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Laurent polynomials, Eulerian numbers,
and Bernstein's theorem



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ABSTRACT

Erman, Smith, and Várilly-Alvarado (2011) showed that the expected number of doubly monic Laurent polynomials $f(z) = z^{-m} + a_{-m+1}z^{-m+1} + \dots + a_{n-1}z^{n-1} + z^n$ whose first $m + n - 1$ powers have vanishing constant term is the Eulerian number $\langle \binom{m+n-1}{m-1} \rangle$, as well as a more refined result about sparse Laurent polynomials. We give an alternate proof of these results using Bernstein's theorem that clarifies the connection between these objects. In the process, we show that a refinement of Eulerian numbers gives a combinatorial interpretation for volumes of certain rational hyperplane sections of the hypercube.

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1. Introduction

Fix positive integers m and n , and let $f(z) = z^{-m} + a_{-m+1}z^{-m+1} + \dots + a_{n-1}z^{n-1} + z^n$ be a doubly monic Laurent polynomial (with complex coefficients). Denote by $\llbracket f^k \rrbracket$, for all positive integers k , the constant coefficient of the k th power of $f(z)$. Then, motivated by a result of Duistermaat and van der Kallen [2] and a related question by Sturmfels [9], Erman, Smith, and Várilly-Alvarado [3] proved the following theorem.

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Theorem 1.1. (See [3, Theorem 2].) *The expected number of Laurent polynomials $f(z)$ such that $\llbracket f^1 \rrbracket = \llbracket f^2 \rrbracket = \dots = \llbracket f^{m+n-1} \rrbracket = 0$ is the Eulerian number $\langle \binom{m+n-1}{m-1} \rangle$.*

Recall that the Eulerian number $\langle \binom{n}{k} \rangle$ is the number of permutations in S_n with exactly k descents. (In [3], the “expected number” is computed as the degree of the ideal

$$I_{m,n} = \langle \llbracket f^1 \rrbracket, \llbracket f^2 \rrbracket, \dots, \llbracket f^{m+n-1} \rrbracket \rangle$$

in $\mathbb{C}[a_{-m+1}, \dots, a_{n-1}]$, assuming this ideal is zero-dimensional.)

Their proof of this theorem involves computing the Chow ring of an appropriate toric variety and then showing that the desired answer satisfies a certain recurrence that is also satisfied by the Eulerian numbers. As a result, the appearance of the Eulerian numbers using this approach is somewhat mysterious.

In this paper, we will attempt to elucidate this connection by giving an alternate proof of this result. We will use Bernstein’s theorem (also known as the BKK Theorem after Bernstein [1], Kušnirenko [5], and Khovanskii [4]) to relate the number of solutions of an appropriate polynomial system to the normalized volume of a hypersimplex, which is known to be enumerated by an Eulerian number in a natural way (see [8]). We will also, again following [3], apply a similar technique to show the analogous result for “sparse” Laurent polynomials. In the process, we will show that the volume of the intersection of the unit hypercube $[0, 1]^n$ with a hyperplane of the form $\sum_{i=1}^n x_i = \frac{c}{d}$ for $\frac{c}{d} \in \mathbb{Q}$ can be interpreted combinatorially in terms of a refinement of the Eulerian numbers. (This refinement is a special case of the one given by the cyclic sieving phenomenon described in [7].)

2. Preliminaries

In this section, we review the two main ingredients needed for our proof, namely Bernstein’s theorem on counting solutions to generic polynomial systems, and the relationship between Eulerian numbers and hypersimplices.

2.1. Bernstein’s theorem

Given a Laurent polynomial $P(x_1, \dots, x_n) = \sum c_a x^a$ (where $x^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$), its *support* is the set of vectors $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ such that c_a is nonzero. The *Newton polytope* $\Delta_P \subset \mathbb{R}^n$ of P is the convex hull of the support of P . Bernstein’s theorem gives a bound on the number of solutions to a polynomial system $P_1 = P_2 = \dots = P_n = 0$ that lie in the algebraic torus $(\mathbb{C}^\times)^n$ in terms of the Newton polytopes $\Delta_{P_1}, \dots, \Delta_{P_n}$.

