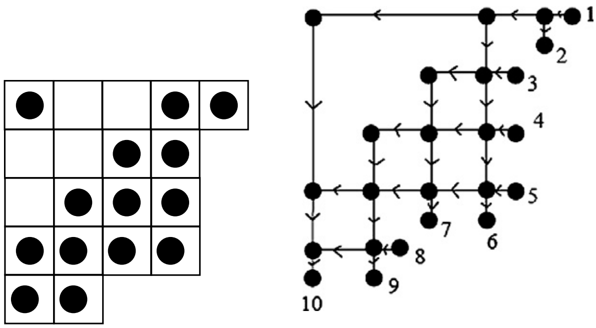

Fig. 1. Example of a ⅃-diagram and a Γ-graph.

			3	2
			4	
			5	
	8	7	6	
10	9			

Fig. 2. Labeling the boxes inside the boundary strip.

the strip. This box is adjacent to a vertical boundary path, and if that path is labeled with j, label the box with $j+1$. Then increase the label as we go downwards and to the left. In other words, if $j \notin I(\lambda)$, then the box labeled j is adjacent to the path labeled j in the boundary path of λ. If $j \in I(\lambda)$, then the box labeled j, is right above the box adjacent to the path labeled j in the boundary path of λ. Fig. 2 shows an example of labeling the boxes inside the boundary strip.

In the following proposition, we will show that a chain rooted at box labeled j represents I_j.

Proposition 14. *Fix a ⅃-diagram L of shape λ and let $\mathcal{I} = (I_1, \ldots, I_n)$ be the Grassmann necklace of $\mathcal{M}_L$. Let (x, y) be the box labeled j in the boundary strip of λ. Then $I_j = J_{(x,y)}$. In particular, if there is no box labeled with j, then $I_j = I(\lambda)$.*

Proof. Let $\mathcal{F}$ be a VD-family that represents I_j. Then $\mathcal{F}$ only contains paths that satisfy $p^s < j \leqslant p^e$. Because if not, then $\mathcal{F} \setminus \{p\}$ represents J such that $J <_j I_j$. So any path $p \in \mathcal{F}$ has to pass through a dot in the region $\{(i, j) \mid i \leqslant x, j \geqslant y\}$.

Let the chain rooted at (x, y) be $(i_t, j_t) \lhd \cdots \lhd (i_1, j_1)$. For each $1 \leqslant r \leqslant t$, denote the hook path at (i_r, j_r) by p_r. Then $\mathcal{F}$ must contain p_1. If not, then $I_j \nleqslant_j I(\lambda) \setminus \{i_1\} \cup \{j_1\}$ because $p^s \leqslant i_1$, $p^e \geqslant j_1$ for all $p \in \mathcal{F}$. If $p_1, \ldots, p_r \in \mathcal{F}$, then we also have $p_{r+1} \in \mathcal{F}$ because if not, we get $I_j \nleqslant_j I(\lambda) \setminus \{i_1, \ldots, i_{r+1}\} \cup \{j_1, \ldots, j_{r+1}\}$ due to the fact that for any path $p \in \mathcal{F} \setminus \{p_1, \ldots, p_r\}$, we have $p^s \leqslant i_{r+1}$ and $p^e \geqslant j_{r+1}$. As a result, we get $\mathcal{F} = \{p_1, \ldots, p_t\}$ and $I_j = J_{(x,y)}$.

In the case where there is no box labeled with j, then there cannot be a path p that satisfies $p^s < j \leqslant p^e$. This implies that I_j is represented by an empty family, so we have $I_j = I(\lambda)$. $\square$

Let us look at an example. In the ⅃-diagram of Fig. 1, I_4 is given by $J_{(3,6)}$. Chain rooted at $(3, 6)$ is given by $(1, 10) \lhd (3, 6)$. So $I_4 = I_1 \setminus \{1, 3\} \cup \{10, 6\} = \{4, 5, 6, 8, 10\}$. I_9 is given by $J_{(8,9)}$. Chain rooted at $(8, 9)$ is given by $(5, 10) \lhd (8, 9)$. So $I_9 = I_1 \setminus \{5, 8\} \cup \{9, 10\} = \{1, 3, 4, 9, 10\}$.

