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Hilbert functions and the finite degree Zariski closure in finite field combinatorial geometry

Zipei Nie^b, Anthony Y. Wang^a

^a Department of Mathematics, University of Chicago, Chicago, IL, 60637, United States

^b Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, 02142, United States

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ABSTRACT

The polynomial method has been used recently to obtain many striking results in combinatorial geometry. In this paper, we use affine Hilbert functions to obtain an estimation theorem in finite field geometry. The most natural way to state the theorem is via a bounded degree Zariski closure operation: given a set, we consider all polynomials of some bounded degree vanishing on that set, and then the common zeros of these polynomials. For example, the degree d closure of $d + 1$ points on a line will contain the whole line, as any polynomial of degree at most d vanishing on the $d + 1$ points must vanish on the line. Our result is a bound on the size of a finite degree closure of a given set. Finally, we adapt our use of Hilbert functions to the method of multiplicities.

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1. Introduction

The polynomial method has recently been applied to problems in combinatorial geometry. The general idea of what is usually termed the polynomial method is to find a polynomial vanishing on a set of interest, and then to use algebraic methods to recover

E-mail addresses: zipei@mit.edu (Z. Nie), anthonyw@math.uchicago.edu (A.Y. Wang).

combinatorial information. Our paper primarily concerns the use of the polynomial method in the geometry over finite fields. Dvir’s proof of the finite field Kakeya conjecture was a recent breakthrough in this area, showing that a Kakeya set in n dimensional space over a finite field \mathbb{F}_q^n must have size at least $c_n q^n$ for some constant c_n [5].

In this paper, we wish to broaden the understanding of the application of the polynomial method to this and related phenomena. We introduce the concept of degree d closure. Let k be a field, and X a subset of k^n . The degree d closure of Y , denoted $\text{cl}_d(Y)$, is defined as

$$\text{cl}_d(Y) = \{x \in k^n \mid \text{every polynomial of degree at most } d \text{ vanishing on } Y \text{ vanishes at } x\}.$$

‘Taking the degree d closure’ seems to be a common operation in applications of the polynomial method to combinatorial geometry, although not in this terminology. For example, saying the degree d closure of $d + 1$ points on a line contains the line is the same as saying every polynomial of degree d vanishing on $d + 1$ points on a line vanishes on the line. The terminology degree d closure was chosen because of its similarity to the Zariski closure of a set. The Zariski closure of Y is

$$\text{cl}(Y) = \{y \in k^n \mid \text{every polynomial vanishing on } Y \text{ vanishes at } y\}.$$

In our paper we show that for subsets Y of \mathbb{F}_q^n , we must have

$$^{\text{aHF}}(\mathbb{F}_q^n, d) |\text{cl}_d(Y)| \leq q^n |Y|,$$

where $^{\text{aHF}}(\mathbb{F}_q^n, d)$ is the affine Hilbert function of the space \mathbb{F}_q^n (the dimension of the polynomials of degree at most d on \mathbb{F}_q^n). In particular, if $d < q$, we have

$$\binom{d+n}{n} |\text{cl}_d(Y)| \leq q^n |Y|.$$

This inequality gives us a bound on the size of the degree d closure of Y over finite fields.

To show the use of our inequality, we note that it will immediately solve the following problem, suggested to us by Larry Guth. Given lines L_1, \dots, L_c in \mathbb{F}_q^3 , pick subsets $\gamma_i \subset L_i$ such that $|\gamma_i| \geq q/2$. Suppose $X = \cup L_i$, and $Y = \cup \gamma_i$. Does there exist a constant C (independent of q) such that $|X| \leq C|Y|$?

We can get such a constant C by noting that the degree $\lfloor (q-1)/2 \rfloor < q/2$ closure of Y contains X , and then applying the inequality. To the extent of our knowledge, the previous best bound was $|X| \leq C \log q |Y|$ as given in [9, Lecture 12]. Indeed, the method can be use to prove variants quite easily. For example, we can get a fixed constant in \mathbb{F}_q^n for any fixed dimension n . Moreover, we can prove a similar bound when instead of X being a union of lines, it is a union of curves of bounded degree:

Theorem. *Given curves $\Gamma_1, \dots, \Gamma_c$ of degree at most Λ in \mathbb{F}_q^n , pick subsets $\gamma_i \subset \Gamma_i$ such that $|\gamma_i| \geq q/2$. Then there is a constant C such that $|\cup \Gamma_i| \leq C |\cup \gamma_i|$.*

This follows from the bound by noting that the degree d closure of $\cup \gamma_i$ contains $\cup \Gamma_i$ for $d < q/(2\Lambda)$. Alternatively, we could have assumed instead of $|\gamma_i| \geq q/2$ that $|\gamma_i| > q^\alpha$ for some $0 < \alpha < 1$ to get

Theorem. *Given lines L_1, \dots, L_c in \mathbb{F}_q^n , pick subsets $\gamma_i \subset \Gamma_i$ such that $|\gamma_i| > q^\alpha$. Then $|\cup L_i| \leq n!q^{n(1-\alpha)}|\cup \gamma_i|$.*

This follows from the inequality by noting that the degree q^α closure of $\cup \gamma_i$ contains $\cup L_i$. We can also restate proofs of old bounds in this language. For example, in Dvir's proof of the finite field Kakeya conjecture [5], part of his argument was establishing that the degree $q - 1$ closure of any Kakeya set is all of \mathbb{F}_q^n . Indeed, he argues that since for any Kakeya set K , there is a line in every direction, any polynomial of degree $\leq q - 1$ vanishing on K must vanish at the hyperplane at infinity. But the only affine polynomial vanishing at the plane at infinity is 0: thus the degree $q - 1$ closure of K must be \mathbb{F}_q^n . We can conclude this version of his proof by applying the bound. The general argument for this and related proofs is to find the smallest degree d of a polynomial vanishing on a set, and then use the fact that for finite sets X , there exists a polynomial of degree $n|X|^{1/n}$ vanishing on it. The bound can be viewed as a sort of generalization of this argument. If d is the smallest degree of a polynomial vanishing on X , then the degree $d - 1$ closure of X will be the whole space. If we are working in the space over a finite field, then our bound will give the same bound on the size of X as the original argument.

To the bound, we first establish that the first d terms of the affine Hilbert function of Y and $\text{cl}_d(Y)$ agree. Then, we use the affine Hilbert function to estimate the size of a set of monomials spanning the polynomial functions on $\text{cl}_d(Y)$. As we are working over a finite field, the set $\text{cl}_d(Y)$ is finite, so the polynomial functions on it are precisely the functions on it and has dimension equal to the size of $\text{cl}_d(Y)$. Thus, we get a bound on $|\text{cl}_d(Y)|$.

We also generalize our results to the method of multiplicities which has been applied to this area by Saraf and Sudan [11], and by Dvir et al. [6]. This method used the fact that polynomials of bounded degree vanishing to high order on a set must vanish to high order on a larger set to get better bounds than was obtained in [5] on the size of a Kakeya set. Similar to before, we can define an operation $\text{cl}_d^{\ell,m}(Y)$ to be the set of points x such that all polynomials of degree at most d vanishing to order m on Y vanish to order ℓ at x . Whereas before, the degree d closure operation of a set was related to the ring of polynomials on the set, this new operation is related to the polynomials on the set up to some order of vanishing. More precisely, just like we can define the ideal $I(Y)$ to be the set of polynomials vanishing at Y , we can define the ideal $I^m(Y)$ to be the set of polynomials vanishing to order m at Y . Now just as the affine Hilbert function $^a\text{HF}(Y, d)$ counts the dimension of polynomials of degree at most d of the ring $k[x_1, \dots, x_n]/I(Y)$, we can define $^a\text{HF}^m(Y, d)$ to count the dimension of polynomials of degree at most d of the ring $k[x_1, \dots, x_n]/I^m(Y)$. Again, the fact that $X \subset \text{cl}_d^{\ell,m}(Y)$ gives information

about the modified Hilbert functions: namely that ${}^a\text{HF}^\ell(X, d) \leq {}^a\text{HF}^m(Y, d)$. We can then use the modified Hilbert functions to recover combinatorial information about X and Y .

