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Refuting conjectures in extremal combinatorics via linear programming ☆

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ABSTRACT

We apply simple linear programming methods and an LP solver to refute a number of open conjectures in extremal combinatorics.

1. Introduction

Conjectures in extremal combinatorics are often intricate – it can be easy to miss a better construction than the one we have, or to misjudge whether the conjecture is true or false for other reasons. Any general method that can tell us whether a statement is likely to be true or false can be extremely useful in practice.

In the present manuscript we argue that the use of linear programming and LP solvers is such a method in extremal combinatorics. Nothing about this method is new, but we use it to resolve a number of open conjectures, questions and problems from a variety of areas. Our new results are given in Section 2:

☆ This is a shortened version of the original tutorial-style paper, which can be found on arXiv [24].

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- In 2.1 we disprove a claim and a conjecture of Frankl on the size of antichains of fixed diameter.
- In 2.2 we disprove two conjectures of Katona, and one conjecture of Frankl et al., on multipartite generalizations of the Erdős–Ko–Rado theorem.
- In 2.3 we solve a problem of Anstee related to a forbidden configuration in set systems.
- In 2.4 we answer a question of Ihringer–Kupavskii on regular set systems achieving a Hoffman-type bound.
- In 2.5 we disprove a conjecture of Frankl–Tokushige related to the Kleitman matching problem.
- In 2.6 we disprove a conjecture of Aharoni–Howard on bipartite graphs without rainbow matchings.
- In 2.7 we improve a construction given by De Silva–Heysse–Kapilow–Schenfisch–Young related to a Turán-type problem.

For the basics on linear programming we refer the reader to e.g. [3,14,24]. We note that sometimes the LP (or even SDP) methods are used to prove a theorem by finding an optimal solution to the corresponding dual problem. Some such examples can be found in [22,23].

2. Main results

2.1. Antichains of fixed diameter

Define the *diameter* $\operatorname{diam}(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$ as $\operatorname{diam}(\mathcal{F}) = \max_{A,B\in\mathcal{F}}\{|(A \setminus B) \cup (B \setminus A)|\}$. Frankl [9] considered the problem of determining the largest size of an antichain in $2^{[n]}$ of diameter at most d.

We phrase this problem as an integer program as follows. We introduce for each set $A \subset [n]$ a 0-1 valued variable x_A that indicates whether $A \in \mathcal{F}$. We can force $\mathcal{F}$ to be an antichain by adding, for each comparable pair $A \subsetneq B$ a linear constraint $x_A + x_B \leq 1$. Next, to ensure that the solution has diameter at most d, for each pair A, B with $|(A \setminus B) \cup (B \setminus A)| > d$ we add a restriction $x_A + x_B \leq 1$. Frankl [9] made the following conjecture:

Conjecture 2.1 *([9]). Let n, d be positive integers, $n > d$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is an antichain with diameter* $\operatorname{diam}(\mathcal{F}) \leq d$. *Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor d/2 \rfloor}.$$

Frankl proved [9] Conjecture 2.1 for $n \geq 6(d+1)^2$. Using an LP solver we were unsuccessful in finding a counterexample to this conjecture. Instead we could establish that it holds for the values $(n, d) = (10, 3), (8, 5), (8, 7)$, thus finding some more support for the conjecture.

Frankl also made a similar conjecture for k-chain free families. A k-chain is a collection of k sets $A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_k$ totally ordered under inclusion. Observe that, similarly to being an antichain, the property of being k-chain-free can be captured by an integer program by adding for each k-chain $A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_k$ a linear constraint $x_{A_1} + x_{A_2} + \ldots + x_{A_k} \leq k - 1$.

Conjecture 2.2 *([9]). Let n, d, ℓ be positive integers, $n > d \geq \ell$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is $\ell + 1$-chain-free with diameter* $\operatorname{diam}(\mathcal{F}) \leq d$*. Then setting $s = \min\{\ell - 1, \lfloor d/2 \rfloor\}$ one has*

$$|\mathcal{F}| \leq \sum_{\lfloor d/2 \rfloor \geq i \geq \lfloor d/2 \rfloor - s} \binom{n}{i}.$$

Frankl noted that the special case $s = \lfloor d/2 \rfloor$ follows directly from Kleitman's diameter theorem [19]. Frankl also wrote [9] that it follows from the methods of his paper that Conjecture 2.2 holds for n large enough. This claim is incorrect, as shown below.

