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EL-labelings, supersolvability and 0-Hecke algebra actions on posets

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Abstract

It is well known that if a finite graded lattice of rank n is supersolvable, then it has an EL-labeling where the labels along any maximal chain form a permutation. We call such a labeling an S_n EL-labeling and we show that a finite graded lattice of rank n is supersolvable if and only if it has such a labeling. We next consider finite graded posets of rank n with $\hat{0}$ and $\hat{1}$ that have an S_n EL-labeling. We describe a type A 0-Hecke algebra action on the maximal chains of such posets. This action is local and gives a representation of these Hecke algebras whose character has characteristic that is closely related to Ehrenborg's flag quasisymmetric function. We ask what other classes of posets have such an action and in particular we show that finite graded lattices of rank n have such an action if and only if they have an S_n EL-labeling.

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1. Introduction

Supersolvable lattices were introduced by Stanley [16] where he showed that the covering relations can be labeled by the integers to give an EL-labeling. We explain and discuss these terms in Section 2. In fact, this EL-labeling of a supersolvable lattice of rank n is seen to have the additional property that the labels along any maximal chain of the lattice form a permutation of $1, 2, \dots, n$. We call this type of labeling an S_n EL-labeling. In Section 5, we prove that the converse result is true: if a finite graded lattice has an S_n EL-labeling then it is supersolvable.

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In Section 3, we describe an action on the maximal chains of an S_n EL-labeled lattice, suggested to the author by Stanley. We show that this action gives a representation of the Hecke algebra of type A at $q = 0$. In [14,18], the Frobenius characteristic of the character of some symmetric group actions is shown to be closely related to Ehrenborg's flag symmetric function. In Section 4, we show that our $\mathcal{H}_n(0)$ action has an analogous property and we follow Simion and Stanley in calling our action a *good* $\mathcal{H}_n(0)$ action. Note that the material of Section 4 is not necessary for the proof of Section 5. Our second main result appears in Section 6. We show that a certain class of posets, which includes finite graded lattices, have a good $\mathcal{H}_n(0)$ action if and only if they have an S_n EL-labeling. It follows that a finite graded lattice is supersolvable if and only if it has a good $\mathcal{H}_n(0)$ action.

2. EL-labelings and supersolvability

Throughout, we let s_i denote the permutation which transposes i and $i + 1$, and composition of permutations will be from right to left. For any positive integer n , write $[n]$ for the set $\{1, 2, \dots, n\}$. Suppose P is a finite graded poset of rank n , with $\hat{0}$ and $\hat{1}$. (For undefined poset terminology, see [17, Chapter 3].) Let rk denote the rank function of P , so $\text{rk}(\hat{0}) = 0$ and $\text{rk}(\hat{1}) = n$. If $x \leq y$ in P , let $\text{rk}(x, y)$ denote $\text{rk}(y) - \text{rk}(x)$. If $x \leq y$ in P and $\text{rk}(x, y) = 1$ then we say that y *covers* x . Let $\mathcal{E}(P) = \{(s, t) : t \text{ covers } s \text{ in } P\}$, the set of edges of the Hasse diagram of P , and let $\mathcal{M}(P)$ denote the set of maximal chains of P .

A function $\lambda : \mathcal{E}(P) \rightarrow \mathbb{Z}$ gives us an *edge-labeling* of P . If $\mathbf{m} : s = s_0 < s_1 < \dots < s_k = t$ is a maximal chain of the interval $[s, t]$, then we write $\lambda(\mathbf{m}) = (\lambda(s_0, s_1), \lambda(s_1, s_2), \dots, \lambda(s_{k-1}, s_k))$. The chain \mathbf{m} is *increasing* if $\lambda(s_0, s_1) \leq \lambda(s_1, s_2) \leq \dots \leq \lambda(s_{k-1}, s_k)$. We let \leq_L denote lexicographic order on finite integer sequences: $(a_1, a_2, \dots, a_k) <_L (b_1, b_2, \dots, b_k)$ if and only if $a_i < b_i$ in the first coordinate where they differ.

Definition 2.1. Let P be a finite graded poset of rank n . An edge-labeling $\lambda : \mathcal{E}(P) \rightarrow \mathbb{Z}$ is called an EL-labeling if the following two conditions are satisfied:

- (i) Every interval $[s, t]$ has exactly one increasing maximal chain \mathbf{m} .
- (ii) Any other maximal chain \mathbf{m}' of $[s, t]$ satisfies $\lambda(\mathbf{m}') >_L \lambda(\mathbf{m})$.

