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A bijective enumeration of labeled trees with given indegree sequence

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ABSTRACT

For a labeled tree on the vertex set $\{1,2,\ldots,n\}$, the local direction of each edge (ij) is from i to j if $i<j$. For a rooted tree, there is also a natural global direction of edges towards the root. The number of edges pointing to a vertex is called its indegree. Thus the local (resp. global) indegree sequence $\lambda = 1^{e_1}2^{e_2}\ldots$ of a tree on the vertex set $\{1,2,\ldots,n\}$ is a partition of $n-1$. We construct a bijection from (unrooted) trees to rooted trees such that the local indegree sequence of a (unrooted) tree equals the global indegree sequence of the corresponding rooted tree. Combining with a Prüfer-like code for rooted labeled trees, we obtain a bijective proof of a recent conjecture by Cotterill and also solve two open problems proposed by Du and Yin. We also prove a q-multisum binomial coefficient identity which confirms another conjecture of Cotterill in a very special case.

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1. Introduction

For an oriented tree T, the *indegree* of a vertex v is the number of edges pointing to it and the sequence $(e_0, e_1, e_2, \ldots)$ is called the *type* of T where e_h is the number of vertices of T with indegree i. Since $\sum_{i\geqslant 0} e_h$ (resp. $\sum_{i\geqslant 0} ie_h$) is the number of vertices (resp. edges) of T, we have $e_0 = 1 + \sum_{i\geqslant 1}(i-1)e_h$. Hence we can ignore e_0 while dealing with types of trees because e_0 is determinated by the others. The partition $\lambda = 1^{e_1}2^{e_2}\ldots$ will be called the *indegree sequence* of T. Throughout this paper, for any partition $\lambda = 1^{m_1}2^{m_2}\ldots$, we denote its length and weight by $\ell(\lambda) = \sum_{i\geqslant 1} m_i$ and $|\lambda| = \sum_{i\geqslant 1} im_i$. Clearly, if λ is an indegree sequence of a tree on $[n] := \{1, \ldots, n\}$, then $|\lambda| = n-1$ and $e_0 = |\lambda| + 1 - \ell(\lambda) = n - \ell(\lambda)$.

Let $\mathcal{T}_n$ be the set of unrooted labeled trees on $[n]$. For any edge (ij) of a tree $T \in \mathcal{T}_n$, there is a *local orientation*, which orients (ij) towards its smaller vertex, i.e., $i \to j$ if $i < j$. Let $\mathcal{T}_n^{(r)}$ be the set of labeled trees on $[n]$ rooted at $r \in [n]$. For any edge (ij) of a tree $T \in \mathcal{T}_n^{(r)}$, there is a *global orientation*, which orients each edge towards the root. It is interesting to note that for a rooted tree each edge has both a global orientation and a local orientation. An example of the local and global orientations is given in Fig. 1.

For any partition λ of $n-1$ and $r \in [n]$, let $\mathcal{T}_{n,\lambda}$ (resp. $\mathcal{T}_{n,\lambda}^{(r)}$) be the subset of trees in $\mathcal{T}_n$ (resp. $\mathcal{T}_n^{(r)}$) with local (resp. global) indegree sequence λ.

The problem of counting the trees with a given indegree sequence was first encountered by Cotterill in his study of algebraic geometry. In particular, Cotterill [2, Eq. (3.34)] made the following conjecture.

Conjecture 1. *Let $\lambda = 1^{e_1}2^{e_2}\ldots$ be a partition of $n-1$ and $e_0 = n - \ell(\lambda)$. Then the cardinality of $\mathcal{T}_{n,\lambda}$ equals*

$$\frac{(n-1)!^2}{e_0!(0!)^{e_0}e_1!(1!)^{e_1}e_2!(2!)^{e_2}\ldots}. \tag{1}$$

This remarkable formula is reminiscent to at least two known enumerative problems. The *type* of a set-partition π is the integer partition $1^{e_1}2^{e_2}\ldots$ if e_h blocks of π have size i, we denote it by type(π). Let $\Pi_{n,\lambda}$ be the set of partitions of an $(n-1)$-element set of *type* $\lambda = 1^{e_1}2^{e_2}\ldots$. Since the cardinality of $\Pi_{n,\lambda}$ is easily seen to equal $(n-1)!/e_1!(1!)^{e_1}e_2!(2!)^{e_2}\ldots$, Stanley (see [3]) noticed that the formula (1) can be written as $|\Pi_{n,\lambda}| \cdot \frac{(n-1)!}{(n-\ell(\lambda))!}$. Based on this factorization a proof of Conjecture 1 was given by Du and Yin [3] by using Möbius inversion formula on the poset of set partitions. Obviously a bijective proof of this result is highly desired. More precisely, for $k \in [n]$, a *k-permutation* of $[n]$ is an ordered sequence of k elements selected from $[n]$, without repetitions. Denote by $\mathcal{S}_{n,k}^{(r)}$ the set of k-permutations $(p_1, \ldots, p_k)$ of $[n]$ with $p_k = r$. The cardinality of $\mathcal{S}_{n,k}^{(r)}$ is equal to

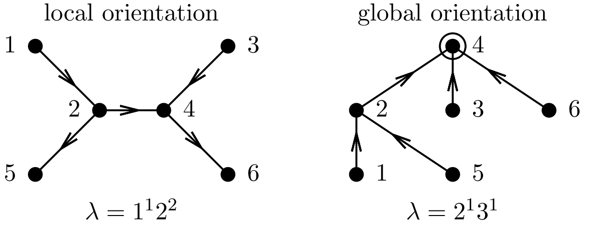


Fig. 1. Local and global indegree sequences.

$(n-1)\dots(n-k+1)=(n-1)!/(n-k)!$. It follows that a bijection between $\mathcal{T}_{n,\lambda}$ and $\Pi_{n,\lambda}\times\mathcal{S}^{(r)}_{n,\ell(\lambda)}$ will give a bijective proof of Conjecture 1. We shall construct such a bijection via labeled rooted trees. Indeed, for a given partition $\lambda=1^{e_1}2^{e_2}\dots$ of $n-1$, the cardinality of $\mathcal{T}^{(r)}_{n,\lambda}$ is independent of the choice $r\in[n]$. From the known formula for the total number of rooted trees on $[n]$ with global indegree sequence of type λ (see, for example, [9, Corollary 5.3.5]) we derive that the cardinality of $\mathcal{T}^{(r)}_{n,\lambda}$ is given by (1). For our purpose, we will first exhibit a *Prüfer-like code* for rooted trees to prove this result.

