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Ramsey numbers of 3-uniform loose paths and loose cycles ☆

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ABSTRACT

The *3-uniform loose cycle*, denoted by $\mathcal{C}^3_n$, is the hypergraph with vertices $v_1, v_2, \ldots, v_{2n}$ and n edges $v_1v_2v_3, v_3v_4v_5, \ldots, v_{2n-1}v_{2n}v_1$. Similarly, the *3-uniform loose path* $\mathcal{P}^3_n$ is the hypergraph with vertices $v_1, v_2, \ldots, v_{2n+1}$ and n edges $v_1v_2v_3, v_3v_4v_5, \ldots, v_{2n-1}v_{2n}v_{2n+1}$. In 2006 Haxell et al. proved that the 2-color Ramsey number of 3-uniform loose cycles on $2n$ vertices is asymptotically $\frac{5n}{2}$. Their proof is based on the method of the Regularity Lemma. Here, without using this method, we generalize their result by determining the exact values of 2-color Ramsey numbers involving loose paths and cycles in 3-uniform hypergraphs. More precisely, we prove that for every $n \geqslant m \geqslant 3$,

$$R\big(\mathcal{P}^3_n, \mathcal{P}^3_m\big) = R\big(\mathcal{P}^3_n, \mathcal{C}^3_m\big) = R\big(\mathcal{C}^3_n, \mathcal{C}^3_m\big) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor,$$

and for every $n > m \geqslant 3$, $R(\mathcal{P}^3_m, \mathcal{C}^3_n) = 2n + \lfloor \frac{m-1}{2} \rfloor$. This gives a positive answer to a recent question of Gyárfás and Raeisi.

1. Introduction

For given k-uniform hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_t$ the *Ramsey number* $R(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_t)$ is the smallest number N such that in every t-coloring of the edges of the complete k-uniform hypergraph on N vertices, $\mathcal{K}^k_N$, there is a monochromatic copy of $\mathcal{H}_i$ in color i, for some $1 \leqslant i \leqslant t$. A *k-uniform*

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loose cycle $\mathcal{C}_n^k$ (shortly, a *cycle of length* n) is a hypergraph with the vertex set $\{v_1, v_2, \ldots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)$, $i = 1, 2, \ldots, n$, where we use mod $n(k-1)$ arithmetic and by adding a number t to a set $H = \{v_1, v_2, \ldots, v_k\}$ we mean a shift, i.e., the set which is obtained by adding t to subscripts of each element of H. Similarly, a *k-uniform loose path* $\mathcal{P}_n^k$ (shortly, a *path of length* n) is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)+1}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)$, $i = 1, 2, \ldots, n$. For an edge e of a given loose path (also a given loose cycle) $\mathcal{K}$, the first vertex and the last vertex are denoted by $v_{\mathcal{K},e}$ and $\hat{v}_{\mathcal{K},e}$, respectively. For $k = 2$ we get the usual definitions of a cycle C_n and a path P_n with n edges.

Determining the exact values of $R(P_n, P_m)$, $R(P_n, C_m)$ and $R(C_n, C_m)$ are classical results; see [1, 6,5,8,16]. Also, the asymptotic value of $R(C_n, C_n, C_n)$ was obtained by Figaj and Luczak [7]. Moreover, Gyárfás et al. [10] determined the value of $R(P_n, P_n, P_n)$ for sufficiently large n. For a survey, including some results on the Ramsey numbers of paths and cycles, see [15].

There are few known results about the Ramsey numbers of hypergraphs. Recently, this topic has received considerable attention. Haxell et al. [12] proved the following result on the Ramsey number of 3-uniform loose cycles, using the Regularity Lemma.

Theorem 1.1. *For all $\eta > 0$ there exists some $n_0 = n_0(\eta)$ such that for every $n > n_0$, every 2-coloring of $\mathcal{K}^3_{5(1+\eta)n/2}$ contains a monochromatic copy of $\mathcal{C}_n^3$.*

Subsequently, Gyárfás, Sárközy and Szemerédi [11] extended this result to k-uniform loose cycles and proved that for any $k \geqslant 3$ and $\eta > 0$ there exists some $n_0 = n_0(\eta)$ such that every 2-coloring of $\mathcal{K}_N^k$ with $N = (1+\eta)\frac{1}{2}(2k-1)n$ contains a monochromatic copy of $\mathcal{C}_n^k$, i.e., $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ is asymptotically equal to $\frac{1}{2}(2k-1)n$. All these proofs are based on the hypergraph regularity method. Recently, Gyárfás and Raeisi [9] determined the exact values of 2-color Ramsey numbers of two k-uniform loose triangles and two k-uniform loose quadrangles. They also posed the following question.

Question 1.2. For every $n \geqslant m \geqslant 3$, is it true that

$$R(\mathcal{P}_n^3, \mathcal{P}_m^3) = R(\mathcal{P}_n^3, \mathcal{C}_m^3) = R(\mathcal{C}_n^3, \mathcal{C}_m^3) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor?$$

In particular, is it true that

$$R(\mathcal{P}_n^3, \mathcal{P}_n^3) = R(\mathcal{C}_n^3, \mathcal{C}_n^3) + 1 = \left\lceil \frac{5n}{2} \right\rceil?$$

In connection with Question 1.2, it is known that for every $n \geqslant \lfloor \frac{5m}{4} \rfloor$, $R(\mathcal{P}_n^3, \mathcal{P}_m^3) = 2n + \lfloor \frac{m+1}{2} \rfloor$; see [13]. In this article, we answer Question 1.2 positively. Our proof involves new ideas (though, it modifies certain ideas from [13] at some points), and does not use the Regularity Lemma. Moreover, we show that $R(\mathcal{P}_m^3, \mathcal{C}_n^3) = 2n + \lfloor \frac{m-1}{2} \rfloor$ for any $n > m \geqslant 3$. Indeed, our results yield Theorem 1.1.

The loose paths and loose cycles are examples of hypergraphs with bounded maximum degree. For this class of hypergraphs, it was conjectured that their Ramsey number is linear in terms of their number of vertices. This conjecture has been established by several authors using the hypergraph regularity method; see [3,4,14]. Recently, Conlon, Fox and Sudakov [2] proved this without using the regularity method.