A poset P with an EL-labeling is said to be *edge-wise lexicographically shellable* or EL-shellable. This definition of a lexicographically shellable poset first appeared in [2] with the motivating examples being from [15,16], which appear as Examples 2.4 and 2.6 below. The ubiquity and usefulness of EL-labelings arises from the fact that if P is EL-shellable, then P is shellable and hence Cohen-Macaulay. Further information on these concepts can be found in [2] and the highly recommended survey article [3]. We will be interested in the following type of EL-labeling:

Definition 2.2. An EL-labeling λ of P is said to be an S_n EL-labeling if, for every maximal chain $m: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ of P , the map sending i to $\lambda(x_{i-1}, x_i)$ is a permutation of $[n]$. In other words, $\lambda(m)$ is a permutation of $[n]$ written in the usual way.

If a poset P has an S_n EL-labeling, or *snelling* for short, then it is said to be S_n EL-shellable, or *snellable* for short. Note that the second condition in the definition of an EL-labeling is redundant in this case.

Example 2.3. Consider the poset B_n , the set of subsets of $[n]$. If y covers x in B_n then $y - x = \{i\}$ for some $i \in [n]$ and we set $\lambda(x, y) = i$. This defines a snelling for B_n .

Example 2.4. Any finite distributive lattice is snellable. Let L be a finite distributive lattice of rank n . By Birkhoff's "Fundamental Theorem of Finite Distributive Lattices" [1, p. 59, Theorem 3], that is equivalent to saying that $L = J(Q)$, the lattice of order ideals of some n -element poset Q . Let $\omega: Q \rightarrow [n]$ be a linear extension of Q , i.e., any bijection labeling the vertices of Q that is order-preserving (if $a < b$ in Q then $\omega(a) < \omega(b)$). This labeling of the vertices of Q defines a labeling of the edges of $J(Q)$ as follows. If y covers x in $J(Q)$, then the order ideal corresponding to y is obtained from the order ideal corresponding to x by adding a single element, labeled by i , say. Then we set $\lambda(x, y) = i$. This gives us a snelling for $L = J(Q)$. Fig. 1 shows a labeled poset and its lattice of order ideals with the appropriate edge-labeling.

Example 2.5. The posets shown in Fig. 2 are seen to be EL-shellable. However, it can be shown that neither of them is snellable. Notice that the second poset, unlike the first, is a lattice. It appears, together with this EL-labeling, in [12].

Example 2.6. The set of supersolvable lattices is our final example and is also the example most relevant to the remainder of the paper. The following definition first appeared in [16].

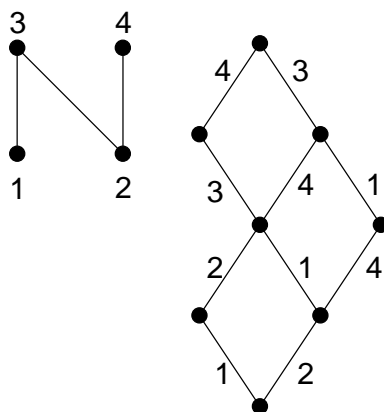


Fig. 1.

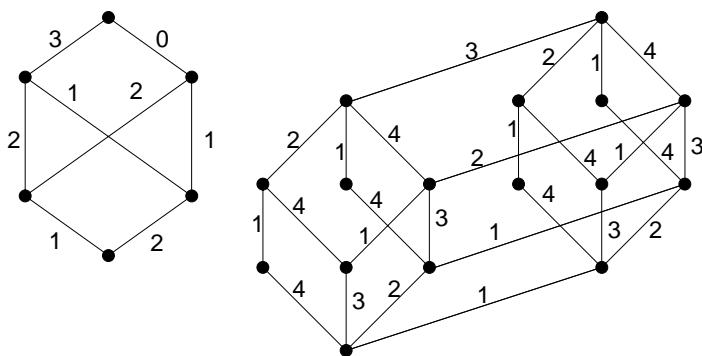


Fig. 2. Two posets that are EL-shellable but not snellable.

Definition 2.7. A finite lattice L is said to be *supersolvable* if it contains a maximal chain, called an *M-chain* of L , which together with any other chain in L generates a distributive sublattice.

We can label each such distributive sublattice by the method described in Example 2.4 in such a way that the M-chain receives the increasing label $(1, 2, \dots, n)$. As shown in [16], this will assign a unique label to each edge of L and the resulting global labeling of L is a snelling.

Examples of supersolvable lattices include distributive lattices, the lattice Π_n of partitions of $[n]$, the lattice NC_n of non-crossing partitions of $[n]$ and the lattice $L(G)$ of subgroups of a supersolvable group G (hence the terminology). The supersolvability of Π_n and $L(G)$ was shown in [16] while [8] contains a proof that NC_n is supersolvable.