Theorem 2. *Let $\lambda=1^{e_1}2^{e_2}\dots$ be a partition of $n-1$ and $r\in[n]$. There is a bijection between $\mathcal{T}^{(r)}_{n,\lambda}$ and $\Pi_{n,\lambda}\times\mathcal{S}^{(r)}_{n,\ell(\lambda)}$.*

Therefore, Cotterill's conjecture will be proved if we can establish a bijection from (unrooted) trees to rooted trees such that the local indegree sequence of a (unrooted) tree equals the global indegree sequence of the corresponding rooted tree. The following is our second main theorem.

Theorem 3. *For any $r\in[n]$, there is a bijection $\Phi_r:\mathcal{T}_{n,\lambda}\to\mathcal{T}^{(r)}_{n,\lambda}$.*

Besides, Cotterill [2, Eq. (3.39)] also conjectured the following formula:

$$\sum_{\substack{|\lambda|=n-1\\ e_0+e_1+\cdots=n}}\frac{(n-1)!}{e_0!e_1!e_2!\dots}\sum_{i\geqslant 0}e_h\binom{i+1}{2}=\binom{2n-1}{n-2}. \tag{2}$$

In a previous version of this paper, we proved

$$\sum_{\substack{|\lambda|=m-1\\ e_0+e_1+\cdots=n}}\binom{n}{e_o,e_1,e_2,\dots}\sum_{i\geqslant 0}e_h\binom{i+p-l}{p}=n\binom{n+m-2+p-l}{n-1+p}, \tag{3}$$

and pointed out that (2) is the $m=n$, $p=2$, and $l=1$ case of (3). After submitting the paper, Ole Warnaar (personal communication) kindly conveyed us with his believe that a q-analogue of (3) must exist and sent us an identity on the Hall–Littlewood functions in the spirit of [10]. Our third aim is to present the q-analogue of (3) derived from Warnaar's original identity. For any partition λ, let $\lambda'=(\lambda'_1,\lambda'_2,\dots)$ be its conjugate and $n(\lambda)=\sum_i\binom{\lambda'_i}{2}$. Note that $\ell(\lambda)=\lambda'_1$. Introduce the q-shifted factorial:

$$(a)_k:=(a;q)_k=(1-a)(1-aq)\cdots\left(1-aq^{k-1}\right)\quad\text{for } k\geqslant 0.$$

The q-binomial and q-multinomial coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\quad\text{and}\quad\begin{bmatrix}n\\e_0,e_1,\dots,e_l\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_{e_0}(q;q)_{e_1}\cdots(q;q)_{e_l}},$$

where $e_0+\cdots+e_l=n$.

Theorem 4. *For nonnegative positive integers m, n, l and p such that $m,n\geqslant 1$, there holds*

$$\sum_{|\lambda|=m-1,\ \ell(\lambda)\leqslant n}q^{(p+1)(m-1)+2n(\lambda)}\begin{bmatrix}n\\e_0,e_1,\dots\end{bmatrix}_q\times\sum_{i\geqslant 0}q^{(1-p)i-2\sum_{k=1}^i\lambda'_k}\begin{bmatrix}i+p-l\\p\end{bmatrix}_q[e_h]_q$$
$$=[n]_q\begin{bmatrix}n+m-2+p-l\\n-1+p\end{bmatrix}_q, \tag{4}$$

where $e_h=\lambda'_i-\lambda'_{i+1}$ with $\lambda'_0=n$.

This paper is organized as follows: In Section 2, we give a Prüfer-like code for rooted labeled trees to prove Theorem 2, and in Section 3, we prove Theorem 3 by constructing a bijection from unrooted labeled trees to rooted labeled trees, which maps local indegree sequence to global indegree sequence. In Section 4, we prove Theorem 4. In the last section, we discuss a connection between Remmel and Williamson's generating function [7] for trees with respect to the indegree type and Coterill's formula (1).

We close this section with some further definitions. Throughout this paper, we denote by $\mathrm{type}_{loc}(T)$ (resp. $\mathrm{type}_{glo}(T)$) the local (resp. global) indegree sequence of a tree T as an integer partition. Let $\Pi_{n,k}^{(r)}$ be the set of partitions of the set $[n]\setminus\{r\}$ with k parts.

2. Proof of Theorem 2

The classical *Prüfer code* for a rooted tree is the sequence obtained by cutting recursively the largest *leave* and recording its parent (see [9, p. 25]). In this section, we shall give an analogous code for rooted trees by replacing leaves by *leaf-groups*.

Given a rooted tree T, a vertex v of T is called a *leaf* if the global indegree of v is 0. If $i \to j$ is an edge of T, then i (resp. j) is called the *child* (resp. *parent*) of j (resp. i). The set of all the children of v is called its *child-group*, denoted by G_v. In particular, a child-group is called *leaf-group* if all the children are leaves. Moreover, we order the leaf-groups by their maximal elements. For example, we have

$$\{5,9,12\} > \{2,11\}. \tag{5}$$

For a fixed $r \in [n]$, let $\mathcal{T}_{n,k}^{(r)}$ be the set of trees on $[n]$ rooted at r with k non-empty child-groups. We first define two preliminary mappings:

The sibship mapping $\boldsymbol{\phi_{glo}} : \boldsymbol{\mathcal{T}_{n,k}^{(r)} \to \Pi_{n,k}^{(r)}}$. For each $T \in \mathcal{T}_{n,k}^{(r)}$, let $\phi_{glo}(T)$ be the set of all child-groups of T.

Clearly, we have $\mathrm{type}_{glo}(T) = \mathrm{type}(\phi_{glo}(T))$, and if $\lambda = \mathrm{type}_{glo}(T)$, then $k = \ell(\lambda)$.

The paternity mapping $\boldsymbol{\psi} : \boldsymbol{\mathcal{T}_{n,k}^{(r)} \to \mathcal{S}_{n,k}^{(r)}}$. Starting from $T_0 = T \in \mathcal{T}_{n,k}^{(r)}$, for $i = 1, \dots, k$, let T_i be the tree obtained from T_{i-1} by deleting the largest leaf-group L_i, set $\psi(T) = (p_1, p_2, \dots, p_k)$, where p_i is the parent of child-group L_i in the tree T_{i-1}.

