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Separation of variables and combinatorics of linearization coefficients of orthogonal polynomials

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ABSTRACT

We propose a new approach to the combinatorial interpretations of linearization coefficient problem of orthogonal polynomials. We first establish a difference system and then solve it combinatorially and analytically using the method of separation of variables. We illustrate our approach by applying it to determine the number of perfect matchings, derangements, and other weighted permutation problems. The separation of variables technique naturally leads to integral representations of combinatorial numbers where the integrand contains a product of one or more types of orthogonal polynomials. This also establishes the positivity of such integrals.

Contents

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[4] This is in appreciation of his fundamental mathematical contributions to special functions and orthogonal polynomials, for his tireless efforts of promoting the subject, and for his outstanding mentoring of younger mathematicians.

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1. Introduction

In the late 1960's Askey formulated several conjectures about the nonnegativity of integrals of products of orthogonal polynomials times certain functions. An excellent survey of the research in this area, which was spearheaded by Askey, is Askey's CBMS lecture notes [3], see also [1]. In the 1970's it was realized that some of the integrals considered by Askey and his coauthors have combinatorial interpretations. Even and Gillis [11] showed that the number of derangements of sets of sizes $n_1, n_2, \dots, n_m$ is

$$(-1)^{n_1+\cdots+n_m} \int_0^\infty e^{-x} \prod_{j=1}^m L_{n_j}(x)\, dx, \tag{1.1}$$

where $L_n(x)$'s are the simple Laguerre polynomials, while Azor, Gillis, and Victor [7] and independently Godsil [16] showed that the number of perfect matchings of sets of sizes $n_1, n_2, \dots, n_m$ is

$$2^{-(n_1+\cdots+n_m)/2} \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} \prod_{j=1}^m H_{n_j}(x)\, dx,$$

where $H_n(x)$'s are the Hermite polynomials. Askey and Ismail [4] used the MacMahon Master theorem to give a systematic combinatorial treatment of the integrals of products of the classical polynomials with respect to certain measures. One of them generalized the Even and Gillis result to Meixner polynomials. Foata and Zeilberger [12] considered the general Laguerre numbers

$$(-1)^{\sum_{j=1}^m n_j} n_1! \cdots n_m! \int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \prod_{j=1}^m L_{n_j}^{(\alpha)}(x)\, dx,$$

where $L_n^{(\alpha)}(x)$'s are the Laguerre polynomials. Zeng and, Kim and Zeng extended this study to all Sheffer polynomials in [24,34,35].

In their combinatorial study of integrals of products of orthogonal polynomials Askey and Ismail [4] pointed out another source of positivity results. Recall that a system $\{Q_n(x)\}$ of birth and death process polynomials [20], [17, §5.2], is generated by

$$Q_0(x) = 1, \qquad Q_1(x) = [b_0 + d_0 - x]/d_0,$$
$$-xQ_n(x) = b_n Q_{n+1}(x) + d_n Q_{n-1}(x) - (b_n + d_n) Q_n(x), \tag{1.2}$$

where $\{b_n\}$ and $\{d_n\}$ are the birth and death rates, respectively, are such that

$$\lambda_n > 0, \quad n \geqslant 0, \quad \text{and} \quad d_n > 0, \quad n > 0, \ d_0 \geqslant 0. \tag{1.3}$$

Karlin and McGregor [20] showed that the probability to go from state (population) m to state (population) n in time t is given by

$$p_{m,n}(t) = \frac{b_0 b_1 \cdots b_{n-1}}{d_1 d_2 \cdots d_n} \int_0^\infty e^{-xt} Q_m(x) Q_n(x) \, d\mu(x), \quad t > 0, \tag{1.4}$$

where μ is the orthogonality measure of $\{Q_n(x)\}$. This proves that

$$\int_0^\infty e^{-xt} Q_m(x) Q_n(x) \, d\mu(x) \geqslant 0. \tag{1.5}$$

The Laguerre polynomials correspond to $b_n = n + 1$, $d_n = n + \alpha$. Thus

$$\int_0^\infty e^{-xt} x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) \, dx > 0, \quad \alpha \geqslant 0, \ t > 0. \tag{1.6}$$

This and the derangement number (1.1) motivated us to consider the combinatorial interpretation of the numbers

$$A^{(\alpha)}(m, n, s) = \frac{(-1)^{m+n}}{\Gamma(\alpha + 1)} \int_0^\infty \frac{x^s}{s!} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} \, dx. \tag{1.7}$$

One important tool used in the combinatorial study of the integrals of orthogonal polynomials is MacMahon's Master theorem and its β-extension due to Foata–Zeilberger [12]. When the β-extension of MacMahon's Master theorem is combined with the exponential formula [30,33], all the known combinatorial interpretations of the linearization coefficients of the orthogonal Sheffer polynomials can be deduced by computing their generating functions. Another way to gain insight into the combinatorial interpretation of the linearization coefficients is from their corresponding moment sequences, see [24,32,35],

$$\mu_n = \int_{\mathbb{R}} x^n \, d\mu(x). \tag{1.8}$$

However the generating function approach fails when one tries to extend the previous results to their q-analogues, even though a conjecture for the combinatorial interpretation is formulated. For example, an important q-analogue for the linearization coefficients of Hermite polynomials was given by Ismail, Stanton and Viennot [18], but their proof remains difficult. We are grateful to a referee for pointing out that Effros and Popa rediscovered the Ismail–Stanton–Viennot result in [10]. Another proof due to Anshelevich [2] uses stochastic processes, and is also far from being elementary. Our paper provides a fresh approach to linearization questions. Indeed, one of the main results of this paper is to give an elementary proof of the Ismail–Stanton–Viennot result.

Separation of variables is a standard technique to solve linear partial differential equations. The idea is to seek solutions which are products of single variables then by the principle of linear superposition the general solution is a linear combination of these products. The only problem left is to use initial and boundary conditions to determine the coefficients. This technique can be used to solve difference or differential equations. One important application of this method is to solve the

Chapman–Kolmogorov equations for birth and death processes, see [17, §5.2]. The latter equations is a system of differential equations in time and partial difference equations in two discrete variables whose solution is given by (1.4).

In this paper we show how the separation of variables gives integral representations for solutions of certain combinatorial problems.

Our approach is explained in detail in Section 2. The integrands in our integral representations are constant multiples of products of orthogonal polynomials times a measure with respect to which the polynomials are orthogonal. The integral representations arise naturally through separation of variables of the solution of systems of difference equations satisfied by the combinatorial numbers. We may reverse the process by starting with integrals of products of orthogonal polynomials times their orthogonality measure and reach the combinatorial numbers. Some of these integrals arose in problems involving linearizations of products of orthogonal polynomials where the focus of attention was their nonnegativity [5,31]. Most of the positivity results originated from work by Askey and his coauthors in the late 1960's and 1970's. For references we refer the interested reader to Askey's monograph [3], and to Ismail's book [17].

The integral representations studied in this work are of the form

$$\int_{\mathbb{R}} f(x) \prod_{j=1}^{m} p_{n_j}(\lambda_j x)\, d\mu(x), \tag{1.9}$$

where μ is a discrete or absolutely continuous measure and f is some integrable function. Ismail and Simeonov [19] studied the large k behavior of integrals of the form (1.9) when the n_j's are all equal. Since the integral in (1.9) represents the number of ways a certain configuration occurs, one can calculate the probability that such configuration occurs. We shall also study integrals of the type (1.9) where the polynomials $p_{n_j}(x)$ come from two different families of orthogonal polynomials. The positivity results which we establish are not only new but seem to be the first of its type.

The rest of this paper is organized as follows. As we already mentioned in the above paragraph our approach is outlined in Section 2, where we characterize the linearization coefficients of orthogonal polynomials as the unique solution of some partial differential equations with boundary conditions. Then we apply the results of Section 2 to various combinatorial problems in Sections 3–8. More precisely, by solving the corresponding partial difference equations combinatorially, we deduce the combinatorial interpretations of Hermite and Charlier polynomials, Laguerre polynomials, Meixner polynomials, Meixner–Pollaczek polynomials, q-Hermite polynomials, q-Charlier polynomials, and q-Laguerre polynomials, respectively. In each case we start with a combinatorial problem involving multisets, deduce a difference equation for the combinatorial numbers involved, then identify the orthogonal polynomials which arise through the machinery developed in Section 2. Furthermore, in Section 9, we extend the previous results to some more general integrals to include the moments, inverse coefficients and linearization coefficients. We also compute the corresponding generating functions for the corresponding integrals of Lagurre and Meixner polynomials and deduce their combinatorial interpretation by applying MacMahon's Master theorem. In Section 10, we give a further extension of the integrals of Laguerre and Meixner polynomials. Finally, in Section 11, we prove the crucial step, Lemma 8.1, towards the combinatorial solution of the partial difference equations of q-Charlier polynomials.

We follow the standard notation for shifted factorials, hypergeometric functions and their q-analogues as in the books [1,14,17]. The work of Koekoek–Lesky–Swarttouw [27] is also a standard reference for formulas involving orthogonal polynomials and their basic analogues.

2. Separation of variables and linearization coefficients

Let $\{p_n(x)\}$ be a sequence of orthogonal polynomials

$$\int_{\mathbb{R}} p_m(x) p_n(x)\, d\mu(x) = \zeta_n \delta_{m,n}, \qquad \zeta_0 = 1. \tag{2.1}$$

The condition $\zeta_0 = 1$ amounts to normalizing total mass of μ to be 1. Then the polynomials $\{p_n(x)\}$ must satisfy a three term recurrence relation of the form

$$p_{n+1}(x) = [A_n x + B_n]p_n(x) - C_n p_{n-1}(x), \quad n > 0, \tag{2.2}$$

and we will always assume $p_0(x) := 1$, $p_1(x) = A_0 x + B_0$. Therefore

$$\zeta_n = \frac{A_0}{A_n} C_1 C_2 \cdots C_n. \tag{2.3}$$

We consider the linearization coefficients in the expansion of $\prod_{j=1}^{m-1} p_{n_j}(\lambda_j x)$ in $\{p_n(x)\}$. Equivalently we consider the numbers

$$I(\boldsymbol{n}) := \int_{\mathbb{R}} \left(\prod_{j=1}^{m-1} p_{n_j}(\lambda_j x) \right) p_{n_m}(x)\, d\mu(x), \tag{2.4}$$

where $\boldsymbol{n} = (n_1, \ldots, n_m)$, n_j is a nonnegative integer for $1 \leqslant j \leqslant m$. We shall use the following notation:

$$I_j^{\pm}(\boldsymbol{n}) = I(n_1, \ldots, n_{j-1}, n_j \pm 1, n_{j+1}, \ldots, n_m).$$

Moreover we assume that $\lambda_m = 1$. It is clear that

$$I_j^{+}(0, \ldots, 0, n) = \lambda_j C_1 \frac{A_0}{A_1} \delta_{n,1} + B_0(1 - \lambda_j)\delta_{n,0}, \quad \text{if } n = 0, 1, \quad \text{and}$$

$$I(0, \ldots, 0) = 1, \qquad I(\boldsymbol{n}) = 0 \quad \text{if } \sum_{j=1}^{m-1} n_j < n. \tag{2.5}$$

Theorem 2.1. *The numbers $I(\boldsymbol{n})$ satisfy the system of difference equations*

$$I_j^{+}(\boldsymbol{n}) - u_{jk}(\boldsymbol{n}) I_k^{+}(\boldsymbol{n}) = [B_{n_j} - u_{jk} B_{n_k}] I(\boldsymbol{n}) - C_{n_j} I_j^{-}(\boldsymbol{n}) + u_{jk}(\boldsymbol{n}) C_{n_k} I_k^{-}(\boldsymbol{n}), \tag{2.6}$$

where $u_{jk}(\boldsymbol{n}) = v_j(\boldsymbol{n})/v_k(\boldsymbol{n})$ and $v_j(\boldsymbol{n}) = A_{n_j}\lambda_j$.

Proof. For $1 \leqslant t \leqslant m$, we have by (2.2)

$$\begin{aligned} I_t^{+}(\boldsymbol{n}) &= \int_{\mathbb{R}} \left[(A_{n_t}\lambda_t x + B_{n_t})\, p_{n_t}(\lambda_t x) - C_{n_t} p_{n_t - 1}(\lambda_t x) \right] \prod_{r \neq t} p_{n_r}(\lambda_r x)\, d\mu(x) \\ &= v_t(\boldsymbol{n}) \int_{\mathbb{R}} x \prod_{r=1}^{m} p_{n_r}(\lambda_r x)\, d\mu(x) + B_{n_t} I(\boldsymbol{n}) - C_{n_t} I_t^{-}(\boldsymbol{n}). \end{aligned}$$

Specializing the above equation at $t = j$ and $t = k$ immediately leads to (2.6). □

Observe that in the system (2.6) we assume $j, k \geqslant 1$. It is more convenient to write (2.6) in the more symmetric form

$$\frac{1}{v_j(\boldsymbol{n})} I_j^{+}(\boldsymbol{n}) - \frac{1}{v_k(\boldsymbol{n})} I_k^{+}(\boldsymbol{n}) = \left[\frac{B_{n_j}}{v_j(\boldsymbol{n})} - \frac{B_{n_k}}{v_k(\boldsymbol{n})} \right] I(\boldsymbol{n}) - \frac{C_{n_j}}{v_j(\boldsymbol{n})} I_j^{-}(\boldsymbol{n}) + \frac{C_{n_k}}{v_k(\boldsymbol{n})} I_k^{-}(\boldsymbol{n}). \tag{2.7}$$

We will show that the system (2.7) describes many combinatorial problems. From now on we will consider different combinatorial problems and derive a system of equations of the type (2.7) for the combinatorial numbers under consideration. Theorems 2.2 and 2.3 identify the combinatorial numbers as integrals of products of orthogonal polynomials.

Theorem 2.2. *One solution to*

$$y_j^+(\boldsymbol{n}) - u_{jk}(\boldsymbol{n}) y_k^+(\boldsymbol{n}) = \left[B_{n_j} - u_{jk}(\boldsymbol{n}) B_{n_k}\right] y(\boldsymbol{n}) - C_{n_j} y_j^-(\boldsymbol{n}) + u_{jk}(\boldsymbol{n}) C_{n_k} y_k^-(\boldsymbol{n}), \tag{2.8}$$

is given by

$$y(\boldsymbol{n}) = \int_{\mathbb{R}} \prod_{j=1}^{m} p_{n_j}(\lambda_j x)\, d\nu(x), \tag{2.9}$$

for any measure ν having finite moments of all orders.

Proof. We try the separation of variables $y(\boldsymbol{n}) = \prod_{j=1}^{m} F_j(n_j)$. When we substitute in (2.8) we get

$$\frac{F_j(n_j+1)}{v_j(\boldsymbol{n}) F_j(n_j)} + \frac{C_{n_j} F_j(n_j-1)}{v_j(\boldsymbol{n}) F_j(n_j)} - \frac{B_{n_j}}{v_j(\boldsymbol{n})} = \frac{F_k(n_k+1)}{v_k(\boldsymbol{n}) F_k(n_k)} + \frac{C_{n_k} F_k(n_k-1)}{v_k(\boldsymbol{n}) F_k(n_k)} - \frac{B_{n_k}}{v_k(\boldsymbol{n})}. \tag{2.10}$$

Thus each side of the above equation is a constant independent of j or k, so we denote the constant by x. This leads to the difference equation

$$F_j(n_j+1) = (\lambda_j A_{n_j} x + B_{n_j}) F_j(n_j) - C_{n_j} F_j(n_j-1),$$

and the F's now depend on x. Comparing with (2.2) and noting that $F_j(-1) = 0$ and $F_j(0) = 1$, we see that the above recurrence relation has a solution given by $F_j(n_j) = p_{n_j}(\lambda_j x)$ and by the principle of linear superposition the function in (2.9) is a solution. □

Theorem 2.3. *The system of Eqs.* (2.8) *and the boundary conditions*

$$y_j^+(0,\dots,0,n) = \lambda_j C_1 \frac{A_0}{A_1}\delta_{n,1} + B_0(1-\lambda_j)\delta_{n,0}, \quad \text{if } n = 0, 1, \quad \text{and}$$
$$y(0,\dots,0) = 1, \quad y(\boldsymbol{n}) = 0 \quad \text{if } \sum_{j=1}^{m-1} n_j < n, \tag{2.11}$$

have a unique solution which is given by (2.4).

Proof. We know that the multisequence (2.4) satisfies the system of Eqs. (2.8) and boundary condition (2.11), hence a solution exists. The second boundary condition defines y for $n \geqslant 0$ and when the rest are zero. The first boundary condition in (2.11) defines y for n and when one other entry $= 1$ and the rest are zero in a unique way. Letting $\boldsymbol{n} = (0,\dots,1,0,\dots,n)$, $n > 0$ in (2.8) with $k = m$ we evaluate $y(0,\dots,2,0,\dots,n)$ and by induction we evaluate $y(0,\dots,n_s,0,\dots,n)$, $1 \leqslant s < m$. Next we use (2.8) to evaluate y for general n_r, n and another nonzero n_s: if $n > n_r + 1$, then $y(\boldsymbol{n})$ with nonzero entries in the positions r, m of $\boldsymbol{n}$ is zero when we have 1 in the position s; if $n \leqslant n_r + 1$, we use (2.8) with $j = s, k = m$ to evaluate y. Thus we can evaluate y inductively when $\boldsymbol{n}$ has three nonzero entries. We continue this argument until we reach any desired general $\boldsymbol{n}$. □

Remark 2.4. It is important to note that (2.8) is satisfied by solutions of the form (2.9) where ν is any probability measure with finite moments. It is the boundary conditions (2.11) that force ν to be an orthogonality measure of $\{p_n(x)\}$.

An important class of orthogonal polynomials is the class of birth and death process polynomials. They are generated by (1.2). These polynomials have only positive zeros so they are orthogonal with respect to a probability measure supported on a subset of $[0,\infty)$. The idea of separation of variables is also used to solve the differential-difference equations describing this model, see §5.2 and Theorem 7.2.1 in [17]. Birth and death processes have many applications in applied probability and queueing theory.

An immediate consequence of Theorem 2.3 is the following result for the polynomials $\{Q_n(x)\}$ generated by (1.2).