Framed in this manner, we can give a speculative reason why considering vanishing to higher multiplicities improves the bounds gotten without considering such higher order vanishings. For this, recall the Schwartz–Zippel lemma with multiplicity, which states that if X is a finite subset of a line L , every polynomial of degree less than $|X|(m - \ell + 1) + \ell - 1$ vanishing to order m on X must vanish to order ℓ on L , i.e. $L \subset \text{cl}_d^{\ell, m}(X)$ for such d . Recall that for proving statements such as the finite field Kakeya and the finite field Nikodym (without multiplicities), we only used lines to show that the degree d closure for some set contains some larger set. The reason using multiplicity gives better bounds is that $\text{cl}_d^{\ell, m}$ allows us to isolate these lines better than cl_d . For example, if Y is a set of points, then $\text{cl}_d(Y)$ must contain all lines passing through at least $d + 1$ of the points of Y . However, it must also contain all lines passing through at least $d + 1$ points of this larger set. And so on. By using $\text{cl}_d^{\ell, m}$ for properly chosen ℓ , m and d , we can reduce the amount of baggage that comes along with the ‘and so on’. On the other hand, if we know the size of X and that $X \subset |\text{cl}_d^{\ell, m}(Y)|$ for properly chosen ℓ , m and d , we know X was gotten from Y ‘with minimal baggage’, and so our bound on the size of Y will be larger.

As an application of our consideration of multiplicities, we apply it to the problem of Statistical Kakeya for Curves in Dvir, Kopparty, Saraf and Sudan’s paper [6]. Namely, we show

Theorem (*Statistical Kakeya for curves*). *Let X and Y be subsets of \mathbb{F}_q^n . Suppose that for every point $x \in X$, there is a curve C_x of degree at most Λ through x which intersects Y in at least τ points. Then*

$$|X| \leq \left(1 + \frac{\Lambda(q-1)}{\tau}\right)^n |Y|.$$

The essential difference in our arguments is that they use a polynomial vanishing to high order on a set to bound the size of the set, while we use information from the modified Hilbert function. Although we achieve the same bound as them for the case where the set X is \mathbb{F}_q^n (which is the case of interest in the applications to the Kakeya problem), our bound is strictly better in nearly all other cases, namely when the dimension is at least 2 and $X \subsetneq \mathbb{F}_q^n$.

We expect that using multiplicities will give better bounds when we know the set $X \subset \text{cl}_d(Y)$ via an argument using lines directly and not some iterated argument. In line with this, the statistical Kakeya for curves will improve the constants in the some of the bounds we got before from just applying the bound for cl_d .

Lastly, we give a bound on $\text{cl}_d^{\ell,m}(Y)$ for $Y \in \mathbb{F}_q^n$. We show that

$${}^{\text{aHF}}(\mathbb{F}_q^n, d) |\text{cl}_d^{\ell,m}(Y)| \leq q^n \binom{m+n-1}{n} |Y|,$$

whose proof follows the same general outline as the proof of the bound on $\text{cl}_d(Y)$.

The general outline of the paper is as follows. Sections 2, 3 and 4 concentrate on preliminaries and establish notation used in the rest of the paper. Section 2 focuses on affine Hilbert functions while Section 3 focuses on graded monomial orders. Section 4 uses the FKG inequality to prove a bound on the size of a subset $Y \subset \mathbb{F}_q^n$ given a value of the affine Hilbert function of Y , namely

$${}^{\text{aHF}}(\mathbb{F}_q^n, d) |Y| \leq {}^{\text{aHF}}(Y, d) q^n.$$

Currently, we do not have an elementary means of proving this inequality without using the FKG inequality. However, we will note that we can skip the use of the FKG inequality at the expense of incurring an extra factor of approximately $n!$ in our estimates of $\text{cl}_d(Y)$. Section 5 focuses on the degree d closure and its applications, while Section 6 focuses on the higher multiplicity versions, and their application.

2. Affine Hilbert functions

In this section, we review preliminaries and set our notation for affine Hilbert functions. (Our reference for this material is [4, Chapter 9, Section 3].)

We work over the ring of polynomials $A = k[x_1, \dots, x_n]$ over a field k . Let $A_{\leq d}$ denote the polynomials of degree at most d . For an ideal I of $k[x_1, \dots, x_n]$, let $I_{\leq d}$ denote the polynomials of degree at most d in I . Note that $A_{\leq d}$ and $I_{\leq d}$ are both vector spaces over k . The **affine Hilbert function** of I , denoted by ${}^{\text{aHF}}_I$ is given by

$${}^{\text{aHF}}_I(d) = \dim A_{\leq d} / I_{\leq d} = \dim A_{\leq d} - \dim I_{\leq d}.$$

As we will only be using the affine Hilbert function throughout this paper, we will typically just call it the Hilbert function. It is clear that ${}^{\text{aHF}}_I(d)$ is nondecreasing in d and that if $I \subset J$ are ideals, then ${}^{\text{aHF}}_I(d) \geq {}^{\text{aHF}}_J(d)$.

Given a set X of k^n , let $I(X)$ denote the ideal of polynomial functions vanishing on X , that is,

$$I(X) = \{P \in k[x_1, \dots, x_n] \mid P(x) = 0 \text{ for all } x \in X\}.$$

The **Hilbert function** of X , which we denote by ${}^{\text{aHF}}(X, d)$ is then defined as the Hilbert function of the ideal $I(X)$. Again, the Hilbert function ${}^{\text{aHF}}(X, d)$ is nondecreasing in d . Moreover, if $X \subset Y$, then $I(X) \supset I(Y)$, so ${}^{\text{aHF}}(X, d) \leq {}^{\text{aHF}}(Y, d)$.

Given a polynomial $P \in k[x_1, \dots, x_n]$ and a finite set $Y = \{y_1, \dots, y_s\} \in k^n$, we can define the evaluation map sending P to its values on Y , i.e.

$$P \mapsto (P(y_1), \dots, P(y_s)) \in k^{|Y|}.$$

This map is clearly linear. The polynomials also surject: the ideals $I(\{y_1\}), \dots, I(\{y_s\})$ are maximal and therefore pairwise coprime. The Chinese remainder theorem then immediately shows that the polynomial functions surject onto $k[x_1, \dots, x_n]/I(Y)$. Moreover, we see the ideal $I(\{y_1, \dots, y_s\}) = I(\{y_1\}) \cdots I(\{y_s\})$.

Lemma 2.1. *If $Y = \{y_1, \dots, y_s\}$ is a finite subset of k^n and $d \geq |Y| - 1$, then*

$${}^a\text{HF}(Y, d) = |Y|.$$

Proof. For $j \geq 2$, it is not difficult to find a polynomial vanishing on y_j but not on y_1 . (Indeed, some component of y_j and y_1 must be different, since otherwise they would be the same point.) By multiplying these polynomials together and normalizing, we can find a polynomial p_1 of degree at most $|Y| - 1$ which vanishes on y_2, \dots, y_s but is equal to 1 on y_1 . Similarly, we can find polynomials p_i which vanish on y_j for $j \neq i$ but is equal to 1 on y_i .

Thus, the polynomials of degree at most $|Y| - 1$ surject onto $k[x_1, \dots, x_n]/I(Y)$, so ${}^a\text{HF}(Y, d) = |Y|$ for $d \geq |Y| - 1$. \square

The bound on d given in the lemma is the best bound that works for all sets Y and all fields k . To see why, one can all Y points to lie on a line. However, for finite fields, we can do better:

Lemma 2.2. *We have $I(\mathbb{F}_q^n) = (x_1^q - x_1, \dots, x_n^q - x_n)$. Thus, when working over the space \mathbb{F}_q^n , the set of monomials*

$$S = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i \leq q - 1\}$$

form a basis for $\mathbb{F}_q[x_1, \dots, x_n]/I(\mathbb{F}_q^n)$.

Proof. Note that the set of monomials

$$S = \{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq q - 1\}$$

form a spanning set of $\mathbb{F}_q[x_1, \dots, x_n]/(x_1^q - x_1, \dots, x_n^q - x_n)$. Now the evaluation map on \mathbb{F}_q^n surjects $A = \mathbb{F}_q[x_1, \dots, x_n]$ onto a vector space of dimension q^n . Clearly the kernel $I(\mathbb{F}_q^n)$ contains $(x_1^q - x_1, \dots, x_n^q - x_n)$. Then

$$q^n = \dim A/I(\mathbb{F}_q^n) \leq A/(x_1^q - x_1, \dots, x_n^q - x_n) \leq |S| = q^n.$$

Thus, the inequalities are equalities and $I(\mathbb{F}_q^n) = (x_1^q - x_1, \dots, x_n^q - x_n)$. We conclude that S forms a basis for $\mathbb{F}_q[x_1, \dots, x_n]/I(\mathbb{F}_q^n)$. \square

Corollary 2.3. *If Y is a subset of \mathbb{F}_q^n and $d \geq n(q-1)$, then*

$${}^a\text{HF}(Y, d) = |Y|.$$

Proof. Now the maximum degree of a polynomial in the set of monomials S of Lemma 2.2 is $n(q-1)$, so we see that $\mathbb{F}_q[x_1, \dots, x_n]_{\leq n(q-1)}$ surjects onto the functions on \mathbb{F}_q^n . In particular, it surjects onto the functions on Y . Therefore,

$${}^a\text{HF}(Y, d) = |Y|,$$

for all $d \geq n(q-1)$. \square

We can use the evaluation map to get another result on Hilbert functions:

Lemma 2.4. *Let X_1, \dots, X_n be subsets of k^n and $X = \bigcup X_i$. Then*

$${}^a\text{HF}(X, d) \leq \sum_{i=1}^n {}^a\text{HF}(X_i, d).$$

Proof. A degree $\leq d$ polynomial P on X is a degree $\leq d$ polynomial P_i on each of the X_i . Now the map $P \mapsto (P_1, \dots, P_n)$ is injective, since if a polynomial is 0 on every X_i , then it is 0 on X . Counting degrees then gives the above bound.