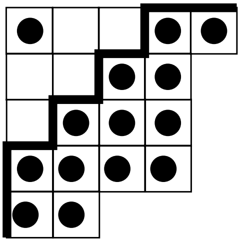

Fig. 3. How the middle path is defined inside a ⅃-diagram.

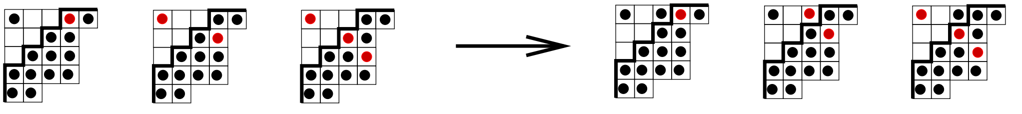

Fig. 4. How the rooted chains change after a new dot is added.

4. Proof of the main theorem

In this section, we will prove the main theorem by showing that for each Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$, we have $\bigcap_{i=1}^{n} \mathrm{SM}^{i}_{I_i} \subseteq \mathcal{M}_{\mathcal{I}}$. To do this, we need to show that each $J \in \bigcap_{i=1}^{n} \mathrm{SM}^{i}_{I_i}$ can be expressed as VD-family inside the Γ-graph of $\mathcal{M}_{\mathcal{I}}$. In order to accomplish this, we will start from a full-⅃-diagram and use induction by increasing the number of empty boxes.

Lemma 15. *Let L be a full ⅃-diagram of shape λ. Then $\mathcal{M}_L = \mathrm{SM}_{I(\lambda)}$.*

Proof. We need to show that for all $J \in \mathrm{SM}_{I(\lambda)}$, we have $J \in \mathcal{M}_L$. Due to the definition of $\leqslant_1$, there is a unique bijection $\phi : I(\lambda) \setminus J \to J \setminus I(\lambda)$ such that for any $a, b \in I(\lambda) \setminus J$, the two intervals $[a, \phi(a)]$ and $[b, \phi(b)]$ do not cross, meaning that they are either disjoint or nested. For each $a \in I(\lambda) \setminus J$, we associate a hook path at $(a, \phi(a))$. Then we get a VD-family representing J. □

Given any ⅃-diagram $L_{\mathcal{I}}$ with associated Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$, we want to add a dot, to obtain a new ⅃-diagram $L_{\mathcal{I}'}$ such that for some $\alpha \in [n]$, we have $|I_\alpha \setminus I'_\alpha| = 1$ and $I'_i = I_i$ whenever $i \neq \alpha$.

Let us first assume that there exists an empty box in the boundary strip of $L_{\mathcal{I}}$. Consider an empty box b in the strip such that there is no empty box to its right or bottom. Then adding a dot to this box b will change exactly one element of the Grassmann necklace, since among the chains rooted at one of the boxes in the boundary strip, only the chain rooted at b is changed. So we only need to consider the case when all the boxes of the boundary strip are filled. We define the ***middle path*** of $L_{\mathcal{I}}$ to be a lattice path inside the diagram such that:

(1) all boxes between the middle path and the boundary path are filled with dots,
(2) the corner boxes of the upper region is empty. ***Upper region*** is the diagram obtained by looking at the boxes above or left of the middle path. A box is a ***corner box*** of a diagram if there are no boxes to its right and below.

Example of a middle path is given as a thick line in Fig. 3.

Now, putting a dot into any corner box of the upper region will work. The reason for this is similar as the previous case, since exactly one chain among the chains rooted at boxes in the boundary strip is going to change, which implies that only one element of the Grassmann necklace is going to be affected by the newly added dot. This phenomenon is illustrated in Fig. 4.