We are now in a position to state our first main result.

Theorem 1. A finite graded lattice of rank n is supersolvable if and only if it is S_n EL-shellable.

We will prove Theorem 1 in Section 5.

3. $\mathcal{H}_n(0)$ actions

Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$. Suppose P has a snelling λ . Then to any maximal chain $\mathfrak{m}: \hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$ we can associate the permutation $\omega_{\mathfrak{m}}$ given by

$$\omega_{\mathfrak{m}} = (\lambda(x_0, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{n-1}, x_n)).$$

It is now natural to define the *descent set* of \mathfrak{m} to be the descent set of $\omega_{\mathfrak{m}}$ and the number of *inversions* of \mathfrak{m} to be the number of inversions of $\omega_{\mathfrak{m}}$. Suppose \mathfrak{m} has a

descent at i . By the snellability of P , there exists exactly one chain $m' : \hat{0} = x_0 < x_1 < \cdots < x_{i-1} < x'_i < x_{i+1} < \cdots < x_n = \hat{1}$ differing only from m at rank i and having no descent at i . This suggests the following definition of functions $U_i : \mathcal{M}(P) \rightarrow \mathcal{M}(P)$.

Definition 3.1. Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$ and with an S_n EL-labeling. Let m be a maximal chain of P . We define $U_1, U_2, \dots, U_{n-1} : \mathcal{M}(P) \rightarrow \mathcal{M}(P)$ by $U_i(m) = m'$, where m' is the unique maximal chain of P differing only from m at possibly rank i and having no descent at i .

Under this definition, we see that the descent set of a maximal chain m of P can also be defined to be the set

$$\{i \in [n-1] : U_i(m) \neq m\}. \quad (3.1)$$

This definition will be used later for posets P where no snelling is defined.

Observe that $\omega_{m'}$ is the same as ω_m except that the i th and $(i+1)$ th elements have been switched. In other words, $\omega_{m'} = \omega_m s_i$. Fig. 3 shows an example for the case $n = 4$. Let m be the maximal chain to the left. It has a descent at 2 and therefore $m' = U_2(m) \neq m$. The labels of m' are forced by the fact that m' does not have a descent at 2. We have that $\omega_{m'} = \omega_m s_2$.

We see that the action of U_1, U_2, \dots, U_{n-1} has the following properties:

1. It is a *local* action, i.e., $U_i(m)$ agrees with m except possibly at the i th rank. Local actions on the maximal chains of a poset have been studied, for example, in [8,14,18,19].
2. $U_i^2 = U_i$ for $i = 1, 2, \dots, n-1$. This differs from most of the local actions in the aforementioned papers which were symmetric group actions and so satisfied $U_i^2 = 1$.

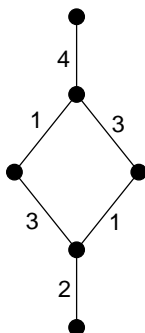


Fig. 3. The parallelogram shape with opposite sides having equal labels appears often as an interval of rank 2 in snellings.

3. $U_i U_j = U_j U_i$ if $|i - j| \geq 2$.
4. $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$ for $i = 1, 2, \dots, n - 2$. This requires the snellable property and is left as an exercise for the reader.

Now we compare this to the definition of the 0-Hecke algebra $\mathcal{H}_n(0)$ as discussed in [5,6,9].

Definition 3.2. The 0-Hecke algebra $\mathcal{H}_n(0)$ of type A_{n-1} is the \mathbb{C} -algebra generated by T_1, T_2, \dots, T_{n-1} with relations

- (i) $T_i^2 = -T_i$ for $i = 1, 2, \dots, n - 1$,
- (ii) $T_i T_j = T_j T_i$ if $|i - j| \geq 2$,
- (iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $i = 1, 2, \dots, n - 2$.

We can extend the action of U_1, U_2, \dots, U_{n-1} on $\mathcal{M}(P)$ to a linear action on $\mathbb{C}\mathcal{M}(P)$, the complex vector space with basis $\mathcal{M}(P)$. If we set $T_i = -U_i$ then U_1, U_2, \dots, U_{n-1} generate the same \mathbb{C} -algebra and so we can now refer to our action on the maximal chains of P as a *local $\mathcal{H}_n(0)$ action*. In [5, Section 3.9], Duchamps, Hivert and Thibon describe the special case of this action on distributive lattices. They work in the language of linear extensions of a poset Q which, as we have seen, correspond to snellings of $J(Q)$.