The more general way to say this is if I_1, \dots, I_n are ideals of a ring A and $I = \bigcap I_i$ is an ideal, then

$${}^a\text{HF}_I(d) \leq \sum_{i=1}^n {}^a\text{HF}_{I_i}(d).$$

Indeed, there is a projection $A_{\leq d}/I_{\leq d} \rightarrow A_{\leq d}/I_{i \leq d}$, and if the image of a polynomial P is zero for all I_i , then the original polynomial must have been in $\bigcap I_i$, i.e. it was 0. Thus, the given map is injective, and the conclusion follows. \square

3. Monomial orders

In this section, we give preliminaries on monomial orders. Our references are Chapter 2, Section 2 and Chapter 9, Section 3 of [4] and Section 15.2 of [7].

First, if $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, then we let x^a denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$. We let $|a| = a_1 + \cdots + a_n$, so that the degree of x^a is $|a|$.

A **monomial order** is a total order on the monomials of the ring $k[x_1, \dots, x_n]$ satisfying the following two conditions:

- (i) $1 \leq x^a$ for all $a \in \mathbb{N}^n$.
- (ii) If $x^a < x^b$, then $x^a x^c < x^b x^c$ for all $c \in \mathbb{N}^n$.

We say a monomial order is **graded** if it refines the partial order on monomials given by degree, that is, if $\deg(x^a) < \deg(x^b)$, then $x^a < x^b$.

Lemma 3.1 (Well ordering). *Given a monomial order, any nonempty subset S of monomials has a least element.*

Proof. Let I be the monomial ideal generated by the elements of S . Since $k[x_1, \dots, x_n]$ is Noetherian, the ideal I is generated by a finite number of elements, which we can take to be monomials (since I is a monomial ideal, each term of a polynomial in I is in I). The smallest generator will then be the smallest element of S . \square

For a nonzero polynomial $P \in k[x_1, \dots, x_n]$, its initial term is the term of P with the largest monomial under some monomial order $>$. If I is an ideal, let $\text{in}(I)$ denote the set of initial terms of polynomials in I under the order $>$, and $\text{out}(I)$ to be the initial terms not in the set $\text{in}(I)$. Note that if x^α is in the set $\text{in}(I)$, then so is every multiple of x^α .

Theorem 3.2 (Macaulay). *Let I be an ideal of $k[x_1, \dots, x_n]$, and $>$ be a monomial order. Then the set $\text{out}(I)$ forms a basis for the ring $k[x_1, \dots, x_n]/I$.*

Proof. First, the elements of $\text{out}(I)$ are linearly independent. For if

$$P = \alpha_1 x^{a_1} + \dots + \alpha_n x^{a_n} \in I,$$

then the initial term of P must be an initial term of I , which is a contradiction if the x^{a_i} are in $\text{out}(I)$.

Next, the elements of $\text{out}(I)$ span the quotient. To show this, consider the span of $\text{out}(I) \cup I$ in the ring $k[x_1, \dots, x_n]$. Suppose the set of polynomials not in this span is nonempty. Then there is a polynomial P not in the span with the smallest initial term. If the initial term of P were in $\text{out}(I)$, we can subtract a multiple of an element of $\text{out}(I)$ to get a polynomial not in the span with smaller initial term. Otherwise the initial term of P is an initial term of a polynomial in I , so we can subtract this polynomial to get a polynomial with smaller initial term. In either case, we contradict our choice of P , so the elements of $\text{out}(I)$ and the polynomials of I span $k[x_1, \dots, x_n]$. \square

Corollary 3.3. *Let I be an ideal of $k[x_1, \dots, x_n]$, and $>$ be a monomial order which is graded. Then ${}^a\text{HF}_I(d)$ is equal to the number of monomials of $\text{out}(I)$ of degree at most d .*

Proof. Let $\text{out}(I)_{\leq d}$ be the set of monomials in $\text{out}(I)$ of degree at most d . Note that for a graded monomial order $<$, the degree of a polynomial and the degree of its initial

term are the same. Using the same proof as in [Theorem 3.2](#), we can show that $\text{out}(I)_{\leq d}$ spans $k[x_1, \dots, x_n]_{\leq d}/I_{\leq d}$. \square

There are many examples of graded monomial orders. We describe the homogeneous lexicographic order. In this order, we first order monomials by degree and then break ties by the lexicographic order. The lexicographic order on \mathbb{N}^n is given by $(a_1, \dots, a_n) < (b_1, \dots, b_n)$ if $a_j < b_j$ for the first index j for which $a_i \neq b_i$. Then the homogeneous lexicographic order is the order given by $x^a < x^b$ if either $|a| < |b|$ or $|a| = |b|$ and $a < b$. It is easy to check that this order is a graded monomial order.

4. A bound via the FKG inequality

Using graded monomial orders, we can reduce properties about the Hilbert function of a subset $Y \subset \mathbb{F}_q^n$ to questions about a set of monomials. For simplicity of notation, we will equivalently work with the lattice \mathbb{N}^n : the monomials of $k[x_1, \dots, x_n]$ are in one-to-one correspondence with \mathbb{N}^n via the map $x_1^{a_1} \cdots x_n^{a_n} \mapsto (a_1, \dots, a_n)$. This map is actually an isomorphism of lattices. The order by divisibility on the monomials gets taken to the order \leq on \mathbb{N}^n given by

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \text{ iff } a_i \leq b_i \text{ for all } i.$$

Moreover, the gcd operation gets taken to taking the min of each component, while the lcm operation gets taken to taking the max of each component. For simplicity of notation, we will identify the set of monomials in $k[x_1, \dots, x_n]$ with \mathbb{N}^n in this section.

Now given an ideal I of $k[x_1, \dots, x_n]$, we see that $\text{in}(I)$ satisfies the property that if $a \in \text{in}(I)$ and $b \geq a$, then $b \in \text{in}(I)$, that is, $\text{in}(I)$ is an upper set. Similarly, if S is the set of monomials not in the set $\text{in}(I)$, then S satisfies the property that if $a \in S$ and $b \leq a$, then $b \in S$, that is S is a lower set.

Consider the situation where Y is a subset of \mathbb{F}_q^n . Let S denote the set of monomials not in the initial terms $\text{in}(I(Y))$. By Macaulay's theorem ([Theorem 3.2](#)), the number of elements of S span the functions on Y , so $|S| = |Y|$. We know that $\text{in}(I(Y))$ contains $\text{in}(I(\mathbb{F}_q^n))$, so by [Lemma 2.2](#), all points of S are contained in the hypercube $\{0, 1, \dots, q-1\}^n$. It is from this set up that we will show

Theorem 4.1. *Let $Y \subset \mathbb{F}_q^n$. Then,*

$${}^a\text{HF}(\mathbb{F}_q^n, d)|Y| \leq {}^a\text{HF}(Y, d)q^n.$$

The proof is an easy application of the FKG inequality. We will state the inequality here, referring the reader to Alon and Spencer's book [[3, Chapter 6](#)] or Fortuin et al. [[8](#)] for a proof.

Let L be a finite distributive lattice. We say that a nonnegative function $\mu : L \rightarrow \mathbb{R}^+$ is log-supermodular if

$$\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$$

for all x, y in L .

