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# Mitosis algorithm for Grothendieck polynomials

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ABSTRACT

The goal of the present paper is to extend the mitosis algorithm, originally developed by Ezra Miller and Allen Knutson for the case of Schubert polynomials, to the case of Grothendieck polynomials. In addition we will also use this algorithm to construct a short combinatorial proof of Fomin–Kirillov's formula for the coefficients of Grothendieck polynomials.



## 0. Introduction

The mitosis algorithm, developed by Ezra Miller and Allen Knutson (see [6], [7]), is a combinatorial rule that allows one to compute the coefficients of Schubert polynomials inductively in terms of so called rc-graphs (originally introduced by Fomin and Kirillow in [4]).

Schubert (and Grothendieck) polynomials are originally defined by downward induction on weak Bruhat order where the induction step is represented by applying the corresponding divided difference operator (or, in the case of Grothendieck polynomi-

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als — the isobaric divided difference operator). At the same time, according to the formula of Billey–Jockusch–Stanley (see [2]), the coefficients of each Schubert polynomial might be obtained from the set of diagrams in an $n \times n$ grid called rc-graphs (or reduced pipe dreams). Mitosis might be considered as an analogue of applying the divided difference operator: namely, if $w$ and $v$ are permutations and $v < w$ in the weak Bruhat order, then by using this algorithm we can obtain the set $\mathcal{RP}(v)$ of reduced pipe dreams corresponding to the Schubert polynomial $\mathfrak{S}_v(x)$ from the set $\mathcal{RP}(w)$. Thus, the algebraic construction of the Schubert polynomials becomes combinatorial. In [7], Ezra Miller provides a short combinatorial proof of this fact, based on Billey–Jockusch–Stanley's formula and adjacent combinatorial properties of reduced pipe dreams ([7, Theorem 15]).

The analogous formula of Fomin and Kirillov (see [3, Theorem 2.3]) establishes the same correlation between the coefficients of Grothendieck polynomials and pipe dreams in general, with its proof being based on the Yang–Baxter equation. The goal of the present paper is to extend the mitosis algorithm to the case of Grothendieck polynomials. While the original paper [7] uses the formula of Billey–Jockush–Stanley for Schubert polynomials in order to justify the mitosis algorithm, we are going to both justify the modified mitosis algorithm and prove Fomin–Kirillov's formula, by using only the combinatorics of pipe dreams.

The paper is structured as follows: in the first section we will give the classical inductive definitions of Schubert and Grothendieck polynomials and introduce the concept of pipe dreams. The second section is devoted to the original mitosis algorithm: we give a brief description of this algorithm and also prove the important lemma about the way mitosis acts on the set of pipe dreams $\mathcal{P}(w)$ corresponding to a permutation $w$. Section 3, analogously to [7], provides a special involution on $\mathcal{P}(w)$. The final section introduces the modified mitosis algorithm and uses it to provide a short combinatorial proof of Fomin–Kirillov's formula.

## 1. Pipe dreams

Denote by $s_i = (i, i+1)$ the corresponding adjacent transposition of $S_n$. It is well known that the set $\{s_i \mid i = 1, \ldots, n-1\}$ generates the group $S_n$ with the following relations:

$$s_i^2 = 1\,,$$

$$s_i s_j = s_j s_i \text{ if } |i-j| > 1\,,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,.$$

Thus, we can say that the set $\{s_i \mid i \in \mathbb{N}\}$ generates the group $S_\infty$ with the relations described above. Let $w$ be an arbitrary element of $S_\infty$. Then we can define a sequence $a_1, \ldots, a_k$ of minimal length such that $w = s_{a_1} \cdots s_{a_k}$. The number $k$ is called *the* *length* of $w$ and is denoted by $l(w)$ (note that the sequence itself can be defined in more than one way: for example, the words $s_i s_{i+1} s_i$ and $s_{i+1} s_i s_{i+1}$ correspond to the same permutation).

Let $w$ be an arbitrary element of $S_\infty$ and $s_i$ be an adjacent transposition. Then we say that $ws_i > w$ if $l(ws_i) = l(w) + 1$. Otherwise we have $l(ws_i) = l(w) - 1$ which means that $l(w) = l((ws_i)s_i) = l(ws_i) + 1$ and, consequently, $w > ws_i$. Now, by using the property of transitivity, we can introduce on $S_\infty$ a partial order (called *the Bruhat order*). Note that $S_n$ ordered in such a way is a Weyl group therefore it has the unique greatest element. This element is called *the order reversing permutation* and is denoted by $w_0^{(n)} = (n\ n-1 \cdots 1)$.

Now we can define Schubert and Grothendieck polynomials inductively. For that we will also need to introduce the set $\{\partial_i\}_{i\in\mathbb{N}}$ of *divided difference operators*, and the set $\{\pi_i\}_{i\in\mathbb{N}}$ of *isobaric divided difference operators*. These linear operators act on the ring $\mathbb{Z}[x_1, x_2, ...]$ as follows:

$$\forall f \in \mathbb{Z}[x_1, x_2, ...] \quad \partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}, \quad \pi_i(f) = \partial_i(f - x_{i+1}f)\,.$$

Here $s_i(f)$ is a polynomial, obtained from $f$ by interchanging the variables $x_i$ and $x_{i+1}$. Note that in both cases the result is also a polynomial with integer coefficients. The term "isobaric" makes sense after the coordinate change $x_i \to 1 - x_i$ for the case where the corresponding $\pi_i$ preserves the degree of homogeneous polynomials.