Our action has one further very desirable property which we now discuss.

4. Good $\mathcal{H}_n(0)$ actions

Before stating the fifth property, we must give some background, much of which is taken from the introduction in [14].

Let P be any finite graded poset of rank n with $\hat{0}$ and $\hat{1}$ and let $S \subseteq [n - 1]$. We let $\alpha_P(S)$ denote the number of chains in P whose elements, other than $\hat{0}$ and $\hat{1}$, have rank set equal to S . In other words,

$$\alpha_P(S) = \#\{\hat{0} < t_1 < \dots < t_{|S|} < \hat{1} : \{\text{rk}(t_1), \dots, \text{rk}(t_{|S|})\} = S\}.$$

The function $\alpha_P : 2^{[n-1]} \rightarrow \mathbb{Z}$ is called the *flag f -vector* of P . It contains equivalent information to that of the *flag h -vector* β_P whose values are given by

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

Ehrenborg in [7, Definition 3] suggested looking at the formal power series (in the variables $x = (x_1, x_2, \dots)$)

$$F_P(x) = \sum_{\hat{0}=t_0 \leq t_1 \leq \dots \leq t_{k-1} < t_k = \hat{1}} x_1^{\text{rk}(t_0, t_1)} x_2^{\text{rk}(t_1, t_2)} \dots x_k^{\text{rk}(t_{k-1}, t_k)},$$

where the sum is over all multichains from $\hat{0}$ to $\hat{1}$ such that $\hat{1}$ occurs exactly once. It is easy to see that the series $F_P(x)$ is homogeneous of degree n and that it is a *quasisymmetric function*, that is, for every sequence n_1, n_2, \dots, n_m of exponents, the monomials $x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m}$ and $x_{j_1}^{n_1} x_{j_2}^{n_2} \cdots x_{j_m}^{n_m}$ appear with equal coefficients whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$. The series $F_P(x)$ can also be rewritten as

$$F_P(x) = \sum_{S \subseteq [n-1]} \beta_P(S) L_{S,n}(x), \quad (4.2)$$

where $L_{S,n}(x)$ denotes Gessel's fundamental quasisymmetric function

$$L_{S,n}(x) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

which constitute a basis for the space of quasisymmetric functions of degree n . The case when F_P is a symmetric function is considered in [14, 18] and we wish, in a sense, to extend this to the case when F_P is a quasisymmetric function. In our brief references to the symmetric function case, we follow the notation of [10]. The usual involution ω on symmetric functions given by $\omega(s_\lambda) = s_{\lambda'}$ can be extended to the ring of quasisymmetric functions by the definition $\omega(L_{S,n}) = L_{[n-1]-S,n}$. As in [20, Exercise 7.94], where this extended definition appears, we leave it as an exercise to check that it restricts to the ring of symmetric functions to give the usual ω .

We now introduce some representation theory related to our local $\mathcal{H}_n(0)$ action. In the symmetric function case, certain classes of posets P have been found to have the property that

$$F_P(x) = \text{ch}(\psi) \quad \text{or} \quad \omega F_P(x) = \text{ch}(\psi),$$

where ψ denotes the character of some local symmetric group action and where $\text{ch}(\psi)$ denotes its Frobenius characteristic as defined in [10, Section I.7]. In extending these concepts to the $\mathcal{H}_n(0)$ case, we follow the definitions in [6, 9]. The representation theory of $\mathcal{H}_n(0)$ is studied by Norton [11]. There are known to be 2^{n-1} irreducible representations, all of dimension 1. Since $T_i^2 = -T_i$, the irreducible representations are obtained by sending a set of generators to -1 and its complement to 0 . We will label these representations by subsets S of $[n-1]$, and then the irreducible representation ψ_S of $\mathcal{H}_n(0)$ is defined by

$$\psi_S(T_i) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Therefore,

$$\psi_S(U_i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Hence the character of ψ_S , denoted by χ_S , is given by

$$\chi_S(U_{i_1} U_{i_2} \cdots U_{i_k}) = \begin{cases} 1 & \text{if } i_j \in S \text{ for } j = 1, 2, \dots, k, \\ 0 & \text{otherwise.} \end{cases}$$

We define its *characteristic* by

$$\text{ch}(\chi_S) = L_{S,n}(x),$$

and we extend it to the set of all characters of representations of $\mathcal{H}_n(0)$ by linearity. We let χ_P denote the character of the defining representation of our local $\mathcal{H}_n(0)$ action on the space $\mathbb{C}\mathcal{M}(P)$.