Theorem 4.2 (FKG inequality). *Let L be a finite distributive lattice. Let $\mu, f, g : L \rightarrow \mathbb{R}^+$ be nonnegative functions on L such that μ is log-supermodular and f, g are either both increasing or both decreasing. Then*

$$\left(\sum_{x \in L} \mu(x)f(x) \right) \left(\sum_{x \in L} \mu(x)g(x) \right) \leq \left(\sum_{x \in L} \mu(x) \right) \left(\sum_{x \in L} \mu(x)f(x)g(x) \right).$$

Proof of Theorem 4.1. Let S be the set of monomials which are not an initial term of $I(Y)$ and $T = \{0, 1, \dots, q-1\}^n$ be the set of monomials not an initial term of $I(\mathbb{F}_q^n)$. Let M denote the set of monomials of degree at most d . We let μ be the indicator function for T , f be the indicator function for S and g be the indicator function for M . It is easy to check that these functions satisfy the conditions of the FKG inequality on $\{0, 1, \dots, q-1\}^n$. Applying the inequality (and noting that $S \subset T$) then gives

$$|S||M \cap T| \leq |T||S \cap M|.$$

Now $|S| = |Y|$, $|M \cap T| = {}^a\text{HF}(\mathbb{F}_q^n, d)$, $|T| = q^n$ and $|S \cap M| = {}^a\text{HF}(Y, d)$. Substituting gives us the desired bound

$${}^a\text{HF}(\mathbb{F}_q^n, d)|Y| \leq {}^a\text{HF}(Y, d)q^n. \quad \square$$

We do not know a proof of the bound without resorting to the FKG inequality. We remark that one may skip the use of FKG and the above inequality in the estimates given below. This will make our bound on the degree d closure worse by a factor of approximately $n!$ where n is the dimension of our space. The factor is smaller in the case of higher multiplicities. In particular, if one is working in a fixed dimension n , avoidance of FKG will not change the asymptotic dependence on the size of the field q .

5. Finite degree closure

In applications of the polynomial method, one often shows a statement of the following form: every polynomial of degree at most d vanishing on a set Y must also vanish on a set X . We can view statements like this in a slightly different light: given a set Y define the **degree d closure** of Y , denoted $\text{cl}_d(Y)$ to be the set of all points x such that every polynomial of degree at most d vanishing on all of Y vanishes must vanish at x also. Equivalently, we can define $\text{cl}_d(Y)$ to be $V(I(Y)_{\leq d})$, where V of a set of polynomials is the set of points in k^n which vanish on all those polynomials. Then the statement at the beginning of this paragraph is equivalent to the statement that $X \subset \text{cl}_d(Y)$. As another example, we can view a large part of Dvir's argument [5] proving the finite field Kakeya

conjecture as an argument establishing that the degree $q - 1$ closure of a Kakeya set is the whole space \mathbb{F}_q^n . The main result of this section is a bound on the size of $\text{cl}_d(Y)$ when working over finite fields.

The reader will note the similarity of the degree d closure with the definition of the Zariski closure. The full Zariski closure is too fine for our purposes: the Zariski closure of a finite point set is just the finite point set. However, if we only allow low degree polynomials, we may be able to get additional structures such as lines, which were absent before.

As a warning, the degree d closure cl_d is a closure operator, but it does not determine a topology. That is, the collection of sets

$$\mathcal{C} = \{X \in k^n \mid X = \text{cl}_d(X)\},$$

is not closed under finite unions, so it is not the collection of closed sets in some topology.

Proposition 5.1. *The degree d closure is a closure operator. That is,*

1. $X \subset \text{cl}_d(X)$,
2. If $X \subset Y$ then $\text{cl}_d(X) \subset \text{cl}_d(Y)$,
3. $\text{cl}_d(\text{cl}_d(X)) = \text{cl}_d(X)$.

Proof. For (1), every polynomial of degree at most d vanishing on X vanishes on X .

For (2), every degree at most d polynomial vanishing on Y vanishes on X . Every degree at most d polynomial vanishing on X vanishes on $\text{cl}_d(X)$. Thus, every degree at most d polynomial vanishing on Y vanishes on $\text{cl}_d(X)$.

For (3), by (1) and (2), it suffices to show that $\text{cl}_d(\text{cl}_d(X)) \subset \text{cl}_d(X)$. Every degree at most d polynomial vanishing on X vanishes on $\text{cl}_d(X)$. Every degree at most d polynomial vanishing on $\text{cl}_d(X)$ vanishes on $\text{cl}_d(\text{cl}_d(X))$. Thus, every degree at most d polynomial vanishing on X vanishes on $\text{cl}_d(\text{cl}_d(X))$. \square

We can use the fact that $X \subset \text{cl}_d(Y)$ to get information on the Hilbert functions of X and Y .

Proposition 5.2. *We have the following:*

1. If $X \subset \text{cl}_d(Y)$, then $I(X)_{\leq d} \supset I(Y)_{\leq d}$.
2. If $X \subset \text{cl}_d(Y)$, then ${}^a\text{HF}(X, m) \leq {}^a\text{HF}(Y, m)$ for all $m \leq d$.
3. If $Y \subset X \subset \text{cl}_d(Y)$, then $I(X)_{\leq d} = I(Y)_{\leq d}$.
4. If $Y \subset X \subset \text{cl}_d(Y)$, then ${}^a\text{HF}(X, m) = {}^a\text{HF}(Y, m)$ for all $m \leq d$.

Proof. For (1), this is exactly the assertion that the degree $\leq d$ polynomials vanishing on Y vanish on X . (2) then follows from (1).

For (3), we apply (1) twice. First, $X \subset \text{cl}_d(Y)$, so $I(X)_{\leq d} \supset I(Y)_{\leq d}$. Moreover, we have $Y \subset X \subset \text{cl}_d(X)$, so $I(Y)_{\leq d} \supset I(X)_{\leq d}$. Thus, $I(X)_{\leq d} = I(Y)_{\leq d}$. (4) then follows from (3). \square

Saying the above in words may be illuminating. We can interpret ${}^a\text{HF}(X, d)$ as the dimension of the degree at most d polynomials on X . If $X \subset \text{cl}_d(Y)$, two different degree at most d polynomials f, g on X must also be different on Y , for otherwise $f - g = 0$ on Y , so $f - g = 0$ on X , a contradiction. Thus, we have that ${}^a\text{HF}(X, d) \leq {}^a\text{HF}(Y, d)$.

One of the things studied in the polynomial method is this: given a set X , what is the minimal degree of the nonzero polynomials vanishing on X . When working over infinite fields, this is the same as asking: what is the largest degree d such that $\text{cl}_d(X)$ is the whole space k^n . The estimate of the fact that for finite sets $|X|$, this degree is at most $n|X|^{1/n}$ can be restated:

Proposition 5.3. *Let X be a finite set in k^n where k is an infinite field. If $\text{cl}_d(X) = k^n$, then*

$$\binom{d+n}{n} \leq |X|.$$

In particular, if $d > n|X|^{1/n}$, then $\text{cl}_d(X) \neq k^n$.

Proof. If $\text{cl}_d(X) = k^n$, then

$$\binom{d+n}{n} = {}^a\text{HF}(k^n, d) = {}^a\text{HF}(X, d) \leq |X|.$$

Using the fact that $(d/n)^n \leq \binom{d+n}{n}$, we get $d \leq n|X|^{1/n}$, as desired. \square

Indeed, we could also do this for X a set of L lines.

Proposition 5.4. *Let X be the union of L lines in k^n where k is infinite. If $\text{cl}_d(X) = k^n$, then*

$$\frac{1}{d+1} \binom{d+n}{n} \leq L.$$

In particular, if $d > nL^{1/(n-1)}$, then $\text{cl}_d(X) \neq k^n$.

Proof. Note that for a line ℓ , we have that ${}^a\text{HF}(\ell, d) = d + 1$. Thus, using [Lemma 2.4](#), we get that ${}^a\text{HF}(X, d) \leq (d + 1)|L|$. The conclusion follows similarly to the previous proposition. \square

We remark that slight care must be taken in finite fields due to the fact that there exist nonzero polynomials which vanish on the whole space \mathbb{F}_q^n .

We also have propositions of the following form:

Proposition 5.5. *Let E_1, \dots, E_n be subsets of k such that $|E_i| > d$ for all i . Then the degree d closure of $E_1 \times \dots \times E_n$ is k^n .*

Proof. This is a weak form of the Combinatorial Nullstellensatz. It is an easy consequence of a lemma in Alon's and Tarsi's paper [2, Lemma 2.1] and in Alon's paper on the Nullstellensatz [1, Lemma 2.1]. \square

As promised, we will give a bound on $\text{cl}_d(Y)$ for finite fields.