**Definition 1.1.** For an arbitrary element $w$ of $S_\infty$ corresponding Schubert and Grothendieck polynomials (denoted by $\mathfrak{S}_w(x)$ and $\mathfrak{G}_w(x)$ respectively) can be defined inductively in compliance with the following rules:

(i) for the order reversing permutation $w_0^{(n)}$, the following equalities hold:

$$\mathfrak{G}_{w_0^{(n)}}(x) = \mathfrak{S}_{w_0^{(n)}}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\,,$$

(ii) if $l(ws_i) = l(w) - 1$ the following equalities hold:

$$\partial_i(\mathfrak{S}_w(x)) = \mathfrak{S}_{ws_i}(x)\,, \quad \pi_i(\mathfrak{G}_w(x)) = \mathfrak{G}_{ws_i}(x)\,.$$

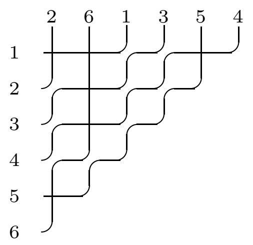
Now we will give the definition of a pipe dream. Consider the direct product $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ represented in the form of a table extending infinitely south and east (the box located in the $i$-th row and $j$-th column is indexed by pair $(i, j)$). Then a *pipe dream* is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, with its elements being marked with the $+$ symbol.

**Example 1.1.** The following pipe dream represents the set $\{(1, 1); (1, 2); (1, 5); (2, 2); (3, 2); (5, 1)\}$.

| | | | | | |
|---|---|---|---|---|---|
| + | + | | | + | |
| | + | | | | |
| | + | | | | |
| | | | | | |
| + | | | | | |
| | | | | | |

Now, in every box with + we put the symbol $+$, and in every empty box we put the symbol . Thus, we obtain the network of strands crossing each other at the positions belonging to the pipe dream and avoiding each other at other positions. A pipe dream is *reduced* if each pair of strands crosses at most once. Now for every $w \in S_\infty$ by $\mathcal{RP}(w)$ we denote the set of reduced pipe dreams such that for each of its elements the strand entering the $i$-th row exits from the $w(i)$-th column.

**Example 1.2.** The depicted pipe dream is reduced and corresponds to the permutation $w = (314652)$.




Note that for simplicity we usually do not draw the "sea" of wavy strands whose entries and exits are equal.

Now Schubert polynomials can be defined combinatorially. Firstly, for an arbitrary pipe dream $D$ we will introduce the following notation

$$x_D = \prod_{(i,j) \in D} x_i \,.$$

The following formula was originally proven by Billey–Jockusch–Stanley in [2] in transparently equivalent language. Later, however, Fomin and Stanley introduced the combinatorial proof in a more fitting form (see [5]).

**Proposition 1.1.** *For an arbitrary element $w$ of $S_\infty$, the following equality holds*

$$\mathfrak{S}_w(x) = \sum_{D \in \mathcal{RP}(w)} x_D \,. \tag{1.1}$$

As a consequence, the coefficients of the polynomial $\mathfrak{S}_w(x)$ are positive.

Consider an arbitrary pipe dream $B$ whose strands can cross each other more than once. We will say that $B$ is *nonreduced*. For each element $(i, j)$ of $B$ we will define its anti-diagonal index by the number $i+j-1$. Then, by moving across the table from right to left and from top to bottom and associating to each element of $B$ its anti-diagonal index, we will obtain a sequence $(i_1, \dots, i_k)$, where $k$ is the number of crosses of $B$. The corresponding permutation $w$ is produced by multiplying the adjacent transpositions $s_{i_1} \cdots s_{i_k}$ in compliance with the following rules:

$$s_i^2 = s_i\,,$$
$$s_i s_j = s_j s_i \text{ if } |i-j| > 1\,,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,.$$

In other words, we omit transpositions that decrease length (for example, the word $s_1 s_1 s_2 s_1 s_1$ gives us the permutation $s_1 s_2 s_1 = (321)$). The corresponding operation is called *the Demazure product*. The set of all pipe dreams whose Demazure products are equal to $w$ is denoted by $\mathcal{P}(w)$. The set $\mathcal{RP}(w)$ is a subset of $\mathcal{P}(w)$. Also, in the case of a reduced pipe dream, the Demazure product is equivalent to the standard group operation, so we can think of $\mathcal{RP}(w)$ as a subset of elements of $\mathcal{P}(w)$ with the minimal number of crosses.

**Example 1.3.** $B$ is a nonreduced pipe dream, belonging to $\mathcal{P}(1423)$.

$$B = \begin{array}{|c|c|c|c|}\hline & + & & \\ \hline + & + & & \\ \hline + & & & \\ \hline & & & \\ \hline\end{array}$$

There is a formula analogous to the equality (1.1) from Proposition 1.1 for the case of Grothendieck polynomials. Namely, for an arbitrary element $w$ of $S_\infty$ we have

$$\mathfrak{G}_w(x) = \sum_{B \in \mathcal{P}(w)} (-1)^{|B|-l(w)} x_B\,. \tag{1.2}$$

Here by $|B|$ we denote the number of crosses of $B$.

This formula can be obtained as a particular case from the more general result of Fomin and Kirillov (see [3, Theorem 2.3]). At the end of the paper we will introduce the alternative combinatorial proof.

