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A note on a series of families constructed over the Cyclic graph

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ABSTRACT

Paul Erdős and László Lovász established by means of an example that there exists a maximal intersecting family of k-sets with $\lfloor (e-1)k! \rfloor$ blocks, where e is the base of natural logarithm. László Lovász conjectured that their example is best known example which has the maximum number of blocks. Later it was disproved. But the quest for such examples remain valid till this date. In this note we compute the transversal size of a certain series of intersecting families of k-sets, which is constructed over the Cyclic graph. It helps to provide an example of maximal intersecting family of k-sets with so many blocks and to present two worthwhile examples which disprove two special cases of one of the conjectures of Frankl et al.

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1. Introduction

By a *family*, we mean a collection (set) of finite sets. For a family $\mathcal{G}$, the members of $\mathcal{G}$ are called its *blocks* and the elements of the blocks are called its *points*. The *point set* of the family $\mathcal{G}$ is defined as $\bigcup_{B\in\mathcal{G}} B$ and is denoted by $\mathrm{P}_{\mathcal{G}}$. A family $\mathcal{G}$ is an *intersecting family* if any two blocks of $\mathcal{G}$ have non empty intersection. By a k-set, where k is a positive integer, we mean a set of size k and we say a family $\mathcal{G}$ is a *family of k-sets*, if all its blocks are k-sets. A *blocking set* of a family $\mathcal{G}$ is a set which intersects every block of $\mathcal{G}$. A minimum size blocking set of $\mathcal{G}$ is called a *transversal* of $\mathcal{G}$. We denote the common size of its transversals by $\mathrm{tr}(\mathcal{G})$ and the family of transversals of $\mathcal{G}$ by $\mathcal{G}^{\top}$.

A family $\mathcal{F}$ is said to be a *maximal intersecting family of k-sets* if ($\mathrm{tr}(\mathcal{F}) = k$ and) $\mathcal{F} = \mathcal{F}^{\top}$. It is not clear from the definition whether such a family has finite number of blocks. Erdős and Lovász proved the surprising result that any such family has at most k^k blocks (see [1, Theorem 7]). This result is of central attraction in the study of intersecting family of k-sets with transversal size t, where $1 \leq t \leq k$. It allows us to define the integer $\mathrm{M}(k)$ to be the maximum number of blocks achievable by any maximal intersecting family of k-sets. We are trying to find an example of a maximal intersecting family of k-sets with $\mathrm{M}(k)$ blocks. It is answered by means of an example that $\mathrm{M}(k) \geq (\frac{k}{2})^{k-1}$. The core part of this example is to produce an intersecting family of k-sets with transversal size $t \leq k-1$. For each t, where $1 \leq t \leq k-1$, such a family of k-sets can be embedded in a maximal intersecting family of k-sets.

In this note, we construct a series of intersecting families of k-sets with transversal size $t \leq k-1$ (namely $\mathbb{F}(k,t)$), such that each such family can be embedded in a maximal intersecting family of k-sets. Such a construction is not entirely new. There are similar type of families, namely $\mathscr{G}$ in [2, §2]. In order to find the similarity between $\mathbb{F}(k,t)$ and $\mathscr{G}$ in [2, §2], it seems quite reasonable to pose the following question. Depending on the negative answer of such question, we may attempt to disprove the uniqueness part of Conjecture 4 in [2] for each large positive integer.

Question. For each large positive integers k and t, are $\mathbb{F}(k,t)$ and $\mathscr{G}$ in [2, § 2] isomorphic?

The compact description of $\mathbb{F}(k,t)$ (given here) is amenable to rigorous arguments. Our purpose of this note is to show that transversal size of $\mathbb{F}(k,t)$ is t (Theorem 1.1) and as a consequence (Theorem 1.2) we have, for any positive integer k,

$$\mathrm{M}(k) \geq |\mathbb{F}(k,k-1)| + |\mathbb{F}^{\top}(k,k-1)|. \tag{$\star$}$$

In this context, we acknowledge that the proof of Theorem 1.1 can be developed by using the blueprint of the proof of Proposition 2 [2]. But we put an warning that such sketchy arguments contain several typographical errors. (For example, we should place $a = |C \cap J_i|$ instead of $a = |C \cap I_i|$ in Claim 2 [2, Page 38].) In a collective way such typographical errors may create a huge confusion to a reader. On accounting this, we present our proof of Theorem 1.1. A reader may consider it as an alternative proof.