Theorem 5.6. *Let Y be a subset of \mathbb{F}_q^n . Then*

$$^{\text{aHF}}(\mathbb{F}_q^n, d) |\text{cl}_d(Y)| \leq q^n |Y|.$$

Proof. We apply Theorem 4.1, Proposition 5.2, and the fact that $^{\text{aHF}}(Y, d) \leq |Y|$:

$$^{\text{aHF}}(\mathbb{F}_q^n, d) |\text{cl}_d(Y)| \leq ^{\text{aHF}}(\text{cl}_d(Y), d) q^n = ^{\text{aHF}}(Y, d) q^n \leq q^n |Y|. \quad \square$$

We remark that we did not use anything special about the field \mathbb{F}_q in proving our bound, merely the fact that $I(\mathbb{F}_q^n) = (x_1^q - x_1, \dots, x_n^q - x_n)$. Indeed, our proof carries over entirely to finite subsets E of k^n where the complement of $\text{in}(I(E))$ in the monomials of $k[x_1, \dots, x_n]$ is a box. For example, for an arbitrary field k , let E_1, \dots, E_n be finite subsets of k . Let $E = E_1 \times \dots \times E_n \subset k^n$. For any subset Y of E , we can get the bound

$$^{\text{aHF}}(E, d) |\text{cl}_d(Y)| \leq |E| |Y|.$$

Indeed, we need to show that the complement of $\text{in}(I(E))$ in the monomials (viewed as a lattice \mathbb{N}^n) is the set

$$B = \{0, \dots, |E_1| - 1\} \times \dots \times \{0, \dots, |E_n| - 1\},$$

which can be done similarly to \mathbb{F}_q^n : by noting that all the other monomials are clearly an initial term of $I(E)$ and then counting to conclude that there are no more. The only other change is to apply the FKG inequality with the set B instead of the set $\{0, \dots, q - 1\}^n$. We leave the details to the reader.

It is perhaps interesting to study the degree d closure in a more general setting. In line with our viewing the ring $\mathbb{F}_q[x_1, \dots, x_n]$ as the polynomial functions on $\mathbb{F}_q^n = \text{Spec } \mathbb{F}_q[x_1, \dots, x_n]/(x_1^q - x_1, \dots, x_n^q - x_n)$, the general framework is probably a ring homomorphism $R = \oplus R_i \rightarrow S$ from a graded ring $R = \oplus R_i$ to a ring S . Here $R = \oplus R_i$ is viewed as the ‘polynomial functions’ on $\text{Spec } S$. We might also want to work with Hilbert functions instead of affine Hilbert functions as we have done in this paper: to get our theory with Hilbert functions we would precompose with the map $k[x_0, x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ sending x_0 to 1.

We remark that it is possible to avoid use of the FKG inequality and use [Corollary 2.3](#) and the estimate from [Theorem 6.10](#).

$$\begin{aligned} \binom{n+d}{n} |\text{cl}_d(Y)| &= \binom{n+d}{n} {}^{\text{a}}\text{HF}(\text{cl}_d(Y), n(q-1)) \\ &\leq \binom{nq}{n} {}^{\text{a}}\text{HF}(\text{cl}_d(Y), d) \\ &= \binom{nq}{n} {}^{\text{a}}\text{HF}(Y, d) \\ &\leq \binom{nq}{n} |Y|. \end{aligned}$$

This estimate is worse than the one given in [Theorem 5.6](#) by a factor of approximately $n!$.

To illustrate possible uses of our theorem, we give a couple of corollaries:

Corollary 5.7. *Let L_1, \dots, L_c be lines in \mathbb{F}_q^n , and $X = \bigcup L_i$. On each L_i , pick a subset γ_i such that $|\gamma_i| \geq q/2$, and let $Y = \bigcup \gamma_i$. Then*

$$|Y| \geq \frac{1}{n! \cdot 2^n} |X|.$$

Proof. The degree $\lfloor (q-1)/2 \rfloor < q/2$ closure of Y contains X . Note that for $d < q$, we have ${}^{\text{a}}\text{HF}(\mathbb{F}_q^n, d) = \binom{d+n}{n} \geq (d+1)^n/n!$. An application of [Theorem 5.6](#) then immediately gives the result. \square

Corollary 5.8 (*Finite field Nikodym conjecture*). *A Nikodym set N in \mathbb{F}_q^n is a set such that for any point x , there is a line through x whose intersection with N has at least $q/2$ elements. For any Nikodym set N , we must have*

$$|N| \geq \frac{q^n}{n! \cdot 2^n}.$$

Proof. We can take $X = \mathbb{F}_q^n$ and $Y = N$ in the previous corollary. \square

The theorem can also be used to prove variants:

Corollary 5.9. *Let L_1, \dots, L_c be lines in \mathbb{F}_q^n , and $X = \bigcup L_i$. On each L_i , pick a subset γ_i such that $|\gamma_i| > q^\alpha$ for some $0 < \alpha < 1$ and let $Y = \bigcup \gamma_i$. Then*

$$|X| \leq n! \cdot q^{n(1-\alpha)} |Y|.$$

Proof. The set X is contained in the degree $\lfloor q^\alpha \rfloor$ closure of Y . Using the fact that for $d < q$, we have ${}^{\text{a}}\text{HF}(\mathbb{F}_q^n, d) \geq (d+1)^n/n!$, we apply [Theorem 5.6](#) to get the result. \square

We remark that the bounds in the corollaries will be improved in the next section by the statistical Kakeya for curves. Note that in the above corollaries, we've only used the fact that the degree d closure of $d + 1$ points on a line must contain the whole line to show that the degree d closure of some smaller set must contain some larger set. In this situation, considering vanishing with multiplicities allows us to get better bounds.

6. Multiplicity

In Dvir et al. [6], the constant in the finite field Kakeya set problem was improved when allowing the polynomials to vanish on sets with higher multiplicity. We pursue this direction of thought in this section.

First, we need to recall what vanishing with multiplicity greater than one at a point means. A polynomial P vanishes with multiplicity m at a point p if its Taylor expansion about p has no terms of degree less than m . Equivalently, letting $\mathfrak{p} = (x_1 - p_1, \dots, x_n - p_n)$ denote the maximal ideal of functions vanishing at the point p , we say P vanishes with multiplicity m at p if $P \in \mathfrak{p}^m$. The order of P at the point p is defined to be the largest m such that $P \in \mathfrak{p}^m$ and is denoted $\text{ord}_p(P)$. By convention, when $P = 0$, we set $\text{ord}_p(P) = \infty$. Note that $\text{ord}_p(PQ) = \text{ord}_p(P) + \text{ord}_p(Q)$.

The multiplicity of a polynomial at a point can also be phrased in terms of Hasse derivatives (for a reference on Hasse derivatives, see for example [10, Section 5.10], or [6, Section 2]). Intuitively, Hasse derivatives are defined so that there is a Taylor expansion about every point $p \in k^n$:

$$P(x) = \sum_{i_1, \dots, i_n} D^{i_1, \dots, i_n} P(p) (x_1 - p_1)^{i_1} \cdots (x_n - p_n)^{i_n}. \quad (6.1)$$

In fields of characteristic 0, the Hasse derivative of order i_1, \dots, i_n will be $1/(i_1! \cdots i_n!)$ the ordinary i_1, \dots, i_n partial derivative. Such a formula does not work in nonzero characteristic because, the number $i_1! \cdots i_n!$ might not be invertible (and if it isn't, the i_1, \dots, i_n partial derivative will be zero). It is clear that if we have the Taylor expansion (6.1), then a polynomial vanishes with multiplicity m at a point p if and only if $D^{i_1, \dots, i_n} P$ vanish at p for all $i_1 + \cdots + i_n < m$.

We define

$$D^{i_1, \dots, i_n} P(x_1, \dots, x_n) = [t^{i_1} \cdots t^{i_n}] P(x_1 + t_1, \dots, x_n + t_n).$$

Letting d be the degree of P and $|i| = i_1 + \cdots + i_n$, this shows that $D^{i_1, \dots, i_n} P$ is a polynomial of degree $d - |i|$. For short, if $i = (i_1, \dots, i_n)$, we let D^i denote D^{i_1, \dots, i_n} , and t^i denote $t_1^{i_1} \cdots t_n^{i_n}$. From the definition,

$$P(x + t) = \sum_{i \in \mathbb{N}^n} D^i P(x) t^i. \quad (6.2)$$

Setting $t = x - p$ and $x = p$ in Eq. (6.2) gives the desired Eq. (6.1).

Let k be a field and X a subset of k^n . We define $I^m(X)$ to be the ideal of all polynomials which vanish to order at least m at each point of X . Note that $I^1(X)$ is just the $I(X)$ which was defined before. It is clear that $I^m(X)$ is decreasing in both m and X : if $m_1 \leq m_2$, then $I^{m_1}(X) \supset I^{m_2}(X)$, and if $X_1 \subset X_2$, then $I^m(X_1) \supset I^m(X_2)$.