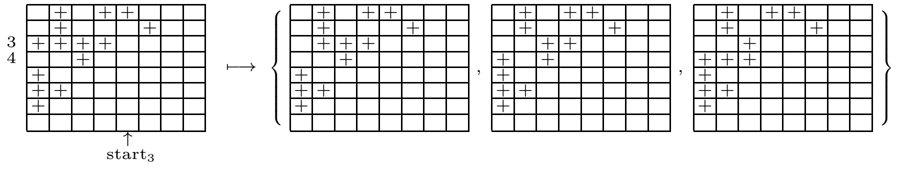
## 2. Mitosis algorithm

Consider an arbitrary pipe dream $D$. By $\mathrm{start}_i(D)$ we will denote the maximal column index such that each box in the $i$-th row located strictly to the left of the corresponding column is marked with a cross. Also, by $\mathcal{J}_i(D)$ we denote the subset of columns $j$ located strictly to the left of $(i, \mathrm{start}_i(D))$ such that the box $(i+1, j)$ is empty.

Now, each element $p$ of $\mathcal{J}_i(D)$ can be associated with a new pipe dream $D_p$, constructed in the following way: first the cross in the box $(i, p)$ is deleted from $D$, then every cross in the $i$-th row located to the left of $(i, p)$ with its column index belonging to $\mathcal{J}_i(D)$ is moved down to the empty box below it.

**Definition 2.1.** The $i$-th mitosis operator (denoted by $\mathrm{mitosis}_i(D)$) sends $D$ to the set $\{D_p \mid p \in \mathcal{J}_i(D)\}$.

**Example 2.1.**




Here $i = 3$, and $\mathcal{J}_i(D)$ contains 1, 2 and 4.

If $\mathcal{C}$ is a set of pipe dreams, then by $\mathrm{mitosis}_i(\mathcal{C})$ we mean the union $\bigcup\limits_{D \in \mathcal{C}} \mathrm{mitosis}_i(D)$ over all elements of $\mathcal{C}$. If $\mathcal{J}_i(D) = \emptyset$, then $\mathrm{mitosis}_i(D) = \emptyset$.

Now let $w$ be an element of $S_\infty$ and $s_i$ be an adjacent transposition such that $l(ws_i) = l(w) - 1$. The main result of the corresponding article by E. Miller ([3]) is the following statement:

**Theorem 2.1.** *Disjoint union* $\bigsqcup\limits_{D \in \mathcal{RP}(w)} \mathrm{mitosis}_i(D)$ *coincides with the set* $\mathcal{RP}(ws_i)$ *of reduced pipe dreams of the permutation* $ws_i$.

Together with the Formula (1.1) this theorem gives us the capability to obtain Schubert polynomials inductively, in terms of pipe dreams. Here we will not give the proof of this theorem (see [7, Theorem 15]). Nevertheless, we will use the same methods and terms, with a slight modification.

For our further reasonings, we will have to introduce the family of operations of the pipe dreams, which, in accordance with the original article [1] by Bergeron–Billey, will be called *chute moves*.

Each chute move applied to a pipe dream interchanges or deletes some of its elements located in two neighbouring rows of the pipe dream. Henceforth, we will distinguish between three types of chute moves.

**Definition 2.2.** All three types of chute moves (which henceforth will be indexed by numbers 1,2 and 3) can be defined graphically in the following way

1)

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & + \\ \hline & + & + & \cdots & + & + & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & \\ \hline + & + & + & \cdots & + & + & \\ \hline\end{array}$$

2)

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & + \\ \hline + & + & + & \cdots & + & + & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & \\ \hline + & + & + & \cdots & + & + & \\ \hline\end{array}$$

3)

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & + \\ \hline + & + & + & \cdots & + & + & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline & + & + & & + & + & + \\ \hline & + & + & \cdots & + & + & \\ \hline\end{array}$$

As we can see, for any chute move we can uniquely describe an inverse one. The following statement is an easy extension of [7, Lemma 9] and is correct for both chute moves and inverse chute moves:

**Lemma 2.1.** *If a pipe dream $D$ belongs to the set $\mathcal{P}(w)$, then the result of applying any of the chute moves 1–3 also belongs to $\mathcal{P}(w)$.*

**Proof.** We will prove the statement only for the cases of chute move 1 and chute move 2, since any chute move 3 can be introduced as a result of consistent applying of chute moves 2 and 1.

For every pipe dream $D$, the corresponding word will be denoted by $\mathrm{word}(D)$. For every word $v$ and pipe dream $D$, the corresponding Demazure product will be denoted by $\mathrm{Demaz}(v)$ ($\mathrm{Demaz}(D)$ accordingly).

1) Suppose that the $m$-th and $(m+1)$-th rows of $D$ look like

$$\begin{array}{cc|c|c|c|c|c|c|c|c} & & \scriptstyle & \scriptstyle (i) & \scriptstyle (i+1) & & & & \scriptstyle (i+k) & \\ \cline{3-9} m & B' & & + & + & & + & + & + & A \\ \cline{3-9} m+1 & A' & & + & + & \cdots & + & + & & B \\ \cline{3-9} & & \scriptstyle (i) & \scriptstyle (i+1) & & & & \scriptstyle (i+k) & & \end{array}$$

so that we can apply chute move 1 (the numbers located under the boxes of the $i$-th row and below the boxes of the $i+1$-th row are the indexes of the corresponding anti-diagonals, and the letters $A, B, A'$ and $B'$ are the corresponding subwords, obtained by "reading" of the pipe dream $D$ in the way described above). Then the subword obtained by reading the $m$-th and $(m+1)$-th rows is the following

$$As_{i+k}s_{i+k-1}\ldots s_{i+1}s_iA'Bs_{i+k}s_{i+k-1}\ldots s_{i+2}s_{i+1}B'.$$

From the properties of the Demazure product, it follows that this subword is equivalent to the following

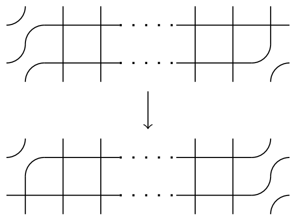
$$AA's_{i+k}s_{i+k-1}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+2}s_{i+1}BB'.$$

This means that for the case of chute move 1, it is enough to prove the equivalence of the words

$$(s_{i+k}s_{i+k-1}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+2}s_{i+1}),$$
$$(s_{i+k-1}s_{i+k-2}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_i)$$

with respect to the Demazure product.

