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The generating function of planar Eulerian orientations ☆

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ABSTRACT

The enumeration of planar maps equipped with an Eulerian orientation has attracted attention in both combinatorics and theoretical physics since at least 2000. The case of 4-valent maps is particularly interesting: these orientations are in bijection with properly 3-coloured quadrangulations, while in physics they correspond to configurations of the ice model.
We solve both problems – namely the enumeration of planar Eulerian orientations and of 4-valent planar Eulerian orientations – by expressing the associated generating functions as the inverses (for the composition of series) of simple hypergeometric series. Using these expressions, we derive the asymptotic behaviour of the number of planar Eulerian orientations, thus proving earlier predictions of Kostov, Zinn-Justin, Elvey Price and Guttmann. This behaviour, $\mu^n/(n \log n)^2$, prevents the associated generating functions from being D-finite. Still, these generating functions are differentially algebraic, as they satisfy non-linear differential equations of order 2. Differential algebraicity has recently been proved for other map problems, in particular for maps equipped with a Potts model.

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Our solutions mix recursive and bijective ingredients. In particular, a preliminary bijection transforms our oriented maps into maps carrying a height function on their vertices. In the 4-valent case, we also observe an unexpected connection with the enumeration of maps equipped with a spanning tree that is *internally inactive* in the sense of Tutte. This connection remains to be explained combinatorially.

1. Introduction

A *planar map* is a connected planar graph embedded in the sphere, and taken up to orientation preserving homeomorphism (see Fig. 1). The enumeration of planar maps is a venerable topic in combinatorics, which was born in the early sixties with the pioneering work of William Tutte [60,61]. Fifteen years later it started a second, independent, life in theoretical physics, where planar maps are seen as a discrete model of *quantum gravity* [22,10]. The enumeration of maps also has connections with factorizations of permutations, and hence representations of the symmetric group [36,37]. Finally, 40 years after the first enumerative results of Tutte, planar maps crossed the border between combinatorics and probability theory, where they are now studied as random metric spaces [2,25,43,48]. The limit behaviour of large planar random maps is now well understood, and gave birth to a variety of limiting objects, either continuous like the Brownian map [26,44,45,51], or discrete like the UIPQ (uniform infinite planar quadrangulation) [2, 24,27,49].

The enumeration of maps equipped with some additional structure (a spanning tree, a proper colouring, a self-avoiding-walk, a configuration of the Ising model...) has attracted the interest of both combinatorialists and theoretical physicists since the early days of this study [28,39,52,63,62]. At the moment, a challenge is to understand the limiting behaviour of maps equipped with one such structure [16,38,41,54,57].

The enumeration of these "decorated" maps, and understanding their structure, remain the very first building blocks towards the resolution of such challenges. Recently, the natural question of counting maps equipped with an *Eulerian orientation* (where all edges are oriented in such a way that every vertex has as many incoming as outgoing edges, see Fig. 1) was raised by Bonichon et al. [14]. They did not solve the problem, but gave sequences of lower bounds and upper bounds on the number of planar Eulerian ori-

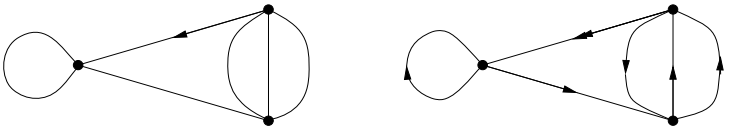

Fig. 1. Left: a rooted planar map, which is 4-valent (or: quartic). Right: the same map, equipped with an Eulerian orientation.

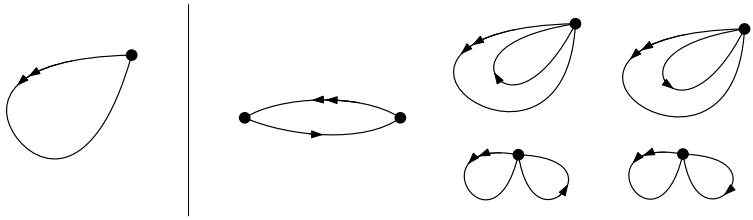

Fig. 2. The planar Eulerian orientations with at most two edges, in agreement with $\mathsf{G}(t) = t+5t^2+O(t^3)$. On the right are the four quartic Eulerian orientations with one vertex, in agreement with $\mathsf{Q}(t) = 4t + O(t^2)$.

entations. They were followed by Elvey Price and Guttmann who, remarkably, were able to write an intricate system of functional equations defining the associated generating function [29]. This allowed them to compute the number g_n of Eulerian orientations with n edges for large values of n, and led them to a conjecture on the asymptotic behaviour of g_n.

Their study also included the special case of 4-valent (or: *quartic*) Eulerian orientations. This problem had already been studied around 2000 in theoretical physics, where it coincides with the *ice model on a random lattice* [42,65]. Another fact that makes this case particularly relevant is that the number of such orientations with n vertices is known to be the number of 3-coloured quadrangulations with n faces [64]. Elvey Price and Guttmann constructed a system of functional equations for this problem as well, and conjectured the asymptotic behaviour of the associated numbers q_n. Their prediction had already been in the physics papers [42,65] for a while, but was probably less accessible to combinatorialists. The more experienced of us observed that the conjectured growth rate, $4\sqrt{3}\pi$, already occurred when counting quartic maps equipped with a certain type of tree [17], and the more optimistic of us looked for, and discovered, an exact (though conjectural) relation between the two problems. This gave an (unpublished) conjecture for the generating function of quartic Eulerian orientations, soon completed by a similar conjecture for general Eulerian orientations. These are the conjectures that we prove in this paper, thus completely solving these two enumeration problems.

Let us now state our main two theorems. As is usual with maps, our orientations are *rooted*, which means that we mark one (oriented) edge (Fig. 1, right). Orientations of small size are shown in Fig. 2.

Theorem 1.1. *Let* $\mathsf{R}(t) \equiv \mathsf{R}$ *be the unique formal power series with constant term* 0 *satisfying*

$$t = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n}\mathsf{R}^{n+1}.$$

Then the generating function of quartic rooted planar Eulerian orientations, counted by vertices, is

$$\mathsf{Q}(t) = \frac{1}{3t^2}\left(t - 3t^2 - \mathsf{R}(t)\right).$$

This is a differentially algebraic series, satisfying a non-linear differential equation of order 2 whose coefficients are polynomials in t. The number q_n of such orientations having n vertices behaves asymptotically as

$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2}$$

where

$$\kappa = 1/18 \qquad \textit{and} \qquad \mu = 4\sqrt{3}\,\pi.$$

The series $\mathsf{Q}(t)$ is not D-finite, which means that it does not satisfy any non-trivial linear *differential equation.*

The first coefficients of R and Q are

$$\mathsf{R}(t) = t - 3t^2 - 12t^3 - 105t^4 - 1206t^5 - \cdots, \qquad \mathsf{Q}(t) = 4t + 35t^2 + 402t^3 + \cdots.$$

Remarks. 1. As we will explain in Section 2.2, the series $\mathsf{Q}(t)$ also counts (by faces) quadrangulations equipped with a proper 3-colouring of the vertices (with prescribed colours on the root edge). It is worth noting that the generating functions of 3-coloured triangulations, and of 3-coloured planar maps, are both algebraic [7] (and thus D-finite), hence in a sense they are much simpler. The corresponding asymptotic estimates are $\kappa\mu^n n^{-5/2}$ in both cases (for other values of μ and κ of course).

2. In Section 5, we will prove that $\mathsf{Q}(t)$ also counts, by edges, Eulerian *partial* orientations of planar maps: that is, only *some* edges are oriented, with the condition that at any vertex there are as many incoming as outgoing edges.

3. As mentioned above, the series $\mathsf{R}(t)$ already occurs in the map literature, and more precisely in the enumeration of quartic maps M weighted by their Tutte polynomial $\mathrm{T}_M(0,1)$. However, our proof does not rely on this observation, and it remains an open problem to understand this connection combinatorially. We refer to the final section for more details.

The counterpart of Theorem 1.1 for all rooted planar Eulerian orientations reads as follows.

Theorem 1.2. *Let $\mathsf{R}(t) \equiv \mathsf{R}$ be the unique formal power series with constant term 0 satisfying*

$$t = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}^{n+1}.$$

Then the generating function of rooted planar Eulerian orientations, counted by edges, is

$$\mathsf{G}(t) = \frac{1}{4t^2}\left(t - 2t^2 - \mathsf{R}(t)\right).$$

This is a differentially algebraic series, satisfying a non-linear differential equation of order 2 whose coefficients are polynomials in t. The number g_n of such orientations having n vertices behaves asymptotically as

$$g_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2}$$

where

$$\kappa = 1/16 \qquad \text{and} \qquad \mu = 4\pi.$$

The series $\mathsf{G}(t)$ is not D-finite.

The first coefficients of R and G are

$$\mathsf{R}(t) = t - 2t^2 - 4t^3 - 20t^4 - 132t^5 - \cdots, \qquad \mathsf{G}(t) = t + 5t^2 + 33t^3 + \cdots.$$

Remark. In Section 5 we will prove that $2\mathsf{G}(t)$ also has an interpretation in terms of 3-coloured maps: it counts (by faces) properly 3-coloured quadrangulations having no bicoloured face. Equivalently, it counts Eulerian orientations of quartic maps with no *alternating vertex* (a vertex where the order of the edges would be in/out/in/out). This is the special case $\alpha = \beta$ of a two matrix model studied in [40], where the point $\alpha = \beta = 1/(4\pi)$ is indeed identified as critical.

Outline of the paper. In Section 2 we begin with basic definitions on maps, orientations, and generating functions. We also discuss various models related to quartic Eulerian orientations. In Section 3 we write a system of functional equations that defines the generating function of quartic Eulerian orientations. We solve it in Section 4, using a guess-and-check approach. Then comes a bijective *intermezzo* in Section 5, where we describe a bijection of Ambjørn and Budd [1]. A specialization of this bijection implies that general Eulerian orientations with n edges are in one-to-two correspondence with certain restricted quartic Eulerian orientations with n vertices. In Section 6 we give a system of equations for these orientations, which we solve in Section 7. In Section 8 we briefly discuss the nature of our generating functions and their singular behaviour, thus proving the asymptotic statements in Theorems 1.1 and 1.2. Section 9 finally raises some open problems.

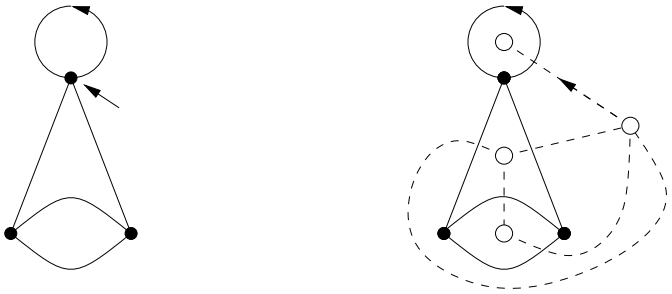

Fig. 3. Left: a rooted planar map, with the root edge and root corner shown. Right: the dual map, in dashed edges.

2. Definitions

2.1. Planar maps

A *planar map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed (Fig. 3). The *faces* of a map are the connected components of its complement. The numbers of vertices, edges and faces of a planar map M, denoted by $\mathrm{v}(M)$, $\mathrm{e}(M)$ and $\mathrm{f}(M)$, are related by Euler's relation $\mathrm{v}(M) + \mathrm{f}(M) = \mathrm{e}(M) + 2$. The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. A *corner* is a sector delimited by two consecutive edges around a vertex; hence a vertex or face of degree k is incident to k corners. The *dual* of a map M, denoted M^*, is the map obtained by placing a vertex of M^* in each face of M and an edge of M^* across each edge of M; see Fig. 3, right. A map is said to be *quartic* if every vertex has degree 4. Duality transforms quartic maps into *quadrangulations*, that is, maps in which every face has degree 4. A planar map is *Eulerian* if every vertex has even degree. Its dual, with even face degrees, is then *bipartite*. We call a face of degree 2 (resp. 4) a *digon* (resp. *quadrangle*).

For counting purposes it is convenient to consider *rooted* maps. A map is rooted by choosing an edge, called the root edge, and orienting it. The starting point of this oriented edge is then the *root vertex*, the other endpoint is the *co-root vertex*. The face to the right of the root edge is the *root face*, and its edges are the *outer edges*. The face to the left of the root edge is the *co-root face*. Equivalently, one can root the map by selecting a corner. The correspondence between these two rooting conventions is that the oriented root edge follows the root corner in anticlockwise order around the root vertex. In figures, we usually choose the root face as the infinite face (Fig. 3). This explains why we often call the root face the *outer face* and its degree the *outer degree* (denoted $\mathrm{od}(M)$). The other faces are called *inner faces*. Similarly, we call the corners of the outer face *outer corners* and all other corners *inner corners*.

From now on, every map is *planar* and *rooted*, and these precisions will often be omitted. Our convention for rooting the dual of a map is illustrated on the right of

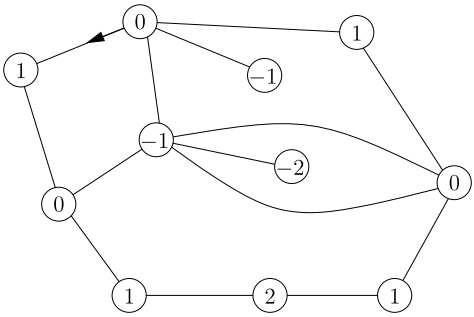

Fig. 4. A labelled map.

Fig. 3. Note that it makes duality of rooted maps a transformation of order 4 rather than 2. By convention, we include among rooted planar maps the *atomic map* having one vertex and no edge.

2.2. Orientations

A (planar) *Eulerian orientation* is a (rooted, planar) map in which all edges are oriented, in such a way that the in- and out-degrees of each vertex are equal. We require that the orientation chosen for the root edge is consistent with its orientation coming from the rooting (Fig. 1, right). Note that the underlying map must be Eulerian. We find it convenient to work with duals of Eulerian orientations, which turn out to be equivalent to certain *labelled maps*.

Definition 2.1. A *labelled map* is a rooted planar map with integer labels on its vertices, such that adjacent labels differ by ± 1 and the root edge is labelled from 0 to 1. Such a map is necessarily bipartite. We also consider the atomic map, with a single vertex (labelled 0), to be a labelled map.

An example is shown in Fig. 4.

Lemma 2.2. *The duality transformation between Eulerian maps and bipartite maps can be extended into a bijection between Eulerian orientations and labelled maps, which preserves the number of edges and exchanges vertex degrees and face degrees.*

The construction was already used in [29, Prop. 2.1]. It is illustrated in Fig. 5. The idea is that an Eulerian orientation of edges of a map gives a height function on its faces, or equivalently, on the vertices of its dual. Height functions on regular grids, like the square lattice, are much studied as models of discrete random surfaces, expected to converge to the Gaussian free field [23,35].

In the case of a quartic Eulerian orientation, the (quadrangular) faces of the dual map can only have two types of labelling, shown in Fig. 6. It is easily shown that, upon replacing every label by its value modulo 3, one obtains a proper 3-colouring of the

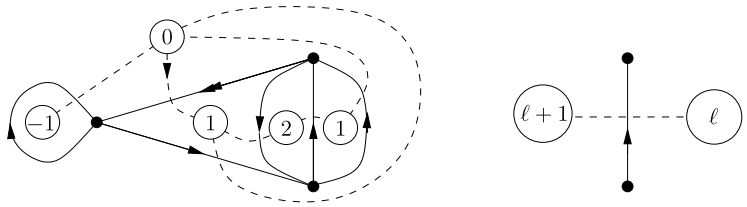

Fig. 5. An Eulerian orientation (solid edges) and the corresponding dual labelled map (dashed edges). The labelling rule is shown on the right.

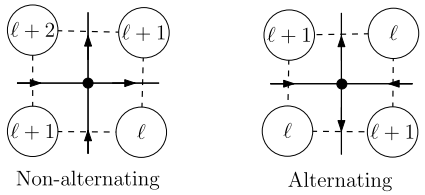

Fig. 6. The two types of vertices in a quartic Eulerian orientation, shown with the associated quadrangles in the dual labelled map. In the six vertex model, alternating vertices are assigned the weight ω.

vertices of the quadrangulation. Conversely, given a 3-coloured quadrangulation such that the root edge is oriented from 0 to 1, all faces must be of one of the types shown in Fig. 6, and one can directly reconstruct an Eulerian orientation of the dual quartic map using the rule of Fig. 5. Then the associated labelled quadrangulation projects on the coloured quadrangulation modulo 3. Hence 4-valent Eulerian orientations with n vertices are in bijection with 3-coloured quadrangulations with n faces (with the root edge oriented from 0 to 1), as claimed below Theorem 1.1. This correspondence between Eulerian orientations of a planar quartic graph and 3-colourings of its dual has been known for a long time. In the more general, non-planar case, the number of Eulerian orientations of a 4-valent graph is given by the value of its *Tutte polynomial* at the point $(0, -2)$, with no interpretation in terms of colourings [64, Sec. 3.6].