For example, the tree T_0 in Fig. 2 is rooted at $r = 4$ and the non-empty child-groups of T_0 are

$$G_4 = \{1,6,13,14\}, \qquad G_6 = \{3,7\}, \qquad G_8 = \{2,11\},$$
$$G_{10} = \{5,9,12\}, \qquad G_{13} = \{10\}, \qquad G_{14} = \{8\},$$

of which only G_6, G_8, and G_{10} are the leaf-groups. Hence

$$\phi_{glo}(T_0) = \{G_4, G_6, G_8, G_{10}, G_{13}, G_{14}\},$$

and the maximal leaf-groups in the trees $T_0, \dots, T_5$ are, respectively,

$$L_1 = G_{10}, \qquad L_2 = G_8, \qquad L_3 = G_{13}, \qquad L_4 = G_{14}, \qquad L_5 = G_6, \qquad L_6 = G_4.$$

So $\psi(T_0) = (10, 8, 13, 14, 6, 4)$.

By construction, we have $\phi_{glo}(T_i) = \phi_{glo}(T_{i-1}) \setminus \{L_i\}$ for all $i \geqslant 0$, so L_i belongs to $\phi_{glo}(T)$ for all i. Since the number of child-groups of $T \in \mathcal{T}_{n,k}^{(r)}$ is equal to $k = \ell(\lambda)$, this implies that $p_k = r$. Because each child-group is deleted only once, the corresponding non-leaf vertex (parent) appears in $\psi(T)$ once and only once. This means that $(p_1, \dots, p_k)$ is a k-permutation in $\mathcal{S}_{n,k}^{(r)}$. The following result shows that the pair of mappings (ϕ_{glo}, ψ) defines a *Prüfer-like algorithm* for rooted labeled trees.

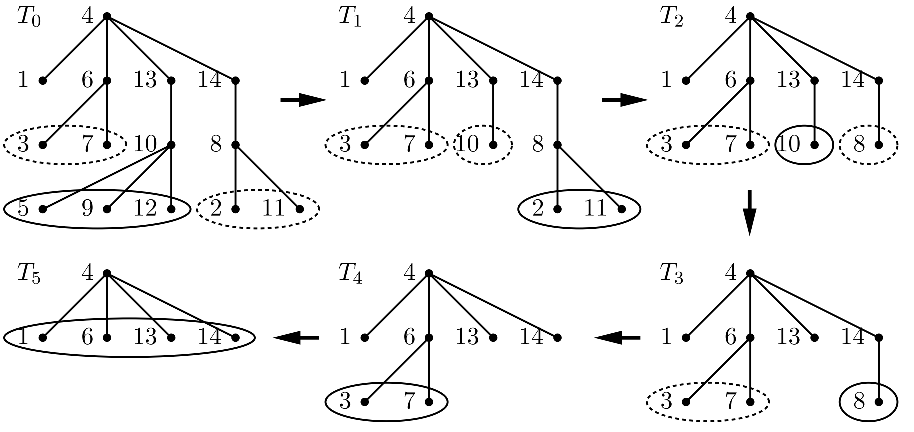


Fig. 2. An example of Prüfer-like algorithm.

Theorem 5. *For all $k \in [n-1]$, the mapping $T \mapsto (\phi_{glo}(T), \psi(T))$ is a bijection from $\mathcal{T}_{n,k}^{(r)}$ to $\Pi_{n,k}^{(r)} \times \mathcal{S}_{n,k}^{(r)}$ such that*

$$\mathrm{type}_{glo}(T) = \mathrm{type}\big(\phi_{glo}(T)\big).$$

Proof. Given a partition $\pi = \{\pi_1, \ldots, \pi_k\} \in \Pi_{n,k}^{(r)}$ and a k-permutation $\mathbf{p} = (p_1, \ldots, p_k) \in \mathcal{S}_{n,k}^{(r)}$, we can construct the tree T in $\mathcal{T}_{n,k}^{(r)}$ as follows. For $i = 1, 2, \ldots, k$:

(a) Order the blocks according to their maximal elements as in (5). Let L_i be the largest block of $\pi \setminus \{L_1, \ldots, L_{i-1}\}$, which does not contain any number in $\{p_i, p_{i+1}, \ldots, p_{k-1}\}$.
(b) Join each vertex in L_i and p_i by an edge.

The existence of the block L_i in (a) can be justified by a counting argument: there remain $k-(i-1)$ blocks in $\pi \setminus \{L_1, \ldots, L_{i-1}\}$ and we have to avoid $k-i$ values in $\{p_i, p_{i+1}, \ldots, p_{k-1}\}$, so there is at least one block without any of those values. □

For example, if $\mathbf{p} = (10, 8, 13, 14, 6, 4) \in \mathcal{S}_{14,6}^{(4)}$ and

$$\pi = \big\{\{1,6,13,14\}, \{5,9,12\}, \{2,11\}, \{10\}, \{8\}, \{3,7\}\big\} \in \Pi_{14,6}^{(4)},$$

then the inverse Prüfer-like algorithm yields $L_1, \ldots, L_6$ as follows

$$L_1 = \{5,9,2\}, \qquad L_2 = \{2,11\}, \qquad L_3 = \{10\},$$
$$L_4 = \{8\}, \qquad L_5 = \{3,7\}, \qquad L_6 = \{1,6,13,14\}.$$

Joining each vertex in L_i with p_i $(1 \leqslant i \leqslant 6)$ by an edge we recover the tree T_0 in Fig. 2.

3. Proof of Theorem 3

Given a tree $T \in \mathcal{T}_n$ and a fixed integer $r \in [n]$, we can turn it as a tree rooted at r by *hanging up* it at r as follows:

- Draw the tree with the vertex r at the top and join r to the vertices incident to r, arranged in increasing order from left to right, by edges.

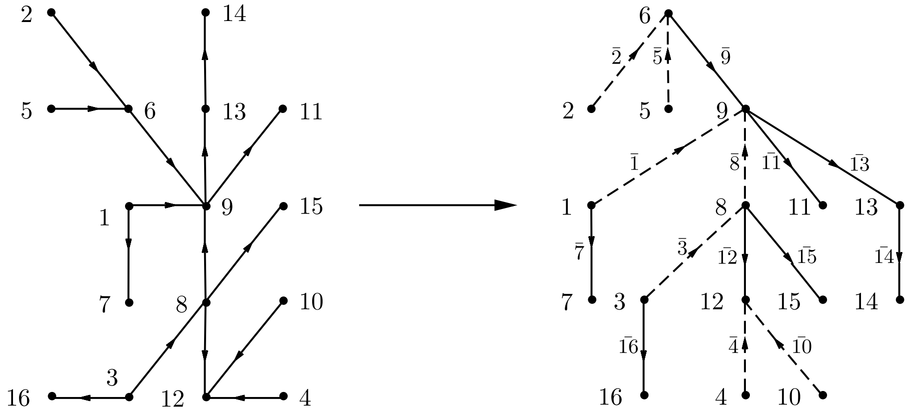


Fig. 3. A tree T hung up at 6.