By solving the IP directly, we obtain a counterexample for $n = 6, d = 5, \ell = 2$. For these parameters the bound given in Conjecture 2.2 is $\binom{n}{2} + \binom{n}{1} = 21$, but in fact a family of size 26 exists:

$$\mathcal{F} = \binom{[6]}{2} \cup \left\{ A \in \binom{[6]}{3} : 1 \in A \right\} \cup \{23456\}.$$

Given this example given by the computer, it is easy to realize that the family given by the entire 2-layer together with a star on the third layer has diameter 5 and is 3-chain-free for any n. (Here a star means all sets containing a fixed element.) It has size $\binom{n}{2} + \binom{n-1}{2} = (n-1)^2$ which is bigger than the construction implied by Conjecture 2.2 for all $n \geq 6$, but as the example shows it is not best possible for $n = 6$. Surprisingly, according to the LP solver this bound is tight for $n = 7, 8$. It is plausible that this is tight for all $n \geq 7$.

For $n = 8, d = 7, \ell = 2$ the LP solver gives that the best solution is to take stars centered at $\{1\}$ in layers 2 and 4, together with all sets avoiding $\{1\}$ on layers 3 and 5, giving a family of size $\binom{7}{1} + \binom{7}{3} + \binom{7}{3} + \binom{7}{5} = 98$ beating the value 84 given by Conjecture 2.2. For $n = 9, d = 7, \ell = 2$ a construction of size 141 is given by taking the full third layer, a star centered on $\{1\}$ on the fourth layer and the single set $\{2, 3, 4, \ldots, 9\}$. It is beyond our computational limits to see if this is best possible for $n = 9, d = 7, \ell = 2$.

In general for d odd one can take the entire $\lfloor d/2 \rfloor$ layer together with a star on the layer above to get a construction of size $\binom{n}{\lfloor d/2 \rfloor} + \binom{n-1}{\lfloor d/2 \rfloor}$ and it is plausible that for $n \geq n_0(d)$ this is the best one can do to avoid 3-chains.

2.2. Multipartite intersecting families

A celebrated theorem of Erdős–Ko–Rado is the following:

Theorem 2.3 *(Erdős–Ko–Rado [7]). Given integers n, k with $k \le n/2$, if $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

Equality in Theorem 2.3 is attained by the family of all k-sets containing a fixed element, which we refer to as a *trivially intersecting family* or a *star*. Hilton and Milner [15] found the largest intersecting, but not trivially intersecting family:

Theorem 2.4 *(Hilton–Milner [15]). If $2k \le n$ and $\mathcal{F}$ is an intersecting but not trivially intersecting family in $\binom{[n]}{k}$ then*

$$|\mathcal{F}| \le 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

Let X_1 and X_2 be disjoint sets of size n_1 and n_2 respectively, and denote by $\binom{X_1,X_2}{k,\ell}$ the family of all sets $S \subset X_1 \dot\cup X_2$ with $|S \cap X_1| = k$ and $|S \cap X_2| = \ell$. Frankl [8] and Katona [17] considered intersecting families in $\binom{X_1,X_2}{k,\ell}$. As before, a family $\mathcal{F} \subset \binom{X_1,X_2}{k,\ell}$ is trivially intersecting if all elements of $\mathcal{F}$ contain a fixed element. Katona [17] observed that if $x \in X_1$ is an arbitrary element and $K \subset X_1 \setminus \{x\}$ is a set of size k then the family

$$\mathcal{F} = \left\{ F \in \binom{X_1, X_2}{k, \ell} : x \in F, F \cap K \neq \varnothing \right\} \cup \{K\}$$

is intersecting but not trivially intersecting. Motivated by this, Katona [17] made the following conjecture.

Conjecture 2.5 *(Katona [17]). If $\mathcal{F}$ is an intersecting but not trivially intersecting subfamily of $\binom{X_1,X_2}{k,\ell}$ then*

$$|\mathcal{F}| \le \max\left\{ \left(1 + \binom{n_1-1}{k-1} - \binom{n_1-k-1}{k-1}\right)\binom{n_2}{\ell}, \right.$$
$$\left. \binom{n_1}{k}\left(1 + \binom{n_2-1}{\ell-1} - \binom{n_2-\ell-1}{\ell-1}\right) \right\}.$$

We assume that the conditions $2k \le n_1$ and $2\ell \le n_2$ are implicitly implied in Conjecture 2.5. Katona also made a conjecture on *two-sided intersecting* families, i.e. intersecting families $\mathcal{F} \subset \binom{X_1,X_2}{k,\ell}$ for which there exist members $F_{11}, F_{12}, F_{21}, F_{22} \in \mathcal{F}$ such that $F_{11} \cap F_{12} \cap X_1 = \varnothing$ and $F_{21} \cap F_{22} \cap X_2 = \varnothing$.