We construct the family $\mathbb{F}(k,t)$ as follows.

Construction. Let k and t be positive integers with $t \le k$. Let X_n $0 \le n \le t-1$, be t pairwise disjoint sets with

$$|X_n| = \begin{cases} k - \lfloor \frac{t}{2} \rfloor & \text{if } 0 \le n \le \lfloor \frac{t-1}{2} \rfloor \\ k - \lfloor \frac{t-1}{2} \rfloor & \text{if } \lfloor \frac{t-1}{2} \rfloor + 1 \le n \le t-1 \end{cases}$$

say $X_n = \{x_p^n : 0 \le p \le |X_n| - 1\}$. Let $\mathbb{F}(k,t)$ be the family of all the k-sets of the form

$$X_n \sqcup \left\{ x_{p_i}^{n+i} : 1 \le i \le k - |X_n| \right\},$$

where $0 \le n \le t-1$, addition in the superscript is modulo t and $\{p_m : m \ge 1\}$ varies over all finite sequences of non negative integers satisfying,

$$p_1 = 0 \text{ or } 1 \text{ and for } m \ge 2,\ p_m = p_{m-1} \text{ or } 1 + p_{m-1}. \qquad (\star\star)$$

In this construction, the pairwise disjoint sets X_n may be thought as arranged along a t-cycle. Since the diameter of a t-cycle is $\lfloor \frac{t}{2} \rfloor$, it is easy to verify that $\mathbb{F}(k,t)$ is an intersecting family of k-sets. The beautiful portion of this note is to establish the following theorem.

Theorem 1.1. $\operatorname{tr}(\mathbb{F}(k,t)) = t$.

We consider a t-set T such that $|T \cap X_i| = 1$ for each i with $0 \le i \le t-1$, then by using Theorem 1.1, we have T is a transversal of $\mathbb{F}(k,t)$. Therefore, there are $\prod_{i=0}^{t-1} |X_i|$ choices for such transversals. But there are other transversals. Hence

$$|\mathbb{F}^\top(k,t)| > \begin{cases} (k-r+1)^{2r-1} & \text{if } t = 2r-1 \\ (k-r)^r (k-r+1)^r & \text{if } t = 2r. \end{cases}$$

Theorem 1.2. *Let $t \le k-1$ and $\mathcal{A}$ be a maximal intersecting family of $(k-t)$-sets with disjoint point set from the point set of $\mathbb{F}(k,t)$. Then $\mathbb{F}(k,t) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k,t))$ is a maximal intersecting family of k-sets. Here $\mathcal{A} \circledast \mathbb{F}^\top(k,t)$ denotes the collection of all sets of the form $A \sqcup T$, where $A \in \mathcal{A}$ and $T \in \mathbb{F}^\top(k,t)$.*

Remark. If we consider the case $t = k-1$ in Theorem 1.2, we have $\mathbb{F}(k,k-1) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k,k-1))$ is a maximal intersecting family of k-sets. As a consequence we deduce ($\star$). We verified that the equality case in ($\star$) holds for $k = 3$. The only clue we have is that the equality case in ($\star$) is false for $k = 5$. Currently, there is no clue on the equality case for $k = 4$ and for $k \ge 6$.