- Suppose that we have drawn all the vertices with distance i to r (counted as the number of edges on the path to r), then join each vertex with distance i to its incident vertices with distance $i+1$ to r, arranged in increasing order from left to right.
- Repeat the process until drawing all vertices.

The hang-up action induces a global orientation of edges of T toward the root r. For a tree T rooted at vertex r we partition the edges in the following manner. An edge is *good*, respectively *bad*, if its local orientation is oriented toward, respectively away from, the root r. We label each edge (vu) by v if its global orientation is $v \to u$. So the set of labels of all edges equals $[n] \setminus \{r\}$ and putting together the labels of edges oriented locally toward to the same vertex yields a partition of $[n] \setminus \{r\}$, denoted by $\phi_{loc}(T)$.

For example, in Fig. 3, a tree is hung up at 6, where the dashed edges are good and the labels of edges are barred to avoid confusion. The corresponding edge-label partition is

$$\phi_{loc}^{(6)}(T) = 1\ 8\ 9/4\ 10\ 12/2\ 5/3/7/11/13/14/15/16,$$

where the blocks are separated by a slash /.

Now we describe a map Φ_r from $\mathcal{T}_n$ to $\mathcal{T}_n^{(r)}$, which will be shown to be a bijection.

3.1. Construction of the mapping Φ_r

We define the mapping Φ_r in three steps.

Step 1: Move out good edges. Starting from a tree $T \in \mathcal{T}_n$, moving out the good edges in T, we get a set of rooted subtrees without any good edges, call them *increasing trees*, $I_T = \{I_1, I_2, \dots, I_d\}$ and a matrix recording the cut good edges

$$D_T = \begin{pmatrix} j_1 & j_2 & \cdots & j_{d-1} \\ i_1 & i_2 & \cdots & i_{d-1} \end{pmatrix},$$

where each column $\binom{j}{i}$ corresponds to a good edge $i \to j$ in T.

Remark. The roots of the d increasing trees are $i_1, \dots, i_{d-1}$ and r.

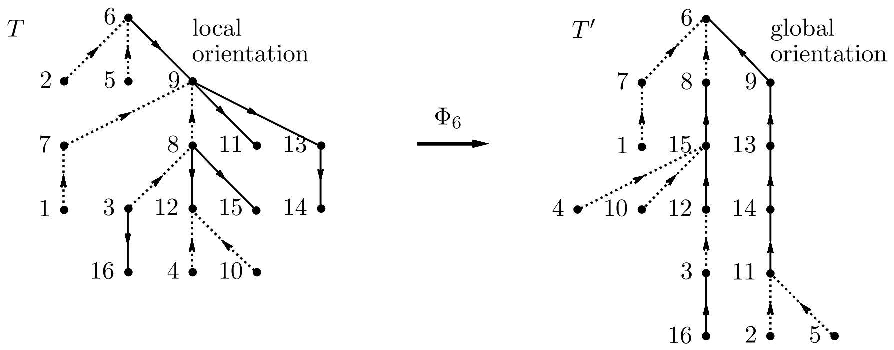


Fig. 4. The bijection $\Phi_6 : \mathcal{T}_{16} \to \mathcal{T}_{16}^{(6)}$ with $\text{type}_{loc}(T) = \text{type}_{glo}(T') = 1^7 2^1 3^2$.

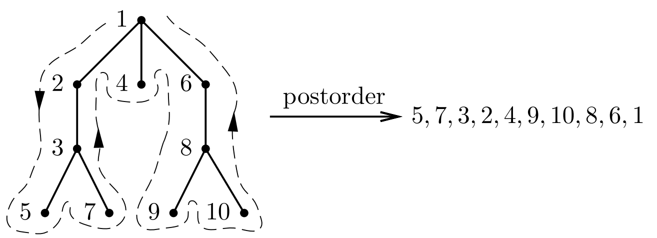
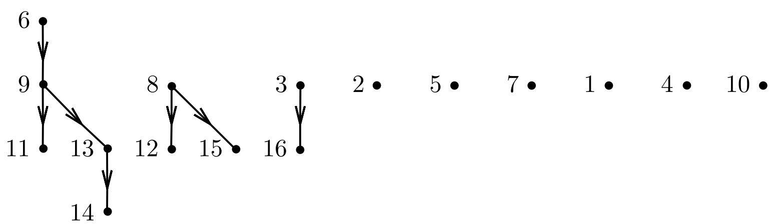


Fig. 5. A increasing tree traversed in postorder.

For example, after cutting the good edges, drawn with dashed arrows, in the tree T of Fig. 4, we get

I_T : (6)



and the matrix recording the eight good edges

$$D_T = \begin{pmatrix} 6 & 6 & 7 & 8 & 9 & 9 & 12 & 12 \\ 2 & 5 & 1 & 3 & 7 & 8 & 4 & 10 \end{pmatrix}. \tag{7}$$

To prepare the second step, we recall a classical linear ordering on the vertices of a tree T, called *postorder*, and denoted $\text{ord}(T)$ (see [4, p. 336]). It is defined recursively as follows: Let v be the root of T and there are subtrees $T_1, \dots, T_k$ connected to v. Order the subtrees $T_1, \dots, T_k$ by their roots, then set

$$\text{ord}(T) = \text{ord}(T_1), \dots, \text{ord}(T_k), v \quad \text{(concatenation of words)}.$$

An example of postorder is given in Fig. 5.