Proposition 16. *Given any Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$, we have $\mathcal{M}_{\mathcal{I}} = \bigcap_{i=1}^n \mathrm{SM}^i_{I_i}$.*

Proof. We will prove the proposition by induction on m, the number of empty boxes inside the ⅃-diagram $L_{\mathcal{I}}$ of $\mathcal{M}_{\mathcal{I}}$. When $m = 0$, this is the full ⅃-diagram case. So assume for the sake of induction that we know the result for ⅃-diagrams having $< m$ empty boxes.

Use the construction above to obtain $L_{\mathcal{I}'}$, where $\mathcal{I}' = (I'_1, \ldots, I'_n)$ and there exists $\alpha \in [n]$ such that $I'_i = I_i$ for all $i \neq \alpha$ and $|I_\alpha \setminus I'_\alpha| = 1$. The induction hypothesis tells us that $\mathcal{M}_{\mathcal{I}'} = \bigcap_{i=1}^n \mathrm{SM}^i_{I'_i}$. It is enough to show $\mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}} \subset \mathrm{SM}^\alpha_{I'_\alpha} \setminus \mathrm{SM}^\alpha_{I_\alpha}$.

Let $(w_{q+r}, z_{q+r}) \lhd \cdots \lhd (w_q, z_q) \lhd \cdots \lhd (w_1, z_1)$ be the chain representing I'_α in $L_{\mathcal{I}'}$, such that (w_q, z_q) is the newly added dot going from $L_{\mathcal{I}}$ to $L_{\mathcal{I}'}$. In $L_{\mathcal{I}'}$, we have dots at (w_a, z_b) for $1 \leqslant a, b \leqslant q$. Any VD-family $\mathcal{F}_J$ representing some $J \in \mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}}$ should contain a path in which (w_q, z_q) is an NW-corner.

In $\mathcal{F}_J$, denote the path going through (w_q, z_q) by p_q. If there is no path in $\mathcal{F}_J$ that passes (w_{q-1}, z_{q-1}), we can perturb the path p_q to go through the points (w_q, z_{q-1}), (w_{q-1}, z_{q-1}), (w_{q-1}, z_q) instead of going through (w_q, z_q). So there must be a path $p_{q-1} \in \mathcal{F}_J$ that passes (w_{q-1}, z_{q-1}). Since (w_q, z_q) is an NW-corner of p_q, (w_{q-1}, z_{q-1}) is also an NW-corner of p_{q-1}. Repeating this argument, we get $p_q, \ldots, p_1 \in \mathcal{F}_J$ each having $(w_q, z_q), \ldots, (w_1, z_1)$ as an NW-corner.

Let $(x_t, y_t) \lhd \cdots \lhd (x_1, y_1)$ be the chain rooted at (w_q, z_q) in $L_{\mathcal{I}}$. Then

$$(x_t, y_t) \lhd \cdots \lhd (x_1, y_1) \lhd (w_{q-1}, z_{q-1}) \lhd \cdots \lhd (w_1, z_1)$$

represents I_j in $L_{\mathcal{I}}$. We have $t \geqslant r$ due to the ⅃-property. We want to show that $J \ngeqslant_\alpha I_\alpha$.

If $p^e_q <_\alpha y_1$ or $p^s_q >_\alpha x_1$, then we have $J \ngeqslant_\alpha I_\alpha$ and we are done. So let us assume $p^e_q \geqslant_\alpha y_1$ and $p^s_q \leqslant_\alpha x_1$. If there is no path going through (x_1, y_1) in $\mathcal{F}_J$, the path p_q can be slightly changed so it goes through (x_1, y_1) and this path cannot have (w_q, z_q) as its NW-corner. So there must be a path p_{q+1} in $\mathcal{F}_J$ that passes through (x_1, y_1).