Theorem 2.5. *The system of difference equations*

$$\frac{b_{n_j}}{\lambda_j}y_j^+(\boldsymbol{n})-\frac{b_{n_k}}{\lambda_k}y_k^+(\boldsymbol{n})=\left[\frac{b_{n_j}d_{n_j}}{\lambda_j}-\frac{b_{n_k}d_{n_k}}{\lambda_k}\right]y(\boldsymbol{n})-\frac{d_{n_j}}{\lambda_j}y_j^-(\boldsymbol{n})+\frac{d_{n_k}}{\lambda_k}y_k^-(\boldsymbol{n}), \tag{2.12}$$

and the boundary conditions

$$y_j^+(0,\dots,0,n)=\lambda_j\frac{d_1}{b_0}\delta_{n,1}+\left(1+\frac{d_0}{b_0}\right)\delta_{n,0},\quad \text{if } n=0,1,\quad \text{and}$$
$$y(0,\dots,0)=1,\qquad y(\boldsymbol{n})=0\quad \text{if}\sum_{j=1}^{m-1}n_j<n, \tag{2.13}$$

have a unique solution which is given by

$$y(\boldsymbol{n})=\int_0^\infty\prod_{j=1}^m Q_{n_j}(\lambda_j x)\,d\mu(x), \tag{2.14}$$

where μ is an orthogonality measure for the polynomials $\{Q_n(x)\}$ in (1.2).

From combinatorial point of view, sometimes it is easier to establish a different kind of difference equations from (2.6). Since $p_1(x)=A_0x+B_0$ and $p_{n+1}(x)=[A_nx+B_n]p_n(x)-C_np_{n-1}(x)$, we have

$$\lambda_j p_1(x)p_{n_j}(\lambda_j x)=\frac{A_0}{A_{n_j}}p_{n_j+1}(\lambda_j x)+\left(\lambda_j B_0-\frac{A_0}{A_{n_j}}B_{n_j}\right)p_{n_j}(\lambda_j x)$$
$$+\frac{A_0}{A_{n_j}}C_{n_j}p_{n_j-1}(\lambda_j x). \tag{2.15}$$

Substituting in (2.4) yields

$$I(1,\boldsymbol{n})=\frac{A_0}{\lambda_j A_{n_j}}I_j^+(\boldsymbol{n})+\left[B_0-\frac{A_0}{A_{n_j}}\frac{B_{n_j}}{\lambda_j}\right]I(\boldsymbol{n})+\frac{A_0}{A_{n_j}}\frac{C_{n_j}}{\lambda_j}I_j^-(\boldsymbol{n}). \tag{2.16}$$

Subtracting (2.16) from itself with j replaced by k, we obtain (2.6). For the Laguerre polynomials, q-Charlier polynomials and q-Laguerre polynomials, we shall first establish combinatorially (2.16) before passing to (2.6). Finally we have the following result.

Theorem 2.6. *Let $\boldsymbol{n}=(n_1,\dots,n_m)$ with $m\geqslant 1$ and $n_1,\dots,n_m\geqslant 0$. Any sequence $I(\boldsymbol{n})$ satisfying the system (2.16) is uniquely determined by its special values at $\boldsymbol{n}=(1,\dots,1)$ and the symmetry with respect to the indices $n_1,\dots,n_m$.*

Remark 2.7. Evaluating the special values of the linearization coefficients at $\boldsymbol{n}=(1,\dots,1)$ amounts to computing the moments of the corresponding orthogonal polynomials, while the boundary condition (2.13) is much easier to check and does not need the knowledge of the moments, though the latter would be a source of inspiration for the linearization coefficients.

3. Linearization coefficients of Hermite and Charlier polynomials

In this section we consider the linearization coefficients of Hermite and Charlier polynomials. We start with some combinatorial setup, which will also be used in the later sections.

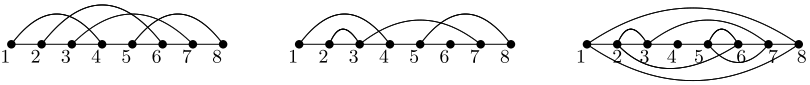


Fig. 1. Diagrams of, from left to right, the matching $M = 14/26/37/58$, the partition $\pi = 14/237/58/6$ and the permutation $\sigma = 83746251$.

Fig. 2. Diagrams of, from left to right, a partition in $\mathcal{P}(2,3,3)$ and permutation in $\mathcal{D}(2,3,3)$.

3.1. Combinatorial definitions

In the sequel, we denote by $\mathcal{M}_n$, Π_n and $\mathfrak{S}_n$ the set of perfect matchings, of partitions and of permutations, respectively, of $[n] := \{1, 2, \ldots, n\}$. Recall that a perfect matching of $[n]$ is just a set partition of $[n]$ the blocks of which have exactly two elements. It is often convenient to represent pictorially set partitions and permutations of $[n]$. We first draw n elements on a line labeled $1, 2, \ldots, n$ in increasing order. Then, the diagram of a partition of $[n]$ is obtained by joining *successive* elements of each block by arcs drawn in the upper half-plane. Here, we say that two elements $i < j$ in the block B are successive, or more precisely that j *follows* i, if there is no element $p \in B$ such that $i < p < j$. We denote by (i, j) the arc whose extremities are i and j. The diagram of a permutation $\sigma \in \mathfrak{S}_n$ is obtained by drawing an arc $i \to \sigma(i)$ above (resp. under) the line if $i < \sigma(i)$ (resp. $i > \sigma(i)$). Arcs are always drawn in a way such that any two arcs cross at most once.

In what follows, we fix an m-tuple of nonnegative integers $\boldsymbol{n} = (n_1, \ldots, n_m)$ such that $n = n_1 + \cdots + n_m$ and partition the n balls $\{1, \ldots, n\}$ into m boxes $S_1, \ldots, S_m$ where $S_j = \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$, $n_0 = 0$, for $j = 1, \ldots, m$. We denote by $[\boldsymbol{n}]$ the set $\{1, \ldots, n\}$ with underlying boxes $S_1, \ldots, S_m$, and the corresponding sets of matching, partitions and permutations by

$$\mathcal{M}(\boldsymbol{n}) := \mathcal{M}_n, \qquad \Pi(\boldsymbol{n}) := \Pi_n \quad \text{and} \quad \mathfrak{S}(\boldsymbol{n}) := \mathfrak{S}_n. \tag{3.1}$$

A partition π of $[\boldsymbol{n}]$ is said to be *inhomogeneous* if each block of π contains at least two elements and no two elements in the same block belong to the same box S_i $(1 \leqslant i \leqslant m)$. Similarly, a permutation σ of $[\boldsymbol{n}]$ is an *inhomogeneous derangement* if $\sigma(S_i) \cap S_i = \emptyset$ for all $i \in [m]$. We let $\mathcal{K}(\boldsymbol{n})$ (resp. $\mathcal{P}(\boldsymbol{n})$ and $\mathcal{D}(\boldsymbol{n})$) denote the set of inhomogeneous perfect matchings (resp. partitions and derangements) of $[\boldsymbol{n}]$. Note that a set partition (resp. permutation) is inhomogeneous if and only in its diagram, there is no isolated vertex and no arc connecting two elements in the same *box* S_j $(1 \leqslant j \leqslant m)$. For instance, if $\boldsymbol{n} = (2, 3, 3)$, then in Fig. 1 the matching drawn is in $\mathcal{K}(\boldsymbol{n})$ while the partition and the permutation are not in $\mathcal{P}(\boldsymbol{n})$ and $\mathcal{D}(\boldsymbol{n})$ (they have isolated points). Inhomogeneous objects are drawn in Fig. 2.

3.2. Hermite polynomials and inhomogeneous matchings

The Hermite polynomials $\{H_n(x)\}_{n \geqslant 0}$ can be defined by one of the following five equivalent conditions:

(1) (Coefficients) $H_n(x) = \sum_{0 \leqslant 2k \leqslant n} (-1)^k \frac{n!}{2^k k!(n-2k)!} (2x)^{n-2k}$.
(2) (Generating function) $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$.
(3) (Orthogonality relation) $\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2}\, dx = 2^n n! \sqrt{\pi} \delta_{mn}$.
(4) (Recurrence relation) $2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$, with $H_{-1}(x) = 0$, $H_0(x) = 1$.
(5) (Moments) $\mu_{2n+1} = 0$, $\mu_{2n} = 1 \cdot 3 \cdots (2n-1)/2^n$.

Let $K(\boldsymbol{n})$ be the number of inhomogeneous perfect matchings of $[\boldsymbol{n}]$.

Lemma 3.1. *For $k, j \in [m]$ and $k \neq j$ the numbers $K(\boldsymbol{n})$ satisfy*

$$K_j^+(\boldsymbol{n}) - K_k^+(\boldsymbol{n}) = n_k K_k^-(\boldsymbol{n}) - n_j K_j^-(\boldsymbol{n}), \tag{3.2}$$

and the boundary condition (2.11) with $\lambda_j = 1$ for all j and $A_0 C_1 = A_1$.

Proof. Let $r \in [m]$ and set $N_r = n_1 + \cdots + n_r$. For any $i \neq r$, the number of matchings in $\mathcal{K}_r^+(\boldsymbol{n})$ in which $N_r + 1$ is matched with an element in S_i is clearly $n_i K_i^-(\boldsymbol{n})$. This implies that for any $r \in [m]$, we have

$$K_r^+(\boldsymbol{n}) = \sum_{\substack{i=1 \\ i \neq r}}^{m} n_i K_i^-(\boldsymbol{n}),$$

from which we immediately deduce (3.2). The boundary conditions in (2.11) are obviously satisfied. □

Theorem 3.2. *The numbers $K(\boldsymbol{n})$ have the following integral representation*

$$K(\boldsymbol{n}) = 2^{-(n_1+\cdots+n_m)/2} \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} \prod_{j=1}^{m} H_{n_j}(x)\, dx. \tag{3.3}$$

Proof. When $\lambda_j = 1$ for all j, and

$$A_n = 1, \qquad B_n = 0, \qquad C_n = n \quad \text{for all } n,$$

by Lemma 3.1, the numbers $K(\boldsymbol{n})$ satisfy (2.8) and (2.11). On the other hand, the corresponding orthogonal polynomials $\tilde{H}_n(x)$ satisfy the recurrence relation

$$x\tilde{H}_n(x) = \tilde{H}_{n+1}(x) + n\tilde{H}_{n-1}(x), \quad \text{with } \tilde{H}_{-1}(x) = 0,\ \tilde{H}_0(x) = 1. \tag{3.4}$$

Hence, these are the normalized Hermite polynomials $\tilde{H}_n(x) = 2^{-n/2} H_n(x/\sqrt{2})$ and their orthogonality relation is

$$\int_{\mathbb{R}} \tilde{H}_m(x)\tilde{H}_n(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = n!\delta_{mn}. \tag{3.5}$$

Therefore

$$K(\boldsymbol{n}) = \int_{\mathbb{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \prod_{j=1}^{m} 2^{-n/2} H_{n_j}(x/\sqrt{2})\, dx,$$

which is equal to (3.3). □

Note that the *exponential formula* (see [30, Corollary 5.1.6]) implies that

$$\sum_{n_1,\ldots,n_m \geqslant 0} K(\boldsymbol{n}) \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} = \exp\Bigg(\sum_{i<j} x_i x_j \Bigg). \tag{3.6}$$

Hence (3.3) can also be proved from the generating function of Hermite polynomials.

3.3. Charlier polynomials and inhomogeneous partitions

The Charlier polynomials $C_n^{(a)}(x)$ can be defined by one of the following five equivalent conditions:

(1) (Explicit formula) $C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!(-a)^{n-k}$.
(2) (Generating function) $\sum_{n=0}^{\infty} C_n^{(a)}(x)\frac{w^n}{n!} = e^{-aw}(1+w)^x$.
(3) (Orthogonality) $\int_0^{\infty} C_n^{(a)}(x) C_m^{(a)}(x)\, d\psi^{(a)}(x) = a^n n!\delta_{mn}$, where $\psi^{(a)}$ is the step function of which the jumps at the points $x = 0, 1, \ldots$ are $\psi^{(a)}(x) = \frac{e^{-a}a^x}{x!}$.
(4) (Recursion relation) $C_{n+1}^{(a)}(x) = (x - n - a)C_n^{(a)}(x) - anC_{n-1}^{(a)}(x)$.
(5) (Moments) $\mu_n = \sum_{k=1}^{n} S(n,k)a^k$, where $S(n,k)$ are the Stirling numbers of the second kind.

The number of blocks of a set partition π is denoted by $\mathrm{bl}(\pi)$. Consider the enumerative polynomial of inhomogeneous partitions

$$C(\boldsymbol{n}; a) = \sum_{\pi \in \mathcal{P}(\boldsymbol{n})} a^{\mathrm{bl}(\pi)}. \tag{3.7}$$

Lemma 3.3. *For $k, j \in [m]$ and $k \neq j$ the polynomials $C(\boldsymbol{n}; a)$ satisfy*

$$C_j^+(\boldsymbol{n}; a) - C_k^+(\boldsymbol{n}; a) = (n_k - n_j)C(\boldsymbol{n}; a) + an_k C_k^-(\boldsymbol{n}; a) - an_j C_j^-(\boldsymbol{n}; a), \tag{3.8}$$

and the boundary condition (2.11) with $\lambda_j = 1$ for all j and $A_0C_1 = A_1$.

Proof. Let $N_j = n_1 + \cdots + n_j$. The partitions of $\mathcal{P}_j^+(\boldsymbol{n})$ can be divided into three categories:

- $N_j + 1$ and one element of S_k form a block of two elements, the corresponding generating function is $an_k C_k^-(\boldsymbol{n}; a)$;
- $N_j + 1$ and one element of S_k belong to a block containing at least one another element, the corresponding generating function is $\sum_{\pi \in \mathcal{P}(\boldsymbol{n})}(n_k - n_{k,j}(\pi))a^{\mathrm{bl}(\pi)}$, where $n_{k,j}(\pi)$ is the number of blocks in π containing both elements of S_j and S_k (clearly $n_{k,j}(\pi) = n_{j,k}(\pi)$);
- $N_j + 1$ is in a block without any element of S_k, let $R_{k,j}(\boldsymbol{n}; a)$ be the corresponding generating function.

Thus we have

$$C_j^+(\boldsymbol{n}; a) = \sum_{\pi \in \mathcal{P}(\boldsymbol{n})} \big(n_k - n_{k,j}(\pi)\big)a^{\mathrm{bl}(\pi)} + an_k C_k^-(\boldsymbol{n}; a) + R_{k,j}(\boldsymbol{n}; a). \tag{3.9}$$

Exchanging k and j in the latter identity and subtracting the resulting identity from the latter identity, we obtain (3.8) in view of the symmetry relation $R_{k,j}(\boldsymbol{n}; a) = R_{j,k}(\boldsymbol{n}; a)$. This relation can be easily proved, for instance by observing that a partition in $\mathcal{P}_j^+(\boldsymbol{n})$ can be seen as an inhomogeneous partition of the union $S_1 \cup S_2 \cup \cdots \cup S_m$ with $S_i = [N_{i-1} + 1, N_i]$ for $i \neq j$ and $S_j = [N_{j-1} + 1, N_j] \cup \{x\}$ where x is any object which is not in $[n]$. □

Remark 3.4. We can also argue as follows. Let $\boldsymbol{n}^* = (n_1, \ldots, n_j, 1, n_{j+1}, \ldots, n_m)$ with $j \in [m]$. It is fairly easy to show that

$$C\big(\boldsymbol{n}^*; a\big) = C_j^+(\boldsymbol{n}; a) + an_j C_j^-(\boldsymbol{n}; a) + n_j C(\boldsymbol{n}; a). \tag{3.10}$$

Subtracting the above identity from (3.10) with $j = k$ yields immediately (3.8). We will use this argument for the Laguerre and Meixner polynomials in the next sections.

We can solve the system (3.8) by applying the method of separation of variables which naturally leads to the Charlier polynomials.

Theorem 3.5. *The polynomials $C(\boldsymbol{n};a)$ have the following integral representation*

$$C(\boldsymbol{n};a)=\int_0^\infty C_{n_1}^{(a)}(x)\cdots C_{n_m}^{(a)}(x)\,d\psi^{(a)}(x). \tag{3.11}$$

Proof. Clearly (2.8) reduces to (3.8) when $\lambda_j=1$ for all j, and

$$A_n=1,\qquad B_n=-n-a,\qquad C_n=an\quad \text{for all } n\geqslant 0.$$

From Lemma 3.3 and Theorem 2.3 we deduce (3.11). □

The above formula was first established by Zeng [34] using the generating function and the exponential formula. A different proof was given by Gessel [15] using rook polynomials.

4. Linearization coefficients of Laguerre polynomials

The shifted factorials are

$$(a)_0=1,\qquad (a)_n=a(a+1)\cdots(a+n-1),\quad n>0. \tag{4.1}$$

The Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!(\alpha+1)_k}x^k, \tag{4.2}$$

and have the generating function

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n=(1-t)^{-\alpha-1}\exp\left(\frac{-xt}{1-t}\right). \tag{4.3}$$

They satisfy the recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x)-(2n+\alpha+1-x)L_n^{(\alpha)}(x)+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0, \tag{4.4}$$

and the orthogonality

$$\int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx=\frac{(\alpha+1)_n}{n!}\,\delta_{m,n}. \tag{4.5}$$

The moments are

$$\mu_n=\frac{1}{\Gamma(\alpha+1)}\int_0^\infty x^{\alpha+n}e^{-x}\,dx=(\alpha+1)_n. \tag{4.6}$$

In this section we shall prove the results of Foata and Zeilberger [12] about the Laguerre polynomials through our method of separation of variables. For $\pi\in\mathfrak{S}(\boldsymbol{n})$, we let $\mathrm{Fix}_i\,\pi=\pi(S_i)\cap S_i$ for $i\in[m]$. For an m-tuple $\boldsymbol{\Lambda}=(\lambda_1,\dots,\lambda_m)$, define

$$L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})=\sum_{\pi\in\mathfrak{S}(\boldsymbol{n})}(\alpha+1)^{\mathrm{cyc}(\pi)}\prod_{i=1}^{m}(\lambda_i-1)^{|\mathrm{Fix}_i\,\pi|}\lambda_i^{|S_i\setminus\mathrm{Fix}_i\,\pi|}, \tag{4.7}$$

where $\mathrm{cyc}(\pi)$ is the number of cycles of π. By definition, for an inhomogeneous permutation $\pi\in\mathcal{D}(\boldsymbol{n})$ we have $|\mathrm{Fix}_i\,\pi|=0$. Hence, when $\boldsymbol{\Lambda}=\mathbf{1}:=(1,\dots,1)$ the summands in (4.7) reduce to $(\alpha+1)^{\mathrm{cyc}(\pi)}$ if $\pi\in D(\boldsymbol{n})$ and 0 otherwise. Thus, we have

$$L(\boldsymbol{n};\alpha,\mathbf{1})=\sum_{\pi\in\mathcal{D}(\boldsymbol{n})}(\alpha+1)^{\mathrm{cyc}(\pi)}. \tag{4.8}$$

Lemma 4.1. *For $j,k\in[m]$ such that $j\neq k$ the polynomials $L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$ satisfy*

$$\begin{aligned}&\lambda_k L_j^+(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})-\lambda_j L_k^+(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})\\&\quad=\big[(2n_k+\alpha+1)\lambda_j-(2n_j+\alpha+1)\lambda_k\big]L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})\\&\qquad+n_k(n_k+\alpha)\lambda_j L_k^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})-n_j(n_j+\alpha)\lambda_k L_j^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda}).\end{aligned} \tag{4.9}$$

Proof. Let $n_0=\lambda_0=1$ and $\boldsymbol{n}^*=(n_0,n_1,\dots,n_m)$ with $S_0=\{1^*\}$. Then $\lambda_j L(\boldsymbol{n}^*;\alpha,\boldsymbol{\Lambda})$ is the generating function of $\sigma\in\mathfrak{S}(\boldsymbol{n}^*)$ such that $\sigma(1^*)\neq 1^*$ and the edge $1^*\to\sigma(1^*)$ is weighted by λ_j. We show that

$$\begin{aligned}\lambda_j L\big(\boldsymbol{n}^*;\alpha,\boldsymbol{\Lambda}\big)&=L_j^+(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})-(\alpha+1)(\lambda_j-1)L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})\\&\quad+2n_jL(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})+n_j(n_j+\alpha)L_j^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda}).\end{aligned} \tag{4.10}$$

To do so, we adjoin the element 1^* to S_j. Thus $(\alpha+1)(\lambda_j-1)L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$ is the generating function of $\sigma\in\mathfrak{S}_j^+(\boldsymbol{n})$ such that $\sigma(1^*)=1^*$. Hence, the difference

$$L_j^+(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})-(\alpha+1)(\lambda_j-1)L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$$

is the generating function of $\sigma\in\mathfrak{S}_j^+(\boldsymbol{n})$ such that $\sigma(1^*)\neq 1^*$, moreover the edge $1^*\to\sigma(1^*)$ is weighted by λ_j-1 if $\sigma(1^*)\in S_j$ and λ_j otherwise. To compensate the over counting, we should add

- the generating function of $\sigma\in\mathfrak{S}_j^+(\boldsymbol{n})$ such that $\sigma(1^*)\in S_j$ and the edge $1^*\to\sigma(1^*)$ is weighted by 1;
- the generating function of $\sigma\in\mathfrak{S}_j^+(\boldsymbol{n})$ such that $\sigma^{-1}(1^*)\in S_j$ and the edge $\sigma^{-1}(1^*)\to 1^*$ is weighted by 1.