More precisely, given polytopes $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$, the volume of the polytope $t_1 \Delta_1 + \dots + t_n \Delta_n$ (where $+$ denotes Minkowski sum) is a polynomial in t_1, \dots, t_n . Then the *mixed volume* $MV(\Delta_1, \dots, \Delta_n)$ is the coefficient of $t_1 t_2 \dots t_n$ in this polynomial. Bernstein’s theorem can then be stated as follows.

Theorem 2.1. (See [1].) Let $P_1, \dots, P_n \in \mathbb{C}[x_1, \dots, x_n]$ with Newton polytopes $\Delta_1, \dots, \Delta_n$. Then the polynomial system $P_1 = P_2 = \dots = P_n = 0$ has at most $MV(\Delta_1, \dots, \Delta_n)$ isolated solutions in $(\mathbb{C}^\times)^n$. Moreover, if P_1, \dots, P_n have generic coefficients given $\Delta_1, \dots, \Delta_n$, then this bound is exact.

Bernstein’s theorem can be proved, for instance, using toric geometry or homotopy continuation methods.

2.2. Hypersimplices

Let k and n be positive integers with $0 < k < n$. The hypersimplex $\Delta_{k,n} \subset \mathbb{R}^n$ is the polytope

$$\Delta_{k,n} = \{(x_1, \dots, x_n) \mid x_1 + x_2 + \dots + x_n = k, x_i \in [0, 1]\}.$$

In other words, $\Delta_{k,n}$ is a slice of the hypercube $[0, 1]^n$ by the hyperplane $\sum x_i = k$. Note that $\Delta_{k,n}$ has dimension $n - 1$, and $\Delta_{1,n}$ is the standard $(n - 1)$ -simplex. Normalizing so that the $(n - 1)$ -dimensional volume of $\Delta_{1,n}$ is 1, we have the following theorem of Laplace.

Theorem 2.2 (Laplace). The normalized volume of $\Delta_{k,n}$ is the Eulerian number $\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle$.

Here the Eulerian number $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$ is the number of permutations in S_n with exactly k descents. In [8], Stanley gives a simple proof of this result by constructing an explicit bijection between permutations in S_{n-1} with $k - 1$ descents and simplices in a unimodular triangulation of $\Delta_{k,n}$ (thereby answering a question of Foata).

3. Proof of the main theorem

We are now ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $N = m + n$, and write

$$f(z) = z^{-m} \prod_{i=1}^N (1 + r_i z),$$

so that $\prod r_i = 1$. Then the map sending $r = (r_1, \dots, r_N)$ to $f(z)$ has degree $N!$, so it suffices to show that the expected number of possible r is $N! \cdot \langle \begin{smallmatrix} N-1 \\ m-1 \end{smallmatrix} \rangle$.

Since the constant term $[[f^k]]$ is the coefficient of z^{mk} in $\prod_{i=1}^N (1 + r_i z)^k$, it can be expressed as a symmetric polynomial $P_k(r_1, \dots, r_N)$. The monomial $r_1^{a_1} \dots r_N^{a_N}$ appears in P_k with coefficient $\prod_{i=1}^N \binom{k}{a_i}$ provided that $\sum a_i = mk$. Hence the support of P_k is the set of all exponent vectors $a = (a_1, \dots, a_N)$ such that $\sum a_i = mk$ and $0 \leq a_i \leq k$. Thus the Newton polytope Δ_{P_k} is just $k\Delta_{m,N}$.