Remark. Erdős and Lovász showed an example of maximal intersecting family of k-sets with $\lfloor (e-1)k! \rfloor$ blocks. Such an example is based on a recursive procedure [1, Construction (c), Page 620]. Such recursive procedure can be deduced from the case $t = 1$ in Theorem 1.2. For this case $\mathbb{F}(k, 1) = \{X_0\}$, where $|X_0| = k$ and consequently, $\mathbb{F}^\top(k, 1) = \{\{x\} : x \in X_0\}$. Now using Theorem 1.2 we deduce the Erdős-Lovász recursive construction. Hence starting with the unique maximal intersecting family of 1-set, we apply this recursive procedure repeatedly to obtain the Erdős-Lovász example of maximal intersecting family of k-sets with $\lfloor (e-1)k! \rfloor$ blocks. Lovász conjectured in [3] that $\mathrm{M}(k) = \lfloor (e-1)k! \rfloor$. He predicted that the Erdős-Lovász example of maximal intersecting family of k-sets is an example of the said family with $\mathrm{M}(k)$ blocks. Later it turns out that this conjecture is false.

There is one more application of Theorem 1.2. In [2], it is conjectured that the construction of Frankl et al. yields the unique maximal intersecting family of k-sets with the largest number of blocks. Both parts of this conjecture are false. Specifically, the uniqueness part is incorrect for $k = 4$, while the optimality part is incorrect for $k = 5$. It is shown that Example 1.3 and Example 1.4 are counter examples to [2, § 3, Conjecture 4] in those special cases.

Example 1.3. By using Theorem 1.2 we have, for $k \geq 3$, $\mathbb{F}(k, 2) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k, 2))$ is a maximal intersecting family of k-sets, where $\mathcal{A}$ is a maximal intersecting family of $(k-2)$-sets. We observe that for $0 \leq p \leq k-2$ and $0 \leq q \leq k-1$, the transversals of $\mathbb{F}(k, 2)$ are $\{x_p^0, x_q^1\}$; $\{x_0^1, x_1^1\}$. Hence there are $k^2 - k + 1$ transversals and 3 blocks in $\mathbb{F}(k, 2)$. So if we consider the case $k = 4$ we have $\mathbb{F}(4, 2)$ has 3 blocks and 13 transversals. Let $\mathcal{A}$ be the unique maximal intersecting family of 2-sets isomorphic to $\{\{a, b\}, \{b, c\}, \{a, c\}\}$ and $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(4,2)} = \emptyset$. Therefore by Theorem 1.2, $\mathbb{F}(4, 2) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(4, 2))$ is a maximal intersecting family 4-sets with 42 blocks and 10 points. In this maximal intersecting family of 4-sets there are 3 points in 26 blocks, 5 points in 14 blocks and 2 points in 10 blocks.

Example 1.4. By using Theorem 1.2 we have, for $k \geq 4$, $\mathbb{F}(k, 3) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k, 3))$ is a maximal intersecting family of k-sets, where $\mathcal{A}$ is a maximal intersecting family of $(k-3)$-sets. We observe that for $0 \leq p, q, r \leq k-2$, the transversals of $\mathbb{F}(k, 3)$ are $\{x_p^0, x_q^1, x_r^2\}$; $\{x_0^0, x_1^0, x_p^1\}$; $\{x_0^1, x_1^1, x_p^2\}$ and $\{x_0^2, x_1^2, x_p^0\}$. Hence there are $(k-1)^3 + 3(k-1)$ transversals and 6 blocks in $\mathbb{F}(k, 3)$. So if we consider the case $k = 4$ and $k = 5$ respectively, we have $\mathbb{F}(4, 3)$ and $\mathbb{F}(5, 3)$ with 36 and 76 transversals respectively. Let $\mathcal{A}$ be the unique maximal intersecting family of 1-set (respectively, unique maximal intersecting family of 2-sets isomorphic to $\{\{a, b\}, \{b, c\}, \{a, c\}\}$) and $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(4,3)} = \emptyset$ (respectively, $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(5,3)} = \emptyset$). By Theorem 1.2, $\mathbb{F}(4, 3) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(4, 3))$ is a maximal intersecting family of 4-sets with 42 blocks (respectively, $\mathbb{F}(5, 3) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(5, 3))$ is a maximal intersecting family of 5-sets with 234 blocks). In this maximal intersecting family of 4-sets there are 1 point in 36 blocks, 6 points in 16 blocks and 3 points in 12 blocks.