This paper is organized as follows. In Section 2, we state the principal results necessary to prove the main results. In Section 3, we determine the exact value of the Ramsey number of loose cycles in 3-uniform hypergraphs; this generalizes Theorem 1.1. In Section 4, we provide the exact value of the Ramsey number of loose paths in 3-uniform hypergraphs, and finally, in Section 5, the Ramsey number of a loose path and a loose cycle in 3-uniform hypergraphs is determined.

Note. It is shown in [9, Lemma 1] that $(k-1)n + \lfloor \frac{m+1}{2} \rfloor$ is a lower bound for $R(\mathcal{P}_n^k, \mathcal{P}_m^k)$, $R(\mathcal{P}_n^k, \mathcal{C}_m^k)$ and $R(\mathcal{C}_n^k, \mathcal{C}_m^k) + 1$, when $n \geqslant m \geqslant 2$ and $k \geqslant 3$. Here we note that for any $n > m$ and $k \geqslant 3$, $R(\mathcal{P}_m^k, \mathcal{C}_n^k) \geqslant$

$(k-1)n + \lfloor \frac{m-1}{2} \rfloor$. To prove this, consider a complete hypergraph whose vertex set is partitioned into two parts A and B, where $|A| = (k-1)n - 1$ and $|B| = \lfloor \frac{m-1}{2} \rfloor$. Color all edges that contain a vertex of B red, and the rest blue. In this coloring, the longest red path has length at most $m-1$ and there is also no blue copy of $\mathcal{C}_n^k$, since such a copy should have $(k-1)n$ vertices. Our main aim in this article is to prove that for $k = 3$ these lower bounds are sharp. Therefore, in this paper, to determine the Ramsey numbers it suffices to verify that the known lower bounds are also upper bounds.

Throughout the paper, we denote by $\mathcal{H}_{\text{red}}$ and $\mathcal{H}_{\text{blue}}$ the induced 3-uniform sub-hypergraph of $\mathcal{H}$ on the edges with color red and blue, respectively. Also, we denote by $|\mathcal{H}|$ and $\|\mathcal{H}\|$ the number of vertices and edges of $\mathcal{H}$, respectively.

2. Preliminaries

In this section, we present results that we will use in the follow up sections. We also recall some results from [9].

Lemma 2.1. *(See [9].) Let $n \geqslant m \geqslant 3$ and $k \geqslant 3$ and let $\mathcal{H} = \mathcal{K}^k_{(k-1)n+\lfloor \frac{m+1}{2} \rfloor}$ be 2-edge colored red and blue. If $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{red}}$, then either $\mathcal{P}_n^k \subseteq \mathcal{H}_{\text{red}}$ or $\mathcal{P}_m^k \subseteq \mathcal{H}_{\text{blue}}$. Also, if $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{red}}$, then either $\mathcal{P}_n^k \subseteq \mathcal{H}_{\text{red}}$ or $\mathcal{C}_m^k \subseteq \mathcal{H}_{\text{blue}}$.*

Theorem 2.2. *(See [9].) For every $k \geqslant 3$,*

(i) $R(\mathcal{P}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{C}_3^k) + 1 = 3k - 1$;
(ii) $R(\mathcal{P}_4^k, \mathcal{P}_4^k) = R(\mathcal{C}_4^k, \mathcal{P}_4^k) = R(\mathcal{C}_4^k, \mathcal{C}_4^k) + 1 = 4k - 2$.

Let $\mathcal{P}$ be a loose path and let W be a set of vertices with $W \cap V(\mathcal{P}) = \emptyset$. By a $\varpi_{\{v_i, v_j, v_k\}}$-*configuration*, we mean a copy of $\mathcal{P}_2^3$ with edges $\{x, v_i, v_j\}$ and $\{v_j, v_k, y\}$ so that the v_l's, where $l \in \{i, j, k\}$, belong to two consecutive edges of $\mathcal{P}$ and $\{x, y\} \subseteq W$. The vertices x and y are called *the end vertices* of this configuration. A ϖ_S-configuration, with $S \subseteq V(e_i) \cup V(e_{i+1})$, is *good* if the last vertex of e_{i+1} is not in S and it is *bad*, otherwise. Let $\mathcal{H} = \mathcal{K}_l^3$ be 2-edge colored red and blue. We say that a red path $\mathcal{P} = e_1 e_2 \ldots e_n$ of length n is *maximal with respect to* $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ (in brief, *maximal w.r.t.* W) if there are no vertices x and y in W so that for some $1 \leqslant i \leqslant n$ either there is a red path $\mathcal{P}' = e_1 e_2 \ldots e_{i-1} e e' e_{i+1} \ldots e_n$ in $\mathcal{H}$ with $v_{\mathcal{P}',e} = v_{\mathcal{P},e_i}$ if $i = 1$ and $\hat{v}_{\mathcal{P}',e'} = \hat{v}_{\mathcal{P},e_i}$ if $i = n$, or a red path $\mathcal{P}' = e_1 e_2 \ldots e_{i-1} e e' e'' e_{i+2} \ldots e_n$ in $\mathcal{H}$ with $v_{\mathcal{P}',e} = v_{\mathcal{P},e_i}$ if $i = 1$ and $\hat{v}_{\mathcal{P}',e''} = \hat{v}_{\mathcal{P},e_{i+1}}$ if $i = n-1$, such that $V(\mathcal{P}') = V(\mathcal{P}) \cup \{x, y\}$. Clearly, if $\mathcal{P}$ is maximal w.r.t. W, then it is maximal w.r.t. every $W' \subseteq W$ and also every loose path $\mathcal{P}'$ which is a sub-hypergraph of $\mathcal{P}$ is again maximal w.r.t. W. We use these definitions to deduce the following lemma.