Conjecture 2.6 *([17]). If $\mathcal{F}$ is a two-sided intersecting subfamily of $\binom{X_1,X_2}{k,\ell}$ then*

$$|\mathcal{F}| \leq \max\left\{ \left(\binom{n_2-1}{\ell-1} - \binom{n_2-\ell-1}{\ell-1}\right)\binom{n_1}{k} + 1 + \binom{n_1}{k} - \binom{n_1-k}{k}, \right.$$
$$\left. \left(\binom{n_1-1}{k-1} - \binom{n_1-k-1}{k-1}\right)\binom{n_2}{\ell} + 1 + \binom{n_2}{\ell} - \binom{n_2-\ell}{\ell} \right\}.$$

Once again we assume the conditions $2k \leq n_1$ and $2\ell \leq n_2$ are implicit. Let us now try to disprove both Conjectures 2.5 and 2.6. We phrase them as IPs as follows. We fix some X_1, X_2, k, ℓ and for each element F of $\binom{X_1,X_2}{k,\ell}$ we introduce an indicator variable x_F. We force $\mathcal{F}$ to be intersecting by adding for each disjoint pair of sets F, G a constraint $x_F + x_G \leq 1$. For Conjecture 2.5 we ensure that $\mathcal{F}$ is not trivially intersecting by adding for each $x \in X_1 \cup X_2$ a constraint

$$\sum_{x \notin F} x_F \geq 1.$$

For Conjecture 2.6 we force the two-sided intersecting property in a similar fashion. We pick two disjoint k-sets $L_1, L_2 \subset X_1$ and two disjoint ℓ-sets $R_1, R_2 \subset X_2$. Then we add for each $S \in \{L_1, L_2, R_1, R_2\}$ the constraint

$$\sum_{S \subset F} x_F \geq 1.$$

Solving the IP directly yields a counterexample for both conjectures for $n_1 = n_2 = 5$, $k = \ell = 2$ in less than a second. Let $X = \{x_1, x_2, \ldots, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_5\}$. Let $\mathcal{F}$ be the following family:

$$\begin{aligned}\mathcal{F} = &\left\{ \{x_1, x_2\} \cup F : F \in \binom{Y}{2}, F \cap \{y_1, y_2\} \neq \varnothing \right\} \\ &\cup \left\{ \{x_1, x_3\} \cup F : F \in \binom{Y}{2}, F \cap \{y_1, y_2\} \neq \varnothing \right\} \\ &\cup \left\{ \{x_1, x_4\} \cup F : F \in \binom{Y}{2} \right\} \cup \left\{ \{x_1, x_5\} \cup F : F \in \binom{Y}{2} \right\} \\ &\cup \{\{x_4, x_5, y_1, y_2\}\}\end{aligned}$$

The size of $\mathcal{F}$ is 35, while the constructions in Conjectures 2.5 and 2.6 have sizes 30 and 28 respectively. This construction generalizes for $\binom{X,Y}{2,2}$. For simplicity assume $|X| = |Y| \geq 5$.

Proposition 2.7. *Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two disjoint sets of size $m \geq 5$. Then there is a $\mathcal{F} \subset \binom{X,Y}{2,2}$ that is two-sided intersecting, with $|F| \geq 3m^2 - 10m + 10$.*

Proof. Let $\mathcal{F}$ be defined as follows.

$$\mathcal{F} = \left\{ G \cup F : G \in \binom{X}{2}, F \in \binom{Y}{2}, x_1 \in G, F \cap \{y_1, y_2\} \neq \varnothing \right\}$$
$$\cup \left\{ \{x_1, x_2\} \cup F : F \in \binom{Y}{2} \right\} \cup \left\{ \{x_1, x_3\} \cup F : F \in \binom{Y}{2} \right\}$$
$$\cup \{\{x_2, x_3, y_1, y_2\}\}$$

The size of $\mathcal{F}$ is then given by

$$|\mathcal{F}| = (m-3)\left(\binom{m}{2} - \binom{m-2}{2}\right) + 2\binom{m}{2} + 1 = 3m^2 - 10m + 10. \quad \square$$

We note that according to the LP solver, the construction in Proposition 2.7 is in fact the largest non-trivially intersecting (but not necessarily two-sided intersecting) family for $m = 5, 6$.

For $(n_1, n_2, k, \ell) = (7, 7, 3, 3)$ we find a two-sided intersecting family of size 514, beating the values of 455 and 452 in Conjectures 2.5 and 2.6 respectively. Based on generalizing the construction given by the LP solver, we have the following bound.