Remark. Example 1.3 and Example 1.4 prove the existence of two non isomorphic maximal intersecting family of 4-sets with 42 blocks. It disproves a special case (case $k = 4$) of Conjecture 4 in [2], which claims such maximal intersecting family of 4-sets is unique up to isomorphism.

Remark. Example 1.4 proves the existence of a maximal intersecting family of 5-sets with 234 blocks. So we have $\mathrm{M}(5) \geq 234$. It disproves a special case (case $k = 5$) of Conjecture 4 in [2], which claims $\mathrm{M}(5) = 228$.

2. Proof of Theorem 1.1

The following remarkable lemma is essentially the case $n = 1$ of [4, Theorem 2.1]. We include a simpler proof for the sake of completeness. Recall that, for any finite sequence $(r_1, \ldots, r_t)$ its cyclic shifts are the t sequences $(r_{1+i}, \ldots, r_{t+i})$ where $0 \leq i \leq t-1$ and the addition in the subscripts is modulo t.

Lemma 2.1 *(Raney). Let $(r_1, r_2, \ldots, r_t)$ be a finite sequence of integers such that $\sum\limits_{i=1}^{t} r_i = 1$. Then, exactly one of the t cyclic shifts of this sequence has all its partial sums strictly positive.*

Proof. For $1 \leq n \leq t$, let $s_n = r_1 + \ldots + r_n - \frac{n}{t}$. Suppose, $s_m = s_n$ for some indexes $1 \leq m < n \leq t$. Then $r_{m+1} + \ldots + r_n = \frac{n-m}{t}$, which is a contradiction, since the left hand side is an integer and the right hand side is a proper fraction. Thus, the t numbers $s_1, s_2, \ldots, s_t$ are distinct. So there is a unique index μ, with $1 \leq \mu \leq t$, for which s_μ is the minimum of these t numbers. Now, for $\mu + 1 \leq m \leq t$,

$$r_{\mu+1} + \ldots + r_m = (s_m - s_\mu) + \frac{m-\mu}{t} > 0$$

and for $1 \leq m \leq \mu$,

$$\begin{aligned} r_{\mu+1} + \ldots + r_t + r_1 + \ldots + r_m &= 1 - (s_\mu + \frac{\mu}{t}) + (s_m + \frac{m}{t}) \\ &= (s_m - s_\mu) + 1 - \frac{\mu - m}{t} > 0. \end{aligned}$$

Thus, the partial sums of $(r_{\mu+1}, \ldots, r_{\mu+t})$ are all strictly positive. This proves the existence.

Conversely, let μ be an index for which the partial sums of $(r_{\mu+1}, \ldots, r_{\mu+t})$ are all strictly positive. Then each of these partial sums is at least 1, so that if we subtract a proper fraction from one of them, then the result remains positive. For $\mu + 1 \leq m \leq t$,

$$s_m - s_\mu = (r_{\mu+1} + \ldots + r_m) - \frac{m-\mu}{t} > 0$$

and for $1 \leq m < \mu$,

$$s_m - s_\mu = (r_{\mu+1} + \ldots + r_t + r_1 + \ldots + r_m) - (1 - \frac{\mu - m}{t}) > 0$$

Thus μ is the unique index for which $s_\mu = \min\{s_i : 1 \leq i \leq t\}$. This proves the uniqueness. □