More orientations. Obviously, quartic Eulerian orientations are orientations of a quartic map with exactly 2 outgoing edges at each vertex. It turns out that the number of oriented *quadrangulations* in which each vertex has outdegree 2 is known. The associated series is D-finite. A simple bijection transforms these orientations into *bipolar* orientations of planar maps (no cycle, one source, one sink, both on the outer face) [30]. We refer the reader to [4,13,30,34], and references therein. Analogous results exist for orientations of triangulations in which every vertex has outdegree 3, called *Schnyder orientations* [6,12]. Let us also mention recent progress regarding bipolar orientations with prescribed face degrees [19].

2.3. The 6-vertex model and fully packed loops

The enumeration of quartic Eulerian orientations has already been considered, and in some sense solved, in the mathematical physics literature, where it is called the ice

model on a 4-valent random lattice [42,65]. In this model an oxygen atom stands at every vertex, while the hydrogen atoms (two per oxygen in a water/ice molecule) lie on the edges, the arrows indicating with which oxygen they go. This is a special case of the *six vertex model*: in that model, the configurations are still Eulerian orientations, but a weight $\omega = 2\cos(\lambda\pi/2)$ is assigned to each vertex from which the two outgoing edges are opposite each other. We call these vertices *alternating* (see Fig. 6, right). The ice model then corresponds to $\omega = 1$, or equivalently, $\lambda = 2/3$. In combinatorial terms, solving the six vertex model on a random lattice is equivalent to determining the refined generating $\mathsf{Q}(t, \omega)$ of quartic Eulerian orientations, where t still counts vertices, and a weight ω is assigned to alternating vertices. Fig. 6 shows that $\mathsf{Q}(t, \omega)$ also counts labelled quadrangulations by faces, with a weight ω per face having only two distinct labels.

Kostov exactly solved the problem for general λ, though his solution was not entirely rigorous [42]. Kostov's solution relied on analysing the limiting eigenvalue distribution of a sequence of matrices, using results from complex analysis to determine this distribution. We had initially overlooked this solution, in part due to the unfamiliar language and techniques used. In a forthcoming paper, the second author and Zinn-Justin provide a rigorous version of Kostov's derivation, while also fixing a mistake, resulting in a much simplified formula for $\mathsf{Q}(t, \omega)$ compared to the (incorrect) formula that one could extract directly from [42]. This new formula is written parametrically in terms of Jacobi theta functions (see [18] for an extended abstract). In another forthcoming paper, the current authors generalise the methods in the present paper to rederive the same formula for $\mathsf{Q}(t, \omega)$, with our new derivation staying (almost) entirely within the world of formal power series. Our exact formula for $\mathsf{Q}(t, 1)$ can also be shown to agree with the general formula for $\mathsf{Q}(t, \omega)$ at $\omega = 1$, though the equivalence is not obvious [18].

The quantity studied by Kostov is not exactly the series $\mathsf{Q}(t, \omega)$, but the *free energy* $Z(t, \omega)$ related to Q by

$$\mathsf{Q}(t, \omega) = 2t \frac{d}{dt} \log(Z(t, \omega)).$$

Kostov, and also Zinn-Justin [65], predicted that for $\lambda \in [0, 2)$, that is, $\omega \in (-2, 2]$, the dominant singularity of $Z(t, \omega)$ occurs at

$$\rho = \frac{1}{8\lambda\pi} \frac{\sin(\lambda\pi/4)}{\cos(\lambda\pi/4)^3}. \tag{1}$$

In terms of ω, this is

$$\rho = \frac{1}{4\arccos(\omega/2)} \frac{\sqrt{2-\omega}}{(2+\omega)^{3/2}}.$$

Moreover, Kostov [42] predicted that the behaviour of the free energy around this singularity is

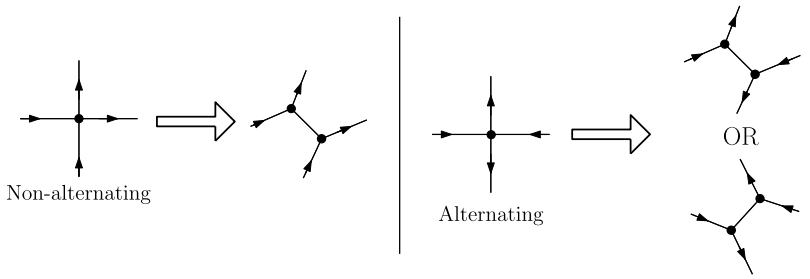

Fig. 7. The transformation from vertices in a quartic Eulerian orientation to pairs of vertices in certain cubic Eulerian partial orientations. This transformation applies when $\omega = 2$.

$$\log(Z(t,\omega)) \sim \frac{(1-t/\rho)^2}{\log(1-t/\rho)},$$

up to some multiplicative constant. This would result in:

$$\mathsf{Q}(t,\omega) \sim \frac{1-t/\rho}{\log(1-t/\rho)},$$

up to some multiplicative constant.

The generating function $\mathsf{Q}(t)$ of Theorem 1.1 is $\mathsf{Q}(t,1)$, so the predictions of Zinn-Justin and Kostov at $\omega = 1$ are verified by Theorem 1.1 (see Proposition 8.2 for the singular behaviour of $\mathsf{Q}(t)$, from which the asymptotic behaviour of the numbers q_n stems). We also claim that our second theorem, Theorem 1.2, solves the case $\omega = 0$ of the six vertex model. Indeed, we will show that the generating function $\mathsf{G}(t)$ of general Eulerian orientations satisfies $2\mathsf{G}(t) = \mathsf{Q}(t,0)$ (see Corollary 5.2). Hence the predictions of Zinn-Justin and Kostov for $\omega = 0$ follow from Theorem 1.2 and Proposition 8.3.

In our forthcoming paper, we analyse the exact formula for $\mathsf{Q}(t,\omega)$. Our analysis strongly suggests that the prediction (1) does not hold on the entire segment $(-2,2]$, but only for $\omega > \omega_c$, where ω_c is around -0.76.

Fully packed loops. The case $\omega = 2$ is also well-understood, and boils down to counting all planar maps weighted by their *Tutte polynomial*, evaluated at the point $(3,3)$. This can be justified as follows: starting from a quartic Eulerian orientation, we first transform each vertex into a pair of vertices with degree 3, as shown in Fig. 7. (This transformation has already been used in, e.g., [42,65].) The two possible choices for each alternating vertex account for the weight $\omega = 2$ assigned to these vertices. The resulting map is a cubic map in which each vertex has one incoming edge, one outgoing edge and one undirected edge. This transformation can be reversed by simply contracting all undirected edges in the cubic map. The oriented edges must then form loops on the cubic map, where each loop is oriented one of two ways — either clockwise or anticlockwise. Moreover, every vertex must be visited by a loop. In [15, Sec. 2.1], this model of fully packed loops on cubic maps is shown to be equivalent to the 4-state Potts model on general planar maps, in which every monochromatic edge gets a weight 3, and every vertex a weight 1/2.

Finally, using the correspondence between the Potts model and the Tutte polynomial (see, e.g., [7, Sec. 3.3]), we conclude that

$$\mathsf{Q}(t,2) = \sum_{M \text{ planar}} t^{\mathrm{e}(M)}\, \mathrm{T}_M(3,3) = 6\,t + 78\,t^2 + 1326\,t^3 + 25992\,t^4 + O(t^5),$$

where $\mathrm{T}_M(\mu,\nu)$ is the Tutte polynomial of M (see [64]). This series was proved to satisfy an (explicit) non-linear differential equation of order 3 (see [8, Thm. 16] for $\beta = 2$), but, to our knowledge, the singular behaviour of $\mathsf{Q}(t,2)$ has not been derived from it. From this differential equation, one can in fact guess-and-prove a smaller differential equation for $\mathsf{Q}(t,2)$, of order 2 and degree 2 (for $\omega = 0$ or 1, we obtain DEs of order 2 but degree 3). Written in terms of the series $S(t) = t^2(1 + \mathsf{Q}(t,2))$ of [8], it reads:

$$(1-32t)\,(6S - 2tS' - t)\,S'' + 2t(1-4S')^2 = 0.$$

2.4. Formal power series

Let A be a commutative ring and x an indeterminate. We denote by $A[x]$ (resp. $A[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in A. If A is a field, then $A(x)$ denotes the field of rational functions in x. We will also consider Laurent series in x, that is, series of the form

$$\sum_{n \geq n_0} a_n x^n,$$

with $n_0 \in \mathbb{Z}$ and $a_n \in A$. The coefficient of x^n in a series $F(x)$ is denoted by $[x^n]F(x)$.

This notation is generalised to polynomials, fractions and series in several indeterminates. For instance, the generating function of Eulerian orientations, counted by edges (variable t) and faces (variable z) belongs to $\mathbb{Q}[[t,z]]$. For a multivariate series, say $F(x,y) \in \mathbb{Q}[[x,y]]$, the notation $[x^i]F(x,y)$ stands for the *series* $F_i(y)$ such that $F(x,y) = \sum_j F_j(y)x^j$. It should not be mixed up with the coefficient of $x^i y^0$ in $F(x,y)$, which we denote by $[x^i y^0]F(x,y)$. If $F(x,x_1,\dots,x_d)$ is a series in the x_i's whose coefficients are Laurent series in x, say

$$F(x,x_1,\dots,x_d) = \sum_{i_1,\dots,i_d} x_1^{i_1}\cdots x_d^{id} \sum_{n \geq n_0(i_1,\dots,i_d)} a(n,i_1,\dots,i_d)x^n,$$

then we define the *non-negative part of F in x* as the following formal power series in $x,x_1,\dots,x_d$:

$$[x^{\geq 0}]F(x,x_1,\dots,x_d) = \sum_{i_1,\dots,i_d} x_1^{i_1}\cdots x_d^{id} \sum_{n \geq 0} a(n,i_1,\dots,i_d)x^n.$$

We define similarly the *positive part* of F in x, denoted $[x^{>0}]F$.

If A is a field, a power series $F(x) \in A[[x]]$ is *algebraic* (over $A(x)$) if it satisfies a non-trivial polynomial equation $P(x, F(x)) = 0$ with coefficients in A. It is *differentially algebraic* (or *D-algebraic*) if it satisfies a non-trivial polynomial differential equation $P(x, F(x), F'(x), \dots, F^{(k)}(x)) = 0$ with coefficients in A. It is *D-finite* if it satisfies a *linear* differential equation with coefficients in $A(x)$. For multivariate series, D-finiteness and D-algebraicity require the existence of a differential equation *in each variable*. We refer to [46,47] for general results on D-finite series, and to [9, Sec. 6.1] for D-algebraic series.

3. Functional equations for quartic Eulerian orientations

In this section we will characterise the generating function $\mathsf{Q}(t)$ of labelled quadrangulations by a system of functional equations.

Theorem 3.1. *There exists a unique* 3*-tuple of series, denoted* $\mathsf{P}(t, y)$, $\mathsf{C}(t, x, y)$ *and* $\mathsf{D}(t, x, y)$*, belonging respectively to* $\mathbb{Q}[[y, t]]$, $\mathbb{Q}[x][[y, t]]$ *and* $\mathbb{Q}[[x, y, t]]$*, and satisfying the following equations:*

$$\begin{aligned}
\mathsf{P}(t, y) &= \frac{1}{y}[x^1]\mathsf{C}(t, x, y), \\
\mathsf{D}(t, x, y) &= \frac{1}{1 - \mathsf{C}\left(t, \frac{1}{1-x}, y\right)}, \\
\mathsf{D}(t, x, y) &= 1 + y\,[x^{\geq 0}]\left(\mathsf{D}(t, x, y)\left([y^1]\mathsf{D}(t, x, y) + \frac{1}{x}\mathsf{P}\left(t, \frac{t}{x}\right)\right)\right),
\end{aligned}$$

together with the initial condition

$$[y^1]\mathsf{D}(t, x, y) = \frac{1}{1 - x}\left(1 + 2t[y^2]\mathsf{D}(t, x, y) - t([y^1]\mathsf{D}(t, x, y))^2\right).$$

The generating function $\mathsf{Q}(t)$ *that counts labelled quadrangulations by faces is*

$$\mathsf{Q}(t) = [y^1]\mathsf{P}(t, y) - 1.$$

By Lemma 2.2, the series $\mathsf{Q}(t)$ *also counts quartic Eulerian orientations by vertices.*

Remarks. 1. With the conditions on the series P, C and D, the operations that occur in the above equations are always well defined:

- the coefficient of x in $\mathsf{C}(t, x, y)$ lies in $\mathbb{Q}[[y, t]]$,
- the series $\mathsf{C}(t, 1/(1 - x), y)$ indeed lies in $\mathbb{Q}[[x, y, t]]$ (upon expanding the powers of $1/(1 - x)$ as series in x),

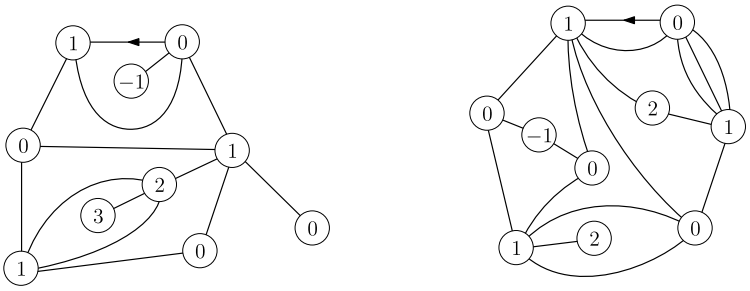

Fig. 8. Left: a patch which contributes t^5y^4 to the generating function $\mathsf{P}(t,y)$. It does not satisfy the first condition of a C-patch. Right: a D-patch which contributes $t^6x^3y^3$ to the generating function $\mathsf{D}(t,x,y)$.

- denoting by $p_{j,n} \in \mathbb{Q}$ the coefficient of $y^j t^n$ in $\mathsf{P}(t,y)$, and by $d_{j,n}(x) \in \mathbb{Q}[[x]]$ the coefficient of $y^j t^n$ in D, the quantity

$$\frac{1}{x}\mathsf{D}(t,x,y)\,\mathsf{P}\left(t,\frac{t}{x}\right) = \left(\sum_{j,n\geq 0} d_{j,n}(x) y^j t^n\right)\left(\sum_{i,m\geq 0} p_{i,m}\frac{1}{x^{i+1}}t^{i+m}\right)$$
$$= \sum_{j,N\geq 0} y^j t^N \sum_{i+m+n=N} p_{i,m}\, d_{j,n}(x)\frac{1}{x^{i+1}}$$

  is a series in y and t whose coefficients are Laurent series in x (because i, m and n are bounded). It thus makes sense to extract its non-negative part in x, which will lie in $\mathbb{Q}[[x,y,t]]$.

2. In [29], another system was given to characterise the series $\mathsf{Q}(t)$. It is more complicated than the one above. In particular, it involves three additional variables (other than the main size variable t) rather than two. We could not solve that complicated system, but we solve the one above in the next section.

The series C, D and P of Theorem 3.1 count certain labelled maps, which we now define. See Fig. 8 for an illustration.

Definition 3.2. A *patch* is a labelled map in which each inner face has degree 4, and the vertices around the outer face are alternately labelled 0 and 1.

A *C-patch* is a patch satisfying two additional conditions: all neighbours of the root vertex are labelled 1, and the root corner is the only outer corner at the root vertex. By convention, the atomic patch is *not* a C-patch.

D-patches resemble patches but may include digons. More precisely, a D-patch is a labelled map in which each inner face has degree 2 or 4, those of degree 2 being incident to the root vertex, and the vertices around the outer face are alternately labelled 0 and 1. We also require that all neighbours of the root vertex are labelled 1.

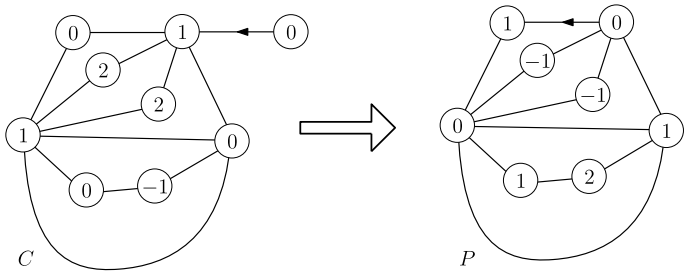

Fig. 9. The transformation of C into P used in the proof of Lemma 3.3.