Proposition 2.8. *Let k, m be integers with $2k \leq m$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two disjoint sets of size m. Then there is a $\mathcal{F} \subset \binom{X,Y}{k,k}$ that is two-sided intersecting, with*

$$|\mathcal{F}| \geq \left(\binom{m-1}{k-1} - \binom{m-k-1}{k-1}\right)\binom{m}{k} + \binom{m-k-1}{k-1}\left(\binom{m}{k} - \binom{m-k}{k}\right) + 1.$$

Proof. Let $K_1 = \{x_2, x_3, \ldots, x_{k+1}\}$, $K_2 = \{y_1, y_2, \ldots, y_k\}$ and define the family as

$$\mathcal{F} = \left\{ F_1 \cup F_2 : F_1 \in \binom{X}{k}, F_2 \in \binom{Y}{k}, x_1 \in F_1, F_1 \cap K_1 \neq \varnothing, \right\}$$
$$\cup \left\{ F_1 \cup F_2 : F_1 \in \binom{X}{k}, F_2 \in \binom{Y}{k}, x_1 \in F_1, F_1 \cap K_1 = \varnothing, F_2 \cap K_2 \neq \varnothing \right\}$$
$$\cup \{K_1 \cup K_2\}. \quad \square$$

It would be interesting to see whether this construction is best possible.

Let us now turn to another related conjecture, by Frankl–Han–Huang–Zhao [10]. We say that a family *has the EKR property* if its largest intersecting subfamily is trivially intersecting.

Conjecture 2.9 *([10]). Suppose $n = n_1 + \ldots + n_d$ and $k \geq k_1 + \ldots + k_d$, where $n_i > k_i \geq 0$ are integers. Let $X_1 \cup \ldots \cup X_d$ be a partition of $[n]$ with $|X_i| = n_i$, and*

$$\mathcal{H} = \left\{ F \subseteq \binom{[n]}{k} : |F \cap X_i| \geq k_i \textit{ for } i = 1, \ldots, d \right\}.$$

If $n_i \geq 2k_i$ *for all* i *and* $n_i > k - \sum_{j=1}^{d} k_j + k_i$ *for all but at most one* $i \in [d]$ *such that* $k_i > 0$*, then* $\mathcal{H}$ *has the EKR property.*

We first observe that if e.g. $d = 2, n_1 = 3, n_2 = 4, k_1 = 1, k_2 = 2, k = 4$ then all conditions of the conjecture are satisfied but $\mathcal{H}$ itself is (non-trivially) intersecting, and so Conjecture 2.9 cannot be true. In particular for this set of parameters $|\mathcal{H}| = 30$ but the largest trivially intersecting subfamily of $\mathcal{H}$ has size 18. We will thus assume that the $n \geq 2k$ condition was intended to be a part of the statement of Conjecture 2.9.

We phrase this problem as an IP in much the same way as before. Fix some values for the parameters, and introduce indicator variables x_F for each $F \in \mathcal{H}$. Then add constraints $x_F + x_H \leq 1$ for each disjoint $F, H \in \mathcal{H}$. Solving the LP yields counterexamples for several sets of parameters. The smallest we could find is for the values $d = 2, n_1 = n_2 = 4, k_1 = 2, k_2 = 1, k = 4$, so that

$$\mathcal{H} = \left\{ F \subseteq \binom{[8]}{4} : |F \cap \{1,2,3,4\}| \geq 2,\ |F \cap \{5,6,7,8\}| \geq 1 \right\}.$$

The largest trivially intersecting family in $\mathcal{H}$ has size 30, but its largest intersecting subfamily has size 34[1]:

$$\begin{aligned}\mathcal{F} = \{&1235, 1236, 1237, 1238, 1245, 1246, 1247, 1248, 1256, 1267, 1268, 1278,\\ &1345, 1346, 1347, 1348, 1358, 1368, 1378, 1467, 1468, 2345, 2346, 2347,\\ &2348, 2356, 2367, 2368, 2378, 2458, 2468, 2478, 3467, 3468\}\end{aligned}$$

2.3. A forbidden trace problem

To introduce the definition of a *forbidden configuration*, we will use the language of matrix theory and identify set systems with their adjacency matrix. An $m \times n$ simple matrix (i.e. with no repeated columns) A with all entries in $\{0, 1\}$ can be thought of as a family $\mathcal{A}$ of n subsets of $[m]$: the rows index the elements of the ground sets and the columns index the subsets. So the number of columns of A is equal to $|\mathcal{A}|$.

Now let F be a $k \times \ell$ matrix with all entries in $\{0, 1\}$. We say that a matrix A *has a configuration* F if a submatrix of A is a row and column permutation of F (this is sometimes called *trace* in the language of sets).

Many classical problems in extremal set theory can be phrased as problems about forbidden configurations. One standard example is bounding the size of a family of VC

[1] An anonymous referee has pointed out that a construction of the same size can be obtained by taking all sets that intersect $\{1, 2, 3\}$ in at least two elements and $\{5, 6, 7, 8\}$ in at least one element. It is plausible that a suitable generalization of this construction is optimal for a large range of the parameters.

dimension at most k. We say that a family $\mathcal{A} \subset 2^{[n]}$ has VC dimension at least k if there exists a set $S \subset [n]$ of size $|S| = k$ such that $|A \cap S : A \in \mathcal{A}| = 2^k$. Hence a family $\mathcal{A}$ has VC dimension less than 3 if and only if the corresponding matrix A does not have configuration F_3, where F_3 is the matrix

$$F_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Denote by $\text{forb}(m, F)$ the maximum number of columns in a matrix A without a configuration F. So for example by a classical theorem of Sauer–Shelah we have $\text{forb}(m, F_3) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2}$. We refer the reader to the excellent survey of Anstee [2] on more background on forbidden configuration problems.