Proposition 4.1. *Let P be a finite snellable graded poset of rank n with $\hat{0}$ and $\hat{1}$. Then the local $\mathcal{H}_n(0)$ action on the maximal chains of P has the property that*

$$\omega F_P(x) = \text{ch}(\chi_P). \quad (4.3)$$

Proof. It is sufficient to show that the coefficient of $L_{S,n}$ for any $S \subseteq [n-1]$ is the same for both sides of (4.3). By (4.2),

$$[L_{S,n}] \omega F_P(x) = \beta_P(S^c),$$

where S^c denotes $[n-1] - S$.

Let $J \subseteq [n-1]$ and let $\{i_1, i_2, \dots, i_k\}$ be a multiset on J where each element of J appears at least once. Let $m \in \mathcal{M}(P)$. If $U_{i_i}(m) \neq m$ for some $i \in [n-1]$ then $U_{i_i}(m)$ has one less inversion than m . It follows that $U_{i_1} U_{i_2} \cdots U_{i_k}(m) = m$ if and only if the descent set of m is disjoint from J . Therefore,

$$\begin{aligned} \chi_P(U_{i_1} U_{i_2} \cdots U_{i_k}) &= \#\{m \in \mathcal{M}(P) : m \text{ has no descents in } J\} \\ &= \sum_{S \supseteq J} \#\{m \in \mathcal{M}(P) : m \text{ has descent set } S^c\} \\ &= \sum_{S \supseteq J} \beta_P(S^c) \text{ by [3, Theorem 2.2]} \\ &= \sum_{S \subseteq [n-1]} \beta_P(S^c) \chi_S(U_{i_1} U_{i_2} \cdots U_{i_k}). \end{aligned}$$

Thus,

$$[L_{S,n}] \text{ch}(\chi_P) = [L_{S,n}] \text{ch} \left(\sum_{S \subseteq [n-1]} \beta_P(S^c) \chi_S \right) = \beta_P(S^c)$$

as required. \square

To summarize, we have that if P is a finite snellable graded poset of rank n , with $\hat{0}$ and $\hat{1}$, then P has a local $\mathcal{H}_n(0)$ action with the property that $\omega F_P(x) = \text{ch}(\chi_P)$. Following [18], we call such an action a *good* $\mathcal{H}_n(0)$ action. It is natural to ask what other types of posets have good $\mathcal{H}_n(0)$ actions.

Example 4.2. Consider the poset P shown in Fig. 4. As stated in Example 2.5, this poset is not snellable. However, it does have a good $\mathcal{H}_n(0)$ action as described in Table 1.

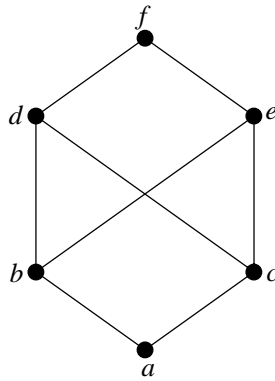


Fig. 4.

Table 1

\mathfrak{m}	$U_1(\mathfrak{m})$	$U_2(\mathfrak{m})$
$\mathfrak{m}_1 : a < b < d < f$	\mathfrak{m}_3	\mathfrak{m}_2
$\mathfrak{m}_2 : a < b < e < f$	\mathfrak{m}_4	\mathfrak{m}_2
$\mathfrak{m}_3 : a < c < d < f$	\mathfrak{m}_3	\mathfrak{m}_4
$\mathfrak{m}_4 : a < c < e < f$	\mathfrak{m}_4	\mathfrak{m}_4

It is easy to check that this gives a local $\mathcal{H}_3(0)$ action. We also have that

$$\omega F_P(x) = L_{\emptyset,3} + L_{\{1\},3} + L_{\{2\},3} + L_{\{1,2\},3} = \text{ch}(\chi_P).$$

Therefore, this poset has a good $\mathcal{H}_n(0)$ action.

Definition 4.3. A graded poset P is said to be *bowtie-free* if it does not contain distinct elements a, b, c and d such that a covers both c and d , and such that b covers both c and d .

In Section 3, we will prove our second main result:

Theorem 2. Let P be a finite graded bowtie-free poset of rank n with $\hat{0}$ and $\hat{1}$. Then P is S_n EL-shellable if and only if P has a good $\mathcal{H}_n(0)$ action.

In particular, since lattices are bowtie-free, we get the following immediate corollary.

Corollary 1. Let L be a finite graded lattice of rank n . Then the following are equivalent:

1. L is supersolvable,

2. L is S_n EL-shellable,
3. L has a good $\mathcal{H}_n(0)$ action.

5. Snellable implies supersolvable

Our main aim for this section is to prove Theorem 1.