By Bernstein’s theorem, the expected number of solutions to the system

$$P_1(r) = \cdots = P_{N-1}(r) = r_1 r_2 \cdots r_N - 1 = 0$$

(noting that we must have $r_i \in \mathbb{C}^\times$) is

$$MV(\Delta_{m,N}, 2\Delta_{m,N}, \dots, (N - 1)\Delta_{m,N}, I),$$

where I is the segment between $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. This is the coefficient of $t_1 t_2 \cdots t_N$ in the volume of

$$\Delta(t_1, \dots, t_N) := (t_1 + 2t_2 + \cdots + (N - 1)t_{N-1})\Delta_{m,N} + t_N I.$$

Let $\Lambda \subset \mathbb{R}^N$ be the hyperplane $\sum a_i = 0$, so that $\Delta_{m,N}$ lies in a hyperplane parallel to Λ and I is orthogonal to Λ . Then $\mathbb{Z}^{N-1} \cap \Lambda$ and $(1, 1, \dots, 1)$ span a lattice that has index N in \mathbb{Z}^N . Thus, by [Theorem 2.2](#), $\text{Vol}(\Delta_{m,N} + I) = \frac{N}{(N-1)!} \langle \begin{smallmatrix} N-1 \\ m-1 \end{smallmatrix} \rangle$, and hence

$$\text{Vol} \Delta(t_1, \dots, t_N) = (t_1 + 2t_2 + \cdots + (N - 1)t_{N-1})^{N-1} t_N \cdot \frac{N}{(N - 1)!} \left\langle \begin{smallmatrix} N - 1 \\ m - 1 \end{smallmatrix} \right\rangle.$$

The coefficient of $t_1 \cdots t_N$ is then $N! \cdot \langle \begin{smallmatrix} N-1 \\ m-1 \end{smallmatrix} \rangle$, as desired. \square

4. Refinement

The authors of [\[3\]](#) also count the expected number of Laurent polynomials as in [Theorem 1.1](#) that are “sparse” in a certain sense. Let us call $f(z)$ d -sparse for some d dividing $N = m + n$ if the coefficient of z^i vanishes unless $i \equiv n \pmod{d}$. (Note that every $f(z)$ is 1-sparse.) We will adapt our technique from the previous section to count the expected number of d -sparse $f(z)$ satisfying [Theorem 1.1](#).

We first define a fractional analogue of the hypersimplex. Given a positive integer n and a rational number $\frac{c}{d}$ with $0 < \frac{c}{d} < n$ and $\text{gcd}(c, d) = 1$, define the rational polytope

$$\Delta_{\frac{c}{d},n} = \left\{ (x_1, \dots, x_n) \mid x_1 + \cdots + x_n = \frac{c}{d}, x_i \in [0, 1] \right\}.$$

In other words, $\Delta_{\frac{c}{d},n}$ is a slice of the hypercube $[0, 1]^n$ by the hyperplane $\sum x_i = \frac{c}{d}$. Although $\Delta_{\frac{c}{d},n}$ is not an integer polytope, $d\Delta_{\frac{c}{d},n}$ is. We will give a combinatorial interpretation for the normalized volume of $d\Delta_{\frac{c}{d},n}$.

We may consider the permutations in S_n as circular permutations of $\{0, 1, \dots, n\}$ (that is, permutations of $\{0, 1, \dots, n\}$ up to cyclic shift). Then a permutation with k descents corresponds to a circular permutation with $k + 1$ (cyclic) descents. For any d dividing $n + 1$, we let $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle_d$ count the number of these circular permutations fixed by adding $\frac{n+1}{d}$ modulo $n + 1$ to each number in the word of the permutation. Note that $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle_1 = \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$.

Example. Let $n = 5$ and $k = d = 2$. Then $\langle \frac{5}{2} \rangle_2$ counts the number of circular permutations of $\{0, \dots, 5\}$ with 3 descents that are fixed upon adding $\frac{6}{2} = 3$ (modulo 6) to each number. There are 6 of these: 042315, 015342, 045312, 021354, 051324, and 024351.

Compare this to $2\Delta_{\frac{3}{2},3}$, the intersection of the cube $[0, 2]^3$ with the plane $x_1 + x_2 + x_3 = 3$, which is a hexagon of normalized volume 6.

Remark. The definition given above is attributed to Postnikov in [3] (cf. [6]). One can also obtain these numbers using the cyclic sieving phenomenon on permutations with fixed cycle type and number of exceedances described in [7] as follows: if $w = w_1 \cdots w_n$ is a permutation in S_n with k ascents, then $(0 w_1 w_2 \cdots w_n)$ is a permutation of $\{0, 1, \dots, n\}$ with cycle type $\lambda = (n + 1)$ and $k + 1$ exceedances. The cyclic action generated by conjugation by $(0 1 2 \cdots n - 1)$ as in [7] then corresponds to the addition of a constant modulo $n + 1$ to each letter of a circular permutation as described in the previous paragraph.