Due to similar reasons, we only need to consider the case when $p^e_{q+1} \geqslant_\alpha y_2$ and $p^s_{q+1} \leqslant_\alpha x_2$. If there is no path going through (x_2, y_2) in $\mathcal{F}_J$, the path p_{q+1} can be slightly changed so it goes through (x_2, y_2). This path cannot pass (x_1, y_1), since we have $x_2 <_\alpha x_1$ and $y_2 >_\alpha y_1$. So there must be a path $p_{q+2} \in \mathcal{F}_J$ that passes through (x_2, y_2). Repeating this argument, we get $p_{q+1}, \ldots, p_{q+t} \in \mathcal{F}_J$. Then $\{p^e_{q+t}, \ldots, p^e_1\} \subset J$ tells us that $J \ngeqslant_\alpha I_\alpha$. (The reason we do this separately from the previous paragraph is because one of $y_1 = z_q$ and $x_1 = w_q$ might be true.)

So we have shown $\mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}} \subset \mathrm{SM}^\alpha_{I'_\alpha} \setminus \mathrm{SM}^\alpha_{I_\alpha}$, and we are finished. □

Let us look at an example on using the main theorem. Let $\mathcal{M}$ be a positroid such that its Grassmann necklace is given by:

$$I_1 = \{1, 2, 4\}, \qquad I_2 = \{2, 4, 5\}, \qquad I_3 = \{3, 4, 5\}, \qquad I_4 = \{4, 5, 2\}, \qquad I_5 = \{5, 1, 2\}.$$

Our main theorem tells us that:

$$\begin{aligned}\mathcal{M} &= \{H \mid H \geqslant_1 I_1,\ H \geqslant_2 I_2, \ldots, H \geqslant_5 I_5\} \\ &= \bigl\{\{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,4,5\}, \{3,4,5\}\bigr\}.\end{aligned}$$

5. Decorated permutations and the upper Grassmann necklace

In the previous section, we have shown that a positroid is a collection given by setting cyclic lower boundaries. In this section, we will show that we can instead set cyclic upper boundaries to get the same collection. Dual Schubert matroids will play the role of setting upper boundaries, just as Schubert matroids played the role of setting the lower boundaries. In other words, in this section, we will show that a positroid is also an intersection of cyclically shifted dual Schubert matroids.

Definition 17. (See [5, Definition 13.3].) A decorated permutation $\pi^{:} = (\pi, col)$ is a permutation $\pi \in S_n$ together with a coloring function *col* from the set of fixed points $\{i \mid \pi(i) = i\}$ to $\{1, -1\}$. That is, a decorated permutation is a permutation with fixed points colored in two colors.

It is easy to see the bijection between necklaces and decorated permutations. To go from a Grassmann necklace $\mathcal{I}$ to a decorated permutation $\pi^{:} = (\pi, col)$,

- if $I_{i+1} = (I_i \backslash \{i\}) \cup \{j\}$, $j \neq i$, then $\pi(i) = j$,
- if $I_{i+1} = I_i$ and $i \notin I_i$ then $\pi(i) = i$, $col(i) = 1$,
- if $I_{i+1} = I_i$ and $i \in I_i$ then $\pi(i) = i$, $col(i) = -1$.

To go from a decorated permutation $\pi^{:} = (\pi, col)$ to a Grassmann necklace $\mathcal{I}$,

$$I_i = \bigl\{ j \in [n] \bigm| j <_i \pi^{-1}(j) \text{ or } \bigl(\pi(j) = j \text{ and } col(j) = -1\bigr) \bigr\}.$$

Let us look at an example. Given a Grassmann necklace $I_1 = \{1, 2, 4\}$, $I_2 = \{2, 4, 5\}$, $I_3 = \{3, 4, 5\}$, $I_4 = \{4, 5, 2\}$, $I_5 = \{5, 1, 2\}$, the associated decorated permutation is $\pi = 53214$.