Steiner triple systems are one of the most classical objects studied in combinatorial design theory, dating back to Kirkman [18]. We say a family of 3-element subsets, called *blocks*, of an n-element set X is a triple system of multiplicity λ if any pair of distinct elements of X are contained in precisely λ blocks. Anstee raised the following problem, which we will disprove:

Problem 2.10 *([2]).* Show that for those m for which a triple system of multiplicity 2 exists,

$$\text{forb}\left(m, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = \frac{5}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}.$$

Denote the forbidden matrix by A. As a triple system of multiplicity one has order $\frac{1}{3}\binom{m}{2}$, it is our guess that the intended construction achieving the bound on the right hand side is $\binom{[m]}{\leq 2} \cup \{[m]\}$ together with a triple system of multiplicity two. However, this construction does in fact contain the forbidden matrix A as a configuration, but removing the single set $\{[m]\}$ would fix the issue. Nevertheless, we are able to find a construction that is larger than the value on the right hand side.

We can phrase this problem as an IP as follows. We introduce for each set $S \subseteq [n]$ a 0-1 valued indicator variable x_S. For any four distinct sets A, B, C, D if there exist three elements of the ground set such that the trace of A, B, C, D on these three elements would give the forbidden matrix A, then we add a constraint $x_A + x_B + x_C + x_D \leq 3$. The objective is then to maximize the sum of all variables.

We begin by noting that there exists a triple system of multiplicity two for $m = 4, 6, 7, 9$, see [4]. Denote by $S_2(m)$ a triple system of multiplicity two and order m, for those m where it exists. Next we observe that we may assume the family $\mathcal{F}$ contains all sets of size 0 or 1 as these do not affect containment of our forbidden configuration, hence we may restrict our search space on $\binom{[m]}{\geq 2}$. Solving the IP directly for $m = 6$ we find that statement of Problem 2.10 is false, the correct answer is 25 rather than 26 – here 25 is given by the natural construction $\mathcal{F} = \binom{[6]}{\leq 2} \cup S_2(6)$.

For $m = 9$ solving the IP was infeasible with the author's laptop. By making the heuristic assumption that the optimal family should contain all sets of size at most two and restricting the search to $\binom{[m]}{3} \cup \binom{[m]}{4}$ we find a construction of size 71 within three minutes – this matches the bound given in Problem 2.10, and hence beats the natural construction of $\mathcal{F} = \binom{[m]}{\le 2} \cup S_2(m)$ by one! The construction given by the LP solver is as follows: take all sets of size at most two, together with a triple system of multiplicity two, which contains the triples $\{123, 124, 134, 234\}$, and add the single set $\{1234\}$. Such a triple system indeed exists, see for example [4]. Hence if we could find a triple system of multiplicity two of some higher order, that contains the triples $\{123, 124, 134, 234, 567, 568, 578, 678\}$ then we could add two 4-sets and beat the bound in Problem 2.10. We will need the following theorem of Colbourn–Hamm–Lindner–Rodger:

Theorem 2.11 *(Colbourn–Hamm–Lindner–Rodger [5]). A partial triple system of order m and multiplicity λ can be embedded in a triple system of multiplicity λ and order at most $4(3\lambda/2 + 1)m + 1$.*

Proposition 2.12. *For every $k \ge 1$ there exists an $m \le 64k + 1$ such that*

$$forb\left(m, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \ge \frac{5}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + k.$$

Proof. Construct a partial triple system of multiplicity 2 by taking, for all $0 \le i \le k-1$, the triples $\{4i+1, 4i+2, 4i+3\}$, $\{4i+1, 4i+2, 4i+4\}$, $\{4i+1, 4i+3, 4i+4\}$ and $\{4i+2, 4i+3, 4i+4\}$. These $4k$ triples form a partial triple system of multiplicity 2 and order $4k$. By Theorem 2.11 these triples are contained in some triple system $\mathcal{F}$ with $\lambda = 2$ and order m, with $m \le 16 \cdot 4k + 1$. Adding to $\mathcal{F}$ all sets of size two or less and the 4-sets $\{4i+1, 4i+2, 4i+3, 4i+4\}$ for all $0 \le i \le k-1$ we obtain a family of the correct size, which does not contain the forbidden configuration given by the matrix A. □

We observe that the bound in Proposition 2.12 is not sharp, in particular any two of the added 4-sets could be allowed to intersect in one element. Indeed, if four sets are witnesses for the configuration A then any two of the four sets intersect in at least two elements. This leads us to an even stronger bound, giving an improvement in the leading coefficient.