Analogously to Theorem 2.2, we have the following theorem.

Theorem 4.1. *The normalized volume of $d\Delta_{\frac{c}{d},n}$ is $\langle \frac{dn-1}{c-1} \rangle_d$.*

Proof. The polytope $d\Delta_{\frac{c}{d},n}$ is defined by $0 \leq x_i \leq d$ for $1 \leq i \leq n$, and $\sum x_i = c$. We may slice this polytope by the hyperplanes $x_i = a$ for all x_i and all integers a . Then each of the resulting regions is a translate of a hypersimplex: the region containing a general point (x_1, \dots, x_n) is a translate of $\Delta_{k,n}$ by $(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$, where $k = c - \sum \lfloor x_i \rfloor$. Hence, using Theorem 2.2, the normalized volume of $d\Delta_{\frac{c}{d},n}$ equals

$$\sum_{k=1}^{n-1} \left\langle \frac{n-1}{k-1} \right\rangle \cdot \#\left\{ (x_1, \dots, x_n) \mid \sum x_i = c - k, x_i \in \{0, 1, \dots, d - 1\} \right\}.$$

In other words, the normalized volume of $d\Delta_{\frac{c}{d},n}$ is the number of pairs (w, x) , where $w \in S_{n-1}$ has $k - 1$ descents and $x = (x_1, \dots, x_n)$ is an integer point with $\sum x_i = c - k$ and $x_i \in [0, d - 1]$. It therefore suffices to show that the number of such pairs (w, x) is $\langle \frac{dn-1}{c-1} \rangle_d$. We consider w and x to be periodic sequences with period n (with $w_0 = 0$).

Let $p = (p_0, p_1, \dots)$ be the sequence such that $p_0 = 0$, and $p_i - nx_i$ is the smallest integer congruent to w_i modulo n that is larger than w_{i-1} . For instance, for $n = 6$, if $w = w_0w_1w_2 \dots = 014352 \dots$ and $x = x_1x_2 \dots = 010021 \dots$, then

$$p = (0, 1, 10, 15, 17, 32, \quad 42, 43, 52, 57, 59, 74, \quad \dots).$$

Note that p is strictly increasing and $p_i - p_{i-1} < n(x_i + 1) \leq dn$ for all i . Moreover, the number of multiples of n in the interval $(p_{i-1}, p_i]$ is x_i if $w_{i-1} < w_i$, and $x_i + 1$ if $w_{i-1} > w_i$. Since the permutation $w \in S_{n-1}$ has exactly $k - 1$ descents (so as a circular permutation it has k descents), we have that $p_n = (k + \sum x_i)n = cn$, so $p_{i+n} = p_i + cn$ for all i .

Then consider the sequence \bar{p} obtained by reducing the terms in the sequence p modulo dn . Since c and d are relatively prime and p_0, \dots, p_{n-1} are all different modulo n , it follows that $\bar{p}_0, \dots, \bar{p}_{dn-1}$ are all distinct (as residues modulo dn). Moreover, since $p_{dn} = cdn$ and $p_i - p_{i-1} < dn$, \bar{p} has exactly c circular descents (corresponding to whenever there is a multiple of dn in the interval $(p_{i-1}, p_i]$). Finally, since $p_{i+n} = p_i + cn$, adding n (modulo dn) to each \bar{p}_i performs a cyclic shift by $c'n$ to \bar{p} , where $c' \equiv c^{-1} \pmod{d}$. It follows that \bar{p} is one of the circular permutations counted by $\langle \binom{dn-1}{c-1} \rangle_d$.