Proof of Theorem 1.1. We recall that if T is a t-set such that $|T \cap X_n| = 1$, for each n with $0 \leq n \leq t-1$, then T is a blocking set of $\mathbb{F}(k,t)$. Hence $\mathrm{tr}(\mathbb{F}(k,t)) \leq |T| = t$. Therefore, it suffices to show that $\mathbb{F}(k,t)$ has no blocking set C of size $t-1$. Let C be a $(t-1)$-subset of $\mathrm{P}_{\mathbb{F}(k,t)}$. For $0 \leq n \leq t-1$, $|C \cap X_n|$ is a non negative integer and $\sum\limits_{i=0}^{t-1} |C \cap X_i| = t-1$. Therefore, if we define the integers $r_{n+1} = 1 - |C \cap X_n|$, where $0 \leq n \leq t-1$, then $\sum\limits_{i=1}^{t} r_i = 1$. So applying Lemma 2.1 to this sequence, we get a unique index μ, with $0 \leq \mu \leq t-1$ such that $\sum\limits_{i=0}^{n} r_{\mu+i} \geq 1$, i.e. $|C \cap (\bigsqcup\limits_{i=0}^{n} X_{\mu+i})| \leq n$, for each n with $0 \leq n \leq t-1$. In particular, C is disjoint from X_μ. For each n with $1 \leq n \leq k - |X_\mu|$, let $l_n = n - \sum\limits_{i=1}^{n} |C \cap X_{\mu+i}|$. Thus $l_n \geq 0$. Let P_n be the set of all integers $p \geq 0$ for which there is a sequence $(p_1, \ldots, p_n)$ satisfying $(\star\star)$ such that $p_n = p$ and for $1 \leq i \leq n$, $x_{p_i}^{\mu+i} \notin C$.

Claim: $|P_n| \geq 1 + l_n$ for $1 \leq n \leq k - |X_\mu|$.

Proof of claim. We prove it by finite induction on n. When $n = 1$,

$$\begin{aligned} |P_n| &= 2 - |C \cap X_{\mu+n}| \\ &= 1 + l_n. \end{aligned}$$

So the claim is true for $n = 1$.

Now let $1 \leq m \leq k - 1 - |X_\mu|$ and suppose that the claim is true for m. Since $|C \cap X_{\mu+m+1}| = 1 + l_m - l_{m+1}$ and clearly

$$P_{m+1} \supseteqq (P_m \cup \{1 + p : p \in P_m\}) \setminus \{i : x_i^{\mu+m+1} \in C \cap X_{\mu+m+1}\},$$

we have

$$\begin{aligned} |P_{m+1}| &\geq |P_m \cup \{1 + p : p \in P_n\}| - |C \cap X_{\mu+m+1}| \\ &\geq 1 + |P_m| - |C \cap X_{\mu+m+1}| \\ &\geq 2 + l_m - (1 + l_m - l_{m+1}) \\ &= 1 + l_{m+1} \end{aligned}$$

This completes the induction and proves the claim.

By the case $n = k - |X_\mu|$ of the claim, $P_{k-|X_\mu|}$ is non empty. Hence there is a sequence $\{p_1, \dots, p_{k-|X_\mu|}\}$ satisfying ($\star\star$) and such that $\{x_{p_i}^{\mu+i} : 1 \le i \le k - |X_\mu|\}$ is disjoint from C. Therefore, the block $X_\mu \sqcup \{x_{p_i}^{\mu+i} : 1 \le i \le k - |X_\mu|\}$ is disjoint from C. Thus C is not a blocking set of $\mathbb{F}(k,t)$. Since C is an arbitrary set of size $t-1$, this shows $\operatorname{tr}(\mathbb{F}(k,t)) \ge t$. □

3. Proof of Theorem 1.2

Let C be a blocking k-set of $\mathbb{F}(k,t) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k,t))$. To prove, $C \in \mathbb{F}(k,t) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(k,t))$. If $C \in \mathbb{F}(k,t)$ we are done. So assume $C \notin \mathbb{F}(k,t)$. To show $C \in \mathcal{A} \circledast \mathbb{F}^\top(k,t)$. We claim the following.

Claim: There exists at least one transversal T of $\mathbb{F}(k,t)$, which is disjoint from C.