Note that in the case of applying chute move 1 to a reduced pipe dream, preservation of the permutation can easily be proven graphically. Indeed, all strands remain "untouched" except the pair involved in the conversion. Also the crossing of two involved strands in the upper right corner of the chutable rectangle is replaced with the crossing in the lower left corner and the exits of these two strands stay the same (see the following figure).



In particular, this proof works for the pipe dream $B$ whose crosses coincide with those depicted on the figure above. This pipe dream is reduced and yields the word $(s_{i+k}s_{i+k-1}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+2}s_{i+1})$. Applying the corresponding chute move 1, we get a reduced pipe dream $B'$ with the word $(s_{i+k-1}s_{i+k-2}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_i)$. Hence, the words word$(B)$ and word$(B')$ are reduced decompositions of the same permutation Demaz$(B)$ = Demaz$(B')$. In particular, word$(B)$ and word$(B')$ are equivalent with respect to the Demazure product.

2) Suppose now that the $m$-th and $(m+1)$-th rows of $D$ look like

| | | | $(i)$ | $(i+1)$ | | | | $(i+k)$ | |
|---|---|---|---|---|---|---|---|---|---|
| $m$ | $B'$ | | + | + | …… | + | + | + | $A$ |
| $m+1$ | $A'$ | + | + | + | …… | + | + | | $B$ |
| | | $(i)$ | $(i+1)$ | | | | $(i+k)$ | | |

so we can apply chute move 2. By carrying out the arguments analogously to 1), we reduce our statement to proving the equivalence of the words

$$(s_{i+k}s_{i+k-1}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_i)$$
$$(s_{i+k-1}s_{i+k-2}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_i).$$

Again, consider a reduced pipe dream $B$. Then we have

$$(s_{i+k}s_{i+k-1}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_i) = \text{word}(B)s_i \sim$$
$$\sim \text{word}(B')s_i \sim (s_{i+k-1}s_{i+k-2}\dots s_{i+1}s_is_{i+k}s_{i+k-1}\dots s_{i+1}s_is_i)\,.$$

Thus, since the subwords $s_is_i$ and $s_i$ are equivalent, the initial two words are also equivalent. Here we also use associativity of the Demazure product. $\square$

Now we can redefine the mitosis algorithm in terms of chute moves.

**Proposition 2.1.** *Let $D$ be a pipe dream. Set $j_{min}$ to be the minimal element of $\mathcal{J}_i(D)$ (on the assumption that it is not empty). Then each $D_p \in \mathrm{mitosis}_i(D)$ is obtained from $D$ by*

*(1) deleting $(i, j_{min})$ and then*
*(2) moving to the right, one by one applying chute moves 1, so that $(i,p)$ is the last cross moved from the $i$-th row to $i+1$-th row.*

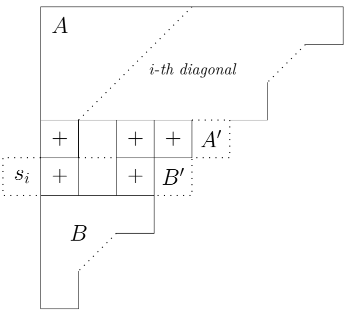
From this proposition we can see that all elements of $\mathrm{mitosis}_i(D)$ correspond to the same permutation because all of them are the results of chute moves applied to $D\backslash(i, j_{min})$.

Now we are ready to prove the main theorem of this section.

**Theorem 2.2.** *Let $w$ be an element of $S_\infty$, $s_i$ be an adjacent transposition, and let $D$ belong to $\mathcal{P}(w)$. Then, if $\mathrm{mitosis}_i(D)$ is not empty, it lies entirely in either $\mathcal{P}(ws_i)$ or $\mathcal{P}(w)$.*

**Proof.** According to Lemma 2.1 and Proposition 2.1 if $D\backslash(i, j_{min})$ belongs to $\mathcal{P}(ws_i)$ (accordingly, to $\mathcal{P}(w)$), then the same is true of all the elements of $\mathrm{mitosis}_i(D)$.

Let $D$ be of the form




Then the following equalities hold

$$w = \mathrm{Demaz}(D) = \mathrm{Demaz}(AA's_{i+j_{min}-1}\dots s_iB's_{i+j_{min}-1}\dots s_{i+1}B)\,.$$

Since $s_i$ commutes with the subword $B$, by applying to the left word a transform analogous to chute move 1 gives

$$\begin{aligned} w &= \mathrm{Demaz}(D) = \mathrm{Demaz}(AA's_{i+j_{min}-2}\dots s_iB's_{i+j_{min}-1}\dots s_{i+1}Bs_i) = \\ &= \mathrm{Demaz}(\mathrm{word}(D')s_i) \end{aligned}$$

where $D'$ is the result of removing $(i, j_{min})$ from $D$.

Denote by $\tilde{w}$ the permutation $\mathrm{Demaz}(D')$. Then according to the definition of the Demazure product two cases are possible:

1) $l(\tilde{w}s_i) = l(\tilde{w}) - 1$. Then we have $w = \mathrm{Demaz}(\mathrm{word}(D')s_i) = \tilde{w}$ and $D'$ belongs to $\mathcal{P}(w)$.
2) $l(\tilde{w}s_i) = l(\tilde{w}) + 1$. Then we have $w = \mathrm{Demaz}(\mathrm{word}(D')s_i)) = \tilde{w}s_i$, which means that $\tilde{w} = ws_i$ and $D'$ belongs to $\mathcal{P}(ws_i)$. $\square$

Note that in both cases $l(ws_i) = l(w) - 1$. In particular, the following corollary takes place.