Definition 18. For $I = (i_1, \ldots, i_k) \in \binom{[n]}{k}$, the ***cyclically shifted dual Schubert matroid*** $\widetilde{\mathrm{SM}}_I^i$ consists of bases $H = (j_1, \ldots, j_k)$ such that $I \geqslant_i H$.

Fix a decorated permutation $\pi^{:} = (\pi, col)$. Let $\mathcal{I}_{\pi^{:}} = (I_1, \ldots, I_n)$ be the corresponding Grassmann necklace and $\mathcal{M}_{\pi^{:}}$ the corresponding positroid.

Lemma 19. *For any $H \in \mathcal{M}_{\pi^{:}}$, we have $H \leqslant_i \pi^{-1}(I_i)$ for all $i \in [n]$.*

Proof. In this proof, we will show that for any $H \in \mathcal{M}_{\pi^{:}}$, we have $H \leqslant_1 \pi^{-1}(I_1)$. The proof for other inequalities is similar. Denote $I_1 = \{i_1, \ldots, i_k\}$ where $i_1, \ldots, i_k$ are labeled in a way that satisfies $\pi^{-1}(i_1) < \cdots < \pi^{-1}(i_k)$.

Denote elements of H by $h_1 < \cdots < h_k$. Let j be the biggest element of $[k]$ such that:

(1) $h_t \leqslant \pi^{-1}(i_t)$ for all $t \in (j, k]$ and
(2) $h_j > \pi^{-1}(i_j)$.

Since $h_j \in (\pi^{-1}(i_j), \pi^{-1}(i_{j+1})]$, we have $\{i_1, \ldots, i_j\} \subset I_{h_j}$. We get $|H \cap [1, h_j)| < |I_{h_j} \cap [1, h_j)|$, but this contradicts $H \geqslant_{h_j} I_{h_j}$. Hence there cannot be a $j \in [k]$ such that $h_j > \pi^{-1}(h_j)$. This implies that $H \leqslant \{\pi^{-1}(j_1), \ldots, \pi^{-1}(j_k)\}$. $\square$

The collection $(J_1 := \pi^{-1}(I_1), \ldots, J_n := \pi^{-1}(I_n))$ forms a necklace in the sense that $J_{i+1} = J_i \setminus \{\pi^{-1}(i)\} \cup \{i\}$ except for i such that $\pi(i) = i$. We will call this the ***upper Grassmann necklace*** of π.

To go from a decorated permutation $\pi^{:} = (\pi, col)$ to an upper Grassmann necklace $\mathcal{J}$,

$$J_r = \bigl\{ i \in [n] \bigm| \pi(i) <_r i \text{ or } \bigl(\pi(i) = i \text{ and } col(i) = -1\bigr) \bigr\}.$$

Define $\widetilde{\mathcal{M}}_{\pi^{:}}$ as:

$$\widetilde{\mathcal{M}}_{\pi^{:}} = \bigcap_{i=1}^{n} \widetilde{\mathrm{SM}}_{J_i}^i.$$

Lemma 19 tells us that $\mathcal{M}_{\pi^{:}} \subseteq \widetilde{\mathcal{M}}_{\pi^{:}}$. The proof of the following lemma is similar to Lemma 19.

Lemma 20. *For any $H \in \widetilde{\mathcal{M}}_{\pi^{:}}$, we have $H \geqslant_i \pi(J_i) = I_i$ for all $i \in [n]$.*

As a consequence of this lemma we obtain the following result:

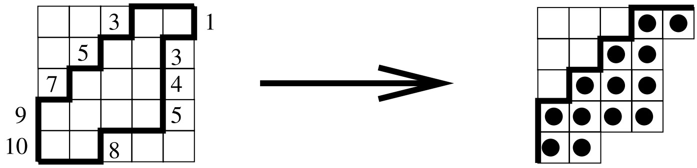

Fig. 5. Example of a lattice path matroid.