We define ${}^a\text{HF}^m(X, d)$ to be the Hilbert function of $I^m(X)$. By the above, we see that ${}^a\text{HF}^m(X, d)$ is increasing in m , X and d .

It is clear from the definitions that if p is a point in k^n , then $I^m(\{p\}) = (I(\{p\}))^m$. Let $Y = \{y_1, \dots, y_s\}$ be a finite subset of k^n . Consider the evaluation function sending a polynomial P to its values and the values of all its Hasse derivatives of order at most $m-1$ on each of the points of Y . This image has dimension $\binom{m+n-1}{n}|Y|$: we wish to show it surjects. Now $I^m(\{y_1\}), \dots, I^m(\{y_s\})$ are pairwise coprime since $I(\{y_i\})$ is coprime with $I(\{y_j\})$ for $i \neq j$ (and if \mathfrak{a} is coprime with \mathfrak{b} and \mathfrak{c} , then \mathfrak{a} is coprime with \mathfrak{bc}). The Chinese Remainder theorem immediately shows that the polynomial functions surject. Moreover, it shows that for finite spaces $I^m(Y) = I^m(\{y_1\}) \cdots I^m(\{y_s\}) = (I(\{y_1\}) \cdots I(\{y_s\}))^m = (I(Y))^m$. By being careful, we can get a degree bound for when ${}^a\text{HF}^m(Y, d)$ stabilizes:

Lemma 6.1. *Let $Y = \{y_1, \dots, y_s\}$ be a finite subset of k^n . Then for all $d \geq 2m|Y| - m - |Y|$,*

$${}^a\text{HF}^m(Y, d) = \binom{m+n-1}{n}|Y|.$$

Proof. For simplicity of notation, let $I^m(y) = I^m(\{y\})$ for points y . By the above discussion, it suffices to show that the polynomials of degree at most $2m|Y| - m - |Y|$ form a spanning set for $\mathbb{F}_q[x_1, \dots, x_n]/I^m(Y)$.

We wish to construct polynomials p_1, \dots, p_n of degree $(s-1)(2m-1)$ such that $p_i \equiv 1 \pmod{I^m(y_i)}$ but $p_i \equiv 0 \pmod{I^m(y_j)}$ for $j \neq i$. We construct p_1 ; the others are constructed similarly. For $j \neq 1$, we can find linear polynomials $f \in I(y_1)$ and $g_j \in I(y_j)$ such that $f + g_j = 1$. Then

$$1 = (f + g_j)^{2m-1} = af^m + bg_j^m,$$

shows that there is a polynomial $h_j = bg_j^m$ of degree $2m-1$ which is in $I^m(y_j)$ and is congruent to $1 \pmod{I^m(y_1)}$. The polynomial $h_2 \cdots h_s$ will then have degree $(s-1)(2m-1)$ and is congruent to $1 \pmod{I^m(y_1)}$ but congruent to $0 \pmod{I^m(y_j)}$ for $j \neq 1$.

It is not difficult to see that the polynomials of degree $\leq m-1$ span the ring $\mathbb{F}_q[x_1, \dots, x_n]/I^m(y_i)$ for all i . Now by the Chinese remainder theorem, an element p of $\mathbb{F}_q[x_1, \dots, x_n]/I^m(Y)$ is determined by the set of its residues in $\mathbb{F}_q[x_1, \dots, x_n]/I^m(y_i)$ for all i . Let r_1, \dots, r_n be polynomials of degree at most $m-1$ such that $p \equiv r_i \pmod{I^m(y_i)}$. Then we see that $p \equiv r_1p_1 + \cdots + r_sp_s \pmod{I^m(Y)}$. The polynomial $r_1p_1 + \cdots + r_sp_s$ has degree at most $(s-1)(2m-1) + m-1 = 2ms - m - s$, thus giving the conclusion. \square

Again, we can do better in the case of finite fields.

Lemma 6.2. *The ideal $I^m(\mathbb{F}_q^n)$ is generated by the set*

$$\{(x_1^q - x_1)^{m_1} \cdots (x_n^q - x_n)^{m_n} \mid m_1 + \cdots + m_n = m\}.$$

In particular, the set

$$\{x_1^{m_1} \cdots x_n^{m_n} \mid \left\lfloor \frac{m_1}{q} \right\rfloor + \cdots + \left\lfloor \frac{m_n}{q} \right\rfloor \leq m - 1\}$$

forms a basis for $\mathbb{F}_q[x_1, \dots, x_n]/I^m(\mathbb{F}_q^n)$.

Proof. The first statement is a consequence of the fact that $I^m(\mathbb{F}_q^n) = (I(\mathbb{F}_q^n))^m$ since \mathbb{F}_q^n is a finite set. For the second statement, note that the set of monomials not among the leading terms of $I^m(\mathbb{F}_q^n)$ must be contained in the above set. This is because it is easy to show all the monomials in the complement are leading terms of $I^m(\mathbb{F}_q^n)$. But then the size of

$$\{x_1^{m_1} \cdots x_n^{m_n} \mid \left\lfloor \frac{m_1}{q} \right\rfloor + \cdots + \left\lfloor \frac{m_n}{q} \right\rfloor \leq m - 1\}$$

is $\binom{m+n-1}{n}|Y|$, so they must span. \square

Lemma 6.3. *Let Y be a set in \mathbb{F}_q^n . If $d \geq n(q-1) + (m-1)q$, then*

$${}^a\text{HF}^m(Y, d) = \binom{m+n-1}{n}|Y|.$$

Proof. The claim follows by noting the basis given in Lemma 6.2 for the ring $\mathbb{F}_q[x_1, \dots, x_n]/I^m(\mathbb{F}_q^n)$ consists only of polynomials of degree at most $n(q-1) + (m-1)q$. \square

If I is a subset of $k[x_1, \dots, x_n]$, define $V^\ell(I)$ to be the set of all points $x \in k^n$ such that every polynomial in I vanishes with multiplicity at least ℓ at x . It is clear that V is decreasing in both ℓ and I . Note that $I \subset I^m(V^m(I))$ and $X \subset V^m(I^m(X))$. Moreover, we have that $V^m(I^m(V^m(I))) = V^m(I)$ and $I^m(V^m(I^m(X))) = I^m(X)$.

Similar to before, we define $\text{cl}_d^{\ell, m}(X) = V^\ell(I^m(X)_{\leq d})$, so that if a polynomial of degree at most d vanishes with multiplicity at least m at each point of X , then it vanishes with multiplicity at least ℓ at each point of $\text{cl}_d^{\ell, m}(X)$. We see that $\text{cl}_d^{\ell, m}(X)$ is decreasing in d and ℓ and increasing in m and X . Using the above, we get $\text{cl}_d^{j, \ell}(\text{cl}_d^{\ell, m}(X)) \subset \text{cl}_d^{j, m}(X)$ and $\text{cl}_d^{\ell, \ell}(\text{cl}_d^{\ell, m}(X)) = \text{cl}_d^{\ell, m}(X) = \text{cl}_d^{\ell, m}(\text{cl}_d^{m, m}(X))$.

We finally get to

Lemma 6.4. *If $X \subset \text{cl}_d^{\ell,m}(Y)$, then*

$${}^a\text{HF}^\ell(X, d) \leq {}^a\text{HF}^m(Y, d).$$

Proof. Indeed,

$$I^\ell(X) \supset I^\ell(\text{cl}_d^{\ell,m}(Y)) = I^\ell(V^\ell(I^m(Y)_{\leq d})) \supset I^m(Y)_{\leq d}. \quad \square$$

Note that the operation $\text{cl}_d^{\ell,m}$ may not be a closure operator unless $\ell = m$, so it is somewhat of a misnomer. (We have used the symbol $\text{cl}_d^{\ell,m}$ because of its obvious relation to cl_d .) Indeed, we always have that if $X \subset Y$ then $\text{cl}_d^{\ell,m}(X) \subset \text{cl}_d^{\ell,m}(Y)$. We can show $X \subset \text{cl}_d^{\ell,m}(X)$ when $\ell \leq m$. We can show $\text{cl}_d^{\ell,m}(\text{cl}_d^{\ell,m}(X)) = \text{cl}_d^{\ell,m}(X)$ when $\ell = m$.

Proposition 6.5 (Schwartz–Zippel with multiplicity [6]). *Let X be a finite subset of k . If $d < |X|(m - \ell + 1) + \ell - 1$, then $\text{cl}_d^{\ell,m}(X) = k$.*

Proof. For notional purposes, let $i = (i_1, \dots, i_n) \in \mathbb{N}^n$ and $|i| = i_1 + \dots + i_n$. Also, let $D^i = D^{i_1, \dots, i_n}$.