**Corollary 2.2.1.** *If $ws_i > w$, then the set $\mathcal{J}_i(D)$ is empty.*

In compliance with the proven theorem for each case $l(ws_i) = l(w) - 1$, we divide $\mathcal{P}(w)$ into three disjoint sets: $\mathcal{P}_{s_i}(w)$ (the set of all pipe dreams which are sent to $\mathcal{P}(ws_i)$), $\mathcal{P}_I(w)$ (the set of all pipe dreams which are sent to $\mathcal{P}(w)$), and $\mathcal{P}_\varnothing(w)$ (the set of all pipe dreams which are sent to the empty set). Here and further by mitosis we mean mitosis$_i$ with condition that $l(ws_i) = l(w) - 1$. This partition will be used later.

**Example 2.2.** Consider the case $w = s_2 = (23)$. Then $s_2s_2 = e$ and the set $\mathcal{P}(s_2)$ consists of three pipe dreams

$$\left\{ \begin{array}{|c|c|}\hline & + \\ \hline + & \\ \hline \end{array}, \begin{array}{|c|c|}\hline & + \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|}\hline & \\ \hline + & \\ \hline \end{array} \right\}$$

Here the first pipe dream belongs to $\mathcal{P}_I(w)$, the second belongs to $\mathcal{P}_\varnothing(w)$ and the third belongs to $\mathcal{P}_{s_2}(w)$.

## 3. Intron mutations

Let $D$ be an arbitrary pipe dream with a fixed row index $i$. We will index the boxes in $i$-th and $(i+1)$-th rows as shown in the following figure

$$\begin{array}{r|c|c|c|c|l} & 1 & 2 & 3 & 4 & \cdots\cdots\cdots \\ \hline (i) & 1 & 3 & 5 & 7 & \\ \cline{1-5} (i+1) & 2 & 4 & 6 & 8 & \cdots\cdots\cdots \\ \hline \end{array}$$

Henceforth, by an intron we mean a $2 \times m$ rectangle, located in these two adjacent rows such that:

1) the first and the last boxes of this rectangle are empty,
2) no $\boxed{\begin{smallmatrix}\cdot\\+\end{smallmatrix}}$ column can be located to the right of a $\boxed{\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}}$ column or a $\boxed{\begin{smallmatrix}+\\\cdot\end{smallmatrix}}$ column and no $\boxed{\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}}$ column can be located to the right of a $\boxed{\begin{smallmatrix}+\\\cdot\end{smallmatrix}}$ column. (The first (last) box of a rectangle is the box with the maximal (minimal) index according to the ordering described above.)

An intron $C$ is maximal, if the empty box with largest index before $C$ (if there is one) is located in the $i+1$-th row and the empty box with smallest index after $C$ is located in the $i$-th row. In other words, an intron is maximal, if in cannot be extended rightwards or leftwards.

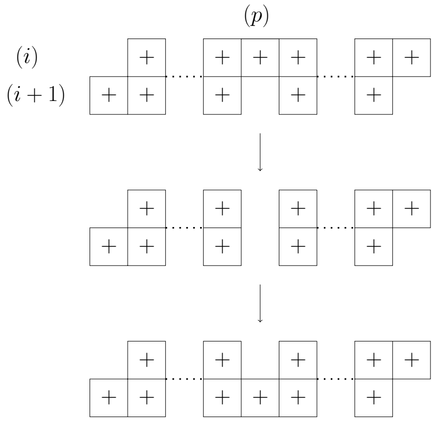
**Lemma 3.1.** *Let $D$ be a pipe dream and $C \subseteq D$ be an intron. Then by applying a sequence of chute moves and inverse chute moves we can transform $C$ to a new intron $\tau(C)$ with the following properties:*

*1) the set of $\boxed{\begin{smallmatrix}+\\+\end{smallmatrix}}$ columns in $C$ coincides with the set of $\boxed{\begin{smallmatrix}+\\+\end{smallmatrix}}$ columns in $\tau(C)$, and*
*2) the number $c_i$ of crosses in the i-row of $C$ coincides with the number $\tilde{c}_{i+1}$ of crosses in the $i+1$-th row of $\tau(C)$ and vice versa.*

**Proof.** Suppose that $c_i \geq \tilde{c}_{i+1}$. Then the proof is by induction on the parameter $c := c_i - \tilde{c}_{i+1}$. In the case $c = 0$ we obviously have $C = \tau(C)$. Now suppose that $c > 0$. Denote the index of the leftmost $\boxed{\begin{smallmatrix}+\\\cdot\end{smallmatrix}}$ column by $p$. Then moving to the left from this column we will, sooner of later, find a column of the type $\boxed{\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}}$ or $\boxed{\begin{smallmatrix}\cdot\\+\end{smallmatrix}}$. If it is a $\boxed{\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}}$ column, then we can apply chute move-1 and thereby chute the cross from the $i$-th row to the $i+1$-th one. Since the result of this conversion will also be an intron, the proof is reduced to the induction hypothesis. If it looks like a $\boxed{\begin{smallmatrix}\cdot\\+\end{smallmatrix}}$ column, then, owing to the fact that $c > 0$ and, thereafter, there is more than one $\boxed{\begin{smallmatrix}+\\\cdot\end{smallmatrix}}$ column in $C$, the corresponding fragment of $C$ will take the form

$$
\begin{array}{cccccccccc}
 & & & & & (p) & & & & \\
(i) & & + & \cdots & + & + & + & \cdots & + & + \\
(i+1) & + & + & & + & & + & & + &
\end{array}
$$

Then by applying the composition of the chute move 2 and the reverse chute move 3 the way it shown on the following figure, we will bring the corresponding fragment of $C$ to a form



and thereby again chute the cross from the $i$-th row to the $i+1$-th one. Thus, the proof is again reduced to the induction hypothesis.