Lemma 2.3. *Assume that $\mathcal{H} = \mathcal{K}_l^3$ is 2-edge colored red and blue. Let $\mathcal{P} \subseteq \mathcal{H}_{\text{red}}$ be maximal w.r.t. $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ with $|W| \geqslant 3$. Let $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and $e_{i+1} = \{v_{2i+1}, v_{2i+2}, v_{2i+3}\}$ be consecutive edges of $\mathcal{P}$. Then there is a good ϖ_S-configuration C in $\mathcal{H}_{\text{blue}}$ with end vertices in W and $S \subseteq V(e_i) \cup V(e_{i+1})$ or there is a bad ϖ_{S_1}-configuration C_1 in $\mathcal{H}_{\text{blue}}$ with end vertices in W and $S_1 \subseteq V(e_i) \cup V(e_{i+1}) \setminus \{v_{2i+2}\}$. If there is no such good configuration C in $\mathcal{H}_{\text{blue}}$ and $e_{i+2} = \{v_{2i+3}, v_{2i+4}, v_{2i+5}\}$ is an edge of $\mathcal{P}$, then there is also a good ϖ_{S_2}-configuration C_2 in $\mathcal{H}_{\text{blue}}$ with $S_2 \subseteq V(e_{i+1}) \cup V(e_{i+2})$ and with end vertices in W and $S_1 \cap S_2 = \emptyset$. Moreover, each vertex of W, with the exception of at most one, can be considered as an end vertex of C if there exists such a configuration and otherwise, each vertex of W can be considered as an end vertex of C_1 and C_2.*

Proof. Assume that $W = \{x_1, \ldots, x_t\} \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$. If $\{v_{2i-1}, v_{2i}, x\}$ (resp. $\{v_{2i+2}, v_{2i+3}, x\}$) is red for some $x \in W$, then the maximality of $\mathcal{P}$ w.r.t. W implies that for arbitrary vertices $x' \neq x'' \in W \setminus \{x\}$ the edges $\{x', v_{2i+1}, v_{2i}\}$ and $\{v_{2i}, v_{2i+2}, x''\}$ (resp. $\{x', v_{2i+1}, v_{2i+2}\}$ and $\{v_{2i+2}, v_{2i}, x''\}$) are blue and there is a good ϖ_S-configuration $C = \{x', v_{2i+1}, v_{2i}\}\{v_{2i}, v_{2i+2}, x''\}$ (resp. $C = \{x', v_{2i+1}, v_{2i+2}\}$ $\{v_{2i+2}, v_{2i}, x''\}$) in $\mathcal{H}_{\text{blue}}$ with $S = \{v_{2i}, v_{2i+1}, v_{2i+2}\} \subseteq V(e_i) \cup V(e_{i+1})$. So we may assume that for every $x \in W$ both edges $\{v_{2i-1}, v_{2i}, x\}$ and $\{v_{2i+2}, v_{2i+3}, x\}$ are blue.

If there is a vertex $y \in W$ such that at least one of the edges $f_1 = \{v_{2i-1}, v_{2i+1}, y\}$, $f_2 = \{v_{2i}, v_{2i+1}, y\}$, $f_3 = \{v_{2i-1}, v_{2i+2}, y\}$ and $f_4 = \{v_{2i}, v_{2i+2}, y\}$, say f, is blue, then there is a good ϖ_S-configuration $C = \{v_{2i-1}, v_{2i}, x\} f$, $x \neq y$, with

$$S = \{v_{2i-1}, v_{2i}\} \cup f \setminus \{y\} \subseteq V(e_i) \cup V(e_{i+1}).$$

Hence, if there is no such good configuration C, we may assume that for every $y \in W$ the edges f_1, f_2, f_3 and f_4 are red. Therefore, the maximality of $\mathcal{P}$ w.r.t. W implies that for every $y' \in W$ the edge $\{v_{2i}, v_{2i+3}, y'\}$ is blue. Otherwise, replacing $e_i e_{i+1}$ by $\{v_{2i-1}, v_{2i+2}, y\}\{y, v_{2i+1}, v_{2i}\}\{v_{2i}, v_{2i+3}, y'\}$, $y \neq y'$, in $\mathcal{P}$ yields a red path $\mathcal{P}'$ greater than $\mathcal{P}$, which is a contradiction. Thus, for every $a \neq b \in W$, $C_1 = \{v_{2i-1}, a, v_{2i}\}\{v_{2i}, v_{2i+3}, b\}$ is a bad ϖ_{S_1}-configuration in $\mathcal{H}_{\text{blue}}$ with the desired properties, so that

$$S_1 = \{v_{2i-1}, v_{2i}, v_{2i+3}\} \subseteq V(e_i) \cup V(e_{i+1}) \setminus \{v_{2i+2}\}.$$

Clearly, when $e_{i+2} = \{v_{2i+3}, v_{2i+4}, v_{2i+5}\}$ is an edge of $\mathcal{P}$, then for every $x \in W$, $\{v_{2i+2}, v_{2i+4}, x\}$ (resp. $\{v_{2i+1}, v_{2i+4}, x\}$) is blue; otherwise, for some $y \in W \setminus \{x\}$, replacing $e_i e_{i+1}$ by $\{v_{2i-1}, v_{2i+1}, y\}\{y, v_{2i}, v_{2i+2}\}\{v_{2i+2}, v_{2i+4}, x\}$ (resp. $e_i e_{i+1}$ by $\{v_{2i-1}, v_{2i+2}, y\}\{y, v_{2i}, v_{2i+1}\}\{v_{2i+1}, v_{2i+4}, x\}$) in $\mathcal{P}$ yields a red path $\mathcal{P}'$ greater than $\mathcal{P}$, which contradicts the maximality of $\mathcal{P}$ w.r.t. W. Thereby, for every $a \neq b \in W$, $C_2 = \{a, v_{2i+2}, v_{2i+4}\}\{v_{2i+4}, v_{2i+1}, b\}$ is a good ϖ_{S_2}-configuration with the desired properties, where

$$S_2 = \{v_{2i+1}, v_{2i+2}, v_{2i+4}\} \subseteq V(e_{i+1}) \cup V(e_{i+2}). \qquad \square$$

The following is an immediate corollary of Lemma 2.3.

Corollary 2.4. *Assume that $\mathcal{H} = \mathcal{K}_l^3$ is 2-edge colored red and blue. Let $\mathcal{P} \subseteq \mathcal{H}_{\text{red}}$ be maximal w.r.t. $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ with $|W| \geqslant 3$. Let $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$, $e_{i+1} = \{v_{2i+1}, v_{2i+2}, v_{2i+3}\}$ and $e_{i+2} = \{v_{2i+3}, v_{2i+4}, v_{2i+5}\}$ be consecutive edges of $\mathcal{P}$. Then either for every distinct vertices x and y of W, except for at most one, there is a blue path $\mathcal{Q} = \{x, \bar{v}_1, \bar{v}_2\}\{\bar{v}_2, \bar{v}_3, y\}$ of length 2 with $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \subseteq V(e_i) \cup V(e_{i+1}) \setminus \{v_{2i+3}\}$ or for every distinct vertices x, y and z of W there is a blue path $\mathcal{Q}' = \{x, v'_1, v'_2\}\{v'_2, v'_3, y\}\{y, v'_4, v'_5\}\{v'_5, v'_6, z\}$ of length 4 with $\{v'_1, v'_2, v'_3\} = \{v_{2i-1}, v_{2i}, v_{2i+3}\}$ and $\{v'_4, v'_5, v'_6\} = \{v_{2i+1}, v_{2i+2}, v_{2i+4}\}$.*