For any $\sigma\in\mathfrak{S}_j^+(\boldsymbol{n})$, we let $a=\sigma(1^*)$ and $b=\sigma^{-1}(1^*)$. There are four cases to consider.

(1) $a\in S_j$ and $b\notin S_j$. We can construct such a permutation σ as follows: starting from a permutation $\tau\in\mathfrak{S}(\boldsymbol{n})$ and choosing a point $\xi\in S_j$, we define $\sigma(x)=\tau(x)$ if $x\neq 1^*,\tau^{-1}(\xi)$, and $\sigma(1^*)=\xi$, $\sigma(\tau^{-1}(\xi))=1^*$ As the weight of the edge $1^*\to\xi$ is 1 and that of $\tau^{-1}(\xi)\to 1^*$ in σ is equal to that of $\tau^{-1}(\xi)\to\xi$ in τ, the weight of σ is equal to that of τ, hence the generating function is $n_jL(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$.
(2) $a\notin S_j$ and $b\in S_j$. Similar to the above case, the generating function is $n_jL(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$.
(3) $a\in S_j$ and $b\in S_j$, but $a\neq b$. Starting from $\sigma\in\mathfrak{S}(\boldsymbol{n}^*)$ we can define the permutation τ on $[n]\setminus\{a\}$ by $\tau(j)=\sigma(j)$ for $j\neq a,b$ and $\tau(b)=\sigma(a)$. Clearly $\mathrm{cyc}(\sigma)=\mathrm{cyc}(\tau)$. Inversely, starting from a permutation $\tau\in\mathfrak{S}_j^-(\boldsymbol{n})$ there are $n_j(n_j-1)$ choices for a and b. Thus, the corresponding generating function is $n_j(n_j-1)L_j^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$.
(4) $a=b\in S_j$. The generating function is $(\alpha+1)n_jL_j^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$.

Summing up the above four cases we obtain (4.10). Now, substituting j by k in (4.10) yields

$$\begin{aligned}\lambda_k L\big(\boldsymbol{n}^*;\alpha,\boldsymbol{\Lambda}\big)&=L_k^+(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})-(\alpha+1)(\lambda_k-1)L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})\\&\quad+2n_kL(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})+n_k(n_k+\alpha)L_k^-(\boldsymbol{n};\alpha,\boldsymbol{\Lambda}).\end{aligned} \tag{4.11}$$

Multiplying (4.10) and (4.11) by λ_k and λ_j, respectively, and then subtracting, we obtain the identity (4.9). □

We need to state some preliminary results before proving the main result of this section. Let A and B be two disjoint sets of cardinality m and n, respectively. An injection f from A to $A\cup B$ can be depicted by a graph on $A\cup B$ such that there is an edge $x\to y$ if and only if $f(x)=y$. Hence the connected components of the graph consists of cycles, i.e., $x\to f(x)\to\cdots\to f^l(x)$ with $f^i(x)\in A$ and $f^l(x)=x$ and paths, i.e., $x\to f(x)\to\cdots\to f^l(x)$ with $f^l(x)\in B$. Let $\mathrm{cyc}(f)$ be the number of cycles of f. Then, Foata and Strehl [13] proved

$$\sum_{f:A\to A\cup B \text{ injection}} \beta^{\mathrm{cyc}(f)} = (\beta+n)_m. \tag{4.12}$$

Theorem 4.2. *The polynomials $L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$ have the following integral representation*

$$L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda}) = (-1)^{n_1+\cdots+n_m}\frac{n_1!\cdots n_m!}{\Gamma(\alpha+1)}\int_0^\infty x^\alpha e^{-x}\prod_{j=1}^m L_{n_j}^{(\alpha)}(\lambda_j x)\,dx. \tag{4.13}$$

Moreover, this formula is equivalent to the special $\boldsymbol{\Lambda}=\mathbf{1}$ case.

Remark 4.3. Let us first explain what we mean by the equivalence of (4.13) and its special $\boldsymbol{\Lambda}=\mathbf{1}$ case. We first prove that the definition (4.7) implies that $L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})$ is given by the integral (4.13). By taking $\boldsymbol{\Lambda}=\mathbf{1}$ in (4.13), the formula reduces to the special $\boldsymbol{\Lambda}=\mathbf{1}$ case. The point is that we shall prove that knowing the equality in (4.13) for $\boldsymbol{\Lambda}=\mathbf{1}$ proves that the two sides of (4.13) are equal via the use of the well-known formula [17, Theorem 4.6.5]:

$$L_n^{(\alpha)}(cx) = (\alpha+1)_n\sum_{k=0}^n \frac{c^k(1-c)^{n-k}}{(n-k)!(\alpha+1)_k}L_k^{(\alpha)}(x). \tag{4.14}$$

The formula (4.13) was first proved by Even and Gillis [11] for $\alpha=0$ and $\boldsymbol{\Lambda}=\mathbf{1}$. Foata and Zeilberger [12] proved the general case of (4.13) by introducing the cycles.

Proof. Clearly (2.8) reduces to (4.9) when $\lambda_m=1$, and

$$A_n=1,\qquad B_n=-(2n+\alpha+1),\qquad C_n=n(n+\alpha)$$

for all n. That is, the orthogonal polynomials are the normalized Laguerre polynomials $p_n(x)=(-1)^n n!L_n^{(\alpha)}(x)$, which satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x)+(2n+\alpha+1)p_n(x)+n(n+\alpha)p_{n-1}(x), \tag{4.15}$$

and the orthogonal relation

$$\int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}p_n(x)p_m(x)\,dx = n!(\alpha+1)_n\delta_{m,n}.$$

From Lemma 4.1 and Theorem 2.3 we deduce (4.13) when $\lambda_m=1$. To recover the general $\lambda_m\neq 1$ case, we can proceed as follows: let E be a subset of S_m with cardinality n_m-k, we consider the permutations π of $\mathfrak{S}(\boldsymbol{n})$ such that $\mathrm{Fix}_m(\pi)=E$. Any such a permutation corresponds to a pair (σ,τ) such that σ is the restriction of π on E, which is an injection from E to S_m, and τ is a permutation on $S_1\cup\cdots\cup S_{m-1}\cup(S_m\setminus E)$ defined by $\tau(x)=\pi(x)$ if $\pi(x)\notin E$ and $\tau(x)=\pi^l(x)$ where l is the minimum integer such that $\pi^l(x)\notin E$. Clearly, the correspondence $\pi\mapsto(\sigma,\tau)$ is a bijection and the generating function of such permutations is

$$(\alpha+1+k)_{n_m-k}(\lambda_m-1)^{n_m-k}\lambda_m^k\,L\big(\boldsymbol{n}_m^*;\alpha,\boldsymbol{\Lambda}^*\big),$$

where $\boldsymbol{n}_m^*=(n_1,\ldots,n_{m-1},k)$ and $\boldsymbol{\Lambda}^*=(\lambda_1,\ldots,\lambda_{m-1},1)$. Applying the result for $\lambda_m=1$ case we obtain

$$
\begin{aligned}
&L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})\\
&=\sum_{k=0}^{n_m}\binom{n_m}{k}(\alpha+1+k)_{n_m-k}(\lambda_m-1)^{n_m-k}\lambda_m^k L\big(\boldsymbol{n}_m^*;\alpha,\boldsymbol{\Lambda}^*\big)\\
&=\prod_{j=1}^{m}(-1)^{n_j}n_j!\int_{\mathbb{R}}\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}\prod_{j=1}^{m-1}L_{n_j}^{(\alpha)}(\lambda_j x)\sum_{k=0}^{n_m}\frac{(\alpha+1+k)_{n_m-k}(1-\lambda_m)^{n_m-k}\lambda_m^k}{k!}L_k^{(\alpha)}(x)\,dx.
\end{aligned}
$$

Now, invoking the known formula (4.14), we deduce (4.13).

It remains to show the special $\Lambda=\mathbf{1}$ case of (4.13) implies (4.13) for general Λ. As in the above argument, instead of operating within the last *box*, applying the same operation to all the boxes and using (4.13) for $\boldsymbol{\Lambda}=\mathbf{1}$ we obtain

$$
\begin{aligned}
L(\boldsymbol{n};\alpha,\boldsymbol{\Lambda})&=\sum_{k_1,\dots,k_m\geqslant 0}\prod_{j=1}^{m}\binom{n_j}{k_j}(\alpha+1+k_j)_{n_j-k_j}(\lambda_j-1)^{n_j-k_k}\lambda_j^{k_j}L(\mathbf{k};\alpha,\mathbf{1})\\
&=\int_{\mathbb{R}}\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}\left(\prod_{j=1}^{m}(-1)^{n_j}n_j!\sum_{k=0}^{n_j}\frac{(\alpha+1+k)_{n_j-k}(1-\lambda_j)^{n_j-k}\lambda_j^k}{(n_j-k)!}L_k^{(\alpha)}(x)\right)dx.
\end{aligned}
$$

Thus, the general formula (4.13) follows by applying the multiplication formula (4.14). □

Remark 4.4. The analogue of (4.13) for Hermite polynomials [25, Proposition 5.1] reads

$$
2^{-(n_1+\cdots+n_m)/2}\int_{\mathbb{R}}\frac{e^{-x^2}}{\sqrt{\pi}}\prod_{j=1}^{m}H_{n_j}(\lambda_j x)\,dx=\sum_{\pi\in\mathcal{M}(\boldsymbol{n})}\prod_{i=1}^{m}\big(\lambda_i^2-1\big)^{\hom_i(\pi)}\lambda_i^{|S_i|-2\hom_i(\pi)},\tag{4.16}
$$

where $\hom_i(\pi)$ denotes the number of homogeneous edges in S_i for $1\leqslant i\leqslant m$. When $\lambda_i=1$, the right-hand side of (4.16) reduces obviously to the number of inhomogeneous matchings of $[\boldsymbol{n}]$, so the formula (4.16) becomes (3.3). As the analogue of (4.14) for Hermite polynomials [17, (4.6.33)] is

$$
H_n(cx)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!(-1)^k}{k!(n-2k)!}\big(1-c^2\big)^k c^{n-2k}H_{n-2k}(x),\tag{4.17}
$$

a similar proof of (4.16) from (3.3) using (4.17) can be given. We leave this to the interested reader.

5. Linearization coefficients of Meixner polynomials

The Meixner polynomials are [17,27]

$$
M_n(x;\beta,c)=(\beta)_n{}_2F_1(-n,-x;\beta;1-1/c),\tag{5.1}
$$

and satisfy the orthogonality relation

$$
\sum_{x=0}^{\infty}M_m(x;\beta,c)M_n(x;\beta,c)\frac{(\beta)_x}{x!}c^x=\frac{(\beta)_n\,n!}{c^n(1-c)^\beta}\,\delta_{m,n},\quad \beta>0,\ 0<c<1.\tag{5.2}
$$

The Meixner polynomials generalize the Laguerre polynomials in the sense

$$
\lim_{c\to 1}M_n\big(x/(1-c);\alpha+1,c\big)=n!L_n^\alpha(x).
$$

They have the generating function

$$
\sum_{n=0}^{\infty}M_n(x;\beta,c)\frac{t^n}{n!}=(1-t/c)^x(1-t)^{-x-\beta}.\tag{5.3}
$$

The notation here is slightly different from [17, Chapter 6]. The three-term recurrence relation is

$$-xM_n(x;\beta,c) = c(1-c)^{-1}M_{n+1}(x;\beta,c) - \big[c(\beta+n)+n\big](1-c)^{-1}M_n(x;\beta,c) + (1-c)^{-1}(\beta+n-1)nM_{n-1}(x;\beta,c). \tag{5.4}$$

The moments are, see [28,32,35],

$$\mu_n(\beta,c) = (1-c)^{\beta}\sum_{k\geqslant 0}k^n c^k\frac{(\beta)_k}{k!} = \frac{\sum_{\pi\in\mathfrak{S}_n}c^{\mathrm{wex}(\pi)}\beta^{\mathrm{cyc}(\pi)}}{(1-c)^n}, \tag{5.5}$$

where $\mathrm{wex}(\pi)$ is the number of *weak excedances* of π, i.e.,

$$\mathrm{wex}(\pi) = \big|\big\{i \;\big|\; 1\leqslant i\leqslant n \text{ and } i\leqslant \pi(i)\big\}\big|. \tag{5.6}$$

Let π be a permutation of $[\boldsymbol{n}]$. We say that π has an excedance (resp. *box-excedance*) at $i\in[n]$ if $i<\pi(i)$ (resp. $i\in S_k$, $\pi(i)\in S_j$ and $j>k$). Denote by $\mathrm{exc}(\pi)$ (resp. $\mathrm{exc}_b(\pi)$) the number of excedances (resp. box-excedances) of π. Clearly, if π is an inhomogeneous derangement, then $\mathrm{exc}(\pi)=\mathrm{exc}_b(\pi)$. Consider the generating function of the derangements with respect to the numbers of cycles and (box-)excedances:

$$M(\boldsymbol{n};\beta,c) = \sum_{\pi\in\mathcal{D}(\boldsymbol{n})}\beta^{\mathrm{cyc}(\pi)}c^{\mathrm{exc}(\pi)}. \tag{5.7}$$

Lemma 5.1. *For any $k,j\in[m]$ such that $k\neq j$ we have*

$$M_j^+(\boldsymbol{n};\beta,c) - M_k^+(\boldsymbol{n};\beta,c) = (c+1)(n_k-n_j)M(\boldsymbol{n};\beta,c) + cn_k(n_k+\beta-1)M_k^-(\boldsymbol{n};\beta,c) - cn_j(n_j+\beta-1)M_j^-(\boldsymbol{n};\beta,c), \tag{5.8}$$

and the boundary condition (2.11) with $\lambda_j=1$ for all j and $A_0C_1=A_1$.

Proof. Let $j\in[m]$ and set $\boldsymbol{n}^*=(n_1,\dots,n_j,1,n_{j+1},\dots,n_m)$. We first show that

$$M\big(\boldsymbol{n}^*;\beta,c\big) = M_j^+(\boldsymbol{n};\beta,c) + n_j(c+1)M(\boldsymbol{n};\beta,c) + n_j(n_j+\beta-1)M_j^-(\boldsymbol{n};\beta,c). \tag{5.9}$$

Let $u=n_1+\cdots+n_j+1$ and for each $\pi\in D(\boldsymbol{n}^*)$, let $a:=\pi(u)$ and $b:=\pi^{-1}(u)$. We partition the derangements in $D(\boldsymbol{n}^*)$ into five categories:

(1) $a\notin S_j$ and $b\notin S_j$. These derangements can be easily identified with the derangements in $D_j^+(\boldsymbol{n})$, so the corresponding enumerative polynomial is $M_j^+(\boldsymbol{n};\beta,c)$.

(2) $a\notin S_j$ and $b\in S_j$. Define the derangement π' on $[\boldsymbol{n}^*]\setminus\{u\}$ by $\pi'(j)=\pi(j)$ for $j\neq b$ and $\pi'(b)=a$. Clearly $\mathrm{cyc}(\pi')=\mathrm{cyc}(\pi)$ and $\mathrm{exc}(\pi')=\mathrm{exc}(\pi)-1$. Conversely, starting with any derangement π' of $[\boldsymbol{n}^*]\setminus\{u\}$, we can recover a derangement $\pi\in D(\boldsymbol{n}^*)$ by choosing any element in S_j as b and breaking the arrow $b\to\pi'(b)$ into $b\to u$ and $u\to\pi'(b)$, so the corresponding enumerative polynomial is $cn_jM(\boldsymbol{n};\beta,c)$.

(3) $a\in S_j$ and $b\notin S_j$. Define the derangement π' on $[\boldsymbol{n}^*]\setminus\{u\}$ by $\pi'(j)=\pi(j)$ for $j\neq b$ and $\pi'(b)=a$. Clearly $\mathrm{cyc}(\pi')=\mathrm{cyc}(\pi)$ and $\mathrm{exc}(\pi')=\mathrm{exc}(\pi)$. As in the case (2), the corresponding enumerative polynomial is $n_jM(\boldsymbol{n};\beta,c)$.

(4) $a=b$ and $a\in S_j$. The corresponding enumerative polynomial is $c\beta n_jM_j^-(\boldsymbol{n};\beta,c)$.

(5) $a\in S_j$, $b\in S_j$, and $a\neq b$. Define the derangement π' on $[\boldsymbol{n}^*]\setminus\{a,u\}$ by $\pi'(j)=\pi(j)$ for $j\neq b$ and $\pi'(b)=\pi(a)$. Clearly $\mathrm{cyc}(\pi')=\mathrm{cyc}(\pi)$ and $\mathrm{exc}(\pi')=\mathrm{exc}(\pi)-1$. Conversely, starting with a derangement on $[\boldsymbol{n}^*]\setminus\{u,a\}$, we can reverse this process by choosing any element in $S_j\setminus\{a\}$ as b. As there are $n_j(n_j-1)$ ways to choose two different elements a and b in S_j, the corresponding enumerative polynomial is $cn_j(n_j-1)M_j^-(\boldsymbol{n};\beta,c)$.