In the case $c_i < \tilde{c}_{i+1}$ we just flip the argument 180°. $\square$

The transformation $\tau$ is called the *intron mutation*. Note that the intron $\tau(C)$ is defined uniquely and by construction $\tau(\tau(C)) = C$, i.e. $\tau$ is an involution.

Now we are ready to make the main statement of this section.

**Theorem 3.1.** *Let $w$ be an element of $S_\infty$. Then for each $i \in \mathbb{N}$ there is an involution $\tau_i : \mathcal{P}(w) \longrightarrow \mathcal{P}(w)$ such that for any $D \in \mathcal{P}(w)$ the following conditions take place:*

*1) $\tau_i(D)$ coincides with $D$ in all rows with indexes different from $i$ and $i+1$,*
*2) $start_i(D) = start_i(\tau_i(D))$ and $\tau_i(D)$ agrees with $D$ in all columns with indexes strictly less than $start_i(D)$,*
*3) $l_i^i(\tau_i(D)) = l_{i+1}^i(D)$ (here $l_r^i(-)$ is the number of crosses in the $r$-th row located to the right or in column with index $start_i(-)$).*

**Proof.** Let $D$ belong to $\mathcal{P}(w)$. Consider all crosses of the union of the $i$-th and $(i+1)$-th rows located to the right or in column with index $start_i(D)$. Then, according to the definition of $start_i(D)$, we can find a minimal rectangle with empty last box starting at the column $start_i(D)$ and containing all these crosses. Since the first box of this rectangle is also empty, it can be uniquely represented in form of a disjoint union of maximal introns and rectangles, completely filled with crosses. Apply to an every maximal intron the corresponding intron mutation. Since every mutation is a sequence of chute moves and reverse chute moves, the obtained pipe dream obviously belongs to $\mathcal{P}(w)$. Owing the fact that each mutation is an involution and that the result of applying a mutation to a maximal intron is also a maximal intron, the obtained transformation is also an involution. Properties *1)–3)* are obvious from the construction scheme. $\square$

**Remark.** Note that the partition $\mathcal{P}(w) = \mathcal{P}_{\varnothing}(w) \sqcup \mathcal{P}_{s_i}(w) \sqcup \mathcal{P}_I(w)$ is invariant under the constructed involution. This fact will be used in the near future.

## 4. Mitosis theorem

In the second section we have defined the way mitosis acts on the set $\mathcal{P}(w)$. Nevertheless, in order to prove the main theorem of this article we will have to slightly modify its initial definition:

**Definition 4.1.** Let $w$ be an element of $S_\infty$, $s_i$ to be an adjacent transposition such that $l(ws_i) = l(w) - 1$. Then if $\mathcal{J}_i(D) = \{j_1, \ldots, j_k\}$ $D$ belong to $\mathcal{P}(w)$ the operator $\mathrm{mitosis}'_i$ sends $D \in \mathcal{P}(w)$ to the set

$$\left\{ D_{j_1}, D_{j_1} \cup D_{j_2}, D_{j_2}, \ldots, D_{j_{k-1}}, D_{j_{k-1}} \cup D_{j_k}, D_{j_k} \right\}.$$

As we can see, the elements of the set $\mathrm{mitosis}'_i(D)$ form some kind of a chain, where links are elements of $D_{j_m}$ and the result of the cohesion of two links is a union of the corresponding pipe dreams.

Let us show that pipe dreams of $\mathrm{mitosis}'_i(D)$ represent the same permutation. Indeed, it's easy to see that $D_{j_m} \cup D_{j_{m+1}}$ is obtained from $D_{j_m}$ by applying the inverse chute move 2. Thus, the partition $\mathcal{P}(w) = \mathcal{P}_s(w) \sqcup \mathcal{P}_I(w) \sqcup \mathcal{P}_{\varnothing}(w)$ constructed for $\mathrm{mitosis}_i$ is also preserved by $\mathrm{mitosis}'_i$.

**Theorem 4.1.** *Let $w$ be an element of $S_\infty$ and $s_i$ to be an adjacent transposition such that $l(ws_i) = l(w) - 1$. Then the following equalities take place:*

*1)* $\displaystyle\bigsqcup_{D \in \mathcal{P}_I(w)} \mathrm{mitosis}'_i(D) = \mathcal{P}_{\varnothing}(w)$

*2)* $\displaystyle\bigsqcup_{D \in \mathcal{P}_{s_i}(w)} \mathrm{mitosis}'_i(D) = \mathcal{P}(ws_i)$.

**Proof.** The disjointness of the unions on the left sides of the equalities *1)* and *2)* is obvious. Indeed, each element of the image of $\mathrm{mitosis}'_i$ agrees with its preimage everywhere except $i$-th and $(i+1)$-th rows. In these two adjacent rows they also coincide to the right from the leftmost column of the type $\begin{array}{|c|}\hline + \\ \hline \cdot \\ \hline\end{array}$ or $\begin{array}{|c|}\hline \cdot \\ \hline \cdot \\ \hline\end{array}$. The rest of the diagram is restored uniquely according to the corresponding algorithm.