Corollary 2.5. *Let $\mathcal{H} = \mathcal{K}_l^3$ be 2-edge colored red and blue and $\mathcal{P} = e_1 e_2 \ldots e_n$, $n \geqslant 3$, be a maximal red path w.r.t. W, where $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ and $|W| \geqslant 3$. Then for some $r \geqslant 0$ and $W' \subseteq W$ there is a blue path $\mathcal{Q} = f_1 f_2 \ldots f_q$ between W' and $\bar{\mathcal{P}} = e_1 e_2 \ldots e_{n-r}$ so that $W' = \{v_{\mathcal{Q}, f_1}\} \cup \{\hat{v}_{\mathcal{Q}, f_{2i}} \mid 1 \leqslant i \leqslant q/2\}$ and $V(\mathcal{Q}) \setminus W' \subseteq \bar{\mathcal{P}}$. Moreover, $\mathcal{Q}$ does not have $\hat{v}_{\mathcal{P}, e_{n-r}}$ as a vertex, $V(f_1 f_2 \ldots f_{q-2}) \cap V(e_{n-r} e_{n-r+1} \ldots e_n) \subseteq \{v_{\mathcal{P}, e_{n-r}}\}$, $\|\mathcal{Q}\| = q = 2(|W'| - 1) \geqslant n - r$ and either $x = |W \setminus W'| \in \{0, 1\}$ or $x \geqslant 2$ and $0 \leqslant r \leqslant 2$.*

Proof. Let $\mathcal{P} = e_1 e_2 \ldots e_n$ be a maximal red path w.r.t. $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ where $e_i = \{v_1, v_2, v_3\} + 2(i-1)$, $i = 1, 2, \ldots, n$.

Step 1: Set $i_1 = 1$, $\mathcal{P}_1 = \mathcal{P}$, $W_1 = W$, $\mathcal{P}'_1 = e_{i_1} e_{i_1+1}$ and $\mathcal{P}''_1 = e_{i_1} e_{i_1+1} e_{i_1+2}$. Since $\mathcal{P}_1$ is maximal w.r.t. W_1, using Corollary 2.4 there is a blue path $\mathcal{Q}_1$ with end vertices x_1 and y_1 in W_1 between $\overline{\mathcal{P}}_1$ and $W'_1 \subseteq W_1$ where $\overline{\mathcal{P}}_1 \in \{\mathcal{P}'_1, \mathcal{P}''_1\}$, $\|\mathcal{Q}_1\| = 2\|\overline{\mathcal{P}}_1\| - 2$ and $\mathcal{Q}_1$ does not contain the last vertex of $\overline{\mathcal{P}}_1$. If $\overline{\mathcal{P}}_1 = \mathcal{P}$, then $\mathcal{Q} = \mathcal{Q}_1$ is a blue path between $W' = W'_1$ and $\bar{\mathcal{P}} = \mathcal{P}$ with the desired properties. Otherwise, continue to Step 2.

Step *k* (*k* $\geqslant$ 2): Set

$$i_k = \min\bigl\{j\colon\ j \in \{i_{k-1} + 2, i_{k-1} + 3\},\ e_j \notin \overline{\mathcal{P}}_{k-1}\bigr\}, \quad \mathcal{P}_k = e_{i_k} e_{i_k+1} \ldots e_n,$$

and $W_k = (W \setminus V(\bigcup_{i=1}^{k-1} \mathcal{Q}_i)) \cup \{x_{k-1}, y_{k-1}\}$. If either $|W_k| \leqslant 3$ or $\|\mathcal{P}_k\| \leqslant 2$, then $\mathcal{Q} = \bigcup_{i=1}^{k-1} \mathcal{Q}_i$ is a blue path between $W' = \bigcup_{i=1}^{k-1} W'_i$ and $\bar{\mathcal{P}} = e_1 e_2 \ldots e_{i_k-1}$ with the desired properties. Otherwise, set

$\mathcal{P}'_k = e_{i_k} e_{i_k+1}$ and $\mathcal{P}''_k = e_{i_k} e_{i_k+1} e_{i_k+2}$. Since $\mathcal{P}_k$ is maximal w.r.t. W_k, using Corollary 2.4 there is a blue path $\mathcal{Q}_k$ between $\overline{\mathcal{P}}_k$, $\overline{\mathcal{P}}_k \in \{\mathcal{P}'_k, \mathcal{P}''_k\}$, and $W'_k \subseteq W_k$ such that $\|\mathcal{Q}_k\| = 2\|\overline{\mathcal{P}}_k\| - 2$, $\mathcal{Q}_k$ does not contain the last vertex of $\overline{\mathcal{P}}_k$ and $\bigcup_{i=1}^{k} \mathcal{Q}_i$ is a blue path with end vertices x_k, y_k in W_k. If the last edge of $\overline{\mathcal{P}}_k$ is e_n, then $\mathcal{Q} = \bigcup_{i=1}^{k} \mathcal{Q}_i$ is a blue path between $W' = \bigcup_{i=1}^{k} W'_i$ and $\bar{\mathcal{P}} = \mathcal{P}$ with the desired properties. Otherwise, continue to Step $k+1$.

Let $t \geqslant 2$ be the minimum integer for which either the last edge of $\overline{\mathcal{P}}_{t-1}$ is e_n or $|W_t| \leqslant 3$ or $\|\mathcal{P}_t\| \leqslant 2$. If the first case holds, then $\mathcal{Q} = \bigcup_{i=1}^{t-1} \mathcal{Q}_i$ is a blue path between $W' = \bigcup_{i=1}^{t-1} W'_i$ and $\bar{\mathcal{P}} = \mathcal{P}$ with the desired properties. Otherwise, let $W' = \bigcup_{i=1}^{t-1} W'_i$. Clearly, either $|W \setminus W'| = 0, 1$ or $|W \setminus W'| \geqslant 2$ and $0 \leqslant \|\mathcal{P}_t\| \leqslant 2$. So $\mathcal{Q} = \bigcup_{i=1}^{t-1} \mathcal{Q}_i$ is a blue path between $\bar{\mathcal{P}} = e_1 e_2 \dots e_{n-r}$ and W' with the desired properties where $r = n - i_t + 1$. $\square$

3. Cycle–cycle Ramsey number in 3-uniform hypergraphs

In this section, we provide the exact value of $R(\mathcal{C}^3_n, \mathcal{C}^3_m)$, when $n \geqslant m \geqslant 3$. Before we proceed we need the following two lemmas.