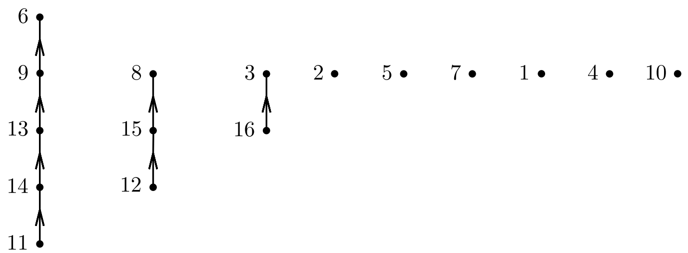
Step 2: Read vertices in increasing trees in postorder. For each increasing tree I_h we construct a *linear tree* $J_h = v_1 \to \cdots \to v_l$, of which every vertex has at most one child, and a cyclic permutation $\sigma_h = (v_1, \dots, v_l)$, where $v_1, \dots, v_l$ are the vertices of I_h ordered by postorder. So the last v_l is the root of the tree I_h and also the minimum in the sequence $v_1, \dots, v_l$. Define $J_T = \{J_1, \dots, J_d\}$ and the matrix

$$\sigma(D_T) = \begin{pmatrix} \sigma(j_1) & \sigma(j_2) & \cdots & \sigma(j_{d-1}) \\ i_1 & i_2 & \cdots & i_{d-1} \end{pmatrix},$$

where $\sigma = \sigma_1 \dots \sigma_d$.

In the above example, we have

$J_T:$



(8)

and three non-identical cyclic permutations corresponding to the first three trees:

$$\sigma_1 = (11, 14, 13, 9, 6), \qquad \sigma_2 = (12, 15, 8), \quad \text{and} \quad \sigma_3 = (16, 3). \tag{9}$$

Applying σ to the matrix (7), we obtain the matrix

$$\sigma(D_T) = \begin{pmatrix} 11 & 11 & 7 & 12 & 6 & 6 & 15 & 15 \\ 2 & 5 & 1 & 3 & 7 & 8 & 4 & 10 \end{pmatrix}. \tag{10}$$

For a graph G, let $V(G)$ be the set of all vertices in G. Define the relation $\sim_G$ on its vertices as follows

$$a \sim_G b \quad \Leftrightarrow \quad a, b \text{ are connected by a path in } G \text{ regardless of an orientation.}$$

By definition, I_T and J_T are graphs with d connected components. We shall identify an edge $i \to j$ with the column $\binom{j}{i}$ in the matrix D_T and $\sigma(D_T)$.

Lemma 6. *In Step* 2*, for any vertex $v \nsim_{J_T} r$, there is a unique sequence of edges $\binom{\sigma(j_1)}{i_1}, \binom{\sigma(j_2)}{i_2}, \dots, \binom{\sigma(j_l)}{i_l}$ in $\sigma(D_T)$ such that*

$$v \sim_{J_T} i_1, \quad \sigma(j_1) \sim_{J_T} i_2, \quad \dots, \quad \sigma(j_{l-1}) \sim_{J_T} i_l, \quad \textit{and} \quad \sigma(j_l) \sim_{J_T} r. \tag{11}$$

Proof. Since two connected components including r in I_T and J_T have the same vertices, $v \nsim_{J_T} r$ implies $v \nsim_{I_T} r$. Since T is a tree (so connected), for any vertex $v \nsim_{I_T} r$, there is a unique sequence of good edges $i_1 \to j_1, i_2 \to j_2, \dots, i_l \to j_l$ such that

$$v \sim_{I_T} i_1, \quad j_1 \sim_{I_T} i_2, \quad \dots, \quad j_{l-1} \sim_{I_T} i_l, \quad \text{and} \quad j_l \sim_{I_T} r.$$

Since $V(I_h) = V(J_h)$ for all h and $j \sim_{J_T} \sigma(j)$ for all j, the edges $\binom{\sigma(j_1)}{i_1}, \binom{\sigma(j_2)}{i_2}, \dots, \binom{\sigma(j_l)}{i_l}$ in $\sigma(D_T)$ satisfy the condition (11). □

Example. In the previous example with $r = 6$, if $v = 10$ then the unique sequence of edges in (10) satisfying (11) is $\binom{15}{10}$ and $\binom{6}{8}$.

Step 3: Construct the rooted tree. By Lemma 6, the linear trees in J_T are connected by edges $i \to j$, where $\binom{j}{i}$ is a column in the matrix $\sigma(D_T)$. This yields a tree $\Phi_r(T)$ rooted at r (with the global orientation).

An example of the map Φ_r with Step 3 is illustrated in Fig. 4, where Steps 1 and 2 are given in (6) and (7), (8) and (10).

Next we have to show that the map Φ_r is a bijection. As suggested by a referee, it is convenient to summarize the key properties of Φ_r before the proof.

3.2. Key properties of Φ_r

We denote by $I_T := (I_h)_h$ the connected components of the graph made up of the bad edges, some components may be reduced to a single vertex. Each component I_h contains a (spanning) tree made up of bad edges that is rooted at the vertex r_h which is at minimal distance to the root r among the vertices of I_h. If $r_h \neq r$, the path from r_h to r starts with an edge e_h called the *rooting edge* of I_h. By definition, an edge is a rooting edge if and only if it is a good edge. Each component I_h defines an edge set C_h made up of the bad edges between two vertices of I_h and the good edges incident to a vertex of I_h, except the rooting edge e_h, if any. The sets $(C_h)_h$ forms a partition of the edges of T: bad edge's endpoints appear in a single I_h and a good edge is the rooting edge of one of its endpoint and thus appears in the component defined by its other endpoint. All edges contributing to the local indegree of a vertex $v \in I_h$ in T belong to C_h. The bijection will be defined independently on each set C_h using only the additional (global) information of the root vertex r_h. The possible components C_h are the trees rooted at r_h where any child with a label lower than the label of its parent is a leaf. For any vertex $v \in I_h$ we denote by

$$L_h(v) := \{w\colon (wv) \in C_h \text{ and } w \notin I_h\}$$

the set of its lower children, since $\forall w \in L_h(v)$, $w < v$. The post-order linear ordering of the vertices of I_h leads to a cyclic permutation σ_h of the vertices of I_h.

The transformation by postorder leads to a graph where for any vertex $v \neq r_h$ in I_h, the vertex v and $L_h(v)$ form the sibship of the vertex $\sigma_h(v)$, so v is the member of this new sibship with the biggest label. Moreover, the local indegree of v was $1 + |L_h(v)|$ and the new global degree of $\sigma_h(v)$ is the same. In the case of r_h of local indegree $0 + |L_h(r_h)|$, its lower children of $L_h(r_h)$ become the sibship of another vertex v_l of I_h whose new global indegree is also $0 + |L_h(r_h)|$. In addition, all the vertices of $L_h(r_h)$, if any, are smaller than r_h in particular the biggest label among $L_h(r_h)$. Thus the distribution local indegrees of vertices of I_h becomes the distribution of global indegrees of vertices of J_h after the transformation.