Summarizing the above five cases leads to (5.9). Specializing (5.9) at $j=k$ and then subtracting the resulted equation from (5.9) ends the proof. □

Theorem 5.2. *We have*

$$M\left(\boldsymbol{n};\beta,c^{-1}\right)=(-1)^{n_1+\cdots+n_m}(1-c)^{\beta}\sum_{x=0}^{\infty}\prod_{j=1}^{m}M_{n_j}(x;\beta,c)\frac{c^x(\beta)_x}{x!}. \tag{5.10}$$

Proof. When $\lambda_j=1$ for all j, and

$$A_n=1-1/c,\qquad B_n=\beta+n+n/c,\qquad C_n=n(\beta+n-1)/c\quad\text{for all } n\geqslant 0,$$

by Lemma 5.1, the polynomials $(-1)^{n_1+\cdots+n_m}M(\boldsymbol{n};\beta,c^{-1})$ satisfy (2.8) and (2.11). Theorem 2.3 implies then (5.10). □

This formula was first given by Askey and Ismail [4] when $\beta=1$, and by Zeng [34] for general β.

6. Linearization coefficients of Meixner–Pollaczek polynomials

The Meixner–Pollaczek polynomials $P_n(x):=P_n(x;\delta,\eta)$ can be defined by [9,27],

$$\sum_{n=0}^{\infty}P_n(x;\delta,\eta)\frac{t^n}{n!}=\left[(1+\delta t)^2+t^2\right]^{-\eta/2}\exp\left[x\arctan\left(\frac{t}{1+\delta t}\right)\right]. \tag{6.1}$$

They satisfy the recurrence relation:

$$P_{n+1}(x;\delta,\eta)=\left(x-(\delta+2n)\eta\right)P_n(x;\delta,\eta)-n(\eta+n-1)\left(1+\delta^2\right)P_{n-1}(x;\delta,\eta). \tag{6.2}$$

The orthogonality relation is

$$\frac{1}{\int_{\mathbb{R}}w(x)\,dx}\int_{\mathbb{R}}P_n(x)P_m(x)w(x)\,dx=\left(\delta^2+1\right)^n n!(\eta)_n\delta_{mn}, \tag{6.3}$$

where $w(x)=x(x;\delta,\eta)$ is given by

$$w(x;\delta,\eta)=\left[\Gamma(\eta/2)\right]^{-2}\left|\Gamma\left(\frac{\eta+\imath x}{2}\right)\right|^2\exp(-x\arctan\delta).$$

Recall that a permutation π of $[\boldsymbol{n}]$ has a *drop* (resp. *box-drop*) at $i\in[n]$ if $i>\pi(i)$ (resp. $i\in S_k$, $\pi(i)\in S_j$ and $j<k$). Denote by $\operatorname{drop}(\pi)$ (resp. $\operatorname{drop}_b(\pi)$) the number of drops (resp. box-drops) of π.

The moments of Meixner–Pollaczek polynomials [35] are

$$\mu_n(\delta,\eta)=\frac{1}{\int_{\mathbb{R}}w(x)\,dx}\int_{\mathbb{R}}x^n w(x)\,dx=\sum_{\sigma\in\mathfrak{S}_n}(\delta+\imath)^{\operatorname{drop}(\sigma)}(\delta-\imath)^{\operatorname{exc}(\sigma)}\eta^{\operatorname{cyc}(\sigma)}, \tag{6.4}$$

where $\imath^2=-1$.

Consider the enumerative polynomial of the inhomogeneous derangements

$$P(\boldsymbol{n};\delta,\eta)=\sum_{\pi\in\mathcal{D}(\boldsymbol{n})}(\delta+\imath)^{\operatorname{drop}(\pi)}(\delta-\imath)^{\operatorname{exc}(\pi)}\eta^{\operatorname{cyc}(\pi)}. \tag{6.5}$$

Lemma 6.1. *For any $k,j\in[m]$ such that $k\neq j$ we have*

$$\begin{aligned}P_j^+(\boldsymbol{n};\delta,\eta)-P_k^+(\boldsymbol{n};\delta,\eta)&=2\delta(n_k-n_j)P(\boldsymbol{n};\delta,\eta)+n_k(n_k+\eta-1)\left(\delta^2+1\right)P_k^-(\boldsymbol{n};\delta,\eta)\\&\quad-n_j(n_j+\eta-1)\left(\delta^2+1\right)P_j^-(\boldsymbol{n};\delta,\eta),\end{aligned} \tag{6.6}$$

and the boundary condition (2.11) with $\lambda_j=1$ for all j and $A_0C_1=A_1$.

Proof. For $j \in [m]$ let $\boldsymbol{n}^* = (n_1, \dots, n_j, 1, n_{j+1}, \dots, n_m)$. Following the proof of Lemma 5.1 we obtain

$$P\big(\boldsymbol{n}^*; \delta, \eta\big) = P_j^+(\boldsymbol{n}; \delta, \eta) + 2n_j\delta P(\boldsymbol{n}; \delta, \eta) + \big(\delta^2 + 1\big)n_j(n_j + \eta - 1)P_j^-(\boldsymbol{n}; \delta, \eta). \tag{6.7}$$

Subtracting the last equation from (6.7) with $j = k$ yields (6.6). □

By the method of separation of variables we can solve (6.6) and obtain the following result.

Theorem 6.2. *We have*

$$P(\boldsymbol{n}; \delta, \eta) = \frac{1}{\int_{\mathbb{R}} w(x)\,dx} \int_{\mathbb{R}} \prod_{j=1}^{m} P_{n_j}(x) w(x)\,dx. \tag{6.8}$$

Proof. Clearly (2.8) reduces to (6.6) when $\lambda_j = 1$ for all j, and

$$A_n = 1, \qquad B_n = -(\delta + 2n)\eta, \qquad C_n = n(\eta + n - 1)\big(1 + \delta^2\big) \quad \text{for all } n \geqslant 0.$$

From Lemma 6.1 and Theorem 2.3 we deduce (6.8). □

This formula was first given by Zeng [35], and later generalized by Kim and Zeng [24].

7. Linearization coefficients of q-Hermite polynomials

The continuous q-Hermite polynomials $H_n(x \mid q)$ are generated by

$$H_0(x \mid q) := 1, \qquad H_1(x \mid q) = 2x, \qquad 2xH_n(x \mid q) = H_{n+1}(x \mid q) + \big(1 - q^n\big)H_{n-1}(x \mid q), \quad n > 0, \tag{7.1}$$

and have the orthogonal relation

$$\int_0^{\pi} H_n(\cos\theta \mid q) H_m(\cos\theta \mid q) v(\cos\theta \mid q)\,d\theta = (q; q)_n \delta_{mn}, \tag{7.2}$$

where

$$v(\cos\theta \mid q) = \frac{(q; q)_\infty}{2\pi}\big(e^{2\imath\theta}, e^{-2\imath\theta}; q\big)_\infty.$$

If we rescale the q-Hermite polynomials by

$$\tilde{H}_n(x \mid q) = H_n\left(\frac{1}{2}ax \,\middle|\, q\right)/a^n, \quad a = \sqrt{1 - q},$$

then (7.1) reads

$$x\tilde{H}_n(x \mid q) = \tilde{H}_{n+1}(x \mid q) + [n]_q \tilde{H}_{n-1}(x \mid q),$$

and the orthogonality relation (7.2) becomes

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} \tilde{H}_n(x \mid q)\tilde{H}_m(x \mid q)\tilde{v}(x \mid q)\,dx = n!_q \delta_{mn}. \tag{7.3}$$

Here $n!_q = (q; q)_n/(1 - q)^n$ and

$$\tilde{v}(x \mid q) = \frac{\sqrt{1 - q}(q; q)_\infty}{\sqrt{1 - (1 - q)x^2/44\pi}} \prod_{k=0}^{\infty}\big\{1 + \big(2 - x^2(1 - q)\big)q^k + q^{2k}\big\}. \tag{7.4}$$

(a) the blocks S_j and S_{j+1} in $\mathcal{K}_j^+(\boldsymbol{n})$ (b) the blocks S_j and S_{j+1} in $\mathcal{K}_{j+1}^+(\boldsymbol{n})$

Fig. 3. Crossings in an inhomogeneous perfect matching.

Given a perfect matching M (or more generally, a set partition), a pair of arcs (e_1, e_2) of M is said to cross if $e_1 = (i, j)$, $e_2 = (k, \ell)$, and $i < k < j < \ell$. The number of arc crossings in M is denoted by $\mathrm{cr}(M)$. For instance, if M is the matching drawn in Fig. 1, we have $\mathrm{cr}(M) = 5$. Let

$$K(\boldsymbol{n} \mid q) = \sum_{M \in \mathcal{K}(\boldsymbol{n})} q^{\mathrm{cr}(M)}. \tag{7.5}$$

For any nonnegative integer n we set

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}. \tag{7.6}$$

Lemma 7.1. *For $k, j \in [m]$ and $k \neq j$ the polynomials $K(\boldsymbol{n} \mid q)$ satisfy*

$$K_j^+(\boldsymbol{n} \mid q) - K_k^+(\boldsymbol{n} \mid q) = [n_k]_q K_k^-(\boldsymbol{n} \mid q) - [n_j]_q K_j^-(\boldsymbol{n} \mid q), \tag{7.7}$$

and the boundary condition (2.11) with $\lambda_j = 1$ for all j and $A_0 C_1 = A_1$.

Proof. Let $u = n_1 + \cdots + n_j + 1$. The matchings in $\mathcal{K}_j^+(\boldsymbol{n})$ (resp. $\mathcal{K}_{j+1}^+(\boldsymbol{n})$) can be divided into two categories:

- the integer $u \in S_j$ (resp, $u \in S_{j+1}$) is matched with the ℓth element $u + \ell$ in S_{j+1} (resp. $u - \ell$ in S_j), from left (resp. right), with $\ell \in [n_{j+1}]$ (resp. $\ell \in [n_j]$), then the corresponding arc crosses each of the $\ell - 1$ arcs of which one vertex is $u + t$ (resp. $u - t$) with $1 \leqslant t \leqslant \ell - 1$. An illustration is given in Fig. 3(a) (resp. Fig. 3(b)). Hence the generating function of such matchings is
$$\left(1 + q + \cdots + q^{n_{j+1}-1}\right) K_{j+1}^-(\boldsymbol{n} | q) \quad \left(\text{resp. } \left(1 + q + \cdots + q^{n_j - 1}\right) K_j^-(\boldsymbol{n} \mid q)\right);$$
- the integer u is matched with an element not in $S_j \cup S_{j+1}$, let $R_u(\boldsymbol{n} \mid q)$ be the generating polynomial of such matchings.

It follows that $K_j^+(\boldsymbol{n} \mid q) = [n_{j+1}]_q K_{j+1}^-(\boldsymbol{n} \mid q) + R_u(\boldsymbol{n} \mid q)$ and $K_{j+1}^+(\boldsymbol{n} \mid q) = [n_j]_q K_j^-(\boldsymbol{n} \mid q) + R_u(\boldsymbol{n} \mid q)$. By subtraction we obtain (7.7) for adjacent k and j. The general case follows from the simple identity $u_k - u_j = \sum_{i=j}^{k-1} (u_{i+1} - u_i)$ for any integers j and k such that $j < k$. □

Theorem 7.2. *We have*

$$K(\boldsymbol{n} \mid q) = \int_{\mathbb{R}} \tilde{v}(x \mid q) \prod_{j=1}^{m} \tilde{H}_{n_j}(x \mid q)\, dx. \tag{7.8}$$

Proof. Clearly (2.8) reduces to (7.7) when $\lambda_j = 1$ for all j, $B_k = 0$, $C_k = [k]_q$ for all k, and A_k is a constant independent of k. From Lemma 7.1 and Theorem 2.3 we deduce (7.8). □

Remark 7.3. The representation (7.8) is due to Ismail, Stanton and Viennot [18]. Three different proofs were later given in [2,10,26]. As we can see, the new proof of (7.8) given above parallels our proof in the case $q = 1$.

Note that $K(\boldsymbol{n}|0)$ is the number of perfect inhomogeneous matchings of $[\boldsymbol{n}]$ without crossings and $H_n(x|0)$ is the n-th Chebyshev polynomial of the second kind $U_n(x)$. Hence, letting $q=0$ in Theorem 7.2 we obtain the following result, due to de Sainte-Catherine and Viennot [8].

Corollary 7.4. *The number of perfect inhomogeneous matchings of* $[\boldsymbol{n}]$ *without crossings is given by*

$$K(\boldsymbol{n}|0)=\frac{2}{\pi}\int_{-1}^{1}U_{n_1}(x)\cdots U_{n_m}(x)\left(1-x^2\right)^{1/2}dx. \tag{7.9}$$

Another generalization of the above corollary was given by Kim and Zeng [25].

8. Linearization coefficients of q-Charlier and q-Laguerre polynomials

8.1. Al-Salam–Chihara polynomials

Since our q-Charlier and q-Laguerre polynomials are two rescaled special Al-Salam–Chihara polynomials, we first recall the definition of these polynomials. The Al-Salam–Chihara polynomials $Q_n(x):=Q_n(x;t_1,t_2\mid q)$ may be defined by the recurrence relation [27, Chapter 3]:

$$\begin{cases} Q_0(x)=1, \qquad Q_{-1}(x)=0,\\ Q_{n+1}(x)=\left(2x-(t_1+t_2)q^n\right)Q_n(x)-\left(1-q^n\right)\left(1-t_1t_2q^{n-1}\right)Q_{n-1}(x), \quad n\geqslant 0. \end{cases} \tag{8.1}$$

Let $Q_n(x)=2^n p_n(x)$ then

$$xp_n(x)=p_{n+1}(x)+\frac{1}{2}(t_1+t_2)q^n p_n(x)+\frac{1}{4}\left(1-q^n\right)\left(1-t_1t_2q^{n-1}\right)p_{n-1}(x). \tag{8.2}$$

They also have the following explicit expressions:

$$\begin{aligned} Q_n(x;t_1,t_2\mid q)&=\frac{(t_1t_2;q)_n}{t_1^n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},t_1u,t_1u^{-1}\\ t_1t_2,0\end{matrix}\,\middle|\,q;q\right)\\ &=(t_1u;q)_n u^{-n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},t_2u^{-1}\\ t_1^{-1}q^{-n+1}u^{-1}\end{matrix}\,\middle|\,q;t_1^{-1}qu\right)\\ &=\left(t_2u^{-1};q\right)_n u^n\,{}_2\phi_1\left(\begin{matrix}q^{-n},t_1u\\ t_2^{-1}q^{-n+1}u\end{matrix}\,\middle|\,q;t_2^{-1}qu^{-1}\right), \end{aligned}$$

where $x=\frac{u+u^{-1}}{2}$ or $x=\cos\theta$ if $u=e^{\imath\theta}$.

The Al-Salam–Chihara polynomials have the following generating function

$$G(t,x)=\sum_{n=0}^{\infty}Q_n(x;t_1,t_2\mid q)\frac{t^n}{(q;q)_n}=\frac{(t_1t,t_2t;q)_\infty}{(te^{\imath\theta},te^{-\imath\theta};q)_\infty}.$$

They are orthogonal with respect to the linear functional $\mathcal{L}$:

$$\mathcal{L}\left(x^n\right)=\frac{1}{2\pi}\int_0^{\pi}\cos^n\theta\,\frac{(q,t_1t_2,e^{2\imath\theta},e^{-2\imath\theta};q)_\infty}{(t_1e^{\imath\theta},t_1e^{-\imath\theta},t_2e^{\imath\theta},t_2e^{-\imath\theta};q)_\infty}\,d\theta, \tag{8.3}$$

where $x=\cos\theta$. Equivalently, the Al-Salam–Chihara polynomials $Q_n(x;t_1,t_2\mid q)$ are orthogonal on $[-1,1]$ with respect to the probability measure

$$\frac{(q,t_1t_2;q)_\infty}{2\pi}\prod_{k=0}^{\infty}\frac{1-2(2x^2-1)q^k+q^{2k}}{[1-2xt_1q^k+t_1^2q^{2k}][1-2xt_2q^k+t_2^2q^{2k}]}\frac{dx}{\sqrt{1-x^2}}. \tag{8.4}$$

As in [2,26], we shall consider the q-Charlier polynomials $C_n(x\mid q) := C_n(x,a,b,c\mid q)$ defined recursively by

$$C_{n+1}(x\mid q) = \big(x - c - b[n]_q\big)C_n(x\mid q) - a[n]_q C_{n-1}(x\mid q), \tag{8.5}$$

where $C_{-1}(x\mid q) = 0$ and $C_0(x\mid q) = 1$. Comparing with (8.1) we see that this is a rescaled version of the Al-Salam–Chihara polynomials:

$$C_n(x\mid q) = \left(\frac{a}{1-q}\right)^{n/2} Q_n\left(\frac{1}{2}\sqrt{\frac{1-q}{a}}\left(x - c - \frac{b}{1-q}\right); \frac{-b}{\sqrt{a(1-q)}}, 0\mid q\right). \tag{8.6}$$

We define $u_1(x)$ and $v_1(x)$ by

$$\begin{aligned} u_1(x) &= \frac{1-q}{2a}x^2 - \frac{c(1-q)+b}{a}x + \frac{b^2 + c^2(1-q)^2 + 2(1-q)(bc-a)}{2a(1-q)},\\ v_1(x) &= \frac{1}{2}\sqrt{\frac{1-q}{a}}\left(x - c - \frac{b}{1-q}\right). \end{aligned} \tag{8.7}$$

The moment functional for $C_n(x\mid q)$ is

$$\begin{aligned} \mathcal{L}_1(f) = {}& \frac{(q;q)_\infty}{2\pi}\frac{1}{2}\sqrt{\frac{1-q}{a}}\\ &\times \int_{A_-}^{A_+} \prod_{k=0}^{\infty} \frac{[1 - 2u_1(x)q^k + q^{2k}]f(x)}{1 + 2v_1(x)q^k/(\sqrt{a(1-q)}) + q^{2k}/a(1-q)} \frac{dx}{\sqrt{1 - v_1(x)^2}}, \end{aligned} \tag{8.8}$$

where

$$A_\pm = c + \frac{b}{1-q} \pm 2\sqrt{\frac{a}{1-q}}.$$

As in [21], we shall consider the q-Laguerre polynomials $L_n(x\mid q) := L_n(x, y\mid q)$ defined by the recurrence:

$$L_{n+1}(x\mid q) = \big(x - y[n+1]_q - [n]_q\big)L_n(x\mid q) - y[n]_q^2 L_{n-1}(x\mid q), \tag{8.9}$$

with the initial condition $L_{-1}(x\mid q) = 0$ and $L_0(x\mid q) = 1$. Hence these are the re-scaled Al-Salam–Chihara polynomials:

$$L_n(x\mid q) = \left(\frac{\sqrt{y}}{q-1}\right)^n Q_n\left(\frac{(q-1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q\mid q\right). \tag{8.10}$$

One deduces then the explicit formula:

$$L_n(x\mid q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} \big(x - \big(1 - yq^{-j}\big)[j]_q\big). \tag{8.11}$$

Define $u_2(x)$ and $v_2(x)$ by

$$u_2(x) = \frac{(1-q)^2}{2y}x^2 - \frac{(1-q)(1+y)}{y}x + \frac{y^2+1}{2y}, \qquad v_2(x) = \frac{q-1}{2\sqrt{y}}x + \frac{y+1}{2\sqrt{y}}. \tag{8.12}$$

Then the moment functional in this case is

$$\mathcal{L}_2(f) = \frac{(q,q;q)_\infty}{2\pi}\frac{1-q}{2\sqrt{y}} \times \int_{B_-}^{B_+} \prod_{k=0}^{\infty} \frac{[1-2u_2(x)q^k+q^{2k}]f(x)}{[1-2v_2(x)q^k/\sqrt{y}+q^{2k}/y][1-2v_2(x)q^{k+1}\sqrt{y}+q^{2k+2}y]} \frac{dx}{\sqrt{1-v_2(x)^2}}, \tag{8.13}$$

where

$$B_\pm = \frac{(1\pm\sqrt{y})^2}{1-q}. \tag{8.14}$$

For the combinatorial approach to the linearization coefficients, the q-Hermite and q-Charlier cases were proved by first combining the combinatorial models for the polynomials and moments to obtain a messy sum, and then using a killing involution to reduce it to some nicer models, [8,18,26]. However, this approach seems difficult to deal with the q-Laguerre case. So, a recursive approach based on the symmetry is used in [21], but such a proof for the q-Charlier polynomials is new.