Lemma 3.1. *Let $n \geqslant m \geqslant 3$, $(n, m) \neq (3, 3), (4, 3), (4, 4)$ and let $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor \frac{m-1}{2} \rfloor}$ be 2-edge colored red and blue. Assume that there is no copy of $\mathcal{C}^3_n$ in $\mathcal{H}_{\text{red}}$ and $\mathcal{C} = \mathcal{C}^3_{n-1}$ is a loose cycle in $\mathcal{H}_{\text{red}}$. Then there is a copy of $\mathcal{C}^3_m$ in $\mathcal{H}_{\text{blue}}$. Moreover, for every $n > m$, there is also a copy of $\mathcal{P}^3_m$ in $\mathcal{H}_{\text{blue}}$.*

Proof. Let $t = 2n + \lfloor \frac{m-1}{2} \rfloor$ and $\mathcal{C} = e_1 e_2 \dots e_{n-1}$ be a copy of $\mathcal{C}^3_{n-1}$ in $\mathcal{H}_{\text{red}}$ with edges $e_i = \{v_1, v_2, v_3\} + 2(i-1) \pmod{2(n-1)}$, $i = 1, \dots, n-1$. Let $W = V(\mathcal{H}) \setminus V(\mathcal{C})$. We consider the following cases.

Case 1. For some $1 \leqslant i \leqslant n-1$, there exist an edge $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and a vertex $z \in W$ such that $\{v_{2i}, v_{2i+1}, z\}$ is red.

Let $\mathcal{P} = e_{i+1} e_{i+2} \dots e_{n-1} e_1 e_2 \dots e_{i-2} e_{i-1}$ and $W_0 = W \setminus \{z\}$.

First, let $m \leqslant 4$. Therefore, $|W_0| = 2$. Let $W_0 = \{u, v\}$. We show that $\mathcal{H}_{\text{blue}}$ contains $\mathcal{C}^3_m$ and $\mathcal{P}^3_m$ for each $m \in \{3, 4\}$. Since $n \geqslant 5$ and there is no red copy of $\mathcal{C}^3_n$, clearly $\{u, v_{2i-2}, v_{2i}\}\{v_{2i}, v, v_{2i-1}\}\{v_{2i-1}, z, u\}$ forms a blue copy of $\mathcal{C}^3_3$. Moreover, $\mathcal{P}' = e_{i-3} e_{i-2} e_{i-1}$ (in mod $(n-1)$ arithmetic) is maximal w.r.t. $W = W_0 \cup \{z\}$. Using Lemma 2.3 there is a configuration in $\mathcal{H}_{\text{blue}}$ (no matter good or bad), say C, between $V(e_{i-3}) \cup V(e_{i-2})$ and W with end vertices in W. Without loss of generality assume that u is an end vertex of C. Clearly, $C\{v, z, v_{2i-1}\}\{v_{2i-1}, v_{2i}, u\}$ is a blue copy of $\mathcal{C}^3_4$. Also, using Corollary 2.4 there is either a blue path $\mathcal{Q}$ of length 4 between $\mathcal{P}'$ and W or a blue path $\mathcal{Q}$ of length 2 between $e_{i-3} e_{i-2}$ and W. In the first case, we have a blue copy of $\mathcal{P}^3_4$ and a blue copy of $\mathcal{P}^3_3$. In the second case, $\mathcal{Q}\{y, z, v_{2i-1}\}\{v_{2i-2}, v_{2i}, w\}$ is a blue copy of $\mathcal{P}^3_4$ where $w = y = \{u, v\} \setminus V(\mathcal{Q})$ if $z \in V(\mathcal{Q})$ and $y = v$, $w = u$, otherwise.

Now, let $m \geqslant 5$. Clearly, $|W_0| \geqslant 3$. Since there is no red copy of $\mathcal{C}^3_n$, $\mathcal{P}$ is a maximal path w.r.t. W_0. Now, using Corollary 2.5, there is a blue path $\mathcal{Q}$ of length l' between $\bar{\mathcal{P}}$ (where $\bar{\mathcal{P}}$ is the path obtained from $\mathcal{P}$ by deleting the last r edges) and W' for some $r \geqslant 0$ and $W' \subseteq W_0$ with properties mentioned in Corollary 2.5. Let x' and y' be the end vertices of $\mathcal{Q}$ in W', $T = W_0 \setminus V(\mathcal{Q})$ and $x = |T|$. We have the following subcases.

Subcase 1: $x = 0$.

Clearly, $l' = 2\lfloor \frac{m-1}{2} \rfloor$. First, let m be even. Hence $l' = m - 2$ and so

$$\{v_{2i-1}, z, x'\}\mathcal{Q}\{y', v_{2i}, v_{2i-1}\},$$

is a blue copy of $\mathcal{C}^3_m$. If $n > m$, then $r \geqslant 1$ and so $\{v_{2i-2}, v_{2i}, x'\}\mathcal{Q}\{y', z, v_{2i-1}\}$ is a blue copy of $\mathcal{P}^3_m$. Now, let m be odd. So $l' = m - 1$. In this case, we remove the last two edges of $\mathcal{Q}$ to make a path $\mathcal{Q}'$ of length $m - 3$. Using Corollary 2.5, $v_{2i-2} \notin \mathcal{Q}'$. Now, we may assume that the vertices x' and $y'' \neq y'$ of W' are the end vertices of $\mathcal{Q}'$. Hence,

$$\mathcal{Q}'\{y'', v_{2i-2}, v_{2i}\}\{v_{2i}, y', v_{2i-1}\}\{v_{2i-1}, z, x'\},$$

is a copy of $\mathcal{C}^3_m$ in $\mathcal{H}_{\text{blue}}$. Also, $\mathcal{Q}\{y', v_{2i-1}, v_{2i}\}$ is a blue copy of $\mathcal{P}^3_m$.

Subcase 2: $x=1$.