We define $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$ to be respectively the generating functions of patches, C-patches and D-patches, where t counts inner quadrangles, y the outer degree (halved), and x either the degree of the root vertex (for C-patches) or the number of inner digons (for D-patches). Comparing with the previous paper giving functional equations for this problem [29], we see that one parameter, namely the degree of the co-root vertex, is no longer involved here. The series P, C and D actually belong to the rings prescribed by Theorem 3.1:

- for P it suffices to observe that there are finitely many patches with n inner quadrangles and outer degree $2j$,
- for D we observe that there are finitely many D-patches with n inner quadrangles, i inner digons and outer degree $2j$,
- finally for C, we note that a C-patch with n inner quadrangles cannot have a root vertex of degree larger than $1+4n$, because all non-root corners at the root vertex must belong to an inner quadrangle (by the second condition of Definition 3.2). This explains the polynomiality of $[t^n]\mathsf{C}$ in x (and yields in fact a smaller ring than $\mathbb{Q}[x][[y,t]]$, namely $(\mathbb{Q}[[y]][x])\,[[t]]$, but this won't be needed).

In the next 5 lemmas, we prove that the series that we have defined satisfy the 5 equations of Theorem 3.1. We will finish the section by proving that the system has a unique solution in the prescribed rings of series.

Lemma 3.3. *The generating functions* $\mathsf{P}(t,y)$ *and* $\mathsf{C}(t,x,y)$ *satisfy the equation*

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y).$$

Proof. Let C be any C-patch counted by $[x^1]\mathsf{C}(t,x,y)$, that is, in which the root vertex has degree 1. We construct a new patch P from C, as illustrated in Fig. 9: we delete the root edge and root vertex of C, replace each label ℓ with $1-\ell$, and finally root P at the outer edge of C following the root edge of C anticlockwise. Then the new labelled map P is indeed a patch. If C contains only one edge then P is the atomic map. The outer

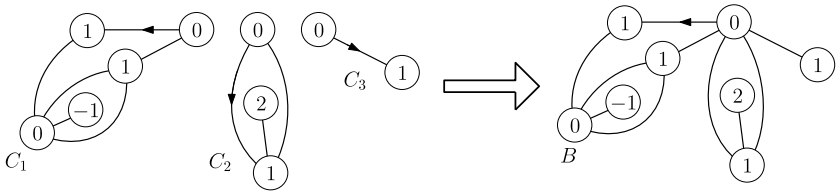

Fig. 10. A sequence of C-patches gives rise to a B-patch, as in Lemma 3.4.

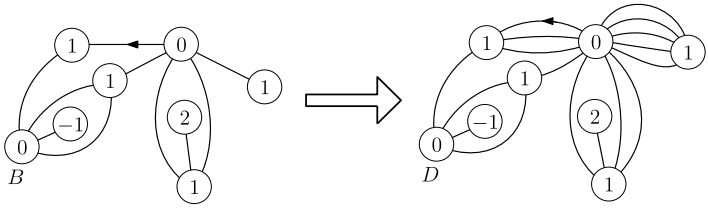

Fig. 11. The transformation from a B-patch to a D-patch, as in Lemma 3.4.

degree has decreased by 2, while the number of inner quadrangles is unchanged. Finally, the transformation from C to P is reversible. This proves the lemma. □

Lemma 3.4. *The generating functions* $\mathsf{D}(t,x,y)$ *and* $\mathsf{C}(t,x,y)$ *satisfy the equation*

$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}.$$

Proof. Recall that C-patches satisfy two conditions: all neighbours of the root vertex have label 1, and the root vertex is only incident once to the root face. By attaching a sequence of C-patches at their root vertex, as shown in Fig. 10, we form a *B-patch*, that is, a patch satisfying only the first of these conditions. The associated generating function is

$$\mathsf{B}(t,x,y) = \frac{1}{1-\mathsf{C}(t,x,y)}.$$

As before, t counts inner quadrangles, x the degree of the root vertex and y the outer degree (halved).

Now in order to construct a D-patch, it suffices to take a B-patch and inflate every edge which is incident to the root vertex into a sequence of digons, as shown in Fig. 11. This explains the transformation $x \mapsto 1/(1-x)$ occurring in the lemma. In this way the variable x now counts digons of D-patches. □

Lemma 3.5. *The generating function* $\mathsf{D}(t,x,y)$ *satisfies the equation*

$$[y^1]\mathsf{D}(t,x,y) = \frac{1}{1-x}\left(1+2t[y^2]\mathsf{D}(t,x,y) - t([y^1]\mathsf{D}(t,x,y))^2\right).$$

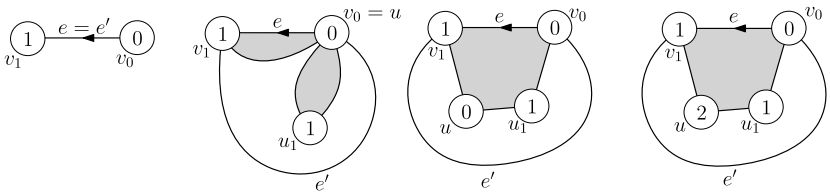

Fig. 12. The four different types of patches which contribute to the generating function $[y^1]\mathsf{C}(t,x,y)$. In the third and fourth cases it is possible that $u_1 = v_1$. The shaded area represents a labelled quadrangulation.

Proof. We will show that

$$[y^1]\mathsf{C}(t,x,y) = x\left(1 + 2t[y^2]\mathsf{C}(t,x,y) + t([y^1]\mathsf{C}(t,x,y))^2\right), \tag{2}$$

from which the desired result follows using Lemma 3.4, while observing that $\mathsf{C}(t,x,0) = 0$. Let C be any C-patch counted by $[y^1]\mathsf{C}(t,x,y)$, that is, having outer degree 2. Let e be the root edge of C, let e' be the other outer edge of C and let v_0 and v_1 be the root vertex and co-root vertex respectively. We consider four cases, illustrated in Fig. 12.

In the first case $e = e'$. Since the outer degree of C is 2, this is only possible if e is the only edge in C, so this case simply contributes x to $[y^1]\mathsf{C}(t,x,y)$. For the other three cases, let Q be the map remaining when e' is removed (this is the shaded area in Fig. 12). Then the outer degree of Q must be 4, so that Q is a quadrangulation. Let the vertices around the outer face of Q be v_0, v_1, u and u_1 in anticlockwise order. Note that v_1 and u_1 must both be labelled 1 since they are adjacent to v_0 and C is a C-patch.

The second case we consider is when $u = v_0$. Then Q can be separated into two C-patches with outer degree 2, hence this case contributes

$$xt([y^1]\mathsf{C}(t,x,y))^2$$

to $[y^1]\mathsf{C}(t,x,y)$. The factor xt appears because the number of inner quadrangles in Q and the degree of the root vertex of Q are each one less than the equivalent numbers in C.

The third case is when $u \neq v_0$, but u is labelled 0. Then Q can be any C-patch with outer degree 4. Hence this case contributes

$$xt[y^2]\mathsf{C}(t,x,y).$$

In the fourth and final case, u is labelled 2, and therefore it cannot be equal to v_0. In this case Q is not a patch because of this label 2 on its outer face. But we construct a new map Q' from Q by replacing every label ℓ in Q with $2-\ell$, except for the label at the root vertex, which remains 0. Then Q' is still a labelled map, all neighbours of the root vertex are still labelled 1, and the root face is only incident once to the root vertex. Hence, Q' can be any C-patch with outer degree 4, so this case contributes

$$xt[y^2]\mathsf{C}(t,x,y).$$

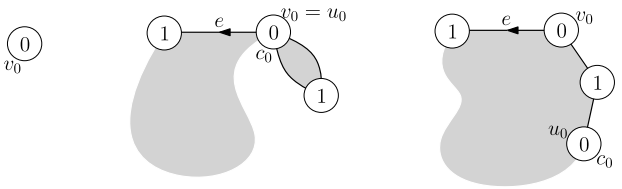

Fig. 13. The three different types of D-patches. In the third case it is possible that the two displayed vertices labelled 1 are the same vertex.

Adding the contributions from the four cases yields (2), which, in turn, yields the desired result using Lemma 3.4. □

In order to prove the most complex equation of our system,

$$\mathsf{D}(t,x,y) = 1 + y\,[x^{\geq 0}]\left(\mathsf{D}(t,x,y)\left([y^1]\mathsf{D}(t,x,y) + \frac{1}{x}\mathsf{P}\left(t,\frac{t}{x}\right)\right)\right), \tag{3}$$

we will consider three types of D-patches, illustrated in Fig. 13, and we will enumerate the D-patches of each type separately. The first type is just the atomic map, which contributes 1 to $\mathsf{D}(t,x,y)$. For any other D-patch D, let v_0 be the root vertex, let c_0 be the outer corner labelled 0 that follows the root corner clockwise around the outer face, and let u_0 be the vertex associated with c_0. We define D-patches of type 2 as those that satisfy $u_0 = v_0$, while D-patches of type 3 satisfy $u_0 \neq v_0$.

Lemma 3.6. *The contribution to $D(t,x,y)$ from D-patches of type 2 is given by*

$$y\left([y^1]\mathsf{D}(t,x,y)\right)\mathsf{D}(t,x,y).$$

Proof. The result follows from the fact that any D-patch of type 2 can be split into two D-patches at v_0 where one has outer degree 2 and the other can be any D-patch. □

Note that this contribution can be written $y[x^{\geq 0}]\left(\mathsf{D}(t,x,y)[y^1]\,\mathsf{D}(t,x,y)\right)$ as in (3). It remains to determine the contribution from D-patches of type 3.

Proposition 3.7. *There is a bijection between D-patches D of type 3 and pairs (P,D') of a patch P and a D-patch D' such that the number of digons in D' is larger than half the outer degree of P. More precisely, if D has n inner quadrangles, d inner digons and outer degree $2j$,*

- *the total number of inner faces of P and D' is $n+d+1$,*
- *the outer degree of D' is $2j-2$,*
- *the number of inner digons in D' is $d+k+1$, where $2k$ is the outer degree of P.*

Before proving the proposition, let us show that it completes the proof of (3).

Corollary 3.8. *The contribution to $D(t,x,y)$ from D-patches of type 3 is given by*

$$\mathsf{D}(t,x,y) = y\,[x^{\geq 0}]\left(\frac{1}{x}\mathsf{D}(t,x,y)\mathsf{P}\left(t,\frac{t}{x}\right)\right).$$

Proof. We use the bijection of Proposition 3.7, and express the statistics of D in terms of those of M and D':

- the outer degree of D is the outer degree of D' plus 2,
- the number d of inner digons in D is the number of inner digons in D', minus half the outer degree of P, minus 1,
- finally, the number of inner quadrangles of D is the sum of the corresponding numbers in P and D', plus half the outer degree of P.

Hence, the contribution from D-patches of type 3 is

$$\frac{y}{x}[x^{>0}]\left(\mathsf{D}(t,x,y)\,\mathsf{P}\left(t,\frac{t}{x}\right)\right) = y\,[x^{\geq 0}]\left(\frac{1}{x}\,\mathsf{D}(t,x,y)\,\mathsf{P}\left(t,\frac{t}{x}\right)\right). \quad \square$$

To prove Proposition 3.7, we need to introduce minus-patches, subpatches and a contraction operation. This contraction operation was already used in [29], on a slightly different class of patches.

Definition 3.9. A *minus-patch* is a map obtained from a patch by replacing each label ℓ with $-\ell$.

Clearly these are equinumerous with patches. We now describe a way to extract a minus-subpatch from a D-patch of type 3. This definition is illustrated on the left of Fig. 14. Recall the notation c_0 for the outer corner labelled 0 that follows the root corner in clockwise order around the outer face, and u_0 for the associated vertex.

Definition 3.10. Let D be a D-patch of type 3. We define the *minus-subpatch of D* as follows. First, let M' be the maximal submap of D that contains u_0 and consists of vertices labelled 0 or less. Let M be the submap of D that contains M' and all edges and vertices within its boundary (assuming the root face is drawn as the infinite face). The map M, which we root at the corner inherited from c_0, is the *minus-subpatch of D*.

The following lemma justifies the terminology *minus-subpatch*.

Lemma 3.11. *The minus-subpatch of a D-patch of type 3 is a minus-patch.*

Proof. In the above definition, it is clear that M and M' share the same outer face. Moreover, all inner faces of M are also inner faces of D. Since the root vertex of D is only adjacent to vertices labelled 1, it cannot be a vertex of M', so it cannot be a

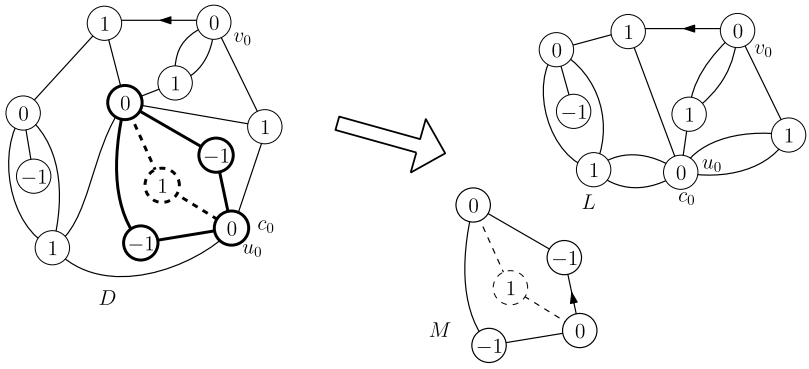

Fig. 14. Left: a D-patch of type 3. The minus-subpatch M of D is highlighted in D and shown separately in the middle. The submap M' is obtained from M by deleting the dashed vertex and edges. Right: the labelled map L constructed from D by contracting M to a single vertex u_0.

vertex of M either. Hence all inner faces of M are quadrangles, since all digons in D are incident to its root. All outer vertices of M must also be outer vertices of M', so they have non-positive labels. Let us prove that these labels can only be 0 and -1. For any outer vertex u of M, there is some face F of D, containing u, which is not a face of M. If F is the outer face of D, with labels 0 and 1, then the label of u, being non-positive, can only be 0. The face F cannot be a digon, otherwise u would be the vertex labelled 0 in this digon, and thus would be the root vertex of D, while we have shown that this vertex is not in M. Finally, if F is an inner quadrangle of D, then it must contain a vertex u' with label at least 1 (otherwise F would be contained in M). Since u and u' are incident to the same quadrangle F, and u has a non-positive label, this label can only be -1 or 0. Hence the outer vertices of M are all labelled 0 or -1, so M is a minus-patch, in the sense of Definition 3.9. □

The patch P associated with a D-patch D of type 3 in Proposition 3.7 will simply be obtained by negating the labels in the minus-patch M.

Let us now describe how D' is constructed. Every edge in D which connects a vertex in M to a vertex not in M must have endpoints labelled 0 (in M) and 1 (not in M). We can thus contract all of M to a single vertex labelled 0, still denoted u_0, to form a new labelled map L (Fig. 14, right). The vertex u_0 is still distinct from the root vertex v_0. Finally, we move u_0 towards v_0 in the outer face of L until these two vertices merge into a new root vertex w_0 (Fig. 15). This creates an extra inner digon at w_0, in addition to those that were incident to u_0 and v_0. Note that we do not merge any edges. This gives a new labelled map, denoted D'.

Lemma 3.12. *The labelled map D' obtained by the above construction is a D-patch. If D has d inner digons, outer degree $2j$, and its minus-patch M has outer degree $2k$, then D' has $d+k+1$ inner digons and outer degree $2j-2$. Finally, D' and M have together one more finite face than D.*

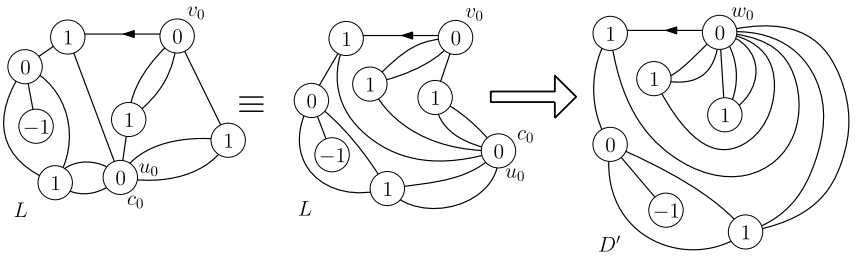

Fig. 15. The transformation of L into a D-patch D'.