Proposition 2.13. *For every sufficiently large m with $m \equiv 1, 4 \pmod{12}$ we have*

$$forb\left(m, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \ge \frac{11}{6}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}.$$

Proof. Given a sufficiently large integer $m \equiv 1, 4 \pmod{12}$, by a theorem of Wilson [25] there exists a family $\mathcal{S}$ of 4-sets in $[m]$ such that any pair $x, y \in [m]$ of distinct elements are covered by precisely one set in $\mathcal{S}$. Note that $|\mathcal{S}| = \binom{m}{2}/6$. Construct a family $\mathcal{F}'$ of 3-sets by including, for each set $S \in \mathcal{S}$, all four 3-subsets of S. Note that since any two sets in $\mathcal{S}$ intersect in at most one element, we have $|\mathcal{F}'| = \frac{2}{3}\binom{m}{2}$ and in fact $\mathcal{F}'$ is a triple system of order m and multiplicity two. Then the family

$$\mathcal{F} = \binom{[m]}{\leq 2} \cup \mathcal{F}' \cup \mathcal{S}$$

does not contain the forbidden configuration and has the correct size. □

2.4. A Hoffman-type eigenvalue bound on regular set systems

We say that a family $\mathcal{F} \subset 2^{[n]}$ is *s-subset-regular* if every set of size s lies in the same number of elements of $\mathcal{F}$. Ihringer and Kupavskii [16] proved the following Hoffman-type eigenvalue upper bound on such regular families:

Theorem 2.14 *(Ihringer–Kupavskii [16]). Fix odd $s \geq 1$. An s-subset-regular k-uniform intersecting family $\mathcal{F}$ on $[n]$ satisfies*

$$|\mathcal{F}| \leq \frac{\binom{n}{k}}{1 + \frac{\binom{n-k}{k}}{\binom{n-k-s-2}{k-s-2}}}.$$

They proved [16] that equality in Theorem 2.14 is achieved with $(n, k, s) = (7, 3, 1)$ and $(9, 4, 1)$. They asked whether there are other values of the parameters with $n \geq 2k+1$ for which Theorem 2.14 is tight. We will show that the answer is yes, by constructing such a family with parameters $(11, 5, 3)$.

We phrase this problem as an IP as follows. We fix some n, k, s. For each $A \in \binom{[n]}{k}$ we introduce a 0-1 variable x_A. We force $\mathcal{F}$ to be intersecting as before, by adding for each disjoint pair of sets A, B a constraint $x_A + x_B \leq 1$. To ensure that $\mathcal{F}$ is s-subset-regular for each $S \subset [n]$ we add a constraint

$$\sum_{[s] \subset A \in \binom{[n]}{k}} x_A - \sum_{S \subset B \in \binom{[n]}{k}} x_B = 0.$$

Solving this IP directly gives the following construction for $(n, k, s) = (11, 5, 3)$ in about 30 seconds:

$$\begin{aligned}\mathcal{F} = &\{\{1,2,3,4,11\},\{1,2,3,5,6\},\{1,2,3,7,8\},\{1,2,3,9,10\},\{1,2,4,5,10\},\\ &\{1,2,4,6,7\},\{1,2,4,8,9\},\{1,2,5,7,9\},\{1,2,5,8,11\},\{1,2,6,8,10\},\\ &\{1,2,6,9,11\},\{1,2,7,10,11\},\{1,3,4,5,7\},\{1,3,4,6,9\},\{1,3,4,8,10\},\\ &\{1,3,5,8,9\},\{1,3,5,10,11\},\{1,3,6,7,10\},\{1,3,6,8,11\},\{1,3,7,9,11\},\\ &\{1,4,5,6,8\},\{1,4,5,9,11\},\{1,4,6,10,11\},\{1,4,7,8,11\},\{1,4,7,9,10\},\end{aligned}$$

$\{1,5,6,7,11\},\{1,5,6,9,10\},\{1,5,7,8,10\},\{1,6,7,8,9\},\{1,8,9,10,11\},$
$\{2,3,4,5,8\},\{2,3,4,6,10\},\{2,3,4,7,9\},\{2,3,5,7,10\},\{2,3,5,9,11\},$
$\{2,3,6,7,11\},\{2,3,6,8,9\},\{2,3,8,10,11\},\{2,4,5,6,9\},\{2,4,5,7,11\},$
$\{2,4,6,8,11\},\{2,4,7,8,10\},\{2,4,9,10,11\},\{2,5,6,7,8\},\{2,5,6,10,11\},$
$\{2,5,8,9,10\},\{2,6,7,9,10\},\{2,7,8,9,11\},\{3,4,5,6,11\},\{3,4,5,9,10\},$
$\{3,4,6,7,8\},\{3,4,7,10,11\},\{3,4,8,9,11\},\{3,5,6,7,9\},\{3,5,6,8,10\},$
$\{3,5,7,8,11\},\{3,6,9,10,11\},\{3,7,8,9,10\},\{4,5,6,7,10\},\{4,5,7,8,9\},$
$\{4,5,8,10,11\},\{4,6,7,9,11\},\{4,6,8,9,10\},\{5,6,8,9,11\},\{5,7,9,10,11\},$
$\{6,7,8,10,11\}\}$