3.3. Construction of the inverse mapping Φ_r^{-1}

Let $T \in \mathcal{T}_n^{(r)}$. First we need to introduce some definitions. If $i \to j$ is an edge of T, we say that the vertex i is a *child* of j. The vertex i is the *eldest child* of j if i is bigger than all other children (if any) of j and the edge $i \to j$ is *eldest* if i is the eldest child of j. Note that deleting all non-eldest edges in T, we obtain a set of linear trees. For a linear tree $v_1 \to \cdots \to v_l$ obtained from T by deleting all non-eldest edges, an edge $i \to j$ is called a *minimal* if i is a right-to-left minimum in the sequence $v_1, \ldots, v_l$. Finally, an edge $i \to j$ of T is *proper* if it is non-eldest or minimal.

For example, for the tree T' in Fig. 4, the proper edges are dashed. Moreover, the edges $7 \to 6$, $8 \to 6$, $4 \to 15$, $10 \to 15$ and $2 \to 11$ are non-eldest, while $3 \to 12$, $1 \to 7$ and $5 \to 11$ are minimal.

Lemma 7. *For a given tree T with its local orientation, every improper edge $i \to j$ in $\Phi_r(T)$ corresponds to a column $\binom{j}{i}$ in $\sigma(D_T)$.*

Proof. Let $i \to j$ be an edge in $\Phi_r(T)$ corresponding to a column $\binom{j}{i}$ in $\sigma(D_T)$. Let $k = \sigma^{-1}(j)$. Since $\binom{j}{i}$ is induced from a good edge $i \to k$, we have $i < k$. Denote by J the linear tree including j obtained from T by Steps 1 and 2.

(1) If j is a non-leaf of J, then k is a child of j. So i cannot be the eldest child of j and the edge $i \to j$ must be proper in $\Phi_r(T)$.
(2) If j is a leaf of J, then $J = j \to \cdots \to k$. Suppose that there exists another column $\binom{j}{i'}$ in $\sigma(D_T)$ such that $i' > i$, then the vertex i cannot be the eldest child of j and the edge $i \to j$ should be proper in $\Phi_r(T)$. Otherwise, since k is also the minimum of J and $i < k$, the vertex i is smaller than all vertices between j and k. That means the edge $i \to j$ is minimal in the linear tree $i \to j \to \cdots \to k$. Thus the edge $i \to j$ should be proper in $\Phi_r(T)$.

Conversely, let $i \to j$ be an edge in $\Phi_r(T)$ such that $\binom{j}{i}$ is not a column in $\sigma(D_T)$. Since the edge $i \to j$ is obtained from some linear tree J, we have $j = \sigma(i)$. If j has another child k in $\Phi_r(T)$, then $\binom{j}{k}$ is a column in $\sigma(D_T)$. Since $\binom{j}{k}$ is induced from a good edge, $k \to i$ implies $k < i$. That means the edge $i \to j$ is always eldest in $\Phi_r(T)$. Since i is also bigger than the root of J, the edge $i \to j$ cannot be minimal. Thus the edge $i \to j$ is not proper. □

The following two lemmas are our main results of this section.

Lemma 8. *The map $\Phi_r : T \mapsto T'$ is a bijection from $\mathcal{T}_n$ to $\mathcal{T}_n^{(r)}$.*

Proof. It suffices to define the inverse procedure. Given a tree $T' \in \mathcal{T}_n^{(r)}$, by cutting out all the proper edges in T', we get a set of linear trees (i.e., trees without any proper edges including singleton vertex) $J_{T'} = \{J_1, J_2, \ldots, J_d\}$ and a matrix recording the cut proper edges

$$P_{T'} = \begin{pmatrix} j_1 & j_2 & \cdots & j_{d-1} \\ i_1 & i_2 & \cdots & i_{d-1} \end{pmatrix}$$

where each column $\binom{j}{i}$ corresponds to a proper edge $i \to j$ in T'. Lemma 7 yields $P_{\Phi_r(T)} = \sigma(D_T)$ for any $T \in \mathcal{T}_n$. For example, for the tree T' in Fig. 4, we obtain the nine linear trees in (8) and the matrix in (10).

To each linear tree $J_h = v_1 \to \cdots \to v_l$ with v_l as root we associate the cyclic permutation $\sigma_h = (v_1, \ldots, v_l)$ and let $\sigma = \sigma_1 \ldots \sigma_d$. For the tree T' in Fig. 4, we get the three non-trivial permutations in (9).

Define the matrix

$$\sigma^{-1}(P_{T'}) = \begin{pmatrix} \sigma^{-1}(j_1) & \sigma^{-1}(j_2) & \cdots & \sigma^{-1}(j_{d-1}) \\ i_1 & i_2 & \cdots & i_{d-1} \end{pmatrix}.$$

Since each column $\binom{j}{i}$ of $P_{T'}$ corresponds to a proper edge $i \to j$, $\sigma^{-1}(j)$ is the eldest child of j or the root of the linear tree containing j. Thus we have $\sigma^{-1}(j) > i$ and the columns of matrix $\sigma^{-1}(P_{T'})$ are decreasing. Continuing above example, we recover the matrix in (7).

Since we read vertices of increasing trees I_h in postorder in Φ_r, every cyclic permutation $\sigma_h = (v_1, \ldots, v_l)$ can also be changed to increasing tree I_h using the *inverse of postorder algorithm*, which is the well-known algorithm (see [8, p. 25]) mapping cyclic permutations to increasing trees as follows: Given a cyclic permutation $\sigma_h = (v_1, \ldots, v_l)$ with v_l as minimum, construct an increasing tree I_h on $v_1, \ldots, v_l$ with the root v_l by defining vertex v_i to be the child of the leftmost vertex v_j in σ_h which follows v_i and which is less than v_i. Since the last v_l is the minimum in all vertices of J_h, there exists such a vertex v_j for all vertex v_i except of v_l. For example, applying the linear trees in (8), we recover the increasing trees in (6).

Finally, merging all increasing trees I_h by the good edges in the matrix $\sigma^{-1}(P_T)$, we recover the tree $\Phi_r^{-1}(T') \in \mathcal{T}_n$, as illustrated in Fig. 4. □

3.4. Further properties of the mapping Φ_r

Define the *sibship* of a vertex v in an oriented tree T hung up r to be the set of labels of edges pointed to v in T and denote it by $\text{sibship}^{(r)}(T; v)$. For instance, $\text{sibship}^{(6)}_{loc}(T; 9) = \{\bar{1}, \bar{8}, \bar{9}\}$ and $\text{sibship}^{(6)}_{glo}(T; 9) = \{\bar{1}, \bar{8}, \bar{11}, \bar{13}\}$ where T is a tree in Fig. 3.