Proof. Since L is obtained by contracting the submap M into a single vertex, its inner faces cannot be bigger than the inner faces of D. Hence they are quadrangles or digons. Moreover, all digons are attached either to v_0 (as in D), or to u_0 (because they result from the contraction of two edges of an inner quadrangle). Hence, once u_0 and v_0 are merged to form the map D', all digons are incident to the new root vertex w_0. Finally, all neighbours of v_0 in L are labelled 1 (as in D), and the same holds for all neighbours of u_0, because they were neighbours of M, and all edges joining a vertex in M to a vertex not in M join label 0 to label 1. Hence, in D', the root vertex is only adjacent to vertices labelled 1, and D' is a D-patch.

Let us now prove the statements dealing with the statistics. Clearly, no outer edge of D lies in M, hence the outer degrees of L and D are the same. The transformation of L into D' reduces the outer degree by 2. The statement involving the number of finite faces is also clear: every finite face of D results in a finite face of M or L, and transforming L into D' creates a new inner digon. The number of inner digons in D' is $d+k+1$, where d is the number of inner digons in D, and k is the number of inner digons attached to the vertex u_0 in the contracted map L. We claim that k is also half the outer degree of M. This comes from the fact that, from every corner c labelled 0 on the outer face of M, there must start (in D) at least one edge ending at a vertex labelled 1. Otherwise, the face of D that contains c would contain two corners labelled -1 and would have degree larger than 4, which is impossible. Hence every pair of two consecutive outer edges of M with labels $0,-1,0$ occurs in a unique quadrangle, outside M, and this quadrangle will be contracted to form a digon of L adjacent to u_0. □

Proof of Proposition 3.7. Starting from a D-patch D of type 3, we construct the minus-patch M, the labelled map L and the D-patch D' as described above. We take for P the patch obtained by negating labels in M. The statements that deal with the statistics of D, M (or P) and D' then follow from Lemma 3.12. We denote $f(D)=(P,D')$.

Conversely, we need to show how to construct a D-patch D of type 3 from a pair (P,D') satisfying the conditions of the proposition. Let $2k$ and $2j-2$ be the outer degrees of P and D', respectively, and let d' be the number of digons of D'. Define $d:=d'-k-1$. This is non-negative by the assumption $d'>k$. Split the root vertex w_0 of D' into two vertices v_0 and u_0 so that exactly d digons remain attached to v_0, and k to u_0. This gives a labelled map L, in which all digons are attached to the root vertex

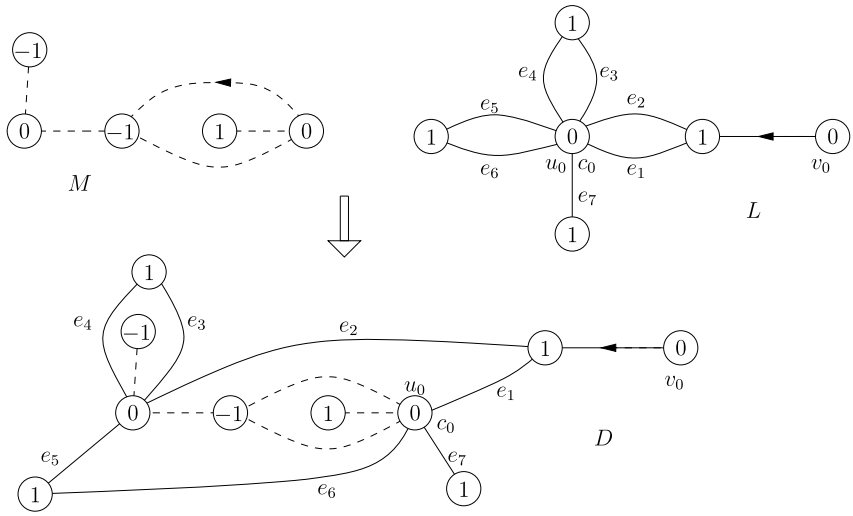

Fig. 16. How to reconstruct the D-patch D from the minus-patch M (dashed edges) and the map L. Here, M has outer degree $2k = 6$, the vertex u_0 of L has degree $\delta = 7$ and is incident to $k = 3$ digons.

v_0 or to u_0. Moreover, all neighbours of these vertices are labelled 1. The outer degree of L is $2j$, and it has one inner face less than D'.

Next, we negate the labels of the patch P to obtain a minus-patch M. We now want to insert M at the vertex u_0 in L to construct a D-patch D. If M is atomic, then we take $D = L$. Otherwise, let c_0 be the outer corner labelled 0 following the root corner in clockwise order around the outer face in L. The vertex at this corner is u_0. Roughly speaking, we need to place the root corner of M at c_0, and to distribute the edges attached to u_0 in L around the minus-patch M. Let $e_1, e_2, \dots, e_\delta$ be the edges of L attached to u_0, in anticlockwise order starting from the corner c_0 (Fig. 16). We now erase the vertex u_0 from L, so that the half-edges e_i are dangling. We connect them to the outer corners of M labelled 0 in the following way: we first attach e_1 to the root corner of M, and then proceed anticlockwise around M, connecting e_{i+1} to the next corner of M labelled 0 if the corner of L at u_0 defined by e_i and e_{i+1} belongs to an inner digon of L (this creates a new quadrangle), and to the same corner as e_i otherwise. Recall that M has outer degree $2k$, so it has k corners labelled 0, which is the same as the number of digons incident to u_0 in L. Hence this construction connects the final edge e_δ to the root corner of M, and we thus obtain a map D, which we define to be $g(P, D')$. Note also that all vertices of M labelled -1 end up on the interior of $g(P, D')$, away from the outer face.

Let us explain why D is a D-patch of type 3. It is clearly a labelled map, and zeroes and ones alternate on its outer face (as in L). When inserting M in L, we have transformed every inner digon that was incident to u_0 in L into an inner quadrangle: hence all inner faces of D have degree 2 or 4. Finally, all neighbours of v_0 in D are labelled 1, as in D' and L. Hence D is a D-patch. Since we have split the vertex w_0 into two distinct vertices v_0 and u_0, it has type 3.

Note that M is the minus-subpatch of D: indeed, it is a minus-patch, it contains u_0, and it is only connected to the rest of D by edges labelled 0 at one end (in M) and 1 at the other end (out of M). This is the key point in proving that $f \circ g(P, D') = (P, D')$.

Finally, to prove that $g \circ f(D) = D$ for any D-patch D of type 3, it suffices to observe that in the application of g, our choices for where to attach the edges $e_1, e_2, \ldots, e_\delta$ (Fig. 16) are the only choices that ensure that the resulting map is a D-patch in which c_0 is contained in the root corner of M. Indeed, the condition on the root corner of M forces e_1 to be attached to this corner, while the rest of the choices are then forced by requirement that the inner faces of $g(P, D')$ that are incident to u_0 must be quadrangles. Hence, when applying g to $(P, D') = f(D)$, we must obtain the map D. □

Lemma 3.13. *The generating function* $\mathsf{Q}(t)$ *is given by*

$$\mathsf{Q}(t) = [y^1]\mathsf{P}(t, y) - 1.$$

Proof. Let Q be any labelled quadrangulation. The outer face may contain a label -1 or 2, hence Q is not necessarily a patch. Let P be the map constructed from Q by adding an edge e' between the root vertex and co-root vertex in the outer face of Q, so that e' and the root edge e are the only outer edges of P. Then P can be any patch with outer degree 2, except for the patch with only one edge. Hence the possible patches P are counted by $([y^1]\mathsf{P}(t, y) - 1)$. Since the number of inner faces of P is equal to the total number of faces of Q, this expression is exactly equal to $\mathsf{Q}(t)$. This concludes the proof. □

Proof of Theorem 3.1. We have now proved the five functional equations. It remains to prove that, together with the conditions on the rings that contain P, C and D, they determine these three series. Let us denote by $p_{j,n}$ the coefficient of $y^j t^n$ in $\mathsf{P}(t, y)$, and similarly for C and D. These quantities should be thought of respectively as elements of $\mathbb{Q}$ (for P), of $\mathbb{Q}[x]$ (for C) and of $\mathbb{Q}[[x]]$ (for D). We will prove by induction on $N \geq 0$ that

- $p_{j,n}$ is completely determined for $j + n < N$,
- $c_{j,n}$ and $d_{j,n}$ are completely determined for $j + n \leq N$.

When $N = 0$, there is nothing to prove for P. The third equation of the system shows that $\mathsf{D} - 1$ is a multiple of y. That is, not only $d_{0,0} = 1$, but in fact we also know that $d_{0,n} = 0$ for $n \geq 1$. The second equation then tells us that C is a multiple of y, so that $c_{0,n} = 0$ for $n \geq 0$. Now assume that the induction hypothesis holds for some $N \geq 0$, and let us prove it for $N + 1$.

We begin with the series D. Of course it suffices to determine the coefficients $d_{j,n}$ for $j + n = N + 1$. We have already explained that $d_{0,N+1} = 0$, so we take $j \geq 1$. The third equation of the system expresses $d_{j,N+1-j}$ in terms of the series $d_{j-1,m}$ (for

$m \leq N+1-j$), $p_{k,\ell}$ (for $k+\ell \leq N+1-j$) and $d_{1,m}$ (for $m \leq N+1-j$). If $j \geq 2$, these series are known, by the induction hypothesis, and thus $d_{j,N+1-j}$ is completely determined. As argued below Theorem 3.1, it belongs to $\mathbb{Q}[[x]]$. To determine the final coefficient $d_{1,N}$, we resort to the fourth equation, which expresses $d_{1,N}$ in terms of $d_{2,N-1}$ (which we have just determined) and the series $d_{1,m}$ for $m \leq N-1$ (which are known by the induction hypothesis). Again, $d_{1,N}$ belongs to $\mathbb{Q}[[x]]$.

Hence for $j+n \leq N+1$, the coefficients $d_{j,n}$ are uniquely determined and hence must count D-patches with outer degree $2j$ and n quadrangles. Since we know that the generating functions of C-patches and D-patches are related by the second equation (see Lemma 3.4), this forces the coefficients $c_{j,n}$, for $j+n \leq N+1$, to count C-patches. Hence they are also fully determined (and are *polynomials* in x). Finally, the first equation of the system shows that the numbers $p_{j,n}$ are also determined for $j+n \leq N$ (we cannot go up to $N+1$ because of the division by y).

This concludes our induction. □

4. Solution for quartic Eulerian orientations

We are now about to solve the system of Theorem 3.1, thus proving, in particular, that the generating function $\mathsf{Q}(t)$ of quartic Eulerian orientations is indeed given by Theorem 1.1. The third equation of the system suggests that we should consider the series $\mathsf{P}(t,ty)$ rather than $\mathsf{P}(t,y)$. In turn, this leads us to apply the same transformation to the series C and D. More precisely, let us consider

$$\mathcal{P}(t,y) = t\,\mathsf{P}(t,ty), \qquad \mathcal{C}(t,x,y) = \mathsf{C}(t,x,ty), \qquad \mathcal{D}(t,x,y) = \mathsf{D}(t,x,ty). \tag{4}$$

Of course, if we determine $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$, then P, C and D are completely determined as well.

The solution below has been *guessed*, and then of course checked. The first step was the discovery of the connection between the generating function Q and the series R coming from [17]. Next, writing the auxiliary series $\mathcal{P}(t,y)$, $\mathcal{C}(t,x,y)$ and $\mathcal{D}(t,x,y)$ as series in R, x and y, we noticed that the coefficients of $\mathcal{P}(t,y)$ were simple products of binomial coefficients. Next, by chance we found that the series $\mathcal{D}(t,0,1)$ appeared in the On-line Encyclopedia of Integer Sequences as the exponential of a much nicer sequence [53, A229452], so we tried taking the log of $\mathcal{D}(t,x,y)$. We were pleasantly surprised to see that $\log(\mathcal{D}(t,x,y))$ had very nice coefficients when written as a series in R, x and y, which allowed us to guess its exact form as well as that of $\mathcal{C}(t,x,y)$. To our knowledge, this is the first time that series of this form appear in combinatorial enumeration.

Theorem 4.1. *Let $\mathsf{R}(t) \equiv \mathsf{R}$ be the unique formal power series with constant term 0 satisfying*

$$t = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n}\mathsf{R}^{n+1}. \tag{5}$$

Then the above series $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$ are:

$$\mathcal{P}(t,y) = \sum_{n\geq 0}\sum_{j=0}^{n} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n-j}{n} y^j \mathsf{R}^{n+1}, \tag{6}$$

$$\mathcal{C}(t,x,y) = 1 - \exp\left(-\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{2n-j} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n-i-j}{n} x^{i+1}y^{j+1}\mathsf{R}^{n+1}\right),$$

$$\mathcal{D}(t,x,y) = \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i\geq 0} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n+i-j+1}{2n-j} x^i y^{j+1}\mathsf{R}^{n+1}\right). \tag{7}$$

The generating function of quartic Eulerian orientations, counted by vertices, is

$$\mathsf{Q}(t) = \frac{1}{3t^2}\left(t - 3t^2 - \mathsf{R}(t)\right).$$

Proof. We take for $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$ the above series, and define P, C and D by (4). Since $\mathsf{R} = O(t)$, these three series are easily seen to belong respectively to the rings $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ and $\mathbb{Q}[[x,y,t]]$, as required by Theorem 3.1. Thus it suffices to check that the first four equations of Theorem 3.1 hold, or, equivalently, that

$$\begin{aligned}
\mathcal{P}(t,y) &= \frac{1}{y}[x^1]\mathcal{C}(t,x,y),\\
\mathcal{D}(t,x,y) &= \frac{1}{1-\mathcal{C}\left(t,\frac{1}{1-x},y\right)},\\
\mathcal{D}(t,x,y) &= 1 + y\,[x^{\geq 0}]\left(\mathcal{D}(t,x,y)\left(\frac{1}{x}\mathcal{P}\left(t,\frac{1}{x}\right) + [y^1]\mathcal{D}(t,x,y)\right)\right), \qquad (8)\\
[y^1]\mathcal{D}(t,x,y) &= \frac{1}{1-x}\left(t + 2[y^2]\mathcal{D}(t,x,y) - ([y^1]\mathcal{D}(t,x,y))^2\right).
\end{aligned}$$

Note that the first three equations do not involve explicitly the variable t: we will prove them without resorting to the definition (5) of R.

The first equation is straightforward. For the second one, it suffices to prove that for all $j \leq n$,

$$\sum_{i=0}^{2n-j}\binom{3n-i-j}{n}\frac{1}{(1-x)^{i+1}} = \sum_{i\geq 0}\binom{3n+i-j+1}{2n-j}x^i.$$

This follows by expanding the left-hand side in x and using the classical identity, taken for $k = 3n-j$:

$$\sum_{i=0}^{k-n}\binom{k-i}{n}\binom{\ell+i}{\ell} = \binom{k+\ell+1}{n+\ell+1} = \binom{k+\ell+1}{k-n}. \tag{9}$$

We now come to the third, and most interesting, equation. Our first observation is that, in the expression (7) of $\mathcal{D}(t,x,y)$, the sum over i is a rational function of x:

$$\sum_{i\geq 0}\binom{3n+i-j+1}{2n-j}x^i = \sum_{k\geq n+1}\binom{2n-j+k}{2n-j}x^{k-n-1}$$
$$= \frac{1}{x^{n+1}(1-x)^{2n-j+1}} - \sum_{\ell=0}^{n}\binom{3n-\ell-j}{2n-j}\frac{1}{x^{\ell+1}}. \qquad (10)$$

We note that the sum over ℓ in the above expression is a polynomial in $1/x$, with no constant term. Let us denote it by $L_{j,n}(1/x)$. The expression of $\mathcal{D}$ thus reads

$$\mathcal{D}(t,x,y) = \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n}\frac{1}{n+1}\binom{2n-j}{n}y^{j+1}\mathsf{R}^{n+1}\left(\frac{1}{x^{n+1}(1-x)^{2n-j+1}} - L_{j,n}(1/x)\right)\right)$$
$$= \exp\big(A(\mathsf{U},z) - B(\mathsf{R},1/x,y)\big), \qquad (11)$$

where

$$\mathsf{U} = \frac{\mathsf{R}}{x(1-x)^2}, \qquad z = (1-x)y,$$
$$A(u,z) = \sum_{n\geq 0}\sum_{j=0}^{n}\frac{1}{n+1}\binom{2n-j}{n}z^{j+1}u^{n+1} \qquad (12)$$

and

$$B(r,1/x,y) = \sum_{n\geq 0}\sum_{j=0}^{n}\frac{1}{n+1}\binom{2n-j}{n}L_{j,n}(1/x)y^{j+1}r^{n+1}.$$

By extracting the coefficient of y from (11), we find

$$[y^1]\mathcal{D}(t,x,y) = (1-x)\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}\mathsf{U}^{n+1} - \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}L_{0,n}(1/x)\mathsf{R}^{n+1},$$
$$= (1-x)\,\mathsf{U}\,\mathrm{Cat}(\mathsf{U}) - \frac{1}{x}\,\mathcal{P}\left(t,\frac{1}{x}\right), \qquad (13)$$

where $\mathrm{Cat}(u)$ is the Catalan series $\sum\binom{2n}{n}\frac{u^n}{n+1}$ and $\mathcal{P}$ is given by (6) (we have used the fact that $\binom{2n}{n}\binom{3n-\ell}{2n} = \binom{2n-\ell}{n}\binom{3n-\ell}{n}$). The identity (8) that we have to prove thus reads

$$1 = [x^{\geq 0}]\big(\mathcal{D}(t,x,y)(1-z\mathsf{U}\,\mathrm{Cat}(\mathsf{U}))\big),$$

where we still denote $z = (1-x)y$. Equivalently, in view of (11):

$$1 = [x^{\geq 0}]\Big(\exp(A(\mathsf{U}, z))\,(1 - z\mathsf{U}\,\mathrm{Cat}(\mathsf{U}))\exp\big(-B\big(\mathsf{R}, 1/x, y\big)\big)\Big).$$

We will prove below in Lemma 4.2 that

$$\exp(A(\mathsf{U}, z))\,(1 - z\mathsf{U}\,\mathrm{Cat}(\mathsf{U})) = 1,$$

which, given that $B(\mathsf{R}, 1/x, y)$ only involves negative powers of x, concludes the proof of the third identity.