8.2. Linearization coefficients of q-Charlier polynomials

Recall that if π is a partition of $[n]$, an arc crossing of π is a pair of arcs (e_1, e_2) such that $e_1 = (i, j)$, $e_2 = (k, \ell)$, and $i < k < j < \ell$. For instance, if π is the partition drawn in Fig. 1 (resp. in Fig. 2), then $\mathrm{cr}(\pi) = 2$ (resp. $\mathrm{cr}(\pi) = 6$). We let $\mathrm{cr}(\pi)$ denote the number of arc crossings in π.

For each partition $\pi \in \Pi_n$ we define the weight

$$w(\pi) = a^{\mathrm{bl}(\pi)} b^{\mathrm{tr}(\pi)} c^{\mathrm{sg}(\pi)} q^{\mathrm{cr}(\pi)}, \tag{8.15}$$

where $\mathrm{bl}(\pi)$, $\mathrm{sg}(\pi)$ and $\mathrm{tr}(\pi)$ are respectively the numbers of blocks, singletons and transients of π. Here, a singleton is just a block of size 1 and a transient is an element which is neither the least nor the greatest element in a block of π.

Consider the enumerative polynomial of inhomogeneous partitions

$$\mathcal{F}(\boldsymbol{n} \mid q) := \mathcal{F}(\boldsymbol{n}; a, b, c \mid q) = \sum_{\pi \in \mathcal{P}(\boldsymbol{n})} w(\pi). \tag{8.16}$$

Note that by the general theory of orthogonal polynomials, the three-term recurrence relation (8.5) and Proposition 4.1 in [22] imply that the linear functional $\mathcal{L}_1$ has the following combinatorial interpretation:

$$\mathcal{L}_1\big(x^n\big) = \sum_{\pi \in \Pi_n} w(\pi). \tag{8.17}$$

To find the partial difference equations satisfied by $\mathcal{F}(\boldsymbol{n} \mid q)$ we need the following key result.

Lemma 8.1. *The polynomials $\mathcal{F}(\boldsymbol{n} \mid q)$ are symmetric with respect to the permutation of indices $n_1, \ldots, n_m$.*

We postpone the proof of this crucial lemma to Section 11.

Lemma 8.2. *For $j \in [m]$, the polynomials $\mathcal{F}(\boldsymbol{n} \mid q)$ satisfy*

$$\mathcal{F}_j^+(\boldsymbol{n} \mid q) = \mathcal{F}\big(\boldsymbol{n}^* \,\big|\, q\big) - b[n_j]_q \mathcal{F}(\boldsymbol{n} \mid q) - a[n_j]_q \mathcal{F}_j^-(\boldsymbol{n} \mid q), \tag{8.18}$$

where $\boldsymbol{n}^* = (1, n_1, \ldots, n_m)$.

Proof. By Lemma 8.1, we can suppose that $j=1$. Hence, it suffices to check that

$$\sum_{\pi\in\mathcal{P}(\boldsymbol{n}^*)} w(\pi)=\mathcal{F}_1^+(\boldsymbol{n}\mid q)+b[n_1]_q\mathcal{F}(\boldsymbol{n}\mid q)+a[n_1]_q\mathcal{F}_1^-(\boldsymbol{n}\mid q), \tag{8.19}$$

where $w(\pi)=a^{\mathrm{bl}(\pi)}b^{\mathrm{tr}(\pi)}q^{\mathrm{cr}(\pi)}$ since $\mathrm{sg}(\pi)=0$ for any $\pi\in\mathcal{P}(\boldsymbol{n})$.

Given a partition $\pi\in\mathcal{P}(\boldsymbol{n}^*)$, we denote by r_1 the integer $i>1$ which is connected to 1 by an arc. We classify the partitions in $\mathcal{P}(\boldsymbol{n}^*)$ into three categories according to the value of r_1 (The reader is suggested to draw diagrams as we do in the proof of Lemma 7.1):

(a) $r_1>n_1+1$; such partitions are exactly the partitions in $\mathcal{P}_1^+(\boldsymbol{n})$, whence the enumerative polynomial of such partitions is $\mathcal{F}_1^+(\boldsymbol{n}\mid q)$.
(b) $2\leqslant r_1\leqslant n_1+1$; then the arc $(1,r_1)$ crosses with each of the r_1-2 arcs of which one vertex is ℓ with $2\leqslant\ell\leqslant r_1-1$. Suppose $\{1,r_1\}$ is a block of π (resp. is not a block of π). Summing over all $r_1=2,3,\ldots,n_1+1$, it is readily seen that the enumerative polynomial of such partitions is $\sum_{r_1=2}^{n_1+1}aq^{r_1-2}\mathcal{F}_1^-(\boldsymbol{n}\mid q)=a[n_1]_q\mathcal{F}_1^-(\boldsymbol{n}\mid q)$ (resp. $\sum_{r_1=2}^{n_1+1}bq^{r_1-2}\mathcal{F}_1(\boldsymbol{n}\mid q)=b[n_1]_q\mathcal{F}(\boldsymbol{n}\mid q)$).

Summing up the above three cases we obtain (8.19). □

The following result is due to Anshelevich [2] and a combinatorial proof was later given by Kim, Stanton and Zeng [26].

Theorem 8.3. *For $m\geqslant 1$ and $n_1,\ldots,n_m\geqslant 0$, we have*

$$\mathcal{F}(\boldsymbol{n}\mid q)=\mathcal{L}_1\big(C_{n_1}(x\mid q)\cdots C_{n_m}(x\mid q)\big). \tag{8.20}$$

Proof. For $j,k\in[m]$ and $j\neq k$ we deduce from (8.18) that

$$\mathcal{F}_j^+(\boldsymbol{n}|q)-\mathcal{F}_k^+(\boldsymbol{n}\mid q)=\big([n_k]_q-[n_j]_q\big)\mathcal{F}(\boldsymbol{n}\mid q)-a[n_j]_q\mathcal{F}_j^-(\boldsymbol{n}\mid q)+a[n_k]_q\mathcal{F}_k^-(\boldsymbol{n}\mid q), \tag{8.21}$$

and the boundary condition (2.11) with $\lambda_j=1$ for all j and $A_0C_1=A_1$. The result then follows by applying Theorem 2.3. □

Remark 8.4. When $q=0$, the polynomials $C_n(x\mid 0)$ are the so-called perturbed Chebyshev polynomials of the second kind and $\mathcal{F}(\boldsymbol{n}\mid 0)$ is the enumerative polynomial of inhomogeneous partitions of $[\boldsymbol{n}]$ without any arc crossings.

Remark 8.5. In view of Lemmas 8.1, 8.2 and Theorem 2.6, we can also prove the above theorem by checking (8.20) for the special $\boldsymbol{n}=1^m:=(1,\ldots,1)$. As $C_1(x;q)=x-c$, the latter identity reads

$$\mathcal{F}\big(1^m\,\big|\,q\big)=\mathcal{L}_1\big((x-c)^m\big)\quad(m\geqslant 1). \tag{8.22}$$

By the binomial formula, this is equivalent to

$$\mathcal{L}_1\big(x^m\big)=\sum_{k=0}^{m}\binom{m}{k}c^k\mathcal{F}\big(1^{m-k}\,\big|\,q\big). \tag{8.23}$$

In view of the combinatorial interpretation of the moments (8.17) and the definition (8.16) the latter identity is obvious if we enumerate the partitions π of $[m]$ by the weight (8.15) and according to the number of singletons.

8.3. Linearization coefficients of q-Laguerre polynomials

For $\sigma \in \mathfrak{S}_n$ the *number of crossings* of σ is defined by

$$\operatorname{cr}(\sigma) = \sum_{i=1}^{n} \#\{j \mid j < i \leqslant \sigma(j) < \sigma(i)\} + \sum_{i=1}^{n} \#\{j \mid j > i > \sigma(j) > \sigma(i)\}. \tag{8.24}$$

Note that the linear functional $\mathcal{L}_2$ has the following combinatorial interpretation [21]:

$$\mathcal{L}_2(x^n) = \sum_{\sigma \in \mathfrak{S}_n} y^{\operatorname{exc}(\sigma)} q^{\operatorname{cr}(\sigma)}. \tag{8.25}$$

Consider the enumerative polynomial of inhomogeneous derangements

$$I(\boldsymbol{n} \mid q) := I(\boldsymbol{n}; y \mid q) = \sum_{\sigma \in \mathcal{D}(\boldsymbol{n})} y^{\operatorname{exc}(\sigma)} q^{\operatorname{cr}(\sigma)}. \tag{8.26}$$

Lemma 8.6. *The polynomials $I(\boldsymbol{n}; y \mid q)$ satisfy*

$$\begin{aligned} &I_j^+(\boldsymbol{n} \mid q) - I_k^+(\boldsymbol{n} \mid q) \\ &\quad = (yq+1)\big([n_k]_q - [n_j]_q\big) I(\boldsymbol{n} \mid q) - y[n_j]_q^2 I_j^-(\boldsymbol{n} \mid q) + y[n_k]_q^2 I_k^-(\boldsymbol{n} \mid q), \end{aligned} \tag{8.27}$$

and the boundary condition (2.11) with $\lambda_j = 1$ for all j and $A_0 C_1 = A_1$.

Proof. It is proved in [21, Eq. (38)] that

$$I_j^+(\boldsymbol{n} \mid q) = I\big(\boldsymbol{n}^* \mid q\big) - (yq+1)[n_j]_q I(\boldsymbol{n} \mid q) - y[n_j]_q^2 I_j^-(\boldsymbol{n} \mid q), \tag{8.28}$$

where $\boldsymbol{n}^* = (1, n_1, \ldots, n_m)$. Replacing j by k in the above equation and then subtracting the resulting equation from the above one we get (8.27). The boundary condition is obvious. □

The following result is due to Kasraoui, Stanton and Zeng [21].

Theorem 8.7. *We have*

$$I(\boldsymbol{n} \mid q) = \mathcal{L}_2\big(L_{n_1}(x \mid q) \cdots L_{n_m}(x \mid q)\big). \tag{8.29}$$

Proof. Clearly (2.8) reduces to (8.27) when $\lambda_j = 1$ for all j, and

$$A_n = 1, \qquad B_n = -\big(y[n+1]_q + [n]_q\big), \qquad C_n = q[n]_q^2, \quad n \geqslant 0.$$

From Lemma 8.6 and Theorem 2.3 we deduce (8.29). □

Remark 8.8. In the above proof, we do not require the combinatorial interpretation of the moments (8.25), which was needed in [21].

9. More integrals of orthogonal polynomials

In this section, for a sequence of orthogonal polynomials $\{p_n(x)\}$, we shall consider integrals of type

$$\int_{\mathbb{R}} x^{n_0} \prod_{j=1}^{m} p_{n_j}(x)\, d\mu(x), \quad n_0 \in \mathbb{N}, \tag{9.1}$$

and

$$\int_{\mathbb{R}} x(x-1)\cdots(x-n_0+1)\prod_{j=1}^{m} p_{n_j}(x)\,d\mu(x), \quad n_0 \in \mathbb{N}, \tag{9.2}$$

where μ is an orthogonal measure for $\{p_n(x)\}$.

One important tool used in this work is MacMahon's Master theorem, [29, vol. 1, pp. 93–98] and its β-extension due to Foata–Zeilberger [12], which we now recall.

Let V_m be the determinant $\det(\delta_{ij} - x_i a_{i,j})$ $(1 \leqslant i, j \leqslant m)$. The MacMahon master theorem asserts that the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in the expansion of V_m^{-1} is equal to the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in

$$\prod_{k=1}^{m}(a_{k,1}x_1 + \cdots + a_{k,m}x_m)^{n_k}. \tag{9.3}$$

It will be convenient to restate this in a slightly different form. Let $\mathcal{C}(\boldsymbol{m})$ be the set of rearrangements of the word $1^{n_1}\ldots m^{n_m}$. For any rearrangement

$$\gamma = \gamma(1,1)\ldots\gamma(1,n_1)\ldots\gamma(m,1)\ldots\gamma(m,n_m) \in \mathcal{C}(\boldsymbol{m}),$$

we associate the weight

$$w(\gamma) = \prod_{i,j} a_{i,\gamma(i,j)} \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i).$$

Then, the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in (9.3) is equal to the sum of all the $w(\gamma)$ with γ running over all the elements in $\mathcal{C}(\boldsymbol{m})$. On the other hand, each sequence $\boldsymbol{n} = (n_1, \ldots, n_m)$ of positive integers defines a unique mapping χ from $[\boldsymbol{n}]$ to $[m]$ given by $\chi(j) = i$ if $j \in S_i$. For each permutation $\pi \in \mathfrak{S}(\boldsymbol{n})$ we let

$$w(\pi) = \prod_{j=1}^{n} a_{\chi(j),\chi(\pi(j))}.$$

Clearly, to each rearrangement γ in $\mathcal{C}(\boldsymbol{m})$, there corresponds exactly $n_1!\cdots n_m!$ permutations π in $\mathfrak{S}(\boldsymbol{n})$ with the property that $w(\pi) = w(\gamma)$. Therefore, the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in (9.3) is also equal to

$$\frac{1}{n_1!\cdots n_m!}\sum_{\pi\in\mathfrak{S}(\boldsymbol{n})} w(\pi).$$

The MacMahon Master theorem can now be restated as

$$\sum_{n_1,\ldots,n_m\geqslant 0} \frac{x_1^{n_1}\cdots x_m^{n_m}}{n_1!\cdots n_m!}\sum_{\pi\in\mathfrak{S}(\boldsymbol{n})} w(\pi) = V_m^{-1}.$$

The β-extension of the MacMahon Master theorem [12] reads as follows.

Theorem 9.1. *We have*

$$\sum_{n_1,\ldots,n_m\geqslant 0} \frac{x_1^{n_1}\cdots x_m^{n_m}}{n_1!\cdots n_m!}\sum_{\pi\in\mathfrak{S}(\boldsymbol{n})} \beta^{\mathrm{cyc}(\pi)} w(\pi) = V_m^{-\beta}. \tag{9.4}$$

Now, we consider the determinant

$$\Delta_{m+1} := \begin{vmatrix} 1 & -cx_1 & \cdots & -cx_1 & -cx_1 \\ -x_2 & 1 & \cdots & -cx_2 & -cx_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_m & -x_m & \cdots & 1 & -cx_m \\ -x_0 & -x_0 & \cdots & -x_0 & 1-x_0 \end{vmatrix}.$$

The proof of the following determinant formula is left to the reader.

Lemma 9.2. *Let a and b be any variables in a commutative ring. Then*

$$\begin{vmatrix} x_1 & a & \cdots & a & a \\ b & x_2 & \cdots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & x_{n-1} & a \\ b & b & \cdots & b & x_n \end{vmatrix} = \frac{a\phi_n(b) - b\phi_n(a)}{a-b},$$

where $\phi_n(x) = (x_1 - x)(x_2 - x)\cdots(x_n - x)$. When $a = b$, the right side should be taken as the limit $\phi_n(a)(1 + a\sum_{j=1}^{m} \frac{1}{x_j - a})$.

Applying the above lemma to Δ_{m+1} we obtain

$$\Delta_{m+1} = \frac{1}{c-1}\big[c(1+x_1)\cdots(1+x_m) - (1+cx_1)\cdots(1+cx_m)\big(1-(1-c)x_0\big)\big]. \tag{9.5}$$

Therefore, denoting the elementary symmetric functions of the indeterminates $x_1, \ldots, x_m$ by $e_1(\boldsymbol{x}), \ldots, e_m(\boldsymbol{x})$, we have

$$\Delta_{m+1} = 1 - \sum_{k=2}^{m}\big(c + \cdots + c^{k-1}\big)e_k(\boldsymbol{x}) - x_0\prod_{j=1}^{m}(1+cx_j). \tag{9.6}$$

Let

$$A^{(\alpha)}(n_0, \boldsymbol{n}) = (-1)^{\sum_{j=1}^{m} n_j}\int_0^{\infty} x^{n_0}\frac{x^{\alpha}e^{-x}}{\Gamma(\alpha+1)}\prod_{j=1}^{m} n_j! L_{n_j}^{(\alpha)}(x)\,dx. \tag{9.7}$$

A main result of this section is the following theorem.

Theorem 9.3. *The integrals $\{A^{(\alpha)}(n_0, \boldsymbol{n})\}$ have the generating function*

$$\sum_{n_0,\ldots,n_m \geqslant 0} A^{(\alpha)}(n_0, \boldsymbol{n})\frac{x_0^{n_0}}{n_0!}\cdots\frac{x_m^{n_m}}{n_m!} = \left[1 - x_0\prod_{j=1}^{m}(1+x_j) - e_2(\boldsymbol{x}) - 2e_3(\boldsymbol{x}) - \cdots - (m-1)e_m(\boldsymbol{x})\right]^{-\alpha-1}. \tag{9.8}$$

Moreover, we have the following combinatorial interpretation:

$$A^{(\alpha)}(n_0, \boldsymbol{n}) = \sum_{\pi \in \mathfrak{S}^*(\boldsymbol{n})} (\alpha+1)^{\mathrm{cyc}(\pi)}, \tag{9.9}$$

where $\mathfrak{S}^*(\boldsymbol{n})$ is the set of permutations of $S_0 \cup \cdots \cup S_m$ such that all the elements in box j should not stay in the original box after permutation for $1 \leqslant j \leqslant m$ and the objects in box 0 are not restricted.