Proof of claim. If for each integer n, with $0 \le n \le t-1$, there exists at least one $x_n \in X_n$ such that $x_n \notin C$, then $\{x_n : 0 \le n \le t-1\}$ is the required T and we are done for this case. Suppose there exists at least one integer n, with $0 \le n \le t-1$, such that $X_n \subsetneq C$. We note that such an integer n is unique. Therefore, for each m with $m \ne n$ and $0 \le m \le t-1$, there exists at least one $x_m \in X_m$ such that $x_m \notin C$. (If not, then there exists one more integer m with $m \ne n$, $0 \le m \le t-1$ such that $X_m \subsetneq C$. This implies that $X_n \sqcup X_m \subset C$; a contradiction arises since $k \ge t+1$.) Hence C is of the form $X_n \sqcup Y$, where Y is a $(k - |X_n|)$-set. We observe that if Y is disjoint from the m-set $T_{n+m-1} := \{x_i^{n+m-1} : 0 \le i \le m-1\}$, for some m with $1 \le m-1 \le k - |X_n|$ and we choose a $(t-m)$-set S such that $|S \cap (X_{n+m+i} \setminus C)| = 1$ for each i with $0 \le i \le t-m-1$. Then $T_{n+m-1} \sqcup S$ is the required transversal disjoint from C and we are done. So we assume that Y intersects $T_{n+\mu}$, for each μ with $1 \le \mu \le k - |X_n|$. We note that $T_{n+\mu} \cap T_{n+\nu} = \emptyset$, for each μ and ν with $1 \le \mu < \nu \le k - |X_n|$ and $|Y| = k - |X_n|$. Therefore for each μ with $1 \le \mu \le k - |X_n|$, $|Y \cap T_{n+\mu}| = 1$. In particular, we note that $|Y \cap \{x_0^{n+1}, x_1^{n+1}\}| = 1$. Suppose $x_{\epsilon_1}^{n+1} \in Y$, where ϵ_1 equals exactly one of 0 or 1. We proceed (inductively) by constructing a finite sequence of non negative integers $\{q_i : i \ge 1\}$ with $x_{q_i}^{n+i} \in Y$ for $i \ge 1$ as follows: $q_1 = \epsilon_1$ and for $m \ge 2$, $q_m = q_{m-1} + \epsilon_m$, where ϵ_m equals exactly one of 0 or 1. Clearly, $\{q_i : i \ge 1\}$ satisfies ($\star\star$). Therefore $\theta := \max\left\{m : \{x_{q_i}^{n+i} : 1 \le i \le m\} \subset Y\right\}$ is a well defined positive integer. Since $C \notin \mathbb{F}(k,t)$, Y is not of the form $\{x_{q_i}^{n+i} : 1 \le i \le k - |X_n|\}$. Consequently, we have $1 \le \theta \le k - |X_n| - 1$. Since θ is maximum, $\{x_{q_\theta}^{n+\theta+1}, x_{1+q_\theta}^{n+\theta+1}\}$ is disjoint from Y. Therefore,

$$\{x_{q_{m-1}+1-\epsilon_m}^{n+m} : 1 \le m \le \theta\} \sqcup \{x_{q_\theta}^{n+\theta+1}, x_{1+q_\theta}^{n+\theta+1}\} \sqcup S$$

is a transversal of $\mathbb{F}(k,t)$, which is disjoint from both Y and X_n (hence from C), where S is a $(t-\theta-2)$-set such that $|S \cap (X_{n+i} \setminus C)| = 1$ for each i with $\theta + 2 \le i \le t-1$. Hence the claim is established.

Using the above claim, suppose T is the required transversal disjoint from C. Particularly, C intersects all the blocks of the form $A \sqcup T$, where $A \in \mathcal{A}$. Therefore $C \cap \mathrm{P}_{\mathcal{A}}$ is a blocking set of $\mathcal{A}$. This implies $|C \cap \mathrm{P}_{\mathcal{A}}| \geq k - t$. We note that C is blocking set of $\mathbb{F}(k, t)$, therefore by using Theorem 1.1, we have $|C \cap \mathrm{P}_{\mathbb{F}(k,t)}| \geq t$. But $|C| = k$, so $|C \cap \mathrm{P}_{\mathcal{A}}| = k - t$ and $|C \cap \mathrm{P}_{\mathbb{F}(k,t)}| = t$. Hence $C \cap \mathrm{P}_{\mathcal{A}} \in \mathcal{A}$ and $C \cap \mathrm{P}_{\mathbb{F}(k,t)} \in \mathbb{F}^{\top}(k, t)$. So $C \in \mathcal{A} \circledast \mathbb{F}^{\top}(k, t)$. This completes the proof of Theorem 1.2.

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