Lemma 9. *For a given tree T hung up at r with the local orientation and for any vertex v of T, the sibship of the vertex v in T is the same as the sibship of the vertex $\sigma(v)$ in $\Phi_r(T)$, i.e.,*

$$\text{sibship}^{(r)}_{loc}(T; v) = \text{sibship}^{(r)}_{glo}\big(T'; \sigma(v)\big)$$

where $T' = \Phi_r(T)$ is a rooted tree with the global orientation. Therefore, $\phi_{loc}(T) = \phi_{glo}(T')$.

Proof. Let T be a tree with the local orientation and $T' = \Phi_r(T)$. Let $\bar{k} \in \mathrm{sibship}_{loc}^{(r)}(T; v)$.

(1) If $k < v$, we find a decreasing edge $k \xrightarrow{\bar{k}} v$. It becomes an edge $k \xrightarrow{\bar{k}} \sigma(v)$ in T' under σ. Thus $\bar{k} \in \mathrm{sibship}_{glo}^{(r)}(T'; \sigma(v))$.

(2) If $k = v$, we find an increasing edge $i \xrightarrow{\bar{v}} v$ for some $i < v$. Since it is an edge in some increasing tree I, v is not the root of I. Then we can find an edge $v \xrightarrow{\bar{v}} \sigma(v)$ in the linear tree corresponding to I. Thus $\bar{v} \in \mathrm{sibship}_{glo}^{(r)}(T'; \sigma(v))$.

(3) If $k > v$, the edge $k \leftarrow v$ points to k which is impossible.

Since any two sibships are disjoint in T', we have

$$\mathrm{sibship}_{loc}^{(r)}(T; v) = \mathrm{sibship}_{glo}^{(r)}\big(T'; \sigma(v)\big)$$

where $T' = \Phi_r(T)$. $\square$

Combining the above two lemmas we obtain Theorem 3.

Remark. Let $r = 1$. Let π be a partition of $\{2, \ldots, n\}$ and $\mathcal{T}_{glo}^{(\pi)}$ (resp. $\mathcal{T}_{loc}^{(\pi)}$) be the set of trees with *sibship set-partition* π induced by the sibship mapping ϕ_{glo} (resp. ϕ_{loc}). Combining two maps Φ_1 and ψ we obtain a bijective proof of Theorem 1.1 in [3]. Indeed, their set T_π in [3] is equal to our set $\mathcal{T}_{loc}^{(\pi)}$, hence

$$\big|\mathcal{T}_{loc}^{(\pi)}\big| \overset{\Phi_1}{=} \big|\mathcal{T}_{glo}^{(\pi)}\big| = \big|(\phi_{glo})^{-1}(\pi)\big| \overset{\psi}{=} \big|\mathcal{S}_{n,\ell(\lambda)}^{(1)}\big| = \frac{(n-1)!}{(n-\ell(\lambda))!}.$$

At the end of their paper [3], Du and Yin also asked for a bijection from $\mathcal{T}_{n,\lambda}$ to $\Pi_{n,\lambda}^{(1)} \times \mathcal{S}_{n,\ell(\lambda)}^{(1)}$ (in our notation). By Theorem 5, the mapping $(\phi_{glo}, \psi) \circ \Phi_1$ provides such a bijection. This is a generalization of Prüfer code for labeled tress, which corresponds to the $\lambda = 1^{n-1}$ case.

4. Proof of Theorem 4

Since $\left[{n \atop e_0, e_1, \ldots}\right]_q [e_h]_q = [n]_q \left[{n-1 \atop e_0, \ldots, e_h - 1, \ldots}\right]_q$, the formula (4) is equivalent to

$$\sum_{i \geqslant 0} \sum_{\substack{|\lambda| = m-1 \\ \ell(\lambda) \leqslant n}} q^{(p+1)(m-i-1)+2n(\lambda)-2\sum_{k=1}^{i}(\lambda'_k - 1)} \times \begin{bmatrix} p+i-l \\ p \end{bmatrix}_q \begin{bmatrix} n-1 \\ e_0, e_1, \ldots, e_h - 1, \ldots \end{bmatrix}_q$$
$$= \begin{bmatrix} n+m-2+p-l \\ n-1+p \end{bmatrix}_q. \tag{12}$$

By using the formula [1, Theorem 3.3]

$$(z; q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j z^j q^{\binom{j}{2}}$$

to expand $(z; q)_N$ and extracting the coefficient of t^k in

$$(-t; q)_{n+k-1} = (-t; q)_{k-1}\big(-tq^{k-1}; q\big)_n,$$

we obtain the q-Chu–Vandermonde identity:

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q = \sum_{r\geqslant 0} q^{r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} k-1 \\ k-r \end{bmatrix}_q.$$

It is well known [5] (see also [10] for some generalizations) that iterating the q-Chu–Vandermonde identity yields

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q = \sum_{|\lambda|=k,\ \ell(\lambda)\leqslant n} q^{2n(\lambda)} \begin{bmatrix} n \\ e_0, e_1, \ldots \end{bmatrix}_q. \tag{13}$$

Using the formula [1, Theorem 3.3]

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix}_q z^j$$

to expand $1/(z;q)_N$ and then extracting the coefficient of x^{m-l-1} in the identity

$$\frac{1}{(x;q)_{p+1}} \frac{1}{(xq^{p+1};q)_{n-1}} = \frac{1}{(x;q)_{p+n}},$$

we obtain

$$\sum_{t\geqslant 0} \begin{bmatrix} p+t \\ t \end{bmatrix}_q \begin{bmatrix} n+m-3-l-t \\ m-1-l-t \end{bmatrix}_q q^{(p+1)(m-1-l-t)} = \begin{bmatrix} n+p+m-2-l \\ m-1-l \end{bmatrix}_q.$$

Shifting t to $t-l$ we get

$$\sum_{t\geqslant 0} \begin{bmatrix} p+t-l \\ p \end{bmatrix}_q \begin{bmatrix} n+m-3-t \\ n-2 \end{bmatrix}_q q^{(p+1)(m-1-t)} = \begin{bmatrix} n+p+m-2-l \\ m-1-l \end{bmatrix}_q. \tag{14}$$