Conversely, starting with a circular permutation \bar{p} counted by $\langle \binom{dn-1}{c-1} \rangle_d$, we can form the sequence p by letting p_i be the smallest integer larger than p_{i-1} congruent to \bar{p}_i modulo dn . Then since \bar{p} has c circular descents, $p_{dn} = cdn$. By the symmetry property of \bar{p} , we must have that $p_{i+n} - p_i = cn$. (Note that for \bar{p} to be a permutation, we must have c relatively prime to d .) It is now easy to reverse the procedure above to recover (w, x) , completing the proof of the bijection. \square

Note that it follows from this proof that $\langle \binom{n}{k} \rangle_d = 0$ if d and $k + 1$ are not relatively prime.

We can now mimic the proof of [Theorem 1.1](#) to get a result about d -sparse Laurent polynomials, as in [\[3\]](#).

Theorem 4.2. (See [\[3, Proposition 5\]](#).) *The expected number of d -sparse Laurent polynomials $f(z)$ such that $\llbracket f^1 \rrbracket = \llbracket f^2 \rrbracket = \dots = \llbracket f^{m+n-1} \rrbracket = 0$ is $\langle \binom{m+n-1}{m-1} \rangle_d$.*

Proof. Let $N = m + n$. If $f(z)$ is d -sparse, then we can write

$$f(z) = z^{-m} \prod_{i=1}^{N/d} (1 + r_i z^d)$$

for some nonzero $r_1, \dots, r_{N/d}$ with product 1, and again, as in [Theorem 1.1](#), it suffices to show that the expected number of possible r is $(\frac{N}{d})! \cdot \langle \binom{N-1}{m-1} \rangle_d$.

As in [Theorem 1.1](#), we wish to find the coefficient of z^{mk} in $\prod_{i=1}^{N/d} (1 + r_i z^d)^k$ for $k = 1, \dots, N - 1$ as a polynomial in the r_i . This polynomial will be nonzero when d divides mk . Hence if d is not relatively prime to m , this will occur for more than $\frac{N}{d} - 1$ values of k , so we would not expect any such f to exist.

If d is relatively prime to m , then for $k = dk'$, let $P_{k'}(r_1, \dots, r_{\frac{N}{d}})$ be the coefficient of z^{mk} in $\prod_{i=1}^{N/d} (1 + r_i z^d)^k$, or equivalently the coefficient of $z^{mk'}$ in $\prod_{i=1}^{N/d} (1 + r_i z^d)^{dk'}$. Then $(a_1, \dots, a_{\frac{N}{d}})$ lies in the support of $P_{k'}$ when $0 \leq a_i \leq dk'$ and $\sum_{i=1}^{N/d} a_i = mk'$. In other words, the Newton polytope of $P_{k'}$ is $k' \cdot d\Delta_{\frac{m}{d}, \frac{N}{d}}$. Thus the expected number of such f is

$$MV\left(d\Delta_{\frac{m}{d}, \frac{N}{d}}, 2 \cdot d\Delta_{\frac{m}{d}, \frac{N}{d}}, \dots, \left(\frac{N}{d} - 1\right) \cdot d\Delta_{\frac{m}{d}, \frac{N}{d}}, I\right),$$

where I is the segment between $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. Then the same argument as in the last paragraph of the proof of [Theorem 1.1](#) shows that this mixed volume is $(\frac{N}{d})!$ times the normalized volume of $d\Delta_{\frac{m}{d}, \frac{N}{d}}$, so the result follows from [Theorem 4.1](#). \square

5. Concluding remarks

Given the apparent connection between Laurent polynomials $f(z)$ that satisfy [Theorem 1.1](#) and Eulerian numbers, it is natural to ask whether there is any direct correspondence. In other words, can one give a bijection between permutations in S_{n-1} with $k-1$ descents and these Laurent polynomials?

It is perhaps worth noting that the proof of [Theorem 2.2](#) given in [\[8\]](#) is bijective: it is shown that a hypersimplex can be triangulated in such a way that there is a natural bijection between the simplices in the triangulation and permutations with a given number of descents. (For more information about this triangulation, see [\[6\]](#).) Moreover, one can prove [Theorem 2.1](#) using homotopy continuation methods, from which one may similarly be able to derive a bijection between the Laurent polynomials $f(z)$ and certain simplices. However, the details behind this line of reasoning have yet to be explored.

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