In order to prove the first equality, it is enough to construct a preimage in $\mathcal{P}_I(w)$ for an arbitrary $D$ from $\mathcal{P}_{\varnothing}(w)$. It can be done in the following way: consider the first column of $D$ which has a restriction on the rows $i, i+1$, different from $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \cdot \\ \hline + \\ \hline\end{array}$. If it looks like $\begin{array}{|c|}\hline \cdot \\ \hline \cdot \\ \hline\end{array}$, then by moving from this column to the left and applying the sequence of inverse chute moves 1 (corresponding to the sequence of the $\begin{array}{|c|}\hline \cdot \\ \hline + \\ \hline\end{array}$-columns, located strictly to the left from our column) we will bring $D$ to a form $B' = B \backslash (i, j_{min})$, where $B$ is a pipe dream with non-empty set $\mathrm{mitosis}_i(B)$ (note that the quantity of applied chute moves may be zero). Applying Theorem 2.2 to $B$ in the corresponding notation, we have $\mathrm{Demaz}(B') = \mathrm{Demaz}(D) = w$ and $l(ws_i) = l(w) - 1$ which means that $\mathrm{Demaz}(B) = w$ and $B$ belongs to $\mathcal{P}_I(w)$. Obviously, $B$ is a preimage of $D$ and also $D$ is a "link" in the $\mathrm{mitosis}'_i(B)$-chain.

If our column look like $\boxed{\substack{+\\ \cdot}}$, then the way to construct a preimage for $D$ is more difficult. Since $D$ belongs to $\mathcal{P}_{\varnothing}(w)$, there must be at least one $\boxed{\substack{\cdot\\ +}}$-column to the left from ours. We will fixate all such columns. Then we will start moving to the left, applying to each $\boxed{\substack{\cdot\\ +}}$-column except the last one the transformation, inverse to the one which is described by the figures in the proof of Lemma 3.1. And finally we will apply chute move 3 to the last $\boxed{\substack{\cdot\\ +}}$-column, again bringing $D$ to a form $B' = B\backslash(i, j_{min})$, where $B$ is a pipe dream, belonging to $\mathcal{P}_I(w)$ and the set $\mathrm{mitosis}_i(B)$ is non-empty. Note, that this time $D$ also belongs to $\mathrm{mitosis}'_i(B)$, but as a cohesion.

The second equality's proof is practically the same, only this time we use the fact that $l(ws_is_i) = l(w) = l(ws_i) + 1$ which, according to the Corollary 2.2.1, means that for each element $D$ of $\mathcal{P}(ws_i)$ the corresponding set $\mathrm{mitosis}'_i(D)$ is empty. $\square$

Thus, Theorem 4.1 gives us the way to list efficiently all pipe dreams of $\mathcal{P}(w)$. Namely, if $l(ws_i) = l(w) - 1$, we have the equality

$$\mathcal{P}(ws_i) = \Big(\bigsqcup_{D\in\mathcal{P}(w)} \mathrm{mitosis}'_i(D)\Big)\Big\backslash \mathcal{P}_{\varnothing}(w)\,.$$

Now denote the corresponding operation of applying $\mathrm{mitosis}'_i$ to all elements of $\mathcal{P}(w)$ and deleting all elements of $\mathcal{P}_{\varnothing}(w)$ by $\mathrm{mitosis}'_i(w)$. Then if $w$ belongs to $S_n$, $w_0^{(n)}$ is corresponding order reversing permutation and $s_{i_1}\cdots s_{i_k}$ is a reduced expression for $w_0^{(n)}w$, we have the equality

$$\mathcal{P}(w) = \mathrm{mitosis}'_{i_k}\ldots\mathrm{mitosis}'_{i_1}(D_0^{(n)})$$

where $D_0^{(n)}$ is the only pipe dream belonging to $\mathcal{P}(w_0^{(n)})$.

Finally, we will give a proof of Formula (1.2).

**Theorem 4.2.** *Let $w$ be an element of $S_\infty$ and $s_i$ be an adjacent transposition such that $l(ws_i) = l(w) - 1$. Then the following equality takes place*

$$\mathfrak{G}_w(x) = \sum_{B\in\mathcal{P}(w)} (-1)^{|B|-l(w)} x_B\,.$$

**Proof.** The proof is by induction on $l(w)$ for each group $S_n$. For each $n$ the inductive basis is the case $w = w_0^{(n)} = (n\; n{-}1 \cdots 1)$. Indeed, the only reduced pipe dream corresponding with this permutation is the one containing all boxes above the $n$-th anti-diagonal (it is the only possible way to connect $i$ with $n+1-i$ for each $i = 1, \cdots, n$). And since $w$ belongs to $S_n$, each element of $\mathcal{P}(w)$ contains only such boxes, which means that

$\mathcal{P}(w)$ contains only one element described above. Corresponding monomial is obviously $x_1^{n-1}x_2^{n-2}\cdots x_{n-1} = \mathfrak{G}_{w_0^{(n)}}(x)$.

Now we want to fulfil the induction step. Let $l(ws_i) = l(w) - 1$ and assume that the induction hypothesis is applicable to $w$. Let $D$ be an arbitrary element of $\mathcal{P}_{s_i}(w)$ with $J = |\mathcal{J}_i(D)|$. Then $x_D = x_i^J x_{D'}$, where $D'$ is a pipe dream obtained from $D$ by removing all crosses in the boxes $(i, j)$, with $j$ belonging to $\mathcal{J}_i(D)$.