If $\lambda = 1^{e_1}2^{e_2}\ldots$, letting $\mu = 1^{e_1}2^{e_2}\ldots i^{e_h-1}\ldots$ be the partition obtained by deleting part i from λ, then

$$n(\lambda) - \sum_{k=1}^{i}(\lambda'_k - 1) = \sum_{k=1}^{i}\binom{\lambda'_k-1}{2} + \sum_{k\geqslant i+1}\binom{\lambda'_k}{2} = n(\mu).$$

Hence, by replacing e_h with e_h+1, the left-hand side of (12) is equal to

$$\sum_i q^{(p+1)(m-1-i)} \begin{bmatrix} p+i-l \\ p \end{bmatrix}_q \sum_{\substack{|\mu|=m-i-1 \\ \ell(\mu)\leqslant n-1}} q^{2n(\mu)} \begin{bmatrix} n-1 \\ e_0, e_1 \ldots \end{bmatrix}_q$$
$$= \sum_i q^{(p+1)(m-1-i)} \begin{bmatrix} p+i-l \\ p \end{bmatrix}_q \begin{bmatrix} n+m-3-i \\ n-2 \end{bmatrix}_q \quad \bigl(\text{by (13)}\bigr),$$

which is the right-hand side of (12) by (14).

Remark. Since the q-Chu–Vandermonde identity can be explained bijectively using Ferrers diagram [1, Chapter 3], we can give a *bijective proof* of (12). Here we just sketch such a proof. Since it is known [1, Theorem 3.1] that

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \sum_{\lambda} q^{|\lambda|},$$

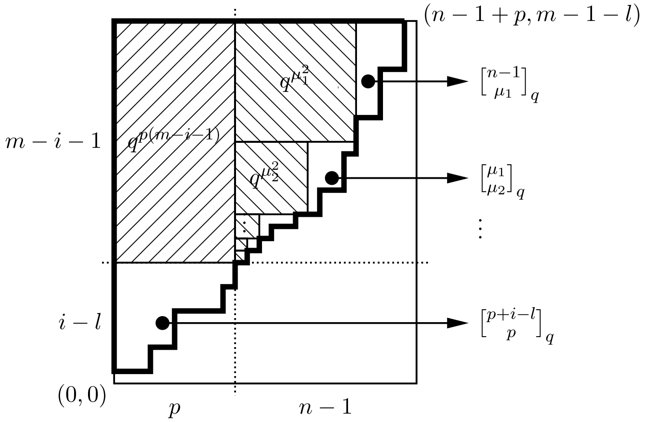


Fig. 6. Decomposition of a partition λ in an $(m-1-l)\times(n-1+p)$ rectangle.

where λ runs over partitions in an $M \times N$ rectangle, the right-hand side of (12) equals the generating function $\sum_\lambda q^{|\lambda|}$ for all partitions λ in an $(m-1-l)\times(n-1+p)$ rectangle. The diagram of such a partition λ can be decomposed as in Fig. 6. Given such a partition λ, defining $i = m - \lambda'_{p+1} - 1$, we take the rectangle of size $(m-i-1)\times p$ from the point $(0, m-1-l)$ in the diagram. And then associate a partition $\mu = (\mu_1, \mu_2, \ldots)$ of $m-i-1$ by taking the lengths μ_j of *successive Durfee squares*, which are started from the point $(p, m-1-l)$ and taken downwards. Given i and μ, the generating function $\sum_\lambda q^{|\lambda|}$ for all corresponding λ is

$$q^{p(m-i-1)+\mu_1^2+\mu_2^2+\mu_3^2+\cdots}\begin{bmatrix} p+i-l \\ p \end{bmatrix}_q \begin{bmatrix} n-1 \\ \mu_1 \end{bmatrix}_q \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_q \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}_q \cdots$$

as indicated by Fig. 6 and it follows that

$$\begin{bmatrix} n+m-2+p-l \\ n-1+p \end{bmatrix}_q = \sum_i \sum_{\substack{n-1 \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots \\ \mu_1+\mu_2+\cdots = m-i-1}} q^{p(m-i-1)+\mu_1^2+\mu_2^2+\mu_3^2+\cdots}\begin{bmatrix} p+i-l \\ p \end{bmatrix}_q \begin{bmatrix} n-1 \\ \mu_1 \end{bmatrix}_q \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_q \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}_q \cdots.$$

Replacing μ_j to $\lambda'_j - 1$ for $j \leqslant i$ (and μ_j to λ'_j for $j > i$), the formula above is equivalent to (12). Hence, the successive Durfee square decomposition of a Ferrers diagram gives a bijective proof of (4), (13), and (14).

5. An open problem

By [7, Eq. (8)] (see also [6, Theorem 4]), we obtain the generating function for trees with respect to local indegree type:

$$P_n(x_1, \ldots, x_n) = \sum_{T \in \mathcal{T}_n} \prod_{i=1}^{n} x_i^{\mathrm{indeg}_T(i)} = x_n \prod_{i=2}^{n-1} (ix_i + x_{i+1} + \cdots + x_n), \tag{15}$$

where $\mathrm{indeg}_T(i)$ is the indegree of vertex i in T with the local orientation. We say that a monomial $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ is of type $\lambda = 1^{e_1} 2^{e_2} \ldots$ if the sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ has e_h i's for $0 < i \leqslant n$. For any partition $\lambda = 1^{e_1} 2^{e_2} \ldots$ of $n-1$ and $e_0 = n - \ell(\lambda)$, from (1) and (15) we derive

$$\sum_{\text{type}(\mathbf{x}^{\alpha})=\lambda} [\mathbf{x}^{\alpha}] P_n(x_1, \ldots, x_n) = \frac{(n-1)!^2}{e_0!(0!)^{e_0} e_1!(1!)^{e_1} e_2!(2!)^{e_2} \ldots}, \tag{16}$$

where $[\mathbf{x}^{\alpha}]P_n(x_1, \ldots, x_n)$ denotes the coefficient of $\mathbf{x}^{\alpha}$ in $P_n(x_1, \ldots, x_n)$.

For example, if $n = 4$, the generating function reads as follows

$$P_4(x_1, x_2, x_3, x_4) = 6x_2x_3x_4 + 2x_2x_4^2 + 3x_3^2x_4 + 4x_3x_4^2 + x_4^3.$$

Clearly, the monomials of type $\lambda = 1^1 2^1$ are $x_2x_4^2$, $x_3^2x_4$ and $x_3x_4^2$ and the sum of their coefficients is $2 + 3 + 4 = 9$, which coincides with the formula (1), i.e., $3!^2/2!^2 = 9$.

Open problem. Find a *direct proof* of the algebraic identity (16).

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