Consider now the fourth equation of the system. Given that the second equation holds, what we need to prove can be rewritten as:

$$[y^1]\mathcal{C}(t, x, y) = x\left(t + 2[y^2]\mathcal{C}(t, x, y) + ([y^1]\mathcal{C}(t, x, y))^2\right).$$

Let us write $\mathcal{C}(t, x, y) = 1 - \exp(-T(t, x, y))$. Then the above identity reads:

$$[y^1]T(t, x, y) - 2x[y^2]T(t, x, y) = tx.$$

A direct calculation gives

$$[y^1]T(t, x, y) - 2x[y^2]T(t, x, y) = x\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}\binom{3n}{n}\mathsf{R}^{n+1},$$

which is precisely xt, by definition (5) of the series R.

We have thus proved the announced expressions of the series $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$, which in turn characterise the generating functions P, C and D of patches of various types (see (4)). We still have to express the generating function $\mathsf{Q}(t)$ of quartic Eulerian orientations in terms of $\mathsf{R}(t)$. The last equation of Theorem 3.1 now reads

$$\begin{aligned}
\mathsf{Q}(t) &= \frac{1}{t^2}[y^1]\mathcal{P}(t, y) - 1 \\
&= \frac{1}{t^2}\sum_{n\geq 1}\frac{1}{n+1}\binom{2n-1}{n}\binom{3n-1}{n}\mathsf{R}^{n+1} - 1 \\
&= \frac{1}{3t^2}\sum_{n\geq 1}\frac{1}{n+1}\binom{2n}{n}\binom{3n}{n}\mathsf{R}^{n+1} - 1 \\
&= \frac{1}{3t^2}\left(t - \mathsf{R} - 3t^2\right)
\end{aligned}$$

by definition of R. □

It remains to prove the following lemma, used in the above proof.

Lemma 4.2. *For any indeterminates u and z, the Catalan series* $\mathrm{Cat}(u) = \sum_{n\geq 0} \binom{2n}{n} u^n/(n+1)$ *and the series* $A(u,z)$ *defined by* (12) *are related by:*

$$\exp(A(u,z))\,(1 - zu\,\mathrm{Cat}(u)) = 1.$$

Proof. Equivalently, what we want to prove reads

$$A(u,z) = \log \frac{1}{1 - zu\,\mathrm{Cat}(u)} = \sum_{j\geq 0} \frac{z^{j+1}}{j+1} (u\,\mathrm{Cat}(u))^{j+1}.$$

Comparing with the expansion in z of $A(u,z)$ (see (12)), what we want to show is

$$[u^{n+1}](u\,\mathrm{Cat}(u))^{j+1} = \frac{j+1}{n+1}\binom{2n-j}{n}.$$

This follows from the Lagrange inversion formula [33, p. 732], applied to

$$F(u) := u\,\mathrm{Cat}(u) = \frac{u}{1 - F(u)}. \quad \square$$

Remark. The above lemma is a special case of a general identity which relates the enumeration of two classes of one-dimensional lattice paths, sharing the same step set, both constrained to end at a non-negative position. For the first class there is no other condition, while for the second class the path is not allowed to visit any negative point. The generating functions of these two classes, counted by the number of steps (variable z) and the final position (variable u) are respectively denoted by $W^+(z,u)$ and $F(z,u)$. Then, on p. 51 of [3], the following identity appears

$$F(z,u) = \exp\left(\int_0^z \left(W^+(t,u) - 1\right) \frac{\mathrm{d}t}{t}\right).$$

When the only allowed steps are $+1$ and -1, this reads, using a standard factorization on non-negative paths into *Dyck paths* (counted by $\mathrm{Cat}(z^2)$):

$$\frac{\mathrm{Cat}(z^2)}{1 - zu\,\mathrm{Cat}(z^2)} = \exp\left(\sum_{n\geq 1}\sum_{j=0}^{n} \frac{1}{2n-j}\binom{2n-j}{n} z^{2n-j} u^j\right).$$

Upon dividing this identity by its specialization at $u = 0$, we obtain

$$\frac{1}{1 - zu\,\mathrm{Cat}(z^2)} = \exp\left(\sum_{n\geq 1}\sum_{j=1}^{n} \frac{1}{2n-j}\binom{2n-j}{n} z^{2n-j} u^j\right).$$

Now some elementary transformations (involving replacing u by uz, then z by $\sqrt{z}$, and finally swapping u and z) shows that this is equivalent to our lemma.

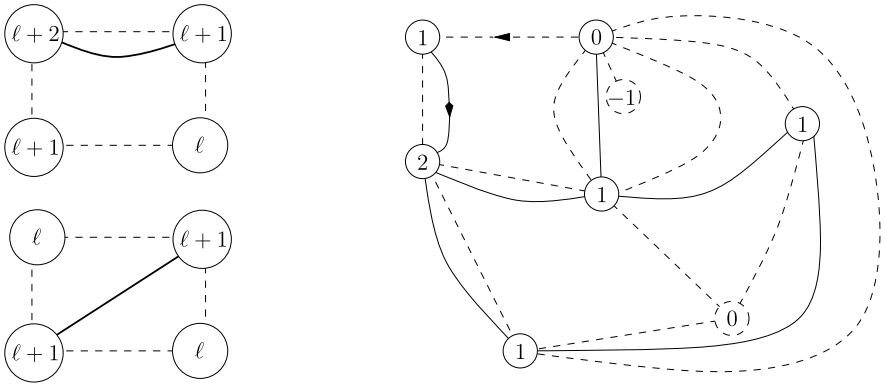

Fig. 17. Left: Construction of Φ in a face of Q. Right: A labelled quadrangulation Q (dashed edges) and the associated map $M = \Phi(Q)$ (solid edges) superimposed. Two white vertices of Q, namely its local minima, shown in dashed disks, disappear when constructing M.

5. A bijection

In this short section, we first recall a bijection of Ambjørn and Budd [1] that sends labelled quadrangulations onto certain maps carrying integer labels on vertices (these maps are more general than the labelled maps of Definition 2.1). A specialization of this bijection sends *certain* labelled quadrangulations (those in which every face contains three labels) onto labelled maps, which, as we have seen, are equinumerous with general Eulerian orientations. This is one of the key steps in the proof of Theorem 1.2. The Ambjørn and Budd bijection, which generalizes the Cori-Vauquelin-Schaeffer bijection between quadrangulations and certain labelled trees [56,25], can also be seen to be equivalent to an earlier bijection of Miermont [50]. We refer to [21] for a rich overview of Schaeffer-like bijections.

The Ambjørn and Budd bijection, which we denote by Φ, starts from a labelled quadrangulation Q. The edges of Q are dashed in our figures. The construction, illustrated on the left of Fig. 17, takes place independently in every face of Q, and in each face, coincides with Schaeffer's construction of labelled trees [25]: a new (solid) edge is created in every face of Q, and its position depends on whether the face contains three of two distinct labels.[1] A complete example is shown on the right of Fig. 17.

Observe that each vertex of Q that is not a local minimum is joined to at least one other vertex. Since the root edge of Q is oriented from 0 to 1, the co-root vertex must be joined to another vertex v by an edge located in the co-root face of Q. We orient this edge towards v: this will be the root edge of the new object. Finally, we delete all edges of Q, and also all vertices of Q that have become isolated: they are those whose label is a local minimum. We denote by $\Phi(Q)$ the resulting object, which is a planar graph embedded in the plane, with a root edge that starts from a vertex labelled 1.

[1] If the outer face has three distinct labels and is drawn as the infinite face, the solid edge that we add still has an edge from $\ell+1$ to $\ell+2$ on its right, now in clockwise order around the outer face.

Proposition 5.1 *(Thm. 1 in [1]). The transformation Φ bijectively sends labelled quadrangulations to planar maps carrying integer labels on vertices, differing by $0, \pm 1$ along edges, having root vertex labelled 1. Moreover, if $\Phi(Q) = M$, then the number of edges and faces in M are given by*

$$\mathrm{e}(M) = \mathrm{f}(Q), \qquad \mathrm{f}(M) = \mathrm{v}_{\min}(Q),$$

where $\mathrm{v}_{\min}(Q)$ denotes the number of local minima in Q. The first identity can be refined as follows: a face of Q in which only two different labels occur gives rise to an edge of M with increment 0, while a face where three different labels occur gives rise to an edge with increment ± 1.

Of particular importance will be labelled quadrangulations in which every face (including the outer one) contains three distinct labels: we call them *colourful*. Take a colourful labelled quadrangulation Q. By the above proposition, the map $M := \Phi(Q)$ has all increments equal to ± 1. Its root vertex is labelled 1, hence the root edge is labelled either from 1 to 0, or from 1 to 2. In the former case, reversing the direction of the root edge gives a labelled map (in the sense of Definition 2.1). In the latter case, subtracting 1 from every label gives a labelled map. Conversely, take a labelled map L, and reverse the orientation of its root edge: this gives a map of the form $\Phi(Q)$, in which the root edge has labels 1 and 0 (Fig. 18, left). Alternatively, one can add 1 to every label of L: the resulting map is of the form $\Phi(Q')$, and its root edge has labels 1 and 2 (Fig. 18, right). This gives a 2-to-1 correspondence between colourful labelled quadrangulations and labelled maps. This will be the key in our enumeration of general Eulerian orientations.

Corollary 5.2. *The number of colourful labelled quadrangulations with n faces and k local minima equals twice the number of labelled maps with n edges and k faces, or equivalently, twice the number of Eulerian orientations with n edges and k vertices.*

If we start from a general labelled quadrangulation Q, possibly containing faces with only two labels, then we can still apply the duality rule of Fig. 5 to the map $\Phi(Q)$ (carrying labels), with the additional rule that we do not orient an edge that lies between two faces with the same label.

In this way we obtain an Eulerian *partial* orientation, that is, a map in which *some* edges are oriented, in such a way that there are as many incoming as outgoing edges at any vertex (Fig. 19).

Corollary 5.3. *There is a bijection between quartic Eulerian orientations with n vertices (in which the root edge is oriented canonically) and Eulerian partial orientations with n edges (with no orientation requirement on the root edge).*

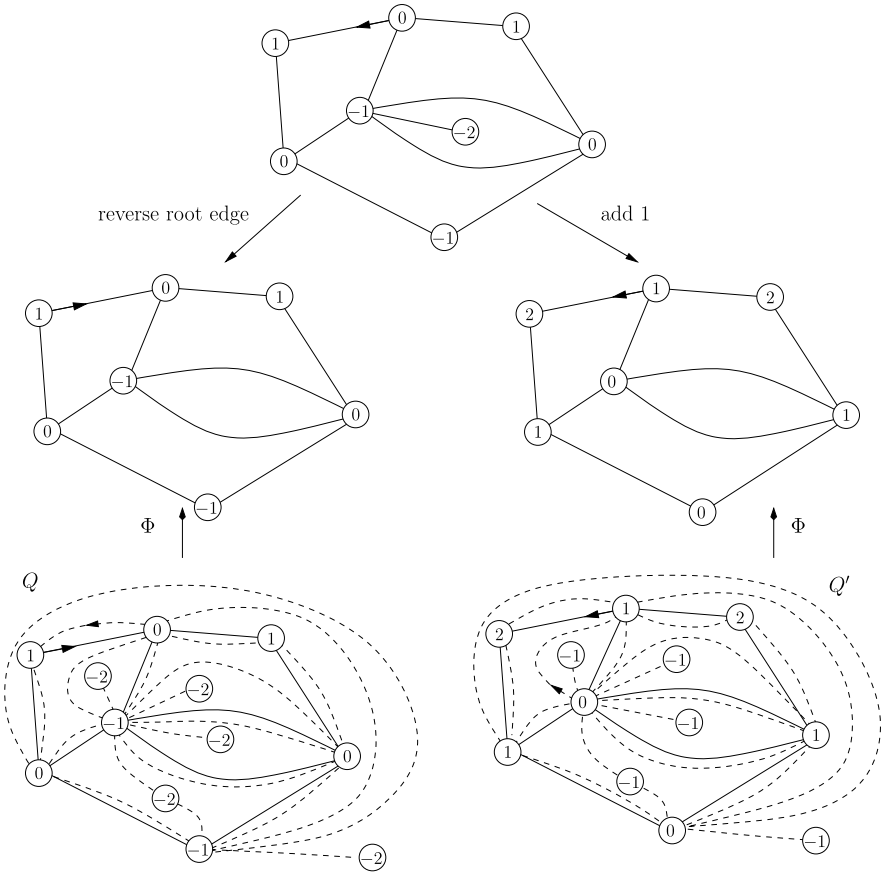

Fig. 18. One labelled map gives two maps with root vertex 1, which are the images by Φ of two colourful labelled quadrangulations Q and Q' (in dashed edges). They only differ by a shift of labels and a change in the root edge.

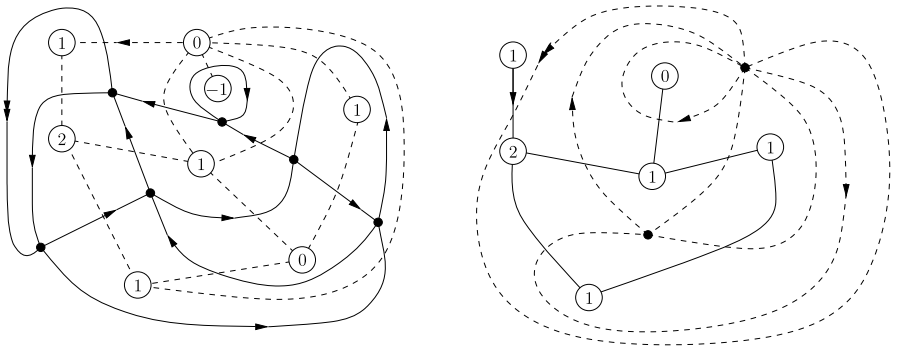

Fig. 19. From quartic Eulerian orientations to Eulerian *partial* orientations. Left: a quartic Eulerian orientation, shown by solid edges, and the dual labelled quadrangulation Q (dashed edges). This is the quadrangulation of Fig. 17, which also shows the map $M = \Phi(Q)$. Right: upon re-applying duality to M (shown in solid lines), one obtains an Eulerian partial orientation of its dual (dashed edges).

6. Functional equations for general Eulerian orientations

In this section we will characterise the generating function $\mathsf{Q}^{\mathsf{c}}(t)$ of colourful labelled quadrangulations (which, by Corollary 5.2, is twice the generating function of Eulerian orientations) by a system of functional equations. As one might expect, we adapt the system of Theorem 3.1 to the colourful setting. However, the third equation and the initial conditions are simpler in the colourful case.

Theorem 6.1. *There exists a unique* 3*-tuple of series, denoted* $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ *and* $\mathsf{D}(t,x,y)$, *belonging respectively to* $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ *and* $\mathbb{Q}[[x,y,t]]$, *and satisfying the following equations:*

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y),$$

$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)},$$

$$\mathsf{C}(t,x,y) = xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)\right),$$

together with the initial condition $\mathsf{P}(t,0)=1$.