Proof. We use (4.3) to see that

$$\sum_{n_0,\ldots,n_m \geqslant 0} A^{(\alpha)}(n_0, \boldsymbol{n})\prod_{j=0}^{m}\frac{x_j^{n_j}}{n_j!}$$

$$= \frac{1}{\Gamma(\alpha+1)} \prod_{j=1}^{m} (1+x_j)^{-\alpha-1} \int_0^\infty \exp\left(-x\left(1-x_0-\sum_{k=1}^{m} x_k/(1+x_k)\right)\right) x^\alpha \, dx$$

$$= \prod_{j=1}^{m} (1+x_j)^{-\alpha-1} \left[1-x_0-\sum_{k=1}^{m} \frac{x_k}{1+x_k}\right]^{-\alpha-1},$$

which reduces to the right-hand side of (9.8) after some simplification using the following identity, which was proved in [6], see also [4, (2.8)],

$$\prod_{j=1}^{m} (1+t_j)\left[1-\sum_{j=1}^{m} \frac{t_j}{1+t_j}\right] = 1-e_2(\boldsymbol{x})-2e_3(\boldsymbol{x})-\cdots-(m-1)e_m(\boldsymbol{x}). \tag{9.10}$$

This proves (9.8). The rest of Theorem 9.3 follows from the β-MacMahon Master theorem and (9.6). □

Remark 9.4. When $\alpha = 0$, $A^{(0)}(n_0, \boldsymbol{n})/n_0!n_1!\cdots n_m!$ can be simply interpreted as follows: we have boxes of sizes $n_0, n_1, \ldots, n_m$ and box j contains n_j indistinguishable elements and we arrange the contents such that no object in box j stays in its original box when $1 \leqslant j \leqslant m$ with no restriction on box number 0. The number of possible rearrangements is $A^{(0)}(n_0, \boldsymbol{n})/n_0!n_1!\cdots n_m!$.

Corollary 9.5. *We have*

$$A^{(0)}(m,n,s) = m!n!s! \sum_{j\geqslant 0} \binom{m}{j} \binom{s}{n+j-m} \binom{s+m-j}{m}. \tag{9.11}$$

Proof. By (9.3) we have the generating function

$$\sum_{m,n,s\geqslant 0} A^{(\alpha)}(m,n,s) \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_0^s}{s!} = \frac{1}{[1-x_0-x_1x_2-x_1x_0-x_2x_0-x_1x_2x_0]^{\alpha+1}}.$$

Since

$$V = \begin{vmatrix} 1 & -x_1 & -x_1 \\ -x_2 & 1 & -x_2 \\ -x_0 & -x_0 & 1-x_0 \end{vmatrix} = 1-x_0-x_1x_2-x_1x_0-x_2x_0-x_1x_2x_0$$

by the MacMahon Master theorem, Theorem 9.1, we see that $A^{(0)}(m,n,s)$ is given by the coefficient of $x_1^m x_2^n x_0^s$ in $(x_2+x_0)^m(x_1+x_0)^n(x_1+x_2+x_0)^s$, which is equal to the claimed expression. □

Motivated by the numbers $A^{(\alpha)}(n_0, \boldsymbol{n})$ we consider the following generalized linearization coefficients of Meixner polynomials:

$$B^{(\beta)}(n_0, \boldsymbol{n}) = (-1)^{\sum_{j=1}^m n_j} c^{-n_0} (1-c)^{\beta+n_0} \times \sum_{x=0}^{\infty} x(x-1)\cdots(x-n_0+1) \frac{c^x (\beta)_x}{x!} \prod_{j=1}^{m} M_{n_j}(x;\beta,c). \tag{9.12}$$

Theorem 9.6. *The integrals* $\{B^{(\beta)}(n_0, \ldots, n_m)\}$ *have the generating function*

$$\sum_{n_0,\ldots,n_m\geqslant 0} B^{(\beta)}(n_0, \boldsymbol{n}) \prod_{j=0}^{m} \frac{x_j^{n_j}}{n_j!} = \left[1-\sum_{k=2}^{m} \frac{1-c^{1-k}}{c(1-c^{-1})} e_k(\boldsymbol{x}) - x_0 \prod_{j=1}^{m} (1+x_j/c)\right]^{-\beta}. \tag{9.13}$$

Moreover, we have the following combinatorial interpretation:

$$B^{(\beta)}(n_0,\boldsymbol{n})=\sum_{\pi\in\mathfrak{S}^*(\boldsymbol{n})}\beta^{\mathrm{cyc}(\pi)}c^{-\mathrm{exc}(\pi)}, \tag{9.14}$$

where $\mathfrak{S}^*(\boldsymbol{n})$ *is the same as in Theorem* 9.3.

Proof. We use (5.3) to see that

$$\begin{aligned}&\sum_{n_0,\dots,n_m\geqslant 0}B^{(\beta)}(n_0,\boldsymbol{n})\frac{x_0^{n_0}}{n_0!}\cdots\frac{x_m^{n_m}}{n_m!}\\&=\sum_{x\geqslant 0}\bigl(1+(1-c)x_0/c\bigr)^x\frac{c^x(\beta)_x}{x!}(1-c)^\beta\prod_{j=1}^m(1+x_j/c)^x(1+x_j)^{-x-\beta}\\&=\left[\frac{\prod_{j=1}^m(1+x_j)-(c+(1-c)x_0)\prod_{j=1}^m(1+x_j/c)}{1-c}\right]^{-\beta}.\end{aligned}$$

This gives (9.13) after simplification.

Comparing with (9.6) we see that the $\beta=1$ case of (9.14) comes from the MacMahon Master theorem associated with the matrix (a_{ij}) with $a_{ii}=0$, $a_{ij}=1/c$ for $j>i$ and $a_{ij}=1$ for $j<i$. The general case follows from using the β-extension of MacMahon's Master theorem. □

Remark 9.7. For the Charlier polynomials we have a similar result for the integral

$$C^{(a)}(n_0,\boldsymbol{n}):=\sum_{x\geqslant 0}x(x-1)\cdots(x-n_0+1)\frac{e^{-a}a^x}{x!}\prod_{j=1}^m C_{n_j}^{(a)}(x). \tag{9.15}$$

A straight computation shows that

$$\begin{aligned}&\sum_{n_0,\dots,n_m\geqslant 0}C^{(a)}(n_0,\boldsymbol{n})\frac{x_0^{n_0}}{n_0!}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_m^{n_m}}{n_m!}\\&\quad=\exp\bigl(a\bigl[x_0+x_0e_1(\boldsymbol{x})+(x_0+1)e_2(\boldsymbol{x})+\cdots+(x_0+1)e_m(\boldsymbol{x})\bigr]\bigr).\end{aligned} \tag{9.16}$$

We apply the exponential formula to see that

$$C^{(a)}(n_0,\boldsymbol{n})=\sum_{\pi\in\mathcal{P}^*(n_0,\boldsymbol{n})}a^{\mathrm{bl}(\pi)}, \tag{9.17}$$

where $\mathcal{P}^*(n_0,\boldsymbol{n})$ is the set of partitions of $S_0\cup S_1\cup\cdots\cup S_m$ such that each block is either a singleton of an element in S_0 or inhomogeneous, i.e., no two elements of S_j $(0\leqslant j\leqslant m)$ can be in the block.

It is clear that Theorem 9.3 is the limit $c\to 1^-$ of Theorem 9.6. Similarly we have the following analogue of Corollary 9.5.

Corollary 9.8. *We have*

$$B^{(1)}(m,n,s)=m!n!s!\sum_{j\geqslant 0}\binom{m}{j}\binom{s}{n+j-m}\binom{s+m-j}{m}c^{n-2m+j}.$$

Corollary 9.9. *We have*

$$x^n=\frac{c^n}{(1-c)^n}\sum_{k=0}^n\binom{n}{k}(\beta+k)_{n-k}(-1)^kM_k(x;\beta,c). \tag{9.18}$$

Proof. Let $x^n = \sum_{k=0}^{n} c(n,k) M_k(x;\beta,c)$. Using the orthogonality (5.2) we obtain

$$(1-c)^{\beta} \sum_{x\geqslant 0} x^n M_k(x;\beta,c) \frac{(\beta)_k}{x!} c^x = c(n,k) \frac{(\beta)_k k!}{c^k}.$$

Comparing with (9.12) we see that the left side is equal to $(-1)^k c^n (1-c)^{-n} B^{(\beta)}(k,n)$. It remains to compute $B^{(\beta)}(k,n)$, which, by Theorem 9.6, is the coefficient of $\frac{x_0^n x_1^k}{n!k!}$ in the expansion

$$\bigl[1 - x_0(1 + x_1/c)\bigr]^{-\beta} = \sum_{n\geqslant 0} \frac{(\beta)_n}{n!} x_0^n (1 + x_1/c)^n = \sum_{n,k\geqslant 0} \frac{n!(\beta)_n}{(n-k)!} c^{-k} \frac{x_0^n x_1^k}{n!k!}.$$

Hence $B^{(\beta)}(k,n) = \frac{n!(\beta)_n}{(n-k)!} c^{-k}$. This yields the desired result. □

Let φ be the linear functional defined by $\varphi(f(x)) = \int_{\mathbb{R}} f(x)\, d\mu(x)$. Then the integral (9.1) contains the following four special cases:

(1) the evaluation of $\varphi(x^n)$ corresponds to the moments,
(2) the evaluation of $\varphi(\prod_{j=1}^{2} p_{n_j}(x))$ corresponds to the orthogonality,
(3) the evaluation of $\varphi(x^n p_k(x))$ combined with the orthogonality corresponds to the coefficient $c_{n,k}$ in the expansion $x^n = \sum_{k=0}^{n} c_{n,k} p_k(x)$,
(4) the evaluation of $\varphi(\prod_{j=1}^{m} p_{n_j}(x))$ corresponds to the linearization coefficients.

Since $A_0 x = p_1(x) - B_0$, we have

$$(A_0 x)^{n_0} = \sum_{l=0}^{n_0} \binom{N}{l} (-B_0)^{N-l} p_1(x)^l, \quad n_0 \in \mathbb{N}.$$

Therefore,

$$\varphi\left((A_0 x)^{n_0} \prod_{j=1}^{m} p_{n_j}(x) \right) = \sum_{l=0}^{n_0} \binom{n_0}{l} (-B_0)^{n_0-l} \varphi\left(p_1(x)^l \prod_{j=1}^{m} p_{n_j}(x) \right). \tag{9.19}$$

We can deduce the combinatorial interpretations of the integrals (9.19) for the orthogonal Sheffer polynomials and the three q-analogues from the combinatorial interpretation of the corresponding linearization coefficients.

For example, as $H_1(x) = 2x$, it follows from Theorem 3.2 that

$$2^{-(n_0+n_1+\cdots+n_m)/2} \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} (2x)^{n_0} \prod_{j=1}^{m} H_{n_j}(x)\, dx \tag{9.20}$$

is the number of perfect inhomogeneous matchings in $\mathcal{K}(\boldsymbol{n})$ with

$$\boldsymbol{n} = (\underbrace{1,\ldots,1}_{n_0}, n_1, n_2, \ldots, n_m).$$

For the Laguerre polynomials we have $x = -L_1^{(\alpha)}(x) + \alpha + 1$, so

$$\begin{aligned} &\int_0^{\infty} \frac{x^{\alpha} e^{-x}}{\Gamma(\alpha+1)} x^{n_0} \prod_{j=1}^{m} (-1)^{n_j} n_j! L_{n_j}^{(\alpha)}(x)\, dx \\ &\quad = \sum_{l=0}^{n_0} \binom{n_0}{l} (\alpha+1)^{n_0-l} \int_0^{\infty} \frac{x^{\alpha} e^{-x}}{\Gamma(\alpha+1)} \bigl(-L_1^{(\alpha)}(x)\bigr)^l \prod_{j=1}^{m} (-1)^{n_j} n_j! L_{n_j}^{(\alpha)}(x)\, dx. \end{aligned} \tag{9.21}$$

We can easily recover the combinatorial interpretation (9.9) in Theorem 9.3 from the above equation and (4.13).

For the Meixner polynomials we have $\frac{1-c}{c}x = \beta - M_1(x;\beta,c)$, so

$$\begin{aligned}\tilde{B}^{(\beta)}(n_0,\boldsymbol{n}) &= c^{-n_0}(1-c)^{\beta+n_0}\sum_{x=0}^{\infty}x^{n_0}\frac{c^x(\beta)_x}{x!}\prod_{j=1}^{m}(-1)^{n_j}M_{n_j}(x;\beta,c)\\ &= \sum_{l=0}^{n_0}\binom{n_0}{l}\beta^{n_0-l}(1-c)^{\beta}\sum_{x=0}^{\infty}\frac{c^x(\beta)_x}{x!}\bigl(-M_1(x;\beta,c)\bigr)^l\\ &\quad\times\prod_{j=1}^{m}(-1)^{n_j}M_{n_j}(x;\beta,c). \end{aligned}\tag{9.22}$$

Using Theorem 5.2, we see the following combinatorial interpretation

$$\tilde{B}^{(\beta)}(n_0,\boldsymbol{n}) = \sum_{\pi\in\mathfrak{S}^*(\boldsymbol{n})}\beta^{\operatorname{cyc}(\pi)}c^{-\operatorname{exc}(\pi)-\operatorname{exc}_0(\pi)},\tag{9.23}$$

where $\mathfrak{S}^*(\boldsymbol{n})$ is the same as in Theorem 9.3 and $\operatorname{exc}_0(\pi)$ is the number of excedances of two elements in S_0, i.e., $\operatorname{exc}_0(\pi) = |\{i\in S_0\colon\ \pi(i)\in S_0 \text{ and } \pi(i)>i\}|$.

10. Laguerre and Meixner polynomials revisited

Recall [17, p. 100] that the Hermite polynomials can be viewed as special Laguerre polynomials since

$$H_{2n+1/2\pm1/2}(x) = (-1)^n 2^{2n} n!(2x)^{1/2\pm1/2}L_n^{(\pm1/2)}\bigl(x^2\bigr).$$

Therefore the integral in (3.3) is a special case of the integral

$$\begin{aligned}&W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})\\ &:= \frac{(-1)^{\sum_{i=1}^{j}m_i+\sum_{r=1}^{k}n_r}}{\Gamma(\alpha+1)}\int_0^{\infty}x^{m+\alpha}e^{-x}\left[\prod_{i=1}^{j}m_i!L_{m_i}^{(\alpha)}(x)\right]\left[\prod_{r=1}^{k}n_r!L_{n_r}^{(\beta)}(x)\right]dx,\end{aligned}\tag{10.1}$$

where $\boldsymbol{m} = (m_1,m_2,\dots,m_j)$ and $\boldsymbol{n} = (n_1,n_2,\dots,n_k)$.

In this section we study the combinatorics of the integrals of the type in (10.1) and their discrete analogues which result by replacing the Laguerre polynomials by Meixner polynomials.

Theorem 10.1. *Let $e_i(\boldsymbol{x})$, $i = 0,1,\dots,j+k$, be the ith elementary symmetric polynomial of $x_1,\dots,x_{j+k}$. The integrals $\{W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})\}$ have the generating function*

$$\begin{aligned}&\sum_{m,m_i,n_r\geqslant 0}W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})\frac{x_0^m}{m!}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_j^{m_j}}{m_j!}\frac{x_{j+1}^{n_1}}{n_1!}\cdots\frac{x_{j+k}^{n_k}}{n_k!}\\ &= \prod_{r=1}^{k}(1+x_{j+r})^{\alpha-\beta}\left[1-x_0\prod_{i=1}^{j+k}(1+x_i)-\sum_{l=2}^{j+k}(l-1)e_l(\boldsymbol{x})\right]^{-\alpha-1}.\end{aligned}\tag{10.2}$$

Proof. Apply the generating function (4.3) to see that the left-hand side of (10.2) is given by

$$\prod_{i=1}^{j}(1+x_i)^{-\alpha-1}\prod_{r=1}^{k}(1+x_{j+r})^{-\beta-1}\int_0^\infty \frac{x^\alpha}{\Gamma(\alpha+1)}\exp\left(-x+xx_0+\sum_{l=1}^{j+k}\frac{xx_l}{1+x_l}\right)dx$$
$$=\prod_{i=1}^{j}(1+x_i)^{-\alpha-1}\prod_{r=1}^{k}(1+x_{j+r})^{-\beta-1}\left[1-x_0-\sum_{l=1}^{j+k}\frac{x_l}{1+x_l}\right]^{-\alpha-1}. \tag{10.3}$$

This establishes (10.2) after some simplification using (9.10). □

Corollary 10.2. *The numbers* $\{W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})\}$ *are positive when* $\alpha>-1$ *and* $\alpha-\beta$ *is a nonnegative integer.*

Assuming that $\alpha-\beta$ is a positive integer N, we can give a combinatorial interpretation for $W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})$. Let $\mathfrak{S}_N^*(\boldsymbol{n})$ be the set of $(k+1)$-tuples $(\pi,f_1,\dots,f_k)$ such that

- σ is an inhomogeneous permutation of $S_0^*\cup S_1\cup\cdots\cup S_j\cup S_{j+1}^*\cup\cdots\cup S_{j+k}^*$, where $S_0^*\subseteq S_0$ and $S_{j+r}^*\subseteq S_{j+r}$ for $r=1,\dots,k$.
- $f_r: S_{j+r}\setminus S_{j+r}^*\to[N]$ is an injection for $r=1,\dots,k$.

From Theorems 9.3 and 10.1 we deduce the following combinatorial interpretation:

$$W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})=\sum_{(\pi,f_1,\dots,f_k)\in\mathfrak{S}_N^*(\boldsymbol{n})}(\alpha+1)^{\mathrm{cyc}(\pi)}. \tag{10.4}$$

Motivated by the numbers $W_{j,k}(m;\alpha,\beta;\boldsymbol{m},\boldsymbol{n})$ we consider the following generalized linearization coefficients of Meixner polynomials:

$$\begin{aligned}Y_{j,k}(m;\alpha,\beta;c;\boldsymbol{m},\boldsymbol{n}):=(-1)^{\sum_{i=1}^{j}m_i+\sum_{r=1}^{k}n_r}c^{-m}(1-c)^{\alpha+m}&\\ \times\sum_{x=0}^{\infty}x(x-1)\cdots(x-m+1)\frac{c^x(\alpha)_x}{x!}&\\ \times\left[\prod_{i=1}^{j}M_{m_i}(x;\alpha,c)\right]\left[\prod_{r=1}^{k}M_{n_r}(x;\beta,c)\right],&\end{aligned} \tag{10.5}$$

where $\boldsymbol{m}=(m_1,m_2,\dots,m_j)$ and $\boldsymbol{n}=(n_1,n_2,\dots,n_k)$.