Recall that if condition $l(ws_i) = l(w) - 1$ is satisfied, then $\mathfrak{G}_{ws_i}(x)$ is obtained from $\mathfrak{G}_w(x)$ by applying the operator $\pi_i$. Thus, we have

$$\sum_{E\in \mathrm{mitosis}'_i(D)} (-1)^{|E|-l(ws_i)} x_E = (-1)^{|D|-l(w)} x_{D'} \sum_{d=1}^{J} x_i^{J-d} x_{i+1}^{d-1} -$$
$$- (-1)^{|D|-l(w)} x_{D'} \sum_{d=1}^{J-1} x_i^{J-d} x_{i+1}^{d} = (-1)^{|D|-l(w)} \pi_i(x_i^J) x_{D'} \,.$$

Note that the partition $\mathcal{P}(w) = \mathcal{P}_\varnothing(w) \sqcup \mathcal{P}_{s_i}(w) \sqcup \mathcal{P}_I(w)$ is preserved by involution $\tau_i$ from Section 3, so we can restrict it to $\mathcal{P}_{s_i}(w)$. Now, if $\tau_i(D) = D$, then it follows from property *3)* of $\tau_i$ that $x_{D'}$ is symmetric in the variables $x_i$ and $x_{i+1}$. Consequently, we have

$$(-1)^{|D|-l(w)} \pi_i(x_i^J) x_{D'} = (-1)^{|D|-l(w)} \pi_i(x_i^J x_{D'}) = \pi_i((-1)^{|D|-l(w)} x_D) \,.$$

On the other hand, if $\tau_i(D) \neq D$, then again, according to property *3)* of $\tau_i$, the sum $x_{D'} + s_i(x_{D'})$ is symmetric in the variables $x_i$ and $x_{i+1}$ and we have

$$(-1)^{|D|-l(w)} \pi_i(x_i^J)(x_{D'} + s_i(x_{D'}))1 = (-1)^{|D|-l(w)} \pi_i(x_i^J(x_{D'} + s_i(x_{D'}))) =$$
$$= \pi_i((-1)^{|D|-l(w)}(x_D + x_{\tau_i(D)})) \,,$$

which, in turn, means that

$$\sum_{E\in \mathrm{mitosis}'_i(D)} (-1)^{|E|-l(ws_i)} x_E + \sum_{E\in \mathrm{mitosis}'_i(\tau_i(D))} (-1)^{|E|-l(ws_i)} x_E =$$
$$= \pi_i((-1)^{|D|-l(w)}(x_D + x_{\tau_i(D)})) \,.$$

Thus, by grouping the elements of $D \in \mathcal{P}_s(w)$ in accordance with the involution $\tau_i$, and taking into account that $\pi_i$ is $\mathbb{R}$-linear, we obtain the following equality

$$\sum_{E\in \mathrm{mitosis}'_i(\mathcal{P}_s(w))} (-1)^{|E|-l(ws_i)} x_E = \pi_i\Big( \sum_{D\in\mathcal{P}_s(w)} (-1)^{|D|-l(w)} x_D \Big) \,. \tag{4.1}$$

There is an analogous equality in the case of $\mathcal{P}_I(w)$

$$\sum_{E\in\mathrm{mitosis}'_i(\mathcal{P}_I(w))}(-1)^{|E|-l(ws_i)}x_E=\pi_i\Big(\sum_{D\in\mathcal{P}_I(w)}(-1)^{|D|-l(w)}x_D\Big)\,. \tag{4.2}$$

Now if $D\in\mathcal{P}_\varnothing(w)$, i.e. $D'=D$ and $\tau_i(D)=D$, then $D$ is symmetric in the variables $x_i$ and $x_{i+1}$ and, consequently, we have

$$(-1)^{|D|-l(w)}\pi(x_D)=(-1)^{|D|-l(w)}x_D\,.$$

Also if $\tau_i(D)\neq D$ then the sum $x_D+x_{\tau_i(D)}$ is symmetric in the variables $x_i$ and $x_{i+1}$ and we have

$$(-1)^{|D|-l(w)}\pi(x_D+x_{\tau_i(D)})=(-1)^{|D|-l(w)}(x_D+x_{\tau_i(D)})\,.$$

Thus, by grouping the elements of $\mathcal{P}_\varnothing(w)$ with accordance with the involution $\tau_i$, we get the equality

$$\sum_{D\in\mathcal{P}_\varnothing(w)}(-1)^{|D|-l(w)}x_D=\pi_i\Big(\sum_{D\in\mathcal{P}_\varnothing(w)}(-1)^{|D|-l(w)}x_D\Big)\,. \tag{4.3}$$

By taking the sum of (4.1), (4.2) and (4.3), we obtain the following result

$$\sum_{E\in\mathrm{mitosis}'_i(\mathcal{P}_{s_i}(w))}(-1)^{|E|-l(ws_i)}x_E=\pi_i(\mathfrak{G}_w(x))=\mathfrak{G}_{ws_i}(x)\,.$$

Indeed, according to Theorem 4.1 *1)*, the right sides of equations (4.2) and (4.3) differ by sign. Also according to Theorem 4.1 *2)*, we have

$$\sum_{E\in\mathrm{mitosis}'_i(\mathcal{P}_{s_i}(w))}(-1)^{|E|-l(ws_i)}x_E=\sum_{D\in\mathcal{P}(ws_i)}(-1)^{|D|-l(ws_i)}x_D\,,$$

which gives us

$$\sum_{D\in\mathcal{P}(ws_i)}(-1)^{|D|-l(ws_i)}x_D=\mathfrak{G}_{ws_i}(x)\,.$$

Thus, the inductive step is fulfilled and the proof is completed. □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