It can be checked that $\mathcal{F}$ covers every 4-set exactly once, and hence this family $\mathcal{F}$ is in fact the (unique) Steiner system with parameters $v = 11, k = 5, t = 4$.

2.5. The Kleitman matching problem

Let $s \geq 3$ be an integer, and let $k(n, s)$ denote the maximum size of a family $\mathcal{F} \subset 2^{[n]}$ without s pairwise disjoint members. Kleitman [20] determined $k(n, s)$ for $n \equiv 0$ or $-1 \pmod s$, see Theorem 2.16. In the case $n \equiv -2 \pmod s$, the value of $k(n, s)$ was determined by Quinn [21] if $s = 3$ and by Frankl and Kupavskii [11,12] for all s.

Recall that $k(n + l, s) \geq 2^l k(n, s)$. Indeed, if $\mathcal{F} \subset 2^n$ has no s pairwise disjoint members, then neither does $\mathcal{F}' = \{F \subset [n+l] : F \cap [n] \in \mathcal{F}\}$. Kleitman showed [20] that $k(n, s) = 2k(n - 1, s)$ if s divides n. Motivated by this, Frankl and Tokushige [13] made the following conjecture:

Conjecture 2.15 *([13], p. 213). Let $s \geq 4$. If $n \equiv 1 \pmod s$, then*

$$k(n, s) = 4k(n - 2, s).$$

Theorem 2.16 *(Kleitman [20]). Let $s \geq 2$ be an integer and $\mathcal{F} \subset 2^{[n]}$ a family without s pairwise disjoint members. Then for $n = s(m + 1) - \ell$ with $\ell \in [s]$ we have*

$$|\mathcal{F}| \leq \frac{\ell - 1}{s}\binom{n}{m} + \sum_{t \geq m+1} \binom{n}{t},$$

and this is sharp for $\ell \in \{1, s\}$.

This gives $k(7, 4) = 120$ and hence in order to disprove Conjecture 2.15 our goal is to show $k(9, 4) \geq 481$. One can formulate this problem as an IP as follows. As before, we introduce a 0-1 valued indicator variable for every $A \subset [n]$. For each quadruple of pairwise disjoint sets A, B, C, D we add the constraint $x_A + x_B + x_C + x_D \leq 3$. Our goal is then simply to maximize the sum of the variables.

To speed up the solution of this IP it helps if one makes the heuristic, though certainly unjustified, assumption that $x_A = 1$ whenever $|A| \geq 4$ and $x_A = 0$ whenever $|A| \leq 1$. Indeed, intuitively it makes sense to include 'large' sets, and so far our family does not even contain three disjoint sets. This restricts the search space to the considerably smaller world $\binom{[n]}{2} \cup \binom{[n]}{3}$.

Note that $480 = 2^9 - 32 = \binom{9}{\geq 4} + 98$. The LP solver finds a family $\mathcal{G}$ of size 99 in $\binom{[9]}{2} \cup \binom{[9]}{3}$ without four pairwise disjoint sets. This gives a counterexample to Conjecture 2.15, as then $\mathcal{G} \cup \binom{[9]}{\geq 4}$ does not contain four pairwise disjoint sets either, and $\left|\mathcal{G} \cup \binom{[9]}{\geq 4}\right| = 4k(7,2) + 1$. The search takes around 2 seconds:

$$\mathcal{G} = \binom{[9]}{3} \cup \left\{ A \in \binom{[9]}{2} : |A \cap [2]| \geq 1 \right\}.$$

As it turns out, this construction has already appeared in a paper by Frankl–Kupavskii [11] but they were not aware of Conjecture 2.15.