Theorem 21. *Pick a decorated permutation $\pi^{:} = (\pi, col)$. Let $\mathcal{I} = (I_1, \ldots, I_n)$ and $\mathcal{J} = (J_1, \ldots, J_n)$ be the corresponding Grassmann necklace and the upper Grassmann necklace. Then $J_i = \pi^{-1}(I_i)$ for all $i \in [n]$. We also have the equality*:

$$\bigcap_{i=1}^{n} \mathrm{SM}_{I_i}^{i} = \bigcap_{i=1}^{n} \widetilde{\mathrm{SM}}_{J_i}^{i}.$$

For example, let us look at the positroid

$$\mathcal{M} = \big\{\{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,4,5\}, \{3,4,5\}\big\},$$

whose associated Grassmann necklace is $I_1 = \{1,2,4\}$, $I_2 = \{2,4,5\}$, $I_3 = \{3,4,5\}$, $I_4 = \{4,5,2\}$, $I_5 = \{5,1,2\}$ and the decorated permutation is $\pi = 53214$. Using $J_i = \pi^{-1}(I_i)$, we get $J_1 = \{3,4,5\}$, $J_2 = \{3,5,1\}$, $J_3 = \{5,1,2\}$, $J_4 = \{5,1,3\}$, $J_5 = \{1,3,4\}$. Theorem 21 tells us that $\mathcal{M} = \{H \mid H \leqslant_1 J_1, \ldots, H \leqslant_5 J_5\}$.

6. Lattice path matroids

Lattice path matroids were defined in [1]. Lattice path matroids turn out to be certain special cases of positroids, which will be proved in this section. We will then show a way to get the decorated permutation of a lattice path matroid.

Definition 22. Pick $I, J \in \binom{[n]}{k}$ such that $I \leqslant J$. The ***lattice path matroid*** is defined as:

$$\mathrm{LP}_{I,J} := \left\{ H \;\middle|\; H \in \binom{[n]}{k},\ I \leqslant H \leqslant J \right\} = \mathrm{SM}_I \cap \widetilde{\mathrm{SM}}_J.$$

Since I, J corresponds to two lattice paths in an $(n-k)$-by-k grid, $\mathrm{LP}_{I,J}$ expresses all the lattice paths between them. The left picture in Fig. 5 shows an example of a lattice path matroid, where I is given by $\{1,3,4,5,8\}$, and J is given by $\{3,5,7,9,10\}$.

Lemma 23. *A lattice path matroid is a positroid.*

Proof. Pick $I = \{i_1 < \cdots < i_k\}$, $J = \{j_1 < \cdots < j_k\}$ such that $I \leqslant J$. Let us construct a k-by-n matrix such that $\Delta_H = 0$ for all $H \in \binom{[n]}{k} \setminus \mathrm{LP}_{I,J}$ and $\Delta_H > 0$ for all $H \in \mathrm{LP}_{I,J}$.

Let $V = (v_{ab})_{a,b=1,1}^{k,n}$ be a k-by-n Vandermonde matrix. Set $v_{ab} = 0$ for all $b \notin [i_a, j_a]$. So V would look like:

$$v_{ab} = \begin{cases} x_a^{b-1} & \text{if } i_a \leqslant b \leqslant j_a, \\ 0 & \text{otherwise.} \end{cases}$$

Assign values to variables $x_1, \ldots, x_k$ such that $x_1 > 1$ and $x_{a+1} = x_a^{k^2}$ for all $a \in [k-1]$. Let us denote $V_{[1,\ldots,a],[c_1,\ldots,c_a]}$ as a submatrix of V by taking rows from 1 to a and columns $c_1, \ldots, c_a$. We have $\Delta_H > 0$ if and only if $V_{[1..k],H}$ has nonzero diagonal entries, which happens if and only if $H \in \mathrm{LP}_{I,J}$. $\square$

Let us try to find $\pi^{:}$ that corresponds to $\mathrm{LP}_{I,J}$. Denote $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$. If $i_t = j_t$ for some $t \in [k]$, this is a coloop in the matroid. The permutation π we are trying to find should satisfy:

- $I = \{i \in [n] \mid i \leqslant \pi^{-1}(i)\}$,
- $J = \{i \in [n] \mid \pi(i) \leqslant i\}$ and
- $\pi(J) = I$.