Let L be a finite graded lattice of rank n . We showed in Example 2.6 that if L is supersolvable, then L is snellable. Now we suppose that L is snellable and we wish to prove that L is supersolvable. We let m_0 denote the unique maximal chain of L labeled by the identity permutation. Taking m_0 to be our candidate M-chain, we let L_c denote the sublattice of L generated by m_0 and any other chain c of L .

It is shown in [1, p. 12] and is easy to see that any sublattice of a distributive lattice is distributive. If c is a chain in L that is not maximal, then we can extend it to a maximal chain m in at least one way. Then L_c is a sublattice of L_m . Therefore, it suffices to show that L_m is distributive for all maximal chains m . Our approach will be to define two new posets, Q_m and $J(P_{\omega_m})$, and to show that

$$L_m = Q_m \cong J(P_{\omega_m}).$$

We have seen that if $U_i(m)$ differs from m , then $U_i(m)$ has one less inversion than m and that $\omega_{U_i(m)} = \omega_m s_i$. It follows that if m has r inversions then we can find a sequence $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ such that $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$. We define \mathcal{M}_m , a subset of $\mathcal{M}(L)$, as follows:

$$\mathcal{M}_m = \{m' \in \mathcal{M}(L) : \exists i_1, i_2, \dots, i_r \text{ such that } m' = U_{i_1} U_{i_2} \cdots U_{i_r}(m)\}.$$

We label the elements of \mathcal{M}_m as they are labeled in L . We define Q_m to be the subposet of L with elements

$$\{u \in L : u \in m' \text{ for some } m' \in \mathcal{M}_m\}$$

and with a partial order inherited from L . Q_m can be thought of as the closure of m in L under the operations U_1, U_2, \dots, U_{n-1} . We should note that it is not obvious that every maximal chain of Q_m is in \mathcal{M}_m . We wish to obtain a clear picture of the structure of Q_m .

We are now ready to start the proof proper of Theorem 1. We break up the argument into a series of small steps, each consisting of an assertion followed by its proof.

Step 1: Let m' and m'' be distinct elements of \mathcal{M}_m . Then $\omega_{m'} \neq \omega_{m''}$.

Suppose that $\omega_{m'} = \omega_{m''}$. Let $U_{i_1}, U_{i_2}, \dots, U_{i_l}$ and $U_{j_1}, U_{j_2}, \dots, U_{j_l}$ be sequences of minimal length such that $m' = U_{i_1} U_{i_2} \cdots U_{i_l}(m)$ and $m'' = U_{j_1} U_{j_2} \cdots U_{j_l}(m)$. Then $s_{i_1} s_{i_2} \cdots s_{i_l}$ and $s_{j_1} s_{j_2} \cdots s_{j_l}$ are both reduced expressions for $\omega_m^{-1} \omega_{m'}$. By Tits' Word Theorem, $s_{i_1} s_{i_2} \cdots s_{i_l}$ can thus be obtained from $s_{j_1} s_{j_2} \cdots s_{j_l}$ by a sequence of braid moves (i.e., replace $s_i s_{i+1} s_i$ by $s_{i+1} s_i s_{i+1}$ or vice versa or replace $s_i s_j$ by $s_j s_i$ if $|i - j| \geq 2$.) But by properties 3 and 4 of the U_i action, $U_{i_1} U_{i_2} \cdots U_{i_l}(m)$ is invariant

under braid moves. We conclude that $m' = m''$, which is a contradiction. Therefore, $\omega_{m'} \neq \omega_{m''}$.

Step 2: Let $u \in Q_m$. Then there is a unique chain $m_u \in \mathcal{M}_m$ that has increasing labels between $\hat{0}$ and u and between u and $\hat{1}$.

Choose any $m' \in \mathcal{M}_m$ such that $u \in m'$. Suppose u has rank i in L . Apply $U_1, U_2, \dots, U_{i-1}, U_{i+1}, \dots, U_{n-1}$ repeatedly to m' to obtain m_u . The chain m_u is unique in \mathcal{M}_m because it is unique in L .

Step 3: To each point u of Q_m we can associate the subset A_u of $[n]$ consisting of the labels on any maximal chain of $[\hat{0}, u]$ in L . Then any two distinct points of Q_m correspond to distinct subsets of $[n]$.

Let u, v be distinct elements of Q_m and suppose $A_u = A_v$. Then $\omega_{m_u} = \omega_{m_v}$, contradicting Step 1.

An important tool for the remainder of the proof will be the weak order on permutations of $[n]$.