The generating function that counts colourful labelled quadrangulations by faces is

$$\mathsf{Q}^{\mathsf{c}}(t) = [y^1]\mathsf{P}(t,y) - 1.$$

By Corollary 5.2, $\mathsf{Q}^{\mathsf{c}}(t) = 2\mathsf{G}(t)$, *where* $\mathsf{G}(t)$ *counts Eulerian orientations by edges.*

Remark. As with the system of Theorem 3.1, the conditions on the series P, C and D make the operations that occur in the above equations well defined. The extraction of the coefficient of x^1, and the replacement of x by $1/(1-x)$, are justified as for the previous system. In the third equation, the term $\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)$ must be seen as a power series in t and y whose coefficients are Laurent series in $1/x$. The extraction of the non-negative part in x then yields an element of $\mathbb{Q}[x][[y,t]]$.

As before, the series P, C and D of Theorem 6.1 count certain labelled maps. Recall the definition of patches, C-patches and D-patches (Definition 3.2). Generalizing the definition of colourful quadrangulations introduced in Section 5, we say that a patch (or a D-patch) is *colourful* if each inner quadrangle contains 3 distinct labels. We define $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$ to be respectively the generating functions of colourful patches, colourful C-patches and colourful D-patches, where t counts inner quadrangles, y the outer degree (halved), and x either the degree of the root vertex (for C-patches) or the number of inner digons (for D-patches). The equations

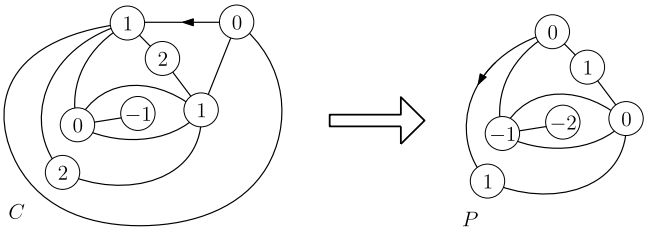

Fig. 20. A colourful C-patch C with outer degree 2 and the corresponding patch P.

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y)$$

$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)},$$

$$\mathsf{Q}^{\mathsf{c}}(t) = [y^1]\mathsf{P}(t,y) - 1,$$

have identical proofs to those in Section 3, except that the patches, D-patches and quadrangulations in the proofs are restricted to being colourful (see Lemmas 3.3, 3.4 and 3.13).

Remark. The third equation of Theorem 3.1,

$$\mathsf{D}(t,x,y) = 1 + y\,[x^{\geq 0}]\left(\mathsf{D}(t,x,y)\left(\frac{1}{x}\mathsf{P}\left(t,\frac{t}{x}\right) + [y^1]\mathsf{D}(t,x,y)\right)\right) \tag{14}$$

also holds in the colourful setting, with the same proof as before (because in the proof of Lemma 3.8, all quadrangles that come from digons are automatically colourful). Its natural complement, which is the fourth equation of Theorem 3.1 (the initial condition) has no clear colourful counterpart: indeed, the relabelling of vertices that we use in Lemma 3.5, and more precisely in the fourth case of Fig. 12, transforms the colourful quadrangles incident to the root vertex into bicoloured quadrangles (and vice versa). We could instead use the initial condition $[y^1]\mathsf{C}(t,x,y) = x\mathsf{P}(t,tx)$, which can be proved by taking a colourful C-patch of outer degree 2 and deleting the root vertex and all incident edges, then decreasing each label by 1 (Fig. 20). However, the third equation of Theorem 6.1 is simpler than (14), and also relies on a simpler construction.

In order to prove the third equation of Theorem 6.1, we need some analogues of minus-patches from Section 3, which we call *shifted patches*.

Definition 6.2. A *shifted patch* is a map obtained from a patch by replacing each label ℓ with $\ell + 1$.

We now describe a way to extract a shifted subpatch from a patch, which parallels the extraction of a minus-patch of Definition 3.10. One minor difference is that we do not

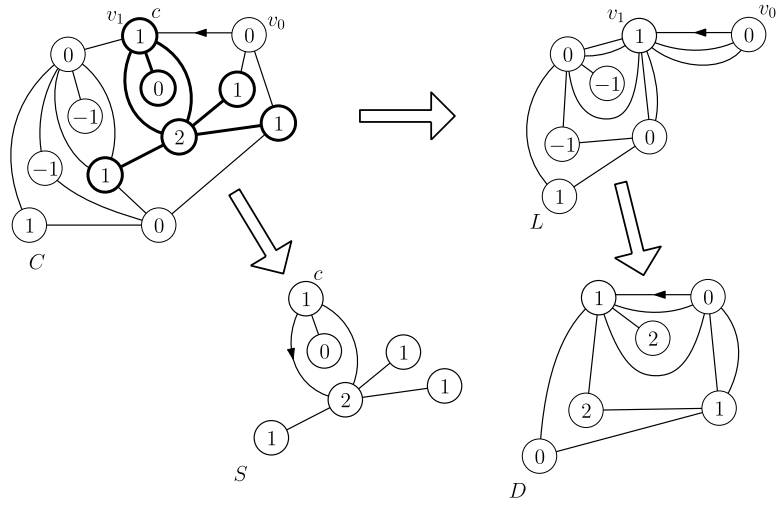

Fig. 21. On the left is a C-patch C. The other maps shown are the shifted subpatch S, the contracted map L and the D-patch D obtained by deleting v_0, as described in the proof of Lemma 6.4.

need conditions on the neighbours of the root vertex, so that we define shifted subpatches for any patch (although we will only extract them from C-patches later).

Definition 6.3. Let P be a patch and let c be an outer corner of P at a vertex v labelled 1. We define the *shifted subpatch of P rooted at c* as follows. First, let S' be the maximal connected submap of P that contains v and consists of vertices labelled 1 or more. Let S be the submap of P that contains S' and all edges and vertices within its boundary (assuming the root face is drawn as the infinite face). The map S, which we root at the corner inherited from c, is the *shifted subpatch of P rooted at c*.

An example is shown on the left of Fig. 21, where the shifted subpatch S is drawn with thick lines. It is easily shown that S is, as it should be, a shifted patch. The argument is the same as for minus-subpatches (in that case, the condition on the neighbours of the root having labels 1 was there to prevent the minus-subpatch to absorb the root vertex; this cannot happen with the shifted subpatch, whose boundary only contains positive labels). Every edge in P that connects a vertex in S to a vertex not in S must have endpoints labelled 1 (in S) and 0 (not in S), and conversely every vertex labelled 1 on the boundary of S is joined to a vertex labelled 0 out of S. We can contract S it into a single vertex v_1 labelled 1. This vertex is only adjacent to vertices labelled 0 in the resulting map L, and the number of digons incident to v_1 is half the outer degree of S. The outer degrees of L and P coincide, because no edge of the boundary of P has been contracted.

As in the case of minus-subpatches, we can uniquely reconstruct the patch P and its marked corner c if we are given the shifted patch S and the contracted map L, together with its outer corner inherited from c. The idea is again to attach the edges incident

to v_1 in L around the shifted patch S, as illustrated (in the case of minus-patches) in Fig. 16.

We are now ready to prove the third equation of Theorem 6.1.

Lemma 6.4. *The generating functions* P, C *and* D *satisfy the equation*

$$\mathsf{C}(t,x,y) = xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)\right).$$

Proof. Let C be any colourful C-patch. Let v_0 and v_1 be the root vertex and co-root vertex of C, and let c be the outer corner of v_1 that is immediately anticlockwise of the root edge (we refer to Fig. 21 for an illustration). Let S be the shifted subpatch of C rooted at c and let L be the labelled map obtained from C by contracting the subpatch S to a single vertex, still denoted v_1. Then in L, the root vertex v_0 is only adjacent to vertices labelled 1 (because this was already true in C), and the co-root vertex v_1 is only adjacent to vertices labelled 0 (because of the contraction).

Recall that all inner faces of L are either digons or quadrangles. We want to prove that in L, the root vertex v_0 is not incident to any inner quadrangle. Assume that such a quadrangle exists. Since v_0 only shares one corner with the outer face of L (this was the case for C already), and since v_0 is adjacent to v_1, one such quadrangle must be incident to v_1. But the above label conditions on the neighbours of v_0 and v_1 force this quadrangle to have labels 0 and 1 only, in L and thus in C. This contradicts the fact that C is colourful. Hence v_0 is only adjacent to inner digons (and to the outer face), and since it shares only one corner with the outer face, its only neighbour in L is v_1 (see Fig. 21).

Let D be the labelled map constructed from L by moving the root edge anticlockwise one place around the outer face, removing the old root vertex v_0 of L and all incident edges, and replacing each vertex label ℓ with $1-\ell$. Now the root vertex has label 0, and all its neighbours have label 1. The outer face still has labels 0 and 1. Each inner quadrangle of D corresponds to an inner quadrangle of C, and is therefore colourful. Hence D is a colourful D-patch.

Let $2j$ be the outer degree of S. Then j is also the number of inner digons in L. Let $i \leq j$ be the number of inner digons left in D after deleting v_0. Then the number of inner digons in L that are incident to v_0 (and v_1) is $j-i$. Therefore, the degree of v_0 in both L and C is $j-i+1$.

Conversely, taking a colourful D-patch D with i inner digons and a colourful shifted patch S of outer degree $2j$ with $j \geq i$ we construct the corresponding C-patch C as follows:

- we first construct a map D' by replacing each label ℓ in D with $1-\ell$,
- next we construct a map L with a new vertex v_0, joined to the root vertex of D' by $j-i+1$ edges,

- finally we insert S into L to create the corresponding patch C (the choice of the corner where the subpatch extraction takes place being canonical).

As already explained, the degree of the root vertex in C is $j-i+1$. The outer degree of C is the outer degree of D plus 2, and the number of inner quadrangles in C is j plus the number of inner quadrangles in S and D. Hence, with the obvious notation,

$$C(t,x,y) = \sum_{\mathrm{od}(S)\geq \mathrm{dig}(D)} x^{\mathrm{od}(S)-\mathrm{dig}(D)+1} y^{1+\mathrm{od}(D)} t^{\mathrm{qu}(S)+\mathrm{qu}(D)+\mathrm{od}(S)},$$

and this gives the equation of the lemma, since shifted patches are counted by $\mathsf{P}(t,y)$. □

Proof of Theorem 6.1. Given that the initial condition $\mathsf{P}(t,0)=1$ is obvious (it accounts for the atomic patch), we have now proved all functional equations. It remains to prove that, together with the conditions on the rings that contain P, C and D, they determine a unique 3-tuple of series. Let us denote by $p_{j,n}$ the coefficient of $y^j t^n$ in $\mathsf{P}(t,y)$, and similarly for C and D. These quantities should be thought of as elements of $\mathbb{Q}$ (for P), of $\mathbb{Q}[x]$ (for C) and of $\mathbb{Q}[[x]]$ (for D). We will prove by induction on $N\geq 0$ that $p_{j,n}$, $c_{j,n}$ and $d_{j,n}$ are completely determined for $j+n\leq N$ — we say *up to order* N.

First take $N=0$. The third equation of the system shows that C is a multiple of y. In particular, $c_{0,0}=0$. The second equation then implies that $\mathsf{D}-1$ is also a multiple of y. In particular, $d_{0,0}=1$. Finally, the initial condition $\mathsf{P}(t,0)=1$ gives $p_{0,0}=1$. Now assume that the induction hypothesis holds for some $N\geq 0$, and let us prove it for $N+1$.

The third equation, with its factor y, allows us to determine $\mathsf{C}(t,x,y)$ up to order $N+1$. By construction, the coefficients $c_{j,N+1-j}$ will be polynomials in x. Then the second equation gives $\mathsf{D}(t,x,y)$ up to the same order. The first equation seems to raise a problem, because of the division by y. But, combined with the third equation, it reads

$$\begin{aligned}\mathsf{P}(t,y) &= [x^0]\mathsf{P}(t,tx)\mathsf{D}(t,1/x,y)\\ &= [x^0]\mathsf{P}(t,tx) + [x^0]\mathsf{P}(t,tx)(\mathsf{D}(t,1/x,y)-1)\\ &= \mathsf{P}(t,0) + [x^0]\mathsf{P}(t,tx)(\mathsf{D}(t,1/x,y)-1).\end{aligned}$$

Now $\mathsf{P}(t,0)=1$ is known, and since $\mathsf{D}(t,1/x,y)-1$ is a multiple of y, knowing $\mathsf{P}(t,tx)$ up to order N and $\mathsf{D}(t,1/x,y)$ up to order $N+1$ suffices to determine $\mathsf{P}(t,y)$ up to order $N+1$. This completes our induction. □

7. Solution for general Eulerian orientations

We are now about to solve the system of Theorem 6.1, thus proving, in particular, that the generating function $\mathsf{G}(t)$ of Eulerian orientations is indeed given by Theorem 1.2. As in Section 4, the third equation of the system leads us to introduce variants of the series P, C and D, defined again by

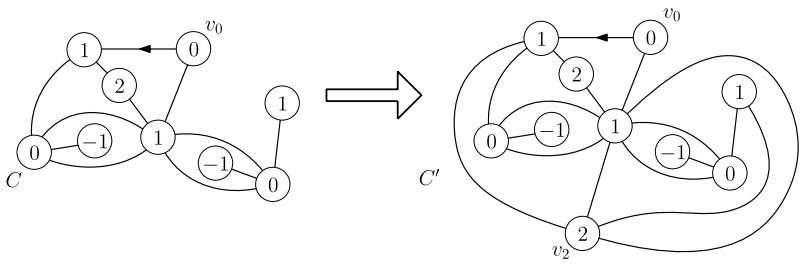

Fig. 22. An example of the transformation from a colourful C-patch C to a colourful labelled quadrangulation C' from the remark below Theorem 7.1.

$$\mathcal{P}(t,y) = t\,\mathsf{P}(t,ty), \qquad \mathcal{C}(t,x,y) = \mathsf{C}(t,x,ty), \qquad \mathcal{D}(t,x,y) = \mathsf{D}(t,x,ty). \tag{15}$$

Of course, if we determine $\mathcal{P}, \mathcal{C}$ and $\mathcal{D}$, then P, C and D are completely determined as well.

Theorem 7.1. *Let* $\mathsf{R}(t) \equiv \mathsf{R}$ *be the unique formal power series with constant term* 0 *satisfying*

$$t = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}^{n+1}.$$

Then the above series $\mathcal{P}$, $\mathcal{C}$ *and* $\mathcal{D}$ *are:*

$$\mathcal{P}(t,y) = \sum_{n\geq 0}\sum_{j=0}^{n} \frac{1}{n+1}\binom{2n}{n}\binom{2n-j}{n} y^j \mathsf{R}^{n+1},$$

$$\mathcal{C}(t,x,y) = 1 - \exp\left(-\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{n} \frac{1}{n+1}\binom{2n-i}{n}\binom{2n-j}{n} x^{i+1}y^{j+1}\mathsf{R}^{n+1}\right),$$

$$\mathcal{D}(t,x,y) = \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i\geq 0} \frac{1}{n+1}\binom{2n-j}{n}\binom{2n+i+1}{n} x^i y^{j+1}\mathsf{R}^{n+1}\right).$$

The generating function of Eulerian orientations, counted by edges, is

$$\mathsf{G}(t) = \frac{1}{4t^2}\left(t - 2t^2 - \mathsf{R}(t)\right).$$

Remark. Observe that the series $\mathcal{C}(t,x,y)$ is symmetric in x and y. Let us give a combinatorial explanation for this, illustrated in Fig. 22. Consider a colourful C-patch C with root vertex v_0, and form a colourful labelled quadrangulation C' by adding a vertex v_2 with label 2 to the outer face of C and joining it to each outer corner of C labelled 1. The generating function $\mathcal{C}(t,x,y) = \mathsf{C}(t,x,ty)$ then counts the possible objects C' by the number of quadrangles (variable t), the degree of the root vertex v_0 (variable x)

and the degree of the new vertex v_2 (variable y). Moreover, the object C' can be any colourful quadrangulation in which the outer face has labels 0, 1, 2, 1 and each vertex that neighbours either v_0 or v_2 is labelled 1. The transformation $\ell \mapsto 2-\ell$ then explains why the generating function $\mathcal{C}(t,x,y)$ is symmetric in x and y.

Proof of Theorem 7.1. We argue as in the proof of Theorem 4.1. Defining $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$ as above, we first observe that the series P, C and D defined by (15) belong respectively to the rings $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ and $\mathbb{Q}[[x,y,t]]$, as prescribed in Theorem 6.1. Hence it suffices to prove that they satisfy the desired system, which reads

$$\begin{aligned}
\mathcal{P}(t,y) &= \frac{1}{y}[x^1]\mathcal{C}(t,x,y),\\
\mathcal{D}(t,x,y) &= \frac{1}{1-\mathcal{C}\left(t,\frac{1}{1-x},y\right)},\\
\mathcal{C}(t,x,y) &= xy[x^{\geq 0}]\mathcal{P}(t,x)\mathcal{D}(t,1/x,y),\\
\mathcal{P}(t,0) &= t.
\end{aligned}$$

Note that the first three equations do not explicitly involve the variable t: we will prove them without resorting to the definition of R. But the fourth equation, namely the initial condition $\mathcal{P}(t,0)=t$, does involve t, and in fact holds precisely by definition of R.