Let $T=\{u\}$. Clearly, $l'=2\lfloor\frac{m-1}{2}\rfloor-2$. Let m be odd. Then $l'=m-3$ and $r\geqslant 1$. Thereby,

$$\mathcal{Q}\{y', v_{2i-2}, v_{2i}\}\{v_{2i}, u, v_{2i-1}\}\{v_{2i-1}, z, x'\},$$

is a blue copy of $\mathcal{C}_m^3$. When $n>m$, then $r\geqslant 2$ and since $\hat{\mathcal{P}}=e_{i-2}e_{i-1}e_i$ is a maximal path w.r.t. $\hat{W}=\{x', y', u, z\}$, using Corollary 2.4 there is either a blue path $\mathcal{Q}'$ of length 2 between $(V(e_{i-2})\cup V(e_{i-1}))\setminus\{v_{2i-1}\}$ and $\hat{W}$ or a blue path $\mathcal{Q}'$ of length 4 between $V(\hat{\mathcal{P}})\setminus\{v_{2i+1}\}$ and $\hat{W}$ so that $\mathcal{Q}''=\mathcal{Q}\cup\mathcal{Q}'$ is a blue path. Now, let l'' be the length of $\mathcal{Q}'$. If $l''=4$, then $\mathcal{Q}''$ is a blue copy of $\mathcal{P}_{m+1}^3$ and so there is a $\mathcal{P}_m^3$ in $\mathcal{H}_{\text{blue}}$. If $l''=2$, the length of $\mathcal{Q}''$ is $m-1$. Without loss of generality let $x', y''\in W$ be the end vertices of $\mathcal{Q}''$. Then $\{v_{2i-1}, v_{2i}, x'\}\mathcal{Q}''$ is a blue copy of $\mathcal{P}_m^3$.

Now, suppose that m is even, so $l'=m-4$ and $r\geqslant 2$. If $\{v_{2i-5}, v_{2i-4}, x'\}$ is red, then

$$\mathcal{Q}\{y', v_{2i-3}, v_{2i-4}\}\{v_{2i-4}, v_{2i-2}, u\}\{u, v_{2i}, v_{2i-1}\}\{v_{2i-1}, z, x'\},$$

is a blue copy of $\mathcal{C}_m^3$. If $\{v_{2i-4}, v_{2i-3}, u\}$ is red, then

$$\mathcal{Q}\{y', v_{2i-1}, v_{2i}\}\{v_{2i}, v_{2i-2}, u\}\{u, z, v_{2i-5}\}\{v_{2i-5}, v_{2i-4}, x'\},$$

is a blue copy of $\mathcal{C}_m^3$. Otherwise,

$$\mathcal{Q}\{y', v_{2i-1}, v_{2i}\}\{v_{2i}, v_{2i-2}, u\}\{u, v_{2i-3}, v_{2i-4}\}\{v_{2i-4}, v_{2i-5}, x'\},$$

is a copy of $\mathcal{C}_m^3$ in $\mathcal{H}_{\text{blue}}$. For $n>m$, clearly $r\geqslant 3$ and since $\hat{\mathcal{P}}=e_{i-3}e_{i-2}e_{i-1}$ is a maximal path w.r.t. $\hat{W}=\{x', y', u, z\}$, using Corollary 2.4 there is a blue path $\mathcal{Q}'$ either of length $l''=2$, between $(V(e_{i-3})\cup V(e_{i-2}))\setminus\{v_{2i-3}\}$ and $\hat{W}$ or of length $l''=4$, between $V(\hat{\mathcal{P}})\setminus\{v_{2i-1}\}$ and $\hat{W}$ such that $\mathcal{Q}''=\mathcal{Q}\cup\mathcal{Q}'$ is a blue path. If $l''=4$, then $\mathcal{Q}''$ is a blue copy of $\mathcal{P}_m^3$. Otherwise, the length of $\mathcal{Q}''$ is $m-2$. We may assume that $x'\in W$ is an end vertex of $\mathcal{Q}''$. Clearly, $\{v_{2i-2}, v_{2i}, x'\}\mathcal{Q}''\{u, z, v_{2i-1}\}$ is a blue copy of $\mathcal{P}_m^3$.

Subcase 3: $x\geqslant 2$.

One can easily check that this implies that $r\geqslant 3$. This subcase does not occur by Corollary 2.5.

Case 2. For some $1\leqslant i\leqslant n-1$, there exist an edge $e_i=\{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and a vertex $z\in W$ such that $\{v_{2i-1}, v_{2i}, z\}$ is red.

In this case, consider the path $\mathcal{P}=e_{i-1}e_{i-2}\ldots e_2e_1e_{n-1}e_{n-2}\ldots e_{i+2}e_{i+1}$ and repeat the proof of Case 1. By an argument similar to the one we have given for Case 1 we can find a blue copy of $\mathcal{C}_m^3$ for any $n\geqslant m$ and a blue copy of $\mathcal{P}_m^3$ for any $n>m$.

Case 3. For every $e_i=\{v_{2i-1}, v_{2i}, v_{2i+1}\}$, $1\leqslant i\leqslant n-1$, and every vertex $z\in W$ the edges $\{v_{2i-1}, v_{2i}, z\}$ and $\{v_{2i}, v_{2i+1}, z\}$ are blue.

Let $W=\{x_1, x_2, \ldots, x_{\lfloor\frac{m-1}{2}\rfloor+2}\}$. For $1\leqslant i\leqslant m$, set

$$f_i=\begin{cases}\{x_{\frac{i+1}{2}}, v_{\frac{3i+1}{2}}, v_{\frac{3i+3}{2}}\} & \text{if } i \text{ is odd},\\ \{v_{\frac{3i}{2}}, v_{\frac{3i}{2}+1}, x_{\frac{i}{2}+1}\} & \text{if } i \text{ is even}.\end{cases}$$

Set $\mathcal{Q}=f_1f_2\ldots f_{m-1}$. For $n=m=5$,

$$\mathcal{Q}f_m=\{x_1, v_2, v_3\}\{v_3, v_4, x_2\}\{x_2, v_5, v_6\}\{v_6, v_7, x_3\}\{x_3, v_8, v_1\},$$

is a blue copy of $\mathcal{P}_m^3$. Otherwise, since $(n,m)\neq(3,3),(4,3),(4,4)$ we have

$$\max\{i\mid v_i\in f_m\}\leqslant 2n-2.$$

Clearly every two non-consecutive f_i's are disjoint and every two consecutive f_i's have exactly one vertex in their intersection. Therefore, $\mathcal{Q}f_m$ is a blue copy of $\mathcal{P}_m^3$. Moreover, depending on whether m is even or odd, $\mathcal{Q}\{x_1, v_{\frac{3m}{2}}, v_{\frac{3m}{2}+1}\}$ or $\mathcal{Q}\{x_{\frac{m+1}{2}}, v_1, v_2\}$ is a blue copy of $\mathcal{C}_m^3$. $\square$

Lemma 3.2. $R(\mathcal{C}_4^3, \mathcal{C}_3^3) = 9$.