Definition 5.1. Let v, w be permutations of $[n]$. We say that $v \leq_R w$ if there exist i_1, i_2, \dots, i_r such that $v = ws_{i_r}s_{i_{r-1}} \cdots s_{i_1}$ and $ws_{i_r} \cdots s_{i_{k+1}}s_{i_k}$ has one less inversion than $ws_{i_r} \cdots s_{i_{k+1}}$ for $k = 1, 2, \dots, r$.

It is known (see, for example, [4, Proposition 2.5]) that $v \leq_R w$ if and only if $\text{INV}(v) \subseteq \text{INV}(w)$, where we define the set of inversions of v , $\text{INV}(v)$, by

$$\text{INV}(v) = \{(v(j), v(i)) \in [n] \times [n] : i < j, v(i) > v(j)\}.$$

Step 4: The labels on the elements of \mathcal{M}_m consist of all those permutations ω satisfying $\omega \leq_R \omega_m$, each occurring exactly once.

Compare the definitions of \mathcal{M}_m and \leq_R . We see that if $m' \in \mathcal{M}_m$ then $\omega_{m'} \leq_R \omega_m$ and if $\omega \leq_R \omega_m$ then there exists $m' \in \mathcal{M}_m$ satisfying $\omega_{m'} = \omega$. The fact that ω occurs only once follows from Step 1.

Step 5: Let $u, v \in Q_m$. We know that if $u \leq v$ then $A_u \subseteq A_v$. Suppose $A_u \subseteq A_v$ for some elements u, v of Q_m . Then $u \leq v$.

Construct a permutation ω as follows:

- Let $\omega(1), \omega(2), \dots, \omega(|A_u|)$ be the elements of A_u taken in increasing order.
- Let $\omega(|A_u| + 1), \dots, \omega(|A_v|)$ be the elements of $A_v - A_u$ taken in increasing order.
- Let $\omega(|A_v| + 1), \dots, \omega(n)$ be the elements of $[n] - A_v$ taken in increasing order.

Then, since $u, v \in Q_m$, we have that $\text{INV}(\omega) \subseteq \text{INV}(\omega_m)$ and so $\omega \leq_R \omega_m$. Let $m_{u,v}$ be the element of \mathcal{M}_m satisfying $\omega_{m_{u,v}} = \omega$. By Step 3, u and v are both elements of $m_{u,v}$. We conclude that $u \leq v$ in Q_m .

We can now exhibit a poset P_{ω_m} such that $Q_m \cong J(P_{\omega_m})$. Construct P_{ω_m} , a poset on $[n]$ with relation \leq defined by $i \leq j$ if and only if $(i, j) \notin \text{INV}(\omega_m)$. For example, if $\omega_m = 2413$ we get the poset on the left in Fig. 1.

Step 6: The map $\phi : Q_m \rightarrow J(P_{\omega_m})$ defined by $\phi(u) = A_u$ is an isomorphism.

Suppose A_u has size k .

$$\begin{aligned}
 u \in Q_m &\Leftrightarrow A_u = \{\omega(1), \omega(2), \dots, \omega(k)\} \text{ for some } \omega \leq_R \omega_m \\
 &\Leftrightarrow A_u = \{\omega(1), \omega(2), \dots, \omega(k)\} \text{ for some } \omega \text{ satisfying} \\
 &\quad \text{INV}(\omega) \subseteq \text{INV}(\omega_m) \\
 &\Leftrightarrow A_u \text{ is an order ideal of } P_{\omega_m} \\
 &\Leftrightarrow A_u \in J(P_{\omega_m}).
 \end{aligned}$$

Therefore, ϕ is a well-defined bijection. If u and v are elements of Q_m , by Step 5,

$$u \leq v \text{ in } Q_m \Leftrightarrow A_u \subseteq A_v \Leftrightarrow A_u \leq A_v \text{ in } J(P_{\omega_m}) \quad (5.4)$$

as required.

It follows from this that Q_m , up to isomorphism, depends only on ω_m and not even on the underlying lattice L .

Step 7: Q_m is a sublattice of L .