Theorem 10.3. *The integrals* $Y_{j,k}(m;\alpha,\beta;c;\boldsymbol{m},\boldsymbol{n})$ *have the generating function*

$$\sum_{m,m_i,n_r\geqslant 0}Y_{j,k}(m;\alpha,\beta;c;\boldsymbol{m},\boldsymbol{n})\frac{x_0^m}{m!}\prod_{i=1}^{j}\frac{x_i^{m_i}}{m_i!}\prod_{r=1}^{k}\frac{x_{j+r}^{n_r}}{n_r!}$$
$$=\prod_{r=1}^{k}(1+x_{j+r})^{\alpha-\beta}\left[1-\sum_{l=2}^{j+k}\frac{1-c^{1-l}}{c(1-c^{-1})}e_l(\boldsymbol{x})-x_0\prod_{i=1}^{j+k}(1+x_i/c)\right]^{-\alpha}. \tag{10.6}$$

Proof. Applying (5.3) to see that the left-hand side of (10.6) is

$$\sum_{x\geqslant 0}\bigl(1+(1-c)x_0/c\bigr)^x\frac{c^x(\alpha)_x}{x!}(1-c)^\alpha$$

$$\times \prod_{i=1}^{j}(1+x_i/c)^{x}(1+x_i)^{-x-\alpha}\prod_{r=1}^{k}(1+x_{j+r}/c)^{x}(1+x_{j+r})^{-x-\beta}$$
$$= (1-c)^{\alpha}\prod_{i=1}^{j}(1+x_i)^{-\alpha}\prod_{r=1}^{k}(1+x_{j+r})^{-\beta}\left[1-\big(c+(1-c)x_0\big)\prod_{i=1}^{j+k}\frac{1+x_i/c}{1+x_i}\right]^{-\alpha}.$$

This establishes (10.6) after some simplification using (9.10). □

In the same vein, assuming that $\alpha-\beta$ is a positive integer N, Theorems 9.6 and 10.3 imply the following combinatorial interpretation:

$$Y_{j,k}(m;\alpha,\beta;c;\boldsymbol{m},\boldsymbol{n}) = \sum_{(\pi,f_1,\dots,f_k)\in\mathfrak{S}_N^*(\boldsymbol{n})}\alpha^{\mathrm{cyc}(\pi)}c^{-\mathrm{exc}(\pi)}. \tag{10.7}$$

Note that Theorem 10.3 shows that the numbers $Y_{j,k}(m;\alpha,\beta;c;\boldsymbol{m},\boldsymbol{n})$ are positive when $\alpha-\beta$ is a nonnegative integer.

11. Proof of Lemma 8.1: Symmetry of $\mathcal{F}(\boldsymbol{n}\mid q)$

Recall that $\boldsymbol{n}=(n_1,\dots,n_m)$ is a sequence of positive integers and $n=n_1+\cdots+n_m$. Clearly we need only to prove the invariance of $\mathcal{F}(\boldsymbol{n}\mid q)$ for the two following permutations of the indices n_j's: the transposition exchanging 1 and 2, and the cyclic permutation mapping i to $i+1 \pmod m$ for $i=1,\dots,m$. Moreover, since $\mathrm{sg}(\pi)=0$ and $\mathrm{tr}(\pi)=n-2\,\mathrm{bl}(\pi)$ for any partition $\pi\in\mathcal{P}(\boldsymbol{n})$, we see that Lemma 8.1 is equivalent to the following result.

Lemma 11.1. *We have*

$$\sum_{\pi\in\mathcal{P}(\boldsymbol{n})}a^{\mathrm{bl}(\pi)}q^{\mathrm{cr}(\pi)} = \sum_{\pi\in\mathcal{P}(n_2,n_3,\dots,n_m,n_1)}a^{\mathrm{bl}(\pi)}q^{\mathrm{cr}(\pi)}, \tag{11.1}$$

$$\sum_{\pi\in\mathcal{P}(\boldsymbol{n})}a^{\mathrm{bl}(\pi)}q^{\mathrm{cr}(\pi)} = \sum_{\pi\in\mathcal{P}(n_2,n_1,n_3\dots,n_m)}a^{\mathrm{bl}(\pi)}q^{\mathrm{cr}(\pi)}. \tag{11.2}$$

For a positive integer k such that $k<n$, we introduce two sets of inhomogeneous partitions:

$${}^{(k)}\mathcal{P}_n := \mathcal{P}(k,\underbrace{1,\dots,1}_{n-k}), \qquad \mathcal{P}_n^{(k)} := \mathcal{P}(\underbrace{1,\dots,1}_{n-k},k).$$

In other words, a partition π of $[n]$ is in ${}^{(k)}\mathcal{P}_n$ (resp. $\mathcal{P}_n^{(k)}$) if and only if it has no singleton and there is no arc in π joining two elements in $[1,k]$ (resp. $[n-k+1,n]$). For instance, the two partitions π_1 and π_2 drawn at the top of Fig. 4 are in ${}^{(4)}\mathcal{P}_{13}$ and $\mathcal{P}_{13}^{(4)}$. We first show that the following result implies (11.1).

Proposition 11.2. *For any positive integer k, there is a bijection $\Phi_{n,k}:{}^{(k)}\mathcal{P}_n\mapsto\mathcal{P}_n^{(k)}$ such that for any $\pi\in{}^{(k)}\mathcal{P}_n$, we have*

(I) *for $k<i<j$, the pair (i,j) is an arc of π if and only if the pair $(i-k,j-k)$ is an arc of $\Phi_{n,k}(\pi)$;*
(II) $\mathrm{bl}(\Phi_{n,k}(\pi))=\mathrm{bl}(\pi)$ *and* $\mathrm{cr}(\Phi_{n,k}(\pi))=\mathrm{cr}(\pi)$.

Indeed, assuming the existence of such a bijection $\Phi_{n,k}$ with $k=n_1$, as $\mathcal{P}(\boldsymbol{n})\subseteq{}^{(n_1)}\mathcal{P}_n$, the property (I) implies that $\Phi_{n,n_1}(\mathcal{P}(\boldsymbol{n}))\subseteq\mathcal{P}(n_2,n_3,\dots,n_m,n_1)$. Since the cardinality of $\mathcal{P}(\boldsymbol{n})$ is invariant by permutations of the n_i's and Φ_{n,n_1} is bijective, we deduce that

$$\Phi_{n,n_1}\big(\mathcal{P}(\boldsymbol{n})\big)=\mathcal{P}(n_2,n_3,\dots,n_m,n_1),$$

and then (11.1) by applying the property (II).

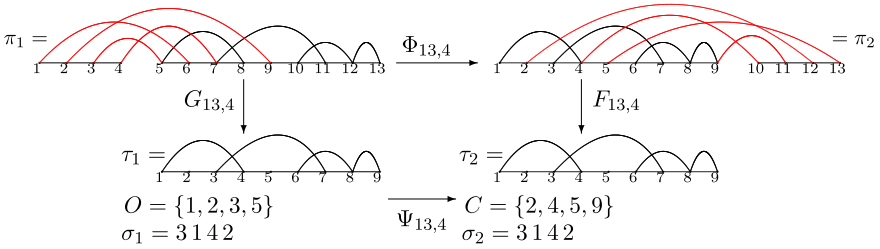


Fig. 4. The mappings $\Phi_{n,k}$, $F_{n,k}$, $G_{n,k}$ and $\Psi_{n,k}$.

We now turn our attention to (11.2). Define the set of inhomogeneous partitions

$$\mathcal{P}_n^{(n_1,n_2)} := \mathcal{P}(n_1, n_2, \underbrace{1, \ldots, 1}_{n-n_1-n_2}).$$

In other words, a partition π of $[n]$ is in $\mathcal{P}_n^{(n_1,n_2)}$ if and only if it has no singleton and there is no arc connecting two integers in $[1, n_1]$ or in $[n_1 + 1, n_1 + n_2]$. For instance, the partitions π_1 and π_2 drawn in Fig. 6 are, respectively, in $\mathcal{P}_{14}^{(3,4)}$ and $\mathcal{P}_{14}^{(4,3)}$. Similarly, we deduce (11.2) from the following result.

Proposition 11.3. *There is a bijection $\Theta_n^{(n_1,n_2)} : \mathcal{P}_n^{(n_1,n_2)} \to \mathcal{P}_n^{(n_2,n_1)}$ such that for any $\pi \in \mathcal{P}_n^{(n_1,n_2)}$, we have*

(I) *for $N_2 < i < j$, the pair (i, j) is an arc of π if and only if the pair (i, j) is an arc of $\Theta_n^{(n_1,n_2)}(\pi)$, where $N_2 := n_1 + n_2$;*
(II) $\mathrm{bl}(\Theta_n^{(n_1,n_2)}(\pi)) = \mathrm{bl}(\pi)$ *and* $\mathrm{cr}(\Theta_n^{(n_1,n_2)}(\pi)) = \mathrm{cr}(\pi)$.

Indeed, since $\mathcal{P}(\boldsymbol{n}) \subseteq \mathcal{P}_n^{(n_1,n_2)}$, the property (I) of $\Theta_n^{(n_1,n_2)}$ implies that $\Theta_n^{(n_1,n_2)}(\mathcal{P}(\boldsymbol{n})) \subseteq \mathcal{P}(n_2, n_1, n_3, \ldots, n_m)$. This, combined with the fact that the cardinality of $\mathcal{P}(\boldsymbol{n})$ is invariant by permutations of the n_i's and $\Theta_n^{(n_1,n_2)}$ is bijective, implies that

$$\Theta_n^{(n_1,n_2)}\big(\mathcal{P}(\boldsymbol{n})\big) = \mathcal{P}(n_2, n_1, n_3, \ldots, n_m).$$

Eq. (11.2) then follows by applying the property (II) of $\Theta_n^{(n_1,n_2)}$.

The next two subsections are dedicated to the proof of Propositions 11.2 and 11.3.

11.1. Construction of the bijection $\Phi_{n,k}$

Given a partition $\pi \in \Pi_n$, an element $i \in [n]$ is said to be *minimal* (resp. *maximal*) if i is the least (resp. largest) element of a block of π. The set of the minimal (resp. maximal) elements in π will be denoted $\min(\pi)$ (resp. $\max(\pi)$). For example, for $\pi = 1\,4\,6/2/3\,5$, $\min(\pi) = \{1, 2, 3\}$ and $\max(\pi) = \{2, 5, 6\}$. Note that $\min(\pi) \cap \max(\pi) = \mathrm{sing}(\pi)$ where $\mathrm{sing}(\pi)$ is for the set of singletons of π. Let S be a subset of X. The *restriction* of a partition $\pi = \{B_1, B_2, \ldots, B_k\}$ of X on S is the partition $\{B_1 \cap S, B_2 \cap S, \ldots, B_k \cap S\}$ of S.

The key idea for the definition of the mapping $\Phi_{n,k}$ is some appropriate decomposition of partitions in ${}^{(k)}\mathcal{P}_n$ and $\mathcal{P}_n^{(k)}$. Let ${}^{(k)}A_n$ (resp. $A_n^{(k)}$) be the set of 3-tuples (τ, R, σ) where

- $\tau \in \Pi_{n-k}$ and $\sigma \in \mathfrak{S}_k$,
- $\mathrm{sing}(\tau) \subseteq R \subseteq \min(\tau)$ (resp. $\mathrm{sing}(\tau) \subseteq R \subseteq \max(\tau)$) and $|R| = k$.

For instance, in Fig. 4, we have $(\tau_1, O, \sigma_1) \in {}^{(4)}A_{13}$ and $(\tau_2, C, \sigma_2) \in A_{13}^{(4)}$.

We first define two simpler mappings $F_{n,k} : \mathcal{P}_n^{(k)} \to A_n^{(k)}$ and $G_{n,k} : {}^{(k)}\mathcal{P}_n \to {}^{(k)}A_n$.

- For $\pi \in \mathcal{P}_n^{(k)}$, set $F_{n,k}(\pi) = (\tau, C, \sigma)$, where
  - τ is the restriction of π on $[n-k]$;
  - C is the set of elements in π which are connected to an element $> n-k$ by an arc;
  - By definition of $\mathcal{P}_n^{(k)}$, we have $|C| = k$. Suppose $C = \{c_1 < c_2 < \cdots < c_k\}$, then σ is the unique permutation in $\mathfrak{S}_k$ such that $(c_1, n-k+\sigma(1))$, $(c_2, n-k+\sigma(2))$, …, $(c_k, n-k+\sigma(k))$ are arcs of π.
- For $\pi \in {}^{(k)}\mathcal{P}_n$, set $G_{n,k}(\pi) = (\tau, O, \sigma)$, where
  - $\tau \in \Pi_{n-k}$ is the partition obtained by subtracting k from each element in the restriction of π on $[k+1, n]$;
  - Let M be the set of elements in π which are connected to an element $j \leqslant k$ by an arc. By definition of $\mathcal{P}_n^{(k)}$, we have $|M| = k$. Suppose $M = \{m_1 < m_2 < \cdots < m_k\}$, then O is obtained by subtracting k from each element of M, i.e., $O = \{m_1 - k, m_2 - k, \ldots, m_k - k\}$;
  - σ is the unique permutation in $\mathfrak{S}_k$ such that $(\sigma(1), m_1)$, $(\sigma(2), m_2)$, …, $(\sigma(k), m_k)$ are arcs of π.

The mappings $F_{n,k}$ and $G_{n,k}$ are illustrated in Fig. 4.

Definition 11.4. Let π be a partition of a set S consisting of positive integers. The *depth* of an element i in π, denoted $\mathrm{dp}_i(\pi)$, is the number of arcs (a, b) in π satisfying $a < i < b$.

Definition 11.5. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation of $[n]$. A pair (i, j), $1 \leqslant i < j \leqslant n$, is said to be a *non-inversion* in σ if $\sigma(i) < \sigma(j)$. The number of non-inversions in σ will be denoted $\mathrm{ninv}(\sigma)$.

Some useful properties of $F_{n,k}$ and $G_{n,k}$ are summarized in the following result.

Proposition 11.6. *The mappings $F_{n,k} : \mathcal{P}_n^{(k)} \to A_n^{(k)}$ and $G_{n,k} : {}^{(k)}\mathcal{P}_n \to {}^{(k)}A_n$ are bijections. Moreover, for any $\pi \in \mathcal{P}_n^{(k)}$, if $F_{n,k}(\pi) = (\tau, C, \sigma)$, then*

$$\mathrm{bl}(\pi) = \mathrm{bl}(\tau) \quad \textit{and} \quad \mathrm{cr}(\pi) = \mathrm{cr}(\tau) + \mathrm{ninv}(\sigma) + \sum_{i \in C} \mathrm{dp}_i(\tau), \tag{11.3}$$

and, for any $\pi \in {}^{(k)}\mathcal{P}_n$, if $G_{n,k}(\pi) = (\tau, O, \sigma)$, then

$$\mathrm{bl}(\pi) = \mathrm{bl}(\tau) \quad \textit{and} \quad \mathrm{cr}(\pi) = \mathrm{cr}(\tau) + \mathrm{ninv}(\sigma) + \sum_{i \in O} \mathrm{dp}_i(\tau). \tag{11.4}$$

Proof. It is easy to see that $F_{n,k}$ (resp. $G_{n,k}$) is a bijection by constructing its inverse (use Fig. 4).

Let S be a finite subset of positive integers. Clearly, if π is a partition of S, then each block B of π is represented by $|B| - 1$ arcs. This easily leads to the following result.

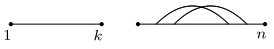
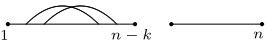
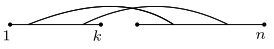
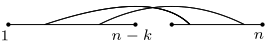
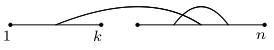
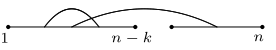
Fact 11.7. *The number of blocks of a partition π of S is equal to $|S| -$ (number of arcs in π).*

The first equation in (11.3) and (11.4) is just a consequence of the above fact. We now turn our attention to the second equation in (11.3) and (11.4). Let $\pi \in \mathcal{P}_n^{(k)}$. Clearly, the arc crossings in the partition π can be divided into three classes $R_1(\pi)$, $R_2(\pi)$ and $R_3(\pi)$ illustrated in Table 1.

They are defined formally as follows:

$$R_1(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 < j_1 < j_2 \leqslant n-k\big\},$$
$$R_2(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 \leqslant n-k < j_1 < j_2\big\},$$
$$R_3(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 < j_1 \leqslant n-k < j_2\big\};$$

Table 1
Sketch of crossings in $L_i(\pi)$ and $R_i(\pi)$.

i	$L_i(\pi)$	$R_i(\pi)$
1		
2		
3		

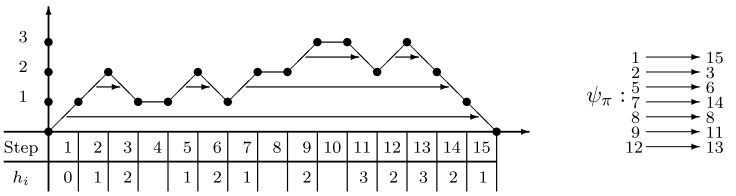


Fig. 5. The Motzkin path associated to the partition $\pi = 1\,4\,15/2\,3/5\,6/7\,10\,13/8/9\,11/12\,14$ and the mapping ψ_π.

and satisfy $\mathrm{cr}(\pi) = |R_1(\pi)| + |R_2(\pi)| + |R_3(\pi)|$. Suppose $F_{n,k}(\pi) = (\tau, C, \sigma)$. Then it is easily checked that $|R_1(\pi)| = \mathrm{cr}(\tau)$, $|R_2(\pi)| = \mathrm{ninv}(\sigma)$ and $|R_3(\pi)| = \sum_{i\in C} \mathrm{dp}_i(\tau)$ (see Fig. 4). This proves the second equation in (11.3).

Similarly, let $\pi \in \mathcal{P}_n^{(k)}$. The arc crossings of the partition π can be divided into three parts $L_1(\pi)$, $L_2(\pi)$ and $L_3(\pi)$ illustrated in Table 1, defined formally as follows:

$$L_1(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; k < i_1 < i_2 < j_1 < j_2 \leqslant n\big\},$$
$$L_2(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 \leqslant k < j_1 < j_2 \leqslant n\big\},$$
$$L_3(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 \leqslant k < i_2 < j_1 < j_2 \leqslant n\big\},$$

and such that $\mathrm{cr}(\pi) = |L_1(\pi)| + |L_2(\pi)| + |L_3(\pi)|$. Suppose $G_{n,k}(\pi) = (\tau, O, \sigma)$. Then it is easily checked that $|L_1(\pi)| = \mathrm{cr}(\tau)$, $|L_2(\pi)| = \mathrm{ninv}(\sigma)$ and $|L_3(\pi)| = \sum_{i\in O} \mathrm{dp}_i(\tau)$ (see Fig. 4). This proves the second equation in (11.4). □

In view of Proposition 11.6, to prove Proposition 11.2, it suffices to prove the following result.