If P is a polynomial of degree d vanishing to order m at a point in X , then $D^i P$ is a polynomial of degree $d - |i|$ vanishing to order $m - |i|$ at that point. If $d - |i| < |X|(m - |i|)$, then $D^i P$ must vanish identically. When $d - \ell + 1 < |X|(m - \ell + 1)$, this is true for all $|i| \leq \ell - 1$, so any polynomial vanishing of order m on X must vanish to order ℓ on all of k , as desired. \square

In Theorem 13 of [6], Dvir et al. prove a theorem called statistical Kakeya for curves. Due to Lemma 6.2, we can improve their bound.

First, a degree Λ curve C in \mathbb{F}_q^n is the set

$$\{(C_1(\lambda), \dots, C_n(\lambda)) \mid \lambda \in F_q, C_1, \dots, C_n \in \mathbb{F}_q[x]_{\leq \Lambda}\},$$

that is, the values a tuple $(C_1(\lambda), \dots, C_n(\lambda))$ takes, where C_1, \dots, C_n are polynomials in one variable of degree at most Λ . Using Proposition 6.5, we see that if $X \subset C$ for some degree Λ curve and $\Lambda d < |X|(m - \ell + 1) + \ell - 1$, then $C \subset \text{cl}_d^{\ell,m}(X)$. We now have

Theorem 6.6 (Statistical Kakeya for curves). *Let X and Y be subsets of \mathbb{F}_q^n . Suppose that for every point $x \in X$, there is a curve C_x of degree at most Λ through x which intersects Y in at least τ points. Then*

$$|X| \leq \left(1 + \frac{\Lambda(q-1)}{\tau}\right)^n |Y|.$$

We remark that this is the Statistical Kakeya for Curves in [6, Theorem 13], with $S = X$, $K = Y$ and $\tau = \eta q$. Our bound is strictly better whenever X is not the whole space \mathbb{F}_q^n and $n \geq 2$ (also, we drop the condition $\tau > \Lambda$).

Proof. Let d , ℓ and m be constants to be chosen later. When $\Lambda d < \tau(m - \ell + 1) + \ell - 1$, we see that $X \subset \text{cl}_d^{\ell, m}(Y)$. If in addition, we have that $d \geq (\ell - 1)q + n(q - 1)$, then

$$\binom{\ell + n - 1}{n} |X| = {}^a\text{HF}^\ell(X, d) \leq {}^a\text{HF}^m(Y, d) \leq \binom{m + n - 1}{n} |Y|.$$

Take $d = (\ell - 1)q + n(q - 1)$. Then the above inequality is true when

$$\Lambda((\ell - 1)q + n(q - 1)) < \tau(m - \ell + 1) + \ell - 1.$$

Rearranging gives

$$\Lambda\ell(q - 1) + \Lambda n(q - 1) - \tau + 1 < \tau(m - \ell) + \ell.$$

Set $m = \lceil (1 + \Lambda(q - 1)/\tau)\ell \rceil$. When $\ell > \Lambda n(q - 1) - \tau + 1$, the above inequality is then satisfied. Taking the limit as $\ell \rightarrow \infty$ in the inequality

$$\binom{\ell + n - 1}{n} |X| \leq \binom{m + n - 1}{n} |Y|,$$

then gives the desired inequality. \square

Corollary 6.7. Let L_1, \dots, L_c be lines in \mathbb{F}_q^n , and $X = \bigcup L_i$. On each L_i , pick a subset γ_i such that $|\gamma_i| \geq q/2$, and let $Y = \bigcup \gamma_i$. Then

$$|X| \leq \left(3 - \frac{2}{q}\right)^n |Y|.$$

Proof. Just apply the statistical Kakeya for curves with $\Lambda = 1$ and $\tau = q/2$. \square

Corollary 6.8 (Finite field Nikodym conjecture). A Nikodym set N in \mathbb{F}_q^n satisfies

$$|N| \geq \frac{q^n}{(3 - \frac{2}{q})^n}.$$

Proof. Set $X = \mathbb{F}_q^n$ and $Y = N$ in the previous corollary. \square

Corollary 6.9. Let L_1, \dots, L_c be lines in \mathbb{F}_q^n , and $X = \bigcup L_i$. On each L_i , pick a subset γ_i such that $|\gamma_i| \geq q^\alpha$ for some $0 < \alpha < 1$ and let $Y = \bigcup \gamma_i$. Then

$$|X| \leq \left(1 + \frac{q - 1}{q^\alpha}\right)^n |Y|.$$

Proof. Apply the statistical Kakeya for curves with $\Lambda = 1$ and $\tau = q^\alpha$. \square

We now work towards our bound on $|\text{cl}_d^{\ell,m}(Y)|$.

Theorem 6.10. *Let I be an ideal of $k[x_1, \dots, x_n]$. Suppose $m_1 \geq m_2$. Then*

$${}^a\text{HF}_I(m_1) \binom{n+m_2}{n} \leq {}^a\text{HF}_I(m_2) \binom{n+m_1}{n}.$$

First, we make some definitions. We give \mathbb{N}^n the usual partial order where $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if $a_i \leq b_i$ for all i . Moreover, for $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, we let $|a|$ denote the sum $a_1 + \dots + a_n$.

Let S be a subset of \mathbb{N}^n . We define $S_{\leq d}$ to be the set of points $a \in S$ with $|a| \leq d$. Similarly, we let $S_{=d}$ to be the set of points $a \in S$ with $|a| = d$. For $v \in \mathbb{N}^n$, we let $S + v$ to be the set of points $a + v$ where $a \in S$. Finally, we define

$$S^+ = S \cup \bigcup_{i=1}^n (S + e_i)$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i -th unit vector. Thus, we think of S^+ as the points of S along with those points which are one unit above them. Recall that S is an upper set if for every $x \in S$ and $y \geq x$, then $y \in S$. It is clear that $S \subset \mathbb{N}^n$ is an upper set if and only if $S = S^+$. We have the following useful lemma:

Lemma 6.11. *For $S \subset \mathbb{N}^n$,*

$$|(S^+)_{\leq d+1}| \geq \frac{n+d+1}{d+1} |S_{\leq d}|.$$

Proof. We prove the lemma by induction on the sum $n + d$. The base cases when $n = 1$ or $d = 0$ are trivial.

Suppose $n \geq 2$ and $d \geq 1$ and that we have proved the lemma for all smaller sums $n + d$. First, we wish to show that $|(S^+)_{=d+1}| \geq (n + d)|S_{=d}|/(d + 1)$. Let V be the projection of $(S^+)_{=d+1}$ onto the first $n - 1$ components, and U be the projection of $S_{=d}$ onto the first $n - 1$ components. It is clear that $|(S^+)_{=d+1}| = |V| = |V_{\leq d+1}|$. Similarly, we have $|S_{=d}| = |U| = |U_{\leq d}|$. Moreover, it is easy to show that $U^+ \subset V$. Applying the inductive hypothesis, we have

$$|(S^+)_{=d+1}| = |V_{\leq d+1}| \geq |(U^+)_{\leq d+1}| \geq \frac{n+d}{d+1} |U_{\leq d}| = \frac{n+d}{d+1} |S_{=d}|,$$

as desired.

To finish the proof, by the induction hypothesis, we have $|(S^+)_{\leq d}| \geq (n + d)|S_{\leq d-1}|/d$. Moreover, since $S^+ \supset S$, we also have $|(S^+)_{\leq d}| \geq |S_{\leq d}|$. Combining these estimates with that of $|(S^+)_{=d+1}|$ gives

$$\begin{aligned}
|(S^+)_{\leq d+1}| &= |(S^+)_{=d+1}| + |(S^+)_{\leq d}| \\
&\geq \frac{n+d}{d+1}|S_{=d}| + \frac{d}{d+1} \cdot \frac{n+d}{d}|S_{\leq d-1}| + \frac{1}{d+1}|S_{\leq d}| \\
&= \frac{n+d}{d+1}|S_{\leq d}| + \frac{1}{d+1}|S_{\leq d}| \\
&= \frac{n+d+1}{d+1}|S_{\leq d}|,
\end{aligned}$$

as claimed. \square

Proof of Theorem 6.10. Let $S = \text{in}(I)$ be the set of monomials which are initial terms of I . Since S is an upper set, we have $S^+ = S$, so $|S_{\leq d+1}| \geq (n+d+1)|S_{\leq d}|/(d+1)$. By induction, we can show that

$$\binom{n+m_2}{n}|S_{\leq m_1}| \geq \binom{n+m_1}{n}|S_{\leq m_1}|$$

for all $m_1 \geq m_2$. Let M be the set of all monomials, and T be the complement of S in M . Then $|M_{\leq d}| = |S_{\leq d}| + |T_{\leq d}|$. By Corollary 3.3, we see that ${}^a\text{HF}_I(d) = |T_{\leq d}|$. Since $|M_{\leq d}| = \binom{n+d}{n}$, it is clear $\binom{n+m_2}{n}|M_{\leq m_1}| = \binom{n+m_1}{n}|M_{\leq m_2}|$. Subtracting the above inequality from this equality, we conclude

$${}^a\text{HF}_I(m_1)\binom{n+m_2}{n} \leq {}^a\text{HF}_I(m_2)\binom{n+m_1}{n},$$

as desired. \square

We remark that the proof works for any subset $T \subset \mathbb{N}^n$ which is a lower set: if $a \in T$ and $b \leq a$, then $b \in T$. For such T , we have $\binom{n+m_2}{n}|T_{\leq m_1}| \leq \binom{n+m_1}{n}|T_{\leq m_2}|$ for all $m_1 \geq m_2$. We shall use this statement in the following theorem.