If π satisfies the above conditions, then $\mathcal{M}_{\pi^{:}}$ is contained in $\mathrm{LP}_{I,J}$. So $\mathrm{LP}_{I,J}$ is the biggest positroid under inclusion inside the collection satisfying the above property. The following lemma is an immediate corollary of [5, Theorem 17.8].

Lemma 24. *If we have $a <_i b <_i \pi(a) <_i \pi(b)$ such that there is no $c \in (a, b)$ that satisfies $\pi(c) \in (\pi(a), \pi(b))$, then $\mathcal{M}_{\mu^{:}} \subset \mathcal{M}_{\pi^{:}}$, where μ is obtained from π by switching $\pi(a)$ and $\pi(b)$ (i.e. $\mu(a) = \pi(b)$ and $\mu(b) = \pi(a)$).*

Combining this with Lemma 23, we get the following result:

Theorem 25. *Choose any $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\} \in \binom{[n]}{k}$ such that $I \leqslant J$. Denote $[n] \setminus J = \{d_1 < \cdots < d_{n-k}\}$ and $[n] \setminus I = \{c_1 < \cdots < c_{n-k}\}$. Then $\mathrm{LP}_{I,J}$ is a positroid and its decorated permutation $\pi^{:} = (\pi, col)$ is given by:*

$$\pi(j_r) = i_r \quad \text{for all } r \in [k],$$
$$\pi(d_r) = c_r \quad \text{for all } r \in [n-k],$$
$$\text{if } \pi(t) = t \quad \text{then } col(t) = \begin{cases} -1 & \text{if } t \in J, \\ 1 & \text{otherwise}. \end{cases}$$

For example, let us look at $\mathrm{LP}_{\{1,3,4,5,8\},\{3,5,7,9,10\}}$. Theorem 25 tells us that this matroid is a positroid, whose decorated permutation is given by $[2, 6, 1, 7, 3, 9, 4, 10, 5, 8]$. The ⅃-diagram of the corresponding positroid is drawn on the right side of Fig. 5.

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References

[1] J. Bonin, A. de Mier, M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials, J. Combin. Theory Ser. A 104 (2003) 63–94.
[2] W. Fulton, Young Tableaux. With Applications to Representation Theory and Geometry, Cambridge University Press, New York, 1997.
[3] A. Knutson, T. Lam, D. Speyer, Positroid varieties I: juggling and geometry, arXiv:0903.3694 [math.AG].
[4] G. Lusztig, Introduction to total positivity, in: J. Hilgert, J.D. Lawson, K.H. Neeb, E.B. Vinberg (Eds.), Positivity in Lie Theory: Open Problems, de Gruyter, Berlin, 1998, pp. 133–145.
[5] A. Postnikov, Total positivity, Grassmannians, and networks, arXiv:math/0609764v1 [math.CO].
[6] K. Rietsch, Total positivity and real flag varieties, PhD dissertation, MIT, 1998.
[7] K. Talaska, Combinatorial formulas for Le-coordinates on a totally nonnegative Grassmannian, J. Combin. Theory Ser. A 118 (1) (2011) 58–66.
[8] M. Yakimov, Cyclicity of Lusztig's stratification of grassmannians and Poisson geometry, in: S. Caenepeel, J. Fuchs, S. Gutt, Ch. Schweigert, A. Stolin, F. van Oystaeyen (Eds.), Noncommutative Structures in Mathematics and Physics, Royal Flemish Academy of Belgium for Sciences and Arts, 2010, pp. 258–262.