Proposition 11.8. *For any partition π, there is a bijection $\psi_\pi : \min(\pi) \mapsto \max(\pi)$ such that $\mathrm{dp}_i(\pi) = \mathrm{dp}_{\psi(i)}(\pi)$ for each $i \in \min(\pi)$ and $\psi_\pi(j) = j$ for $j \in \mathrm{sing}(\pi)$.*

Proof. It is worth noting that such a bijection was already described in the literature (e.g., see Remark 7.2 in [23]). For reader's convenience we recall the construction of ψ_π. The mapping ψ_π can be nicely illustrated using *Motzkin paths*. Recall that a Motzkin path of length n is a lattice path in the plane of integer lattice $\mathbb{Z}^2$ from $(0, 0)$ to $(n, 0)$, consisting of NE-steps $(1, 1)$, E-steps $(1, 0)$ and SE-steps $(1, -1)$, which never passes below the x-axis. The usual way to associate a set partition to a Motzkin path works as follows: to a partition π of $[n]$ we associate the Motzkin path M of length n whose i-th step is NE if $i \in \min(\pi) \setminus \mathrm{sing}(\pi)$, SE if $i \in \max(\pi) \setminus \mathrm{sing}(\pi)$ and E otherwise. An illustration of this correspondence is given in Fig. 5.

A basic property of the above correspondence is the following fact [22].

Fact 11.9. *Suppose M is the Motzkin path associated to a partition π and let h_i be the height of the i-th step of M, i.e., the ordinate of its originate point. Then, $\mathrm{dp}_i(\pi) = h_i$ if the i-th step of M is NE and $\mathrm{dp}_i(\pi) = h_i - 1$ if the i-th step of M is SE.*

We can now describe the mapping ψ_π. We first set $\psi_\pi(j) = j$ for $j \in \mathrm{sing}(\pi)$. Suppose $\mathcal{O}(\pi) := \min(\pi) \setminus \mathrm{sing}(\pi) = \{o_1 < o_2 < \cdots < o_r\}$, $\mathcal{C}(\pi) := \max(\pi) \setminus \mathrm{sing}(\pi) = \{c_1 < c_2 < \cdots < c_r\}$ and let M be the Motzkin path associated to π. Note that the NE (resp. SE) steps in M are exactly the steps indexed by $\mathcal{O}(\pi)$ (resp. $\mathcal{C}(\pi)$). We then pair the NE-steps with SE-steps in M two by two in the following way. Suppose the i-th NE-step (i.e., the o_i-th step) of M is at height h. Then, if the first SE-step to its right at height $h+1$ is the j-th SE step (i.e., the c_j-th step) in M, then we set $\psi_\pi(o_i) = c_j$. An illustration is given in Fig. 5. From the construction of ψ_π and Fact 11.9 it is easy to see that ψ_π is the desired bijection. □

For $(\tau, O, \sigma) \in {}^{(k)}A_n$, we set $\Psi_{n,k}(\tau, O, \sigma) := (\tau, \psi_\tau(O), \sigma)$. Clearly $\Psi_{n,k}$ is a mapping from ${}^{(k)}A_n$ to $A_n^{(k)}$. An illustration is given in Fig. 4. From Proposition 11.8 we immediately deduce the following result.

Proposition 11.10. *The mapping $\Psi_{n,k} : {}^{(k)}A_n \to A_n^{(k)}$ is a bijection. Moreover, if $(\tau, O, \sigma) \in {}^{(k)}A_n$ and $\Psi_{n,k}(\tau, O, \sigma) = (\tau, C, \sigma)$, then we have*

$$\sum_{i \in C} \mathrm{dp}_i(\tau) = \sum_{i \in O} \mathrm{dp}_i(\tau).$$

Finally, we define the mapping $\Phi_{n,k} : {}^{(k)}\mathcal{P}_n \to \mathcal{P}_n^{(k)}$ by

$$\Phi_{n,k} := F_{n,k}^{-1} \circ \Psi_{n,k} \circ G_{n,k}. \tag{11.5}$$

This mapping is illustrated in Fig. 4. Combining Propositions 11.6 and 11.10, we conclude that the mapping $\Phi_{n,k}$ satisfies the requirements of Proposition 11.2.

11.2. Construction of the bijection $\Theta_n^{(n_1,n_2)}$

The key idea for the definition of the mapping $\Theta_n^{(n_1,n_2)}$ is some appropriate decomposition of partitions in $\mathcal{P}_n^{(n_1,n_2)}$. We first introduce some further definitions. For any set K, let $\Pi(K)$ be the set of partitions of K.

Definition 11.11. For two positive integers r, s, we denote by $\mathcal{P}^*(r,s)$ the set of all partitions π of $[r+s]$ such that there is no arc in π connecting two integers in $[1,r]$ or in $[r+1, r+s]$ but π can have singletons. Thus, we have $\mathcal{P}(r,s) \subsetneq \mathcal{P}^*(r,s)$.

Definition 11.12. Let $A_n^{(n_1,n_2)}$ be the set of 3-tuples $((\tau, A), (\gamma, B), \sigma)$ where

- τ is a partition in $\mathcal{P}^*(n_1, n_2)$ and A is a set satisfying $\mathrm{sing}(\tau) \subseteq A \subseteq \max(\tau)$;
- γ is a partition in $\Pi([N_2+1, n])$ and B is a set satisfying $\mathrm{sing}(\gamma) \subseteq B \subseteq \min(\gamma)$;
- the sets A and B have the same cardinality. If $k = |A| = |B|$, then σ is in $\mathfrak{S}_k$.

For instance, in Fig. 6, we have $((\tau_1, A_1), (\gamma_1, B_1), \sigma_1) \in A_{14}^{(3,4)}$ and $((\tau_2, A_2), (\gamma_2, B_2), \sigma_2) \in A_{14}^{(4,3)}$. For $\pi \in \mathcal{P}_n^{(n_1,n_2)}$, we set $H_n^{(n_1,n_2)}(\pi) = ((\tau, A), (\gamma, B), \sigma)$ where

- τ is the restriction of π on $[1, N_2]$ and A is the set of elements $\leqslant N_2$ in π which are connected to an element $> N_2$ by an arc;
- γ is the restriction of π on $[N_2+1, n]$ and B is the set of elements $> N_2$ in π which are connected to an element $\leqslant N_2$ by an arc;
- Suppose $A = \{a_1 < a_2 < \cdots < a_k\}$ and $B = \{b_1 < b_2 < \cdots < b_k\}$. Then, σ is the (unique) permutation in $\mathfrak{S}_k$ such that $(a_1, b_{\sigma(1)})$, $(a_2, b_{\sigma(2)})$, …, $(a_k, b_{\sigma(k)})$ are arcs of π.

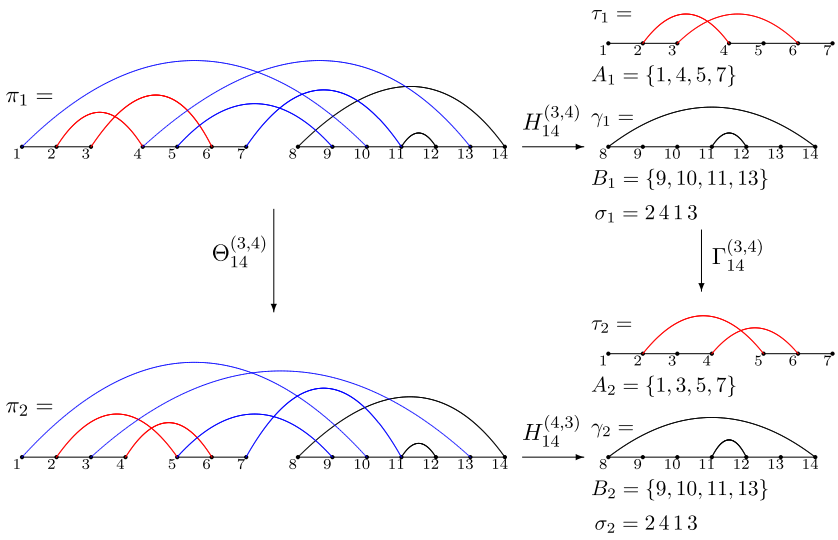
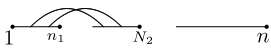
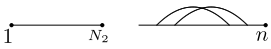
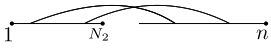
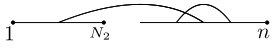
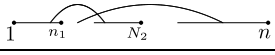


Fig. 6. The mappings $H_n^{(n_1,n_2)}$, $\Gamma_n^{(n_1,n_2)}$ and $\Theta_n^{(n_1,n_2)}$.

Table 2
Sketchs of crossings in $C_i(\pi)$.

i	$C_i(\pi)$
1	
2	
3	
4	
5	

Clearly, $H_n^{(n_1,n_2)}$ is a mapping from $\mathcal{P}_n^{(n_1,n_2)}$ to $A_n^{(n_1,n_2)}$. Two illustrations are given in Fig. 6.

Proposition 11.13. *The mapping $H_n^{(n_1,n_2)} : \mathcal{P}_n^{(n_1,n_2)} \to A_n^{(n_1,n_2)}$ is a bijection. Moreover, for any $\pi \in \mathcal{P}_n^{(n_1,n_2)}$, if $H_n^{(n_1,n_2)}(\pi) = ((\tau, A), (\gamma, B), \sigma)$ and $k = |A|$, then*

(i) $\mathrm{bl}(\pi) = \mathrm{bl}(\tau) + \mathrm{bl}(\gamma) - k$,
(ii) $\mathrm{cr}(\pi) = \mathrm{cr}(\tau) + \mathrm{cr}(\gamma) + \mathrm{ninv}(\sigma) + \sum_{i\in A} \mathrm{dp}_i(\tau) + \sum_{i\in B} \mathrm{dp}_i(\gamma)$.

Proof. It is easy to see that $H_n^{(n_1,n_2)}$ establishes a bijection from $\mathcal{P}_n^{(n_1,n_2)}$ to $A_n^{(n_1,n_2)}$ by constructing its inverse (use Fig. 6), and Property (i) is a direct consequence of Fact 11.7. Let $\pi \in \mathcal{P}_n^{(n_1,n_2)}$. The arc crossings of the partition π can be divided into five parts $C_i(\pi)$, $1 \leqslant i \leqslant 5$, illustrated in Table 2. They are defined formally as follows:

$$C_1(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 < j_1 < j_2 \leqslant N_2\big\},$$
$$C_2(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; N_2 < i_1 < i_2 < j_1 < j_2 \leqslant n\big\},$$

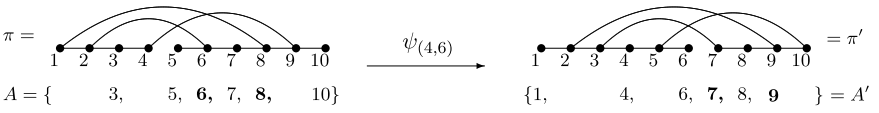


Fig. 7. The mapping $\psi_{(n_1,n_2)}$.

$$C_3(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 \leqslant N_2 < j_1 < j_2 \leqslant n\big\},$$
$$C_4(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 \leqslant N_2 < i_2 < j_1 < j_2 \leqslant n\big\},$$
$$C_5(\pi) = \big\{(i_1, j_1)(i_2, j_2) \in \pi \;\big|\; 1 \leqslant i_1 < i_2 < j_1 \leqslant N_2 < j_2 \leqslant n\big\},$$

and satisfy $\mathrm{cr}(\pi) = \sum_{i=1}^{5} |C_i(\pi)|$. Suppose $H_n^{(n_1,n_2)}(\pi) = ((\tau, A), (\gamma, B), \sigma)$. It is easily checked (use Fig. 6) that $|C_1(\pi)| = \mathrm{cr}(\tau)$, $|C_2(\pi)| = \mathrm{cr}(\gamma)$, $|C_3(\pi)| = \mathrm{ninv}(\sigma)$, $|C_4(\pi)| = \sum_{i\in B} \mathrm{dp}_i(\gamma)$ and $|C_5(\pi)| = \sum_{i\in A} \mathrm{dp}_i(\tau)$. Altogether, this leads to Property (ii). □

Let

$$R(n_1, n_2) := \big\{(\pi, A)\colon \pi \in \mathcal{P}^*(n_1, n_2) \text{ and } \mathrm{sing}(\pi) \subseteq A \subseteq \max(\pi)\big\}. \tag{11.6}$$

For instance, the elements (π, A) and (π, A') drawn in Fig. 7 are, respectively, in $R(4, 6)$ and $R(6, 4)$.

In view of Proposition 11.13, to prove Proposition 11.3, it suffices to demonstrate the following result.

Proposition 11.14. *There is a bijection $\psi_{(n_1,n_2)} : R(n_1, n_2) \to R(n_2, n_1)$ such that, for $(\pi, A) \in R(n_1, n_2)$, if $\psi_{(n_1,n_2)}(\pi, A) = (\pi', A')$, then*

$$\mathrm{cr}(\pi') = \mathrm{cr}(\pi), \qquad |A'| = |A|, \qquad \sum_{i\in A'} \mathrm{dp}_i(\pi') = \sum_{i\in A} \mathrm{dp}_i(\pi). \tag{11.7}$$

Proof. To any $(\pi, A) \in R(n_1, n_2)$ we associate an element (π', A') in $R(n_2, n_1)$ as follows:

- By definition of $\mathcal{P}^*(n_1, n_2)$, the arcs of π are $(i_1, j_{\rho(1)})$, $(i_2, j_{\rho(2)})$, …, $(i_k, j_{\rho(k)})$ for some integers $k \geqslant 0$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n_1$, $n_1 + 1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant N_2$ and some permutation $\rho \in \mathfrak{S}_k$. We use $\bar{i}$ for the complement of i in $[1, N_2]$, i.e., $\bar{i} = N_2 + 1 - i$. Then, we define π' as the partition of $[1, N_2]$ which consists of the arcs $(\bar{j}_r, \bar{i}_{\rho(r)})$ for $1 \leqslant r \leqslant k$. It is clear that $\pi' \in \mathcal{P}^*(n_2, n_1)$. Moreover, we have $\mathrm{cr}(\pi') = \mathrm{ninv}(\rho)$ and $\mathrm{cr}(\pi) = \mathrm{ninv}(\rho)$ whence $\mathrm{cr}(\pi') = \mathrm{cr}(\pi)$.
- Since $\mathrm{sing}(\pi) \subseteq A \subseteq \max(\pi)$, we have $A = \mathrm{sing}(\pi) \cup B$ with $B = \{j_{\ell(1)} < j_{\ell(2)} < \cdots < j_{\ell(t)}\}$ for some increasing sequence $(\ell(s))_{1\leqslant s\leqslant t}$. Suppose $\bar{I} := \{\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_k\} = \{u_1 < u_2 < \cdots < u_k\}$. We then set $A' := \mathrm{sing}(\pi') \cup B'$ with $B' = \{u_{\ell(1)} < u_{\ell(2)} < \cdots < u_{\ell(t)}\}$. Clearly, we have $\mathrm{sing}(\pi') \subseteq A' \subseteq \max(\pi')$ and $|A'| = |A|$. It is also easily checked that $d_{u_{\ell(t)}}(\pi') = d_{j_{\ell(t)}}(\pi)$ for $s = 1, 2, \ldots, t$ whence $\sum_{i\in B'} \mathrm{dp}_i(\pi') = \sum_{i\in B} \mathrm{dp}_i(\pi)$. Moreover, since $\mathrm{sing}(\pi') = \overline{\mathrm{sing}(\pi)}$ and $d_{\bar{i}}(\pi') = d_i(\pi)$ for $i \in \mathrm{sing}(\pi)$, we see that $\sum_{i\in\mathrm{sing}(\pi')} \mathrm{dp}_i(\pi') = \sum_{i\in\mathrm{sing}(\pi)} \mathrm{dp}_i(\pi)$. Altogether, this implies that $\sum_{i\in A'} \mathrm{dp}_i(\pi') = \sum_{i\in A} \mathrm{dp}_i(\pi)$.

Set $\psi_{(n_1,n_2)}(\pi, A) = (\pi', A')$. Then $\psi_{(n_1,n_2)}$ is a well-defined map from $R(n_1, n_2)$ to $R(n_2, n_1)$ and satisfies (11.7). An illustration is given in Fig. 7. Besides, it is easy to see that the composition $\psi_{(n_2,n_1)} \circ \psi_{(n_1,n_2)}$ is the identity mapping. This proves that $\psi_{(n_1,n_2)}$ is a bijection. □

For $((\pi, A), (\gamma, B), \sigma) \in A_n^{(n_1,n_2)}$, we set

$$\Gamma_n^{(n_1,n_2)}\big((\pi, A), (\gamma, B), \sigma\big) := \big(\psi_{(n_1,n_2)}(\tau, A), (\gamma, B), \sigma\big).$$

Clearly $\Gamma_n^{(n_1,n_2)}$ is a mapping from $A_n^{(n_1,n_2)}$ to $A_n^{(n_2,n_1)}$. An illustration is given in Fig. 6. From Proposition 11.14 we deduce the following result.

Proposition 11.15. *The mapping* $\Gamma_n^{(n_1,n_2)} : A_n^{(n_1,n_2)} \to A_n^{(n_2,n_1)}$ *is a bijection. Moreover, if* $((\tau, A), (\gamma, B), \sigma) \in A_n^{(n_1,n_2)}$ *and* $\Gamma_n^{(n_1,n_2)}((\pi, A), (\gamma, B), \sigma) = ((\tau', A'), (\gamma, B), \sigma)$, *then we have*

$$\operatorname{cr}(\tau') = \operatorname{cr}(\tau), \qquad |A'| = |A|, \qquad \sum_{i\in A'} \operatorname{dp}_i(\tau') = \sum_{i\in A} \operatorname{dp}_i(\tau).$$

Finally, we define the mapping $\Theta_n^{(n_1,n_2)} : \mathcal{P}_n^{(n_1,n_2)} \to \mathcal{P}_n^{(n_2,n_1)}$ by

$$\Theta_n^{(n_1,n_2)} := \left(H_n^{(n_2,n_1)}\right)^{-1} \circ \Gamma_n^{(n_1,n_2)} \circ H_n^{(n_1,n_2)}. \tag{11.8}$$

An illustration is given in Fig. 6. Combining Propositions 11.13 and 11.15, we conclude that the mapping $\Theta_n^{(n_1,n_2)}$ satisfies the requirements of Proposition 11.3.

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