Proof. Let $\mathcal{H} = \mathcal{K}_9^3$ be 2-edge colored red and blue. Suppose that there is no red copy of $\mathcal{C}_4^3$ and no blue copy of $\mathcal{C}_3^3$. Using Theorem 2.2 we may assume that there is a blue copy of $\mathcal{C}_4^3$. Let $\mathcal{C} = e_1e_2e_3e_4$ be a copy of $\mathcal{C}_4^3$ in $\mathcal{H}_{\text{blue}}$ with edges $e_i = \{v_1, v_2, v_3\} + 2(i-1) \pmod 8$, $i = 1, \dots, 4$. Let $v \in V(\mathcal{H}) \setminus V(\mathcal{C})$. Since there is no blue copy of $\mathcal{C}_3^3$,

$$\{v_1, v_6, v\}\{v, v_3, v_8\}\{v_8, v_4, v_2\}\{v_2, v_5, v_1\},$$

is a red copy of $\mathcal{C}_4^3$. This is a contradiction. □

The main result of this section is the following result on the Ramsey number of loose cycles in 3-uniform hypergraphs.

Theorem 3.3. *For every $n \geqslant m \geqslant 3$,*

$$R\big(\mathcal{C}_n^3, \mathcal{C}_m^3\big) = 2n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Proof. We prove this theorem by induction on $m+n$. The case $n = m = 3$ holds by Theorem 2.2. Using Theorem 2.2 and Lemma 3.2 we may assume that $n \geqslant 5$. Suppose for a contradiction that the edges of $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor\frac{m-1}{2}\rfloor}$ can be colored red and blue with no red copy of $\mathcal{C}_n^3$ and no blue copy of $\mathcal{C}_m^3$. We consider the following cases.

Case 1: $n = m$. By the induction hypothesis,

$$R\big(\mathcal{C}_{n-1}^3, \mathcal{C}_{n-1}^3\big) = 2(n-1) + \left\lfloor \frac{n-2}{2} \right\rfloor < 2n + \left\lfloor \frac{n-1}{2} \right\rfloor.$$

So we may assume that there is a red copy of $\mathcal{C}_{n-1}^3$ in $\mathcal{H}$. Using Lemma 3.1 we have a blue copy of $\mathcal{C}_n^3$; a contradiction.

Case 2: $n > m$. In this case, we have $n-1 \geqslant m$ and since

$$R\big(\mathcal{C}_{n-1}^3, \mathcal{C}_m^3\big) = 2(n-1) + \left\lfloor \frac{m-1}{2} \right\rfloor < 2n + \left\lfloor \frac{m-1}{2} \right\rfloor,$$

we may assume that $\mathcal{C}_{n-1}^3 \subseteq \mathcal{H}_{\text{red}}$. Using Lemma 3.1 we have a blue copy of $\mathcal{C}_m^3$; a contradiction. □

4. Path–path Ramsey number in 3-uniform hypergraphs

In this section, we determine the exact value of $R(\mathcal{P}_n^3, \mathcal{P}_m^3)$, for $n \geqslant m \geqslant 3$. For this purpose we need the following lemmas.

Lemma 4.1. *Let $n \geqslant m \geqslant 3$, $(n, m) \neq (3,3), (4,3), (4,4)$ and let $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor\frac{m+1}{2}\rfloor}$ be 2-edge colored red and blue. If $\mathcal{P} = \mathcal{P}_{n-1}^3$ is the maximum red path, then there is a copy of $\mathcal{P}_m^3$ in $\mathcal{H}_{\text{blue}}$.*

Proof. Let $t = 2n + \lfloor\frac{m+1}{2}\rfloor$, and $\mathcal{P} = e_1e_2\dots e_{n-1}$ be a red copy of $\mathcal{P}_{n-1}^3$ with edges $e_i = \{v_1, v_2, v_3\} + 2(i-1)$, $i = 1, 2, \dots, n-1$, and let $W = V(\mathcal{K}_t^3) \setminus V(\mathcal{P})$. One can easily check that $|W| \geqslant 3$. By Lemma 2.1 we may assume that there is no copy of $\mathcal{C}_n^3$ in $\mathcal{H}_{\text{red}}$. Since $\mathcal{P} = \mathcal{P}_{n-1}^3$ is the maximum red path, $\mathcal{P}' = e_2 \dots e_{n-1}$ is a maximal red path w.r.t. W. By Corollary 2.5, for some $r \geqslant 0$, there is a blue path $\mathcal{Q}$ (where $\mathcal{Q}$ does not contain $v_{2n-1-2r}$) of length $l' = 2(|W'| - 1)$ between $\bar{\mathcal{P}}' = e_2e_3\dots e_{n-1-r}$ and W' for some $W' \subseteq W$ as in Corollary 2.5. Let y and z be the end vertices of $\mathcal{Q}$ in W', $T = W \setminus W'$ and $x = |T|$. We have one of the following cases.

Case 1: $x = 0$. One can easily check that $l' = 2\lfloor \frac{m+1}{2} \rfloor$. Since $l' \geqslant m$, there is a blue copy of $\mathcal{P}_m^3$ and we are done.

Case 2: $x = 1$. Let $T = \{u\}$. It is easy to see that $l' = 2\lfloor \frac{m+1}{2} \rfloor - 2$. If m is odd, then $l' = m - 1$ and clearly $\mathcal{Q}\{y, v_1, u\}$ is a blue copy of $\mathcal{P}_m^3$. If m is even, then $l' = m - 2$. Since v_{2n-1} is not a vertex of $\mathcal{Q}$ and there is no red copy of $\mathcal{C}_n^3$, clearly $\{v_1, u, y\}\mathcal{Q}\{z, v_2, v_{2n-1}\}$ is a blue copy of $\mathcal{P}_m^3$.