2.6. Rainbow matchings

Aharoni and Howard [1] considered problems related to rainbow matchings in hypergraphs. Given a collection $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ of hypergraphs, a choice of disjoint edges, one from each $\mathcal{F}_i$, is called a *rainbow matching* for $\mathcal{F}$. They made the following conjecture:

Conjecture 2.17 *([1]). Let $d > 1$, and let $F_1, \ldots, F_k$ be bipartite graphs on the same ground set, satisfying $\Delta(F_i) \leq d$ and $|F_i| > (k-1)d$. Then the system $F_1, \ldots, F_k$ has a rainbow matching.*

To disprove Conjecture 2.17 we need to find a collection of bipartite graphs satisfying the bounds above, without a rainbow matching. We phrase this problem as an IP as follows. First, we fix parameters n, k, d. Next we fix the partite sets (L_i, R_i) for each of the k bipartite graphs,[2] so that $L_i \dot\cup R_i = [n]$ for each $1 \leq i \leq k$. We introduce indicator variables $x_{ab}^{(i)}$ for each $1 \leq i \leq k$ and pair of vertices $a \in L_i, b \in R_i$. The maximum degree condition is then a collection of nk simple linear constraints, one for each vertex and each $1 \leq i \leq k$. To ensure the system does not have a rainbow matching, for all k-tuple of disjoint edges $(a_i, b_i) \in (L_i, R_i)$ for $1 \leq i \leq k$, we add a constraint $\sum_{i=1}^{k} x_{a_i,b_i}^{(i)} \leq k-1$. For the sizes of the graphs, we add linear constraints for all $2 \leq i \leq k$ saying that $\sum_{a \in L_i, b \in R_i} x_{ab}^{(i)} \geq (k-1)d + 1$. Our goal is then to maximize $\sum_{a \in L_1, b \in R_1} x_{ab}^{(1)}$ and hope that the value of this maximum is greater than $(k-1)d$.

Solving this IP with $n = 6$, $k = 3$, $d = 2$, and partite set $L_1 = \{1,2,3\}, L_2 = \{2,3,4\}$ and $L_3 = \{3,4,5\}$ gives the following counterexample to Conjecture 2.17:

[2] We are fairly certain that the phrase "same ground set" in Conjecture 2.17 only means same vertex set, given the original context in [1].

$$F_1 = \{15, 16, 24, 26, 34, 35\}, \quad F_2 = \{14, 25, 26, 35, 36\}, \quad F_3 = \{13, 23, 25, 46, 56\}.$$

Here we have $|F_2| = |F_3| = (k-1)d + 1$ and $|F_1| = (k-1)d + 2$.

2.7. A Turán-type problem in multipartite graphs

For graphs G and H denote by $\mathrm{ex}(G, H)$ the maximum number of edges in a subgraph of G that contains no copy of H. For integers k, r let kK_r denote k vertex-disjoint copies of K_r. De Silva et al. considered [6] the problem of determining $\mathrm{ex}(G, H)$ where $H = kK_r$ and G is a complete multi-partite graph. They completely solved this problem when the number of partite sets in G is equal to r:

Theorem 2.18 *(De Silva et al. [6]). For any integers $k \le n_1 \le n_2 \le \ldots \le n_r$,*

$$\mathrm{ex}(K_{n_1,\ldots,n_r}, kK_r) = \left(\sum_{1 \le i < j \le r} n_i n_j \right) - n_1 n_2 + n_2(k-1).$$

De Silva et al. [6] observed that the graph

$$((n_1 + n_2 - k + 1)K_1 \cup K_{k-1,n_3}) + K_4$$

does not contain kK_3, hence

$$\mathrm{ex}(K_{n_1,n_2,n_3,n_4}, kK_3) \ge (n_1 + n_2 + n_3)n_4 + (k-1)n_3.$$

They stated that it is not clear that this is an extremal construction. Using our methods we will show that their intuition was correct, and there exist better constructions.

We phrase the problem as an IP in the standard way. We fix some n_1, n_2, n_3, n_4, k and for each edge e of K_{n_1,n_2,n_3,n_4} we introduce an indicator variable x_e. For every collection of $3k$ edges $e_1, \ldots, e_{3k}$ forming a kK_3 we include the constraint $\sum_{i=1}^{3k} e_i \le 3k - 1$.

Solving this IP directly we find that already in the $n_1 = n_2 = n_3 = n_4$ case there exist better constructions. Generalizing the constructions given by the IP solver, we have the following result:

Proposition 2.19. *For all integers $k \le n$, we have*

$$\mathrm{ex}(K_{n,n,n,n}, kK_3) \ge 4n^2 + (k-1)n.$$

Proof. Let the four partite sets of size n be A, B, C, D. Remove all $2n^2$ edges between the pairs $A - B$ and between $C - D$. Between C and D add a copy of $K_{k-1,n}$. $\square$

3. Concluding remarks

In this paper we presented a general method that can be used to quickly check whether a conjecture has small counterexamples. Nothing about the method itself is new. We hope to have convinced the reader of the usefulness and versatility of this technique in combinatorics with the number of counterexamples to open conjectures in Section 2. In practice, the main advantage of writing linear programs is the time saved – small counterexamples are always found eventually, but it is better to find them in a few minutes rather than a few weeks.

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