The first equation is again straightforward, and the second follows from (9) again. Now consider the third one. Since it is more natural to handle series in x rather than in $1/x$, we will show instead that

$$\begin{aligned}
\mathcal{C}(t,1/x,y) &= y/x[x^{\leq 0}]\mathcal{P}(t,1/x)\mathcal{D}(t,x,y)\\
&= [x^{<0}]\left(y/x\,\mathcal{P}(t,1/x)\mathcal{D}(t,x,y)\right).
\end{aligned} \tag{16}$$

In order to prove (16), we use identities that are similar to those used in the proof of (8). The counterpart of (10) is:

$$\sum_{i\geq 0}\binom{2n+i+1}{n}x^i = \frac{1}{x^{n+1}(1-x)^{n+1}} - \sum_{\ell=0}^{n}\binom{2n-\ell}{n}\frac{1}{x^{\ell+1}}.$$

The counterpart of (11) is:

$$\mathcal{D}(t,x,y) = \exp\left(A(\mathsf{U},y)\right)\left(1-\mathcal{C}(t,1/x,y)\right), \tag{17}$$

with $\mathsf{U}=\frac{\mathsf{R}}{x(1-x)}$ and $A(u,y)$ is still given by (12). This is indeed an analogue of (11), since

$$1-\mathcal{C}(t,1/x,y) = \exp\left(-\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{n}\frac{1}{n+1}\binom{2n-i}{n}\binom{2n-j}{n}\frac{1}{x^{i+1}}y^{j+1}\mathsf{R}^{n+1}\right)$$

can be written as $\exp(-B(\mathsf{R}, 1/x, y))$ where $B(\mathsf{R}, 1/x, y)$ only involves negative powers of x. By extracting the coefficient of y from (17), we find the counterpart of (13):

$$[y]\mathcal{D}(t,x,y) = \mathsf{U}\,\mathrm{Cat}(\mathsf{U}) - \frac{1}{x}\mathcal{P}\left(t, \frac{1}{x}\right), \tag{18}$$

where $\mathrm{Cat}(u)$ is still the Catalan series $\sum_{n\geq 0} \frac{u^n}{n+1}\binom{2n}{n}$.

With these identities at hand, we can now prove (16):

$$\begin{aligned}
[x^{<0}]\,(y/x\,\mathcal{P}(t,1/x)\mathcal{D}(t,x,y)) &= [x^{<0}]\,(y\mathcal{D}(t,x,y)\,(\mathsf{U}\,\mathrm{Cat}(\mathsf{U}) - [y]\mathcal{D}(t,x,y))) \quad \text{by (18)},\\
&= [x^{<0}]\,(y\mathcal{D}(t,x,y)\ \mathsf{U}\,\mathrm{Cat}(\mathsf{U}))\\
&= [x^{<0}]\,(-\mathcal{D}(t,x,y)(1 - y\mathsf{U}\,\mathrm{Cat}(\mathsf{U})))\\
&= [x^{<0}]\,(-\mathcal{D}(t,x,y)\exp(-A(\mathsf{U},y))) \qquad \text{by Lemma 4.2},\\
&= [x^{<0}]\,(-1 + \mathcal{C}(t,1/x,y)) \qquad \text{by (17)},\\
&= \mathcal{C}(t,1/x,y).
\end{aligned}$$

We have thus proved the announced expressions of $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$, which in turn characterise the generating functions P, C and D of colourful patches of various types. Now the last equation of Theorem 6.1 gives

$$\begin{aligned}
2\mathsf{G}(t) = \mathsf{Q}^{\mathsf{c}}(t) &= \frac{1}{t^2}[y^1]\mathcal{P}(t,y) - 1\\
&= \frac{1}{t^2}\sum_{n\geq 1}\frac{1}{n+1}\binom{2n}{n}\binom{2n-1}{n}\mathsf{R}^{n+1} - 1\\
&= \frac{1}{2t^2}\sum_{n\geq 1}\frac{1}{n+1}\binom{2n}{n}^2\mathsf{R}^{n+1} - 1\\
&= \frac{1}{2t^2}\left(t - \mathsf{R} - 2t^2\right).
\end{aligned}$$

The expression given in Theorem 7.1 (and in Theorem 1.2) for $\mathsf{G}(t)$ follows. □

8. Nature of the series and asymptotics

8.1. Nature of the series

We begin by proving that the series $\mathsf{Q}(t)$ and $\mathsf{G}(t)$ that count respectively quartic and general Eulerian orientations satisfy non-linear differential equations of order 2, as claimed in Theorems 1.1 and 1.2. Both series are expressed in terms of a series R that satisfies

$$\Omega(\mathsf{R}) = t,$$

for some hypergeometric series Ω. In the quartic case (Theorem 1.1),

$$\Omega(r) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n} r^{n+1} \tag{19}$$

satisfies

$$6\Omega(r) + r(27r-1)\Omega''(r) = 0,$$

from which we derive that

$$\mathsf{R}(27\,\mathsf{R}-1)\mathsf{R}'' = 6t\,\mathsf{R}'^{\,3}.$$

Using $3t^2\mathsf{Q}(t) = t - 3t^2 - \mathsf{R}(t)$, this gives indeed a second order DE for $\mathsf{Q}(t)$, of degree 3.

For general Eulerian orientations (Theorem 1.2), we still have $\Omega(\mathsf{R}) = t$, with

$$\Omega(r) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}^2 r^{n+1} \tag{20}$$

satisfies

$$4\Omega(r) + r(16r-1)\Omega''(r) = 0,$$

from which we derive that

$$\mathsf{R}(16\,\mathsf{R}-1)\mathsf{R}'' = 4t\,\mathsf{R}'^{\,3}.$$

Using $4t^2\mathsf{G}(t) = t - 2t^2 - \mathsf{R}(t)$, this gives a second order DE for $\mathsf{G}(t)$, of degree 3.

The fact that neither $\mathsf{Q}(t)$ nor $\mathsf{G}(t)$ solve a non-trivial *linear* DE will follow from the asymptotic behaviour of their coefficients, established in the next subsection: indeed, the logarithm occurring at the denominator prevents this behaviour from being that of the coefficients of a D-finite series [33, p. 520 and 582].

We can also describe the nature of the multivariate series counting patches.

Proposition 8.1. *The generating functions* $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ *and* $\mathsf{D}(t,x,y)$ *counting patches of various types, and expressed in Theorem 4.1 through the identities* (4)*, are D-algebraic. The same holds for their colourful counterparts, expressed in Theorem 7.1.*

Proof. This follows by composition of D-algebraic series (see, e.g., [9, Prop. 29]). □

Note that both series $\mathcal{P}(t,y)$ (in the general and colourful cases) are even D-finite *as functions of* y *and* R. The other two series $\mathcal{C}(t,x,y)$ and $\mathcal{D}(t,x,y)$ are clearly D-algebraic as functions of x, y and R, and it is natural to wonder if they might be D-finite. After all,

in Lemma 4.2 we have met a series that is written as the exponential of a hypergeometric series and is not only D-finite, but even algebraic.

In the one-variable setting, it is known that if $F(t)$ is D-finite, then $\exp(\int F(t))$ is D-finite if and only if $F(t)$ is in fact algebraic [58]. We can use this criterion to prove, for instance, that the series $\mathcal{D}(t,0,1)$ of Theorem 4.1 is not D-finite as a function of R. Indeed, $\mathcal{D}(t,0,1) = D(\mathsf{R})$ with

$$\begin{aligned} D(r) &= \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n-j+1}{2n-j} r^{n+1}\right) \\ &= \exp\left(\sum_{n\geq 0} \frac{3n+2}{2(n+1)^2}\binom{2n}{n}\binom{3n+1}{2n} r^{n+1}\right) \\ &= \exp\left(\int F(r)\right) \end{aligned}$$

where

$$F(r) = \sum_{n\geq 0} \frac{3n+2}{2(n+1)}\binom{2n}{n}\binom{3n+1}{2n} r^n.$$

Then $D(r)$ is D-finite if and only if $F(r)$ is algebraic. But this is not the case, as the coefficient of r^n in $F(r)$ is asymptotic to $c\,27^n/n$, which reveals a logarithmic singularity in $F(r)$ (see [31]). The same argument proves that $\mathcal{C}(t,1,1)$ is not a D-finite function of R.

In the colourful case (Theorem 7.1), we have

$$\mathcal{D}(t,0,1) = \exp\left(\sum_{n\geq 0} \frac{1}{n+1}\binom{2n+1}{n}^2 \mathsf{R}^{n+1}\right),$$

and a similar asymptotic argument proves that this cannot be a D-finite function of R. The same holds for $\mathcal{C}(t,1,1)$.

8.2. Asymptotics

As mentioned in the introduction, the series R of Theorem 1.1 already occurred in the map literature, more precisely in the enumeration of quartic maps equipped with a spanning forest [17]. Its singular structure has been studied in details, and the first part of the following result is the case $u=-1$ of [17, Prop. 8.4]. As in [33, Def. VI.1, p. 389], we call Δ-*domain of radius* ρ any domain of the form

$$\{z : |z| < r, z \neq \rho \text{ and } |\operatorname{Arg}(z-\rho)| > \phi\}$$

for some $r > \rho$ and $\phi \in (0, \pi/2)$.

Proposition 8.2. *The series* R *of Theorem 1.1 has radius* $\rho = \frac{\sqrt{3}}{12\pi}$. *It is analytic in a* Δ*-domain of radius* ρ, *and the following estimate holds in this domain, as* $t \to \rho$:

$$\mathsf{R}(t) - \frac{1}{27} \sim \frac{1}{6} \frac{1 - t/\rho}{\log(1 - t/\rho)}.$$

Consequently, the n*th coefficient of* R *satisfies, as* $n \to \infty$,

$$r_n := [t^n]\mathsf{R} \sim -\frac{1}{6} \frac{\mu^n}{n^2 \log^2 n}$$

with $\mu = 1/\rho = 4\sqrt{3}\pi$.

Observe that this provides the asymptotic behaviour of the numbers q_n of Theorem 1.1 since $q_n = -r_{n+2}/3$. The correspondence between the singular behaviour of $\mathsf{R}(t)$ near its dominant singularity ρ and the asymptotic behaviour of its coefficients relies on Flajolet and Odlyzko's *singularity analysis* of generating functions [32,33]. The singular behaviour of R near ρ is obtained using the inversion relation $\Omega(\mathsf{R}(t)) = t$, where the series Ω, given by (19), has radius $1/27$ and satisfies

$$\Omega\left(\frac{1}{27}(1 - \varepsilon)\right) = \frac{\sqrt{3}}{12\pi} + \frac{\sqrt{3}}{54\pi}\varepsilon \log \varepsilon + O(\varepsilon)$$

as $\varepsilon \to 0$.

For general Eulerian orientations, we have a similar result.

Proposition 8.3. *The series* R *of Theorem 1.2 has radius* $\rho = \frac{1}{4\pi}$. *It is analytic in a* Δ*-domain of radius* ρ, *and the following estimate holds in this domain, as* $t \to \rho$:

$$\mathsf{R}(t) - \frac{1}{16} \sim \frac{1}{4} \frac{1 - t/\rho}{\log(1 - t/\rho)}.$$

Consequently, the n*th coefficient of* R *satisfies, as* $n \to \infty$,

$$r_n := [t^n]\mathsf{R} \sim -\frac{1}{4} \frac{\mu^n}{n^2 \log^2 n}$$

with $\mu = 1/\rho = 4\pi$.

As above, this gives the asymptotic behaviour of the numbers g_n of Theorem 1.1 since $g_n = -r_{n+2}/4$.

The proof closely follows the proof of Proposition 8.2 given in [17, Sec. 8.3], and we will not give any details. The series Ω is now given by (20), has radius of convergence $1/16$ and satisfies

$$\Omega\left(\frac{1}{16}(1-\varepsilon)\right) = \frac{1}{4\pi} + \frac{1}{16\pi}\varepsilon\log\varepsilon + O(\varepsilon).$$

One key ingredient is that $t - \mathsf{R}(t)$ has non-negative coefficients, which simply follows from the fact that this series equals $2t^2 + 4t^2\mathsf{G}(t)$, by Theorem 1.2.

9. Final comments and perspectives

We have exactly solved the problem of counting planar Eulerian orientations, both in the general and in the quartic case. Our proof, based on a guess-and-check approach, should not stay the only proof. One should seek a better combinatorial understanding of our results. Can one explain why the series R of Theorem 1.1 also appears in the enumeration of quartic maps M weighted by their Tutte polynomial $\mathrm{T}_M(0,1)$? Can one explain the forms of the series C and D in Theorems 4.1 and 7.1? What about more general vertex degrees? Can one interpolate between the results of Theorems 1.1 and 1.2, given that the second also counts a subclass of quartic Eulerian orientations (those with no alternating vertex)? In this final section we discuss the quest for bijections, and some aspects of interpolation.

9.1. Bijections

Our results reveal an unexpected connection between Eulerian orientations of quartic maps (counted by the specialization $|\mathrm{T}(0,-2)|$ of their Tutte polynomial if we do not force the orientation of the root edge [64, Sec. 3.6]) and the specialization $\mathrm{T}(0,1)$ of slightly larger maps. Let us be more precise: one of the series considered in [17] is

$$F(t) = \sum_{M \text{ quartic}} t^{\mathrm{f}(M)}\,\mathrm{T}_M(0,1),$$

which, in the classical interpretation of the Tutte polynomial [11,59], counts quartic maps M equipped with an *internally inactive* spanning tree. Observe that t records here the number of faces, which exceeds the number of vertices by 2. Then it is proved that

$$F'(t) = 4\sum_{i\geq 1} \frac{1}{i+1}\binom{3i}{i-1}\binom{2i+1}{i}\mathsf{R}^{i+1},$$

where R is the series of Theorem 1.1. There is also an interpretation of $F(t)$ in terms of spanning forests rather than spanning trees, but then some forests have a negative contribution:

$$F(t) = \sum_{\substack{M \text{ quartic} \\ F \text{ forest}}} t^{\mathrm{f}(M)}(-1)^{\mathrm{c}(F)-1}$$

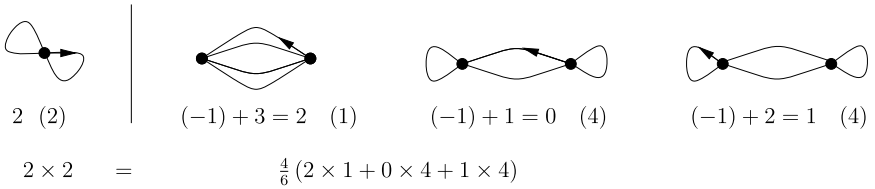

Fig. 23. Left: the only rooted quartic graph with 1 vertex has 2 Eulerian orientations (the orientation of the root is forced) and 2 embeddings as a rooted map (with 3 faces). Right: the three rooted quartic graphs with 2 vertices, shown with the (signed) number of spanning forests avoiding the root edge. The number of embeddings as rooted planar maps is shown between parentheses.

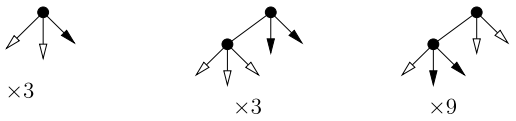

Fig. 24. The trees with 2 and 3 white leaves involved in the expansion of $t - \mathsf{R} = 3t^2 + 12t^3 + O(t^4)$, for the series R of Theorem 1.1. The multiplicities indicate the number of embeddings in the plane.

where $\mathrm{c}(F)$ is the number of connected components of the forest F. One of the advantages of this description in terms of forests is that it gives a direct interpretation of $t-\mathsf{R}$. Indeed, if we restrict the summation to forests not containing the root edge, then we obtain a new series, denoted $H(t)$ in [17], which satisfies

$$H'(t) = 2(t - \mathsf{R}).$$

Comparing with Theorem 1.1 leads to the following statement: the number of Eulerian orientations of quartic maps with n faces is $(n+1)/6$ times the (signed) number of quartic maps with $n+1$ faces equipped with a spanning forest not containing the root edge, every forest F being weighted by $(-1)^{\mathrm{c}(F)-1}$. This is illustrated in Fig. 23 for $n = 3$.

There is also an interpretation (and generalization) of $t - \mathsf{R}(t)$ in terms of certain trees [17, Sec. 5.1]. It involves a parameter u, which is -1 for our series R. In the forest setting, u counts the number of connected components (minus 1).