Theorem 6.12. Let $Y \subset \mathbb{F}_q^n$. Then

$${}^a\text{HF}^m(\mathbb{F}_q^n, d)|Y| \leq {}^a\text{HF}^m(Y, d)q^n.$$

Proof. Define a function $\varphi : \mathbb{N}^n \rightarrow \mathbb{N}^n$ given by

$$\varphi(a_1, \dots, a_n) = \left(\left\lfloor \frac{a_1}{q} \right\rfloor, \dots, \left\lfloor \frac{a_n}{q} \right\rfloor \right).$$

For $p \in \mathbb{N}^n$, define $A_p = \varphi^{-1}(p)$. It is clear that $|A_p| = q^n$.

Let R denote the set of monomials not in the initial terms $\text{in}(I^m(F_q^n))$, viewed as a subset of \mathbb{N}^n . Let M denote the set of all monomials. From the proof of Lemma 6.2, we see that $R = \varphi^{-1}(M_{\leq m-1})$.

Let $S \subset R$ be a lower set. We wish to show that $|S||R_{\leq d}| \leq |S_{\leq d}||R| = |S_{\leq d}| \binom{m+n-1}{n} q^n$. The idea is to use φ to ‘contract’ the situation to one in which we can apply [Theorem 6.10](#).

For any subset $T \subset \mathbb{N}^n$ define

$$T(i) = \{p \in \mathbb{N}^n \mid A_p \cap T \geq i\}.$$

Now given $p \in \mathbb{N}^n$, the point p appears in $T(i)$ for $i = 1, \dots, |A_p \cap T|$. Thus,

$$\sum_{i=1}^{q^n} |T(i)| = \sum_p |A_p \cap T| = |T|.$$

Similarly, if T_1 and T_2 are subsets of \mathbb{N}^n , then p appears in $T_1(i) \cap T_2(j)$ iff $1 \leq i \leq |A_p \cap T_1|$ and $1 \leq j \leq |A_p \cap T_2|$. Thus

$$\sum_{i,j=1}^{q^n} |T_1(i) \cap T_2(j)| = \sum_p |A_p \cap T_1| |A_p \cap T_2|.$$

Now we claim that if T is a lower set, then so is $T(i)$ for each i . Indeed, if $p_1 \geq p_2$, then we can find a bijection $\lambda : A_{p_1} \rightarrow A_{p_2}$ such that $\lambda(x) \leq x$ for all $x \in A_{p_1}$. Thus, if T is a lower set, then $x \in T$ implies $\lambda(x) \in T$, so that $|T \cap A_{p_1}| \leq |T \cap A_{p_2}|$. Thus $T(i)$ is also a lower set.

Going back to the problem, consider the sets $R_{\leq d}(j)$. By the symmetry properties of R , we see that $|R_{\leq d} \cap A_p|$ depends only on the degree d and $|p|$, the sum of the coordinates of p . Thus, $R_{\leq d}(j) = M_{\leq f_d(j)}$ for some function $f_d : \mathbb{N} \rightarrow \mathbb{N}$. Applying the variant of [Theorem 6.10](#), we see that for any lower set T ,

$$\begin{aligned} |T_{\leq m-1}| |R_{\leq d}(j)| &= |T_{\leq m-1}| \binom{n + f_d(j)}{n} \\ &\leq \binom{n + m - 1}{n} |T_{\leq f_d(j)}| \\ &= \binom{n + m - 1}{n} |T \cap R_{\leq d}(j)|. \end{aligned}$$

Since $S \subset R$ is a lower set, $S(i) \subset M_{\leq m-1}$ are all lower sets. We expand

$$\begin{aligned} |S| |R_{\leq d}| &= \sum_{i,j=1}^{q^n} |S(i)_{\leq m-1}| |R_{\leq d}(j)| \\ &\leq \binom{n + m - 1}{n} \sum_{i,j=1}^{q^n} |S(i) \cap R_{\leq d}(j)| \end{aligned}$$

$$= \binom{n+m-1}{n} \sum_p |S \cap A_p| |R_{\leq d} \cap A_p|.$$

Using the FKG inequality, we can show

$$q^n |(S \cap A_p)_{\leq d}| \geq |(A_p)_{\leq d}| |S \cap A_p|.$$

Noting that $|R_{\leq d} \cap A_p|$ is either 0 or $|(A_p)_{\leq d}|$, we see that

$$\begin{aligned} |S| |R_{\leq d}| &\leq q^n \binom{n+m-1}{n} \sum_p |S_{\leq d} \cap A_p| \\ &= q^n \binom{n+m-1}{n} |S_{\leq d}| \\ &= |S_{\leq d}| |R|, \end{aligned}$$

as desired.

Finally, to prove the theorem, let S be the monomials not among the initial terms of $I^m(Y)$. We then have

$${}^{\text{aHF}}m(\mathbb{F}_q^n, d) |S| \leq {}^{\text{aHF}}m(Y, d) q^n \binom{n+m-1}{n}.$$

By [Lemma 6.2](#), we have $|S| = \binom{n+m-1}{n} |Y|$, so

$${}^{\text{aHF}}m(\mathbb{F}_q^n, d) |Y| \leq {}^{\text{aHF}}m(Y, d) q^n,$$

as claimed. \square

Theorem 6.13. *Let $Y \subset \mathbb{F}_q^n$. Let $X = \text{cl}_d^{\ell, m}(Y)$. Then*

$${}^{\text{aHF}}\ell(\mathbb{F}_q^n, d) |X| \leq q^n \binom{m+n-1}{n} |Y|.$$

Proof. By [Theorem 6.12](#), [Lemma 6.4](#), we get

$${}^{\text{aHF}}\ell(\mathbb{F}_q^n, d) |X| \leq {}^{\text{aHF}}\ell(X, d) q^n \leq {}^{\text{aHF}}m(Y, d) q^n \leq q^n \binom{n+m-1}{n} |Y|,$$

which was what we wanted. \square

We remark that, again, there is nothing special about the field \mathbb{F}_q . If E_1, \dots, E_n are finite subsets of k and $E = E_1 \times \dots \times E_n$, then we can follow the same proof as above to show that for $Y \subset E$,

$${}^{\text{a}}\text{HF}^{\ell}(E, d) |\text{cl}_d^{\ell, m}(Y)| \leq |E| \binom{n+m-1}{n} |Y|.$$

Let $|E_i| = s_i$. Then the set of monomials not an initial term of $I^m(E)$ is

$$\{(m_1, \dots, m_n) \mid \left\lfloor \frac{m_1}{s_1} \right\rfloor + \dots + \left\lfloor \frac{m_n}{s_n} \right\rfloor \leq m-1\},$$

as can be shown similarly to Lemma 6.2 by noting that all the others are an initial term of $I^m(E)$ and then counting. The only other change is to use the FKG inequality on boxes of size $s_1 \times \dots \times s_n$ instead of size $q \times \dots \times q$ boxes in Lemma 6.12.

It is possible to avoid the use of the FKG inequality and get a worse bound that has a slightly shorter proof. Using Lemma 6.3, Lemma 6.4, and Theorem 6.10,

$$\begin{aligned} \binom{n+d}{n} |X| &= \binom{n+d}{n} {}^{\text{a}}\text{HF}^{\ell}(X, n(q-1) + (\ell-1)q) \\ &\leq \binom{nq + (\ell-1)q}{n} {}^{\text{a}}\text{HF}^{\ell}(X, d) \\ &\leq \binom{nq + (\ell-1)q}{n} {}^{\text{a}}\text{HF}^m(Y, d) \\ &\leq \binom{nq + (\ell-1)q}{n} \binom{n+m-1}{n} |Y|. \end{aligned}$$

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