Case 3: $x = 2$. Let $T = \{u, v\}$. In this case, $l' = 2\lfloor \frac{m+1}{2} \rfloor - 4$. Clearly, if m is odd, then $l' = m - 3$ and $r \geqslant 1$. One can easily check that

$$\mathcal{Q}\{z, v_2, v_{2n-2}\}\{v_{2n-2}, u, v\}\{v, v_{2n-1}, v_1\},$$

is a blue copy of $\mathcal{P}_m^3$. If m is even, then clearly $l' = m - 4$ and $r \geqslant 2$. Since $\hat{\mathcal{P}} = e_{n-2}e_{n-1}$ is maximal w.r.t. $\hat{W} = \{y, z, u, v\}$, by Lemma 2.3 there is a blue ϖ_S-configuration, say $\mathcal{P}''$, with $S \subseteq V(e_{n-2}) \cup V(e_{n-1})$ so that $\mathcal{Q}' = \mathcal{Q} \cup \mathcal{P}''$ is a blue path of length $m - 2$ and at least one of v_{2n-2} and v_{2n-1}, say w, is not in $V(\mathcal{Q}')$. Without loss of generality assume that y and v are the end vertices of $\mathcal{Q}'$. Thus, $\{u, v_1, y\}\mathcal{Q}'\{v, w, v_2\}$ is a blue copy of $\mathcal{P}_m^3$.

Case 4: $x \geqslant 3$. One can easily check that this implies that $r \geqslant 3$. This case does not occur by Corollary 2.5. □

Using Theorem 2.2 we have $R(\mathcal{P}_4^3, \mathcal{P}_4^3) = 10$. Since $R(\mathcal{P}_4^3, \mathcal{P}_3^3) \leqslant R(\mathcal{P}_4^3, \mathcal{P}_4^3)$, we have the following lemma.

Lemma 4.2. $R(\mathcal{P}_4^3, \mathcal{P}_3^3) = 10$.

The following theorem on the Ramsey number of 3-uniform loose paths is the main result of this section.

Theorem 4.3. *For every $n \geqslant m \geqslant 3$,*

$$R\left(\mathcal{P}_n^3, \mathcal{P}_m^3\right) = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Proof. We give a proof by induction on $m + n$. Using Theorem 2.2 the case $n = m = 3$ is trivial. Using Theorem 2.2 and Lemma 4.2 we may assume that $n \geqslant 5$. Let $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ be 2-edge colored red and blue with no red copy of $\mathcal{P}_n^3$ and no blue copy of $\mathcal{P}_m^3$. Consider the following cases.

Case 1: $n = m$. By the induction hypothesis,

$$R\left(\mathcal{P}_{n-1}^3, \mathcal{P}_{n-1}^3\right) = 2(n-1) + \left\lfloor \frac{n}{2} \right\rfloor < 2n + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Therefore, we may assume that there is a red copy of $\mathcal{P}_{n-1}^3$. Using Lemma 4.1 we have a blue copy of $\mathcal{P}_n^3$ in $\mathcal{H}$. This is a contradiction.

Case 2: $n > m$. In this case, $n - 1 \geqslant m$. Since

$$R\left(\mathcal{P}_{n-1}^3, \mathcal{P}_m^3\right) = 2(n-1) + \left\lfloor \frac{m+1}{2} \right\rfloor < 2n + \left\lfloor \frac{m+1}{2} \right\rfloor,$$

we may assume that there is a copy of $\mathcal{P}_{n-1}^3$ in $\mathcal{H}_{\text{red}}$. Using Lemma 4.1 we have a blue copy of $\mathcal{P}_m^3$ in $\mathcal{H}$; a contradiction. □

5. Path–cycle Ramsey number in 3-uniform hypergraphs

In this section, the Ramsey number of a loose path and a loose cycle in 3-uniform hypergraphs is determined.

It is worth noting that we can conclude that $R(\mathcal{P}_n^3, \mathcal{C}_m^3) \leqslant 2n + \lfloor \frac{m+1}{2} \rfloor$, for any $n \geqslant m \geqslant 3$. To see this, assume that $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ is 2-edge colored red and blue with no red copy of $\mathcal{P}_n^3$ and no blue copy of $\mathcal{C}_m^3$. Since, using Theorem 3.3,

$$R\left(\mathcal{C}_n^3, \mathcal{C}_m^3\right) = 2n + \left\lfloor \frac{m-1}{2} \right\rfloor < 2n + \left\lfloor \frac{m+1}{2} \right\rfloor,$$

we have a red copy of $\mathcal{C}_n^3$ in $\mathcal{H}$. By Lemma 2.1 this contradicts our assumptions. Thereby, the following theorem holds.

Theorem 5.1. *For every $n \geqslant m \geqslant 3$,*

$$R\left(\mathcal{P}_n^3, \mathcal{C}_m^3\right) = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Combination Theorems 3.3, 4.3 and 5.1 gives a positive answer to Question 1.2. Next, we determine $R(\mathcal{P}_m^3, \mathcal{C}_n^3)$ when $n > m \geqslant 3$.

Lemma 5.2. $R(\mathcal{P}_3^3, \mathcal{C}_4^3) = 9$.

Proof. Using Theorem 2.2 we have $R(\mathcal{C}_4^3, \mathcal{C}_4^3) = 9$. On the other hand, $R(\mathcal{P}_3^3, \mathcal{C}_4^3) \leqslant R(\mathcal{C}_4^3, \mathcal{C}_4^3)$. $\square$

Theorem 5.3. *For every $n > m \geqslant 3$,*

$$R\left(\mathcal{P}_m^3, \mathcal{C}_n^3\right) = 2n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Proof. We prove the theorem by induction on $m+n$. By Lemma 5.2 the case when $m = 3$ and $n = 4$ is trivial. Suppose to the contrary that $\mathcal{H} = \mathcal{K}^3_{2n+\lfloor \frac{m-1}{2} \rfloor}$ is 2-edge colored red and blue with no red copy of $\mathcal{P}_m^3$ and no blue copy of $\mathcal{C}_n^3$ in $\mathcal{H}$. For $n = m+1$ by Theorem 5.1 and for $n > m+1$ by the induction hypothesis we have

$$R\left(\mathcal{P}_m^3, \mathcal{C}_{n-1}^3\right) < 2n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Since there is no red copy of $\mathcal{P}_m^3$, we have a copy of $\mathcal{C}_{n-1}^3$ in $\mathcal{H}_{\text{blue}}$. By using Lemma 3.1 we have a red copy of $\mathcal{P}_m^3$; a contradiction. $\square$

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