Proposition 9.1. *Consider rooted plane ternary trees with leaves of two colours (say black and white, see Fig. 24). Define the* charge *of such a tree to be the number of white leaves minus the number of black leaves. Call a tree of charge* 1 balanced. *Then the series* $t - \mathsf{R}(t)$ *of Theorem 1.1 counts, by the number of white leaves, balanced trees in which no proper subtree is balanced.*

More generally, let $R(t, u) \equiv R$ *be the only power series in* t *with constant term* 0 *satisfying*

$$R = t + u \sum_{n \geq 1} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} R^{n+1},$$

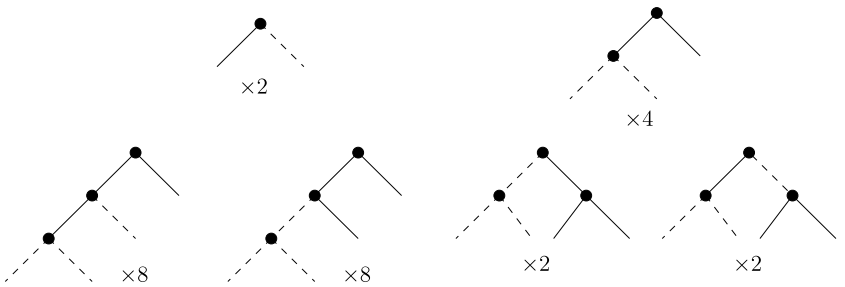

Fig. 25. The trees with at most 4 leaves involved in the expansion of $t - \mathsf{R} = 2t^2 + 4t^3 + 20t^4 + O(t^5)$, for the series R of Theorem 1.2. The multiplicities take into account the number of embeddings in the plane and the exchange of the two colours.

so that $R(t, -1) = \mathsf{R}(t)$. *Then* $(R - t)/u$ *counts balanced trees by the number of white leaves, with an additional weight* $(u + 1)$ *per proper balanced subtree.*

Here, a *subtree* of a tree T consists of a vertex of T and all its descendants, and is *proper* if the chosen vertex is neither the root of T nor a leaf. This proposition is illustrated in Fig. 24.

In the case of general Eulerian orientations (Theorem 1.2), we also have a similar combinatorial interpretation and generalization of $t - \mathsf{R}(t)$.

Proposition 9.2. *Consider rooted plane binary trees with edges of two colours (say solid and dashed, see Fig. 25). Define the* charge *of such a tree to be the number of solid edges minus the number of dashed edges. Call a tree of charge* 0 balanced. *Then the series* $t - \mathsf{R}(t)$ *of Theorem 1.2 counts, by leaves, balanced trees in which no proper subtree is balanced.*

More generally, let $R(t, u) \equiv R$ *be the only power series in* t *with constant term* 0 *satisfying*

$$R = t + u \sum_{n \geq 1} \frac{1}{n+1} \binom{2n}{n}^2 R^{n+1}, \tag{21}$$

so that $R(t, -1) = \mathsf{R}(t)$. *Then* $(R - t)/u$ *counts balanced trees by the number of leaves, with an additional weight* $(u + 1)$ *per proper balanced subtree.*

This proposition is illustrated in Fig. 25.

Proof. Define a *marked* balanced tree as a balanced tree in which a number of inner vertices are marked, in such a way that:

- the root vertex is marked (unless the tree consists of a single vertex)
- the subtree attached at any marked vertex is balanced.

Let $\bar{R}(t,u)$ be the generating function of marked balanced trees with a weight t per leaf and a weight u per marked vertex. We claim that $\bar{R}$ satisfies (21). Indeed, take a marked balanced tree with at least one edge, and consider the tree obtained by deleting all subtrees attached to a (non-root) marked vertex. Then this tree must be balanced. If it has n inner vertices, it can be chosen and coloured in

$$\frac{1}{n+1}\binom{2n}{n}\binom{2n}{n}$$

ways: the Catalan number accounts for the choice of the tree, and the second binomial coefficient for the colouring of its $2n$ edges. To reconstruct the marked balanced tree, we now need to attach to each of the $n+1$ leaves a marked balanced tree, and this gives (21). Hence the series $\bar{R}(t,u)$ coincides with $R(t,u)$.

Now consider a balanced tree. The total weight of all marked trees that can be constructed from it by marking certain vertices is $u(u+1)^b$, where b is the number of proper balanced subtrees. This completes the proof. □

The problem of understanding these equidistributions bijectively is wide open. Let us mention that deep connections are known to exist between certain families of orientations (e.g., acyclic) of a graph and certain families of subgraphs (e.g., spanning forests) of the same graph (see [5] for a survey, and references therein).

Let us finish with another bijective question. There exist two main bijections that transform Eulerian maps into trees: one of them takes the dual bipartite map, and transforms it into a mobile-tree using the distance labelling of the vertices [20]. This is the Bouttier–Di-Francesco–Guitter bijection that we have generalised in Section 5 to more general labellings, and thus to Eulerian *orientations* (rather than Eulerian maps). The second classical bijection, due to Schaeffer [55], transforms Eulerian maps into *blossoming trees*. Underlying this construction is a canonical Eulerian orientation of the map. Is there an extension of this bijection to all Eulerian orientations?

9.2. Interpolating between quartic Eulerian orientations and general Eulerian orientations

Given that the form of our solution for general Eulerian orientations is so similar to that for quartic Eulerian orientations, one may wonder whether these are two special cases of a more general series. In a forthcoming paper we describe two possible ways to simultaneously generalise $\mathsf{G}(t)$ and $\mathsf{Q}(t)$. The first series that we consider counts general Eulerian orientations by edges and vertices. This is an obvious generalisation of $\mathsf{G}(t)$, which only records the number of edges. Moreover, $\mathsf{Q}(t)$ can be extracted from this refined generating function by directly utilising the fact that quartic Eulerian orientations form a subclass of general Eulerian orientations (those having, in a sense, many vertices). The second generalisation concerns labelled quadrangulations, and interpolates between the

series $\mathsf{Q}(t)$ and $\mathsf{Q}^c(t) = 2\mathsf{G}(t)$ by keeping track of the number of quadrangles that only contain two labels (such quadrangles are forbidden in colourful quadrangulations). This corresponds to the six vertex model discussed in Section 2.3.

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References

[1] J. Ambjørn, T.G. Budd, Trees and spatial topology change in causal dynamical triangulations, J. Phys. A 46 (31) (2013) 315201.
[2] O. Angel, O. Schramm, Uniform infinite planar triangulations, Commun. Math. Phys. 241 (2–3) (2003) 191–213, arXiv:math/0207153.
[3] C. Banderier, P. Flajolet, Basic analytic combinatorics of directed lattice paths, Theor. Comput. Sci. 281 (1–2) (2002) 37–80.
[4] R.J. Baxter, Dichromatic polynomials and Potts models summed over rooted maps, Ann. Comb. 5 (1) (2001) 17–36.
[5] O. Bernardi, Tutte polynomial, subgraphs, orientations and sandpile model: new connections via embeddings, Electron. J. Comb. 15 (1) (2008), Research Paper 109, 53 pp.
[6] O. Bernardi, N. Bonichon, Intervals in Catalan lattices and realizers of triangulations, J. Comb. Theory, Ser. A 116 (1) (2009) 55–75, arXiv:0704.3731.
[7] O. Bernardi, M. Bousquet-Mélou, Counting colored planar maps: algebraicity results, J. Comb. Theory, Ser. B 101 (5) (2011) 315–377, arXiv:0909.1695.
[8] O. Bernardi, M. Bousquet-Mélou, Counting coloured planar maps: differential equations, Commun. Math. Phys. 354 (1) (2017) 31–84, arXiv:1507.02391.
[9] O. Bernardi, M. Bousquet-Mélou, K. Raschel, Counting quadrant walks via Tutte's invariant method, arXiv:1708.08215.
[10] D. Bessis, C. Itzykson, J.B. Zuber, Quantum field theory techniques in graphical enumeration, Adv. Appl. Math. 1 (2) (1980) 109–157.
[11] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998.
[12] N. Bonichon, A bijection between realizers of maximal plane graphs and pairs of non-crossing Dyck paths, Discrete Math. 298 (1–3) (2005) 104–114.
[13] N. Bonichon, M. Bousquet-Mélou, É. Fusy, Baxter permutations and plane bipolar orientations, Sémin. Lothar. Comb. 61A (2009/11) B61Ah.
[14] N. Bonichon, M. Bousquet-Mélou, P. Dorbec, C. Pennarun, On the number of planar Eulerian orientations, Eur. J. Comb. 65 (2017) 59–91, arXiv:1610.09837.
[15] G. Borot, J. Bouttier, E. Guitter, Loop models on random maps via nested loops: case of domain symmetry breaking and application to the Potts model, J. Phys. A 45 (2012) 494017, arXiv:1207.4878.
[16] G. Borot, J. Bouttier, B. Duplantier, Nesting statistics in the $O(n)$ loop model on random planar maps, arXiv:1605.02239.
[17] M. Bousquet-Mélou, J. Courtiel, Spanning forests in regular planar maps, J. Comb. Theory, Ser. A 135 (2015) 1–59, arXiv:1306.4536.

[18] M. Bousquet-Mélou, A. Elvey Price, P. Zinn-Justin, Eulerian orientations and the six-vertex model on planar maps, in: Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics, Sémin. Lothar. Comb. 82B (2019) #70, arXiv:1902.07369.
[19] M. Bousquet-Mélou, É. Fusy, K. Raschel, Bipolar orientations and quadrant walks, arXiv:1905.04256.
[20] J. Bouttier, P. Di Francesco, E. Guitter, Planar maps as labeled mobiles, Electron. J. Comb. 11 (1) (2004), Research Paper 69, 27 pp. (electronic).
[21] J. Bouttier, É. Fusy, E. Guitter, On the two-point function of general planar maps and hypermaps, Ann. Inst. Henri Poincaré D 1 (3) (2014) 265–306.
[22] E. Brézin, C. Itzykson, G. Parisi, J.B. Zuber, Planar diagrams, Commun. Math. Phys. 59 (1) (1978) 35–51.
[23] N. Chandgotia, R. Peled, S. Sheffield, M. Tassy, Delocalization of uniform graph homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}$, arXiv:1810.10124.
[24] P. Chassaing, B. Durhuus, Local limit of labeled trees and expected volume growth in a random quadrangulation, Ann. Probab. 34 (3) (2006) 879–917, arXiv:math/0311532.
[25] P. Chassaing, G. Schaeffer, Random planar lattices and integrated superBrownian excursion, Probab. Theory Relat. Fields 128 (2) (2004) 161–212, arXiv:math/0205226.
[26] N. Curien, J.-F. Le Gall, The Brownian plane, J. Theor. Probab. 27 (4) (2014) 1249–1291, arXiv:1204.5921.
[27] N. Curien, G. Miermont, Uniform infinite planar quadrangulations with a boundary, Random Struct. Algorithms 47 (1) (2015) 30–58, arXiv:1202.5452.
[28] B. Duplantier, I. Kostov, Conformal spectra of polymers on a random surface, Phys. Rev. Lett. 61 (13) (1988) 1433–1437.
[29] A. Elvey Price, A.J. Guttmann, Counting planar Eulerian orientations, Eur. J. Comb. 71 (2018) 73–98, arXiv:1707.09120.
[30] S. Felsner, É. Fusy, M. Noy, D. Orden, Bijections for Baxter families and related objects, J. Comb. Theory, Ser. A 118 (3) (2011) 993–1020, arXiv:0803.1546.
[31] P. Flajolet, Analytic models and ambiguity of context-free languages, Theor. Comput. Sci. 49 (2–3) (1987) 283–309.
[32] P. Flajolet, A. Odlyzko, Singularity analysis of generating functions, SIAM J. Discrete Math. 3 (2) (1990) 216–240.
[33] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, 2009.
[34] É. Fusy, D. Poulalhon, G. Schaeffer, Bijective counting of plane bipolar orientations and Schnyder woods, Eur. J. Comb. 30 (7) (2009) 1646–1658, arXiv:0803.0400.
[35] A. Glazman, I. Manolescu, Uniform Lipschitz functions on the triangular lattice have logarithmic variations, arXiv:1810.05592.
[36] D.M. Jackson, Counting cycles in permutations by group characters, with application to a topological problem, Trans. Am. Math. Soc. 299 (2) (1987) 785–801.
[37] D.M. Jackson, T.I. Visentin, A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus, Trans. Am. Math. Soc. 322 (1) (1990) 343–363.
[38] A. Kassel, D.B. Wilson, The looping rate and sandpile density of planar graphs, Am. Math. Mon. 123 (1) (2016) 19–39.
[39] V.A. Kazakov, Ising model on a dynamical planar random lattice: exact solution, Phys. Lett. A 119 (3) (1986) 140–144.
[40] V.A. Kazakov, P. Zinn-Justin, Two-matrix model with $ABAB$ interaction, Nucl. Phys. B 546 (3) (1999) 647–668.
[41] R. Kenyon, J. Miller, S. Sheffield, D.B. Wilson, Bipolar orientations on planar maps and SLE_{12}, arXiv:1511.04068, 2015.
[42] I.K. Kostov, Exact solution of the six-vertex model on a random lattice, Nucl. Phys. B 575 (3) (2000) 513–534.
[43] J.-F. Le Gall, The topological structure of scaling limits of large planar maps, Invent. Math. 169 (3) (2007) 621–670, arXiv:math/0607567.
[44] J.-F. Le Gall, Random geometry on the sphere, in: S.Y. Jang, Y.R. Kim, D.-W. Lee, I. Yie (Eds.), International Congress of Mathematicians, Plenary Lectures and Ceremonies, vol. 1, Seoul, Korea, 2014, pp. 421–442.
[45] J.-F. Le Gall, G. Miermont, Scaling limits of random planar maps with large faces, Ann. Probab. 39 (1) (2011) 1–69, arXiv:0907.3262.
[46] L. Lipshitz, The diagonal of a D-finite power series is D-finite, J. Algebra 113 (2) (1988) 373–378.
[47] L. Lipshitz, D-finite power series, J. Algebra 122 (1989) 353–373.

[48] J.-F. Marckert, A. Mokkadem, Limit of normalized quadrangulations: the Brownian map, Ann. Probab. 34 (6) (2006) 2144–2202, arXiv:math/0403398.
[49] L. Ménard, The two uniform infinite quadrangulations of the plane have the same law, Ann. Inst. Henri Poincaré Probab. Stat. 46 (1) (2010) 190–208, arXiv:0812.0965.
[50] G. Miermont, Tessellations of random maps of arbitrary genus, Ann. Sci. Éc. Norm. Supér. (4) 42 (5) (2009) 725–781.
[51] G. Miermont, The Brownian map is the scaling limit of uniform random plane quadrangulations, Acta Math. 210 (2) (2013) 319–401, arXiv:1104.1606.
[52] R.C. Mullin, On the enumeration of tree-rooted maps, Can. J. Math. 19 (1967) 174–183.
[53] OEIS Foundation Inc., The on-line encyclopedia of integer sequences, http://oeis.org.
[54] L. Richier, Universal aspects of critical percolation on random half-planar maps, Electron. J. Probab. 20 (2015) 129.
[55] G. Schaeffer, Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees, Electron. J. Comb. 4 (1) (1997), Research Paper 20, 14 pp. (electronic).
[56] G. Schaeffer, Conjugaison d'arbres et cartes combinatoires aléatoires, PhD thesis, LaBRI, Université Bordeaux 1, 1999.
[57] S. Sheffield, Quantum gravity and inventory accumulation, Ann. Probab. 44 (6) (2016) 3804–3848, arXiv:1108.2241.
[58] M.F. Singer, Algebraic relations among solutions of linear differential equations, Trans. Am. Math. Soc. 295 (2) (1986) 753–763.
[59] W.T. Tutte, A contribution to the theory of chromatic polynomials, Can. J. Math. 6 (1954) 80–91.
[60] W.T. Tutte, A census of planar triangulations, Can. J. Math. 14 (1962) 21–38.
[61] W.T. Tutte, A census of planar maps, Can. J. Math. 15 (1963) 249–271.
[62] W.T. Tutte, Dichromatic sums for rooted planar maps, in: Combinatorics, Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968, Amer. Math. Soc., Providence, RI, 1971, pp. 235–245.
[63] W.T. Tutte, Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$, Can. J. Math. 25 (1973) 426–447.
[64] D.J.A. Welsh, Complexity: Knots, Colourings and Counting, London Mathematical Society Lecture Note Series, vol. 186, Cambridge University Press, Cambridge, 1993.
[65] P. Zinn-Justin, The six-vertex model on random lattices, Europhys. Lett. 50 (1) (2000) 15–21.