Let $u, v \in Q_m$ with corresponding subsets A_u and A_v , respectively. Let $u \vee_L v$ denote the join of u and v in L and let $u \vee_{Q_m} v$ denote the join of u and v in Q_m , which we now know is a lattice. In L we have that

$$u \vee_{Q_m} v \geq u \vee_L v$$

since Q_m is a subposet of L . But by (5.4),

$$\text{rk}(u \vee_{Q_m} v) = |A_u \cup A_v| \leq \text{rk}(u \vee_L v)$$

since there are maximal chains of $[\hat{0}, u \vee_L v]$ going through u and others going through v . Thus,

$$u \vee_{Q_m} v = u \vee_L v.$$

Similarly,

$$u \wedge_{Q_m} v = u \wedge_L v.$$

We have shown that Q_m is a distributive sublattice of L . Furthermore, L_m is a sublattice of Q_m since L_m is a sublattice of L and Q_m contains m and m_0 . We conclude that L_m is also distributive and hence L is supersolvable. \square

The astute reader will notice that, while we have shown that L is supersolvable and that $L_m \subseteq Q_m$, we have not fulfilled our promise to show that $L_m = Q_m$. However, this follows from the following lemma.

Lemma 5.2. *For each element m' of \mathcal{M}_m , we have $Q_{m'} = L_{m'}$.*

Proof. Let m' be an element of \mathcal{M}_m such that $\omega_{m'}$ has l inversions. The proof is by induction on l with the result being trivially true for $l = 0$. Since we know that $L_{m'} \subseteq Q_{m'} \subseteq Q_m$, it suffices to restrict our attention to Q_m . We will label the elements of Q_m by their corresponding subsets of $[n]$. By (5.4), join and meet in Q_m are just set union and set intersection, respectively.

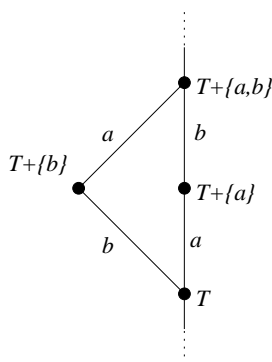


Fig. 5.

Referring to Fig. 5, suppose \mathfrak{m}' is the vertical chain. Suppose that $|T| = i - 1$ and $a > b$ so that \mathfrak{m}' has a descent at rank i . Now

$$T + \{b\} = ((T + \{a, b\}) \cap (\{1, 2, \dots, a - 1\})) \cup T$$

and $\{1, 2, \dots, a - 1\} \in \mathfrak{m}_0$. Therefore, $T + \{b\} \in L_{\mathfrak{m}'}$ and so we get that $L_{U_i(\mathfrak{m}')} \subseteq L_{\mathfrak{m}'}$ as sets. Suppose the descents of \mathfrak{m}' are at ranks i_1, i_2, \dots, i_k . Then, as sets,

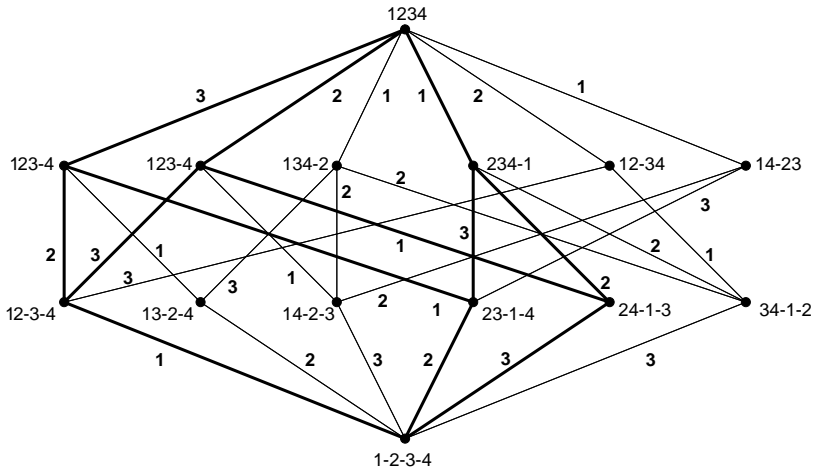
$$\begin{aligned} Q_{\mathfrak{m}'} &= Q_{U_{i_1}(\mathfrak{m}')} \cup Q_{U_{i_2}(\mathfrak{m}')} \cup \dots \cup Q_{U_{i_k}(\mathfrak{m}')} \cup \mathfrak{m}' \\ &= L_{U_{i_1}(\mathfrak{m}')} \cup L_{U_{i_2}(\mathfrak{m}')} \cup \dots \cup L_{U_{i_k}(\mathfrak{m}')} \cup \mathfrak{m}' \quad \text{by induction} \\ &\subseteq L_{\mathfrak{m}'}. \quad \square \end{aligned}$$

Example 5.3. A *non-crossing partition* of $[n]$ is a partition of $[n]$ into blocks with the property that if some block B contains i and k and some block B' contains j and l with $i < j < k < l$ then $B = B'$. We order the set of non-crossing partitions by *refinement*: if μ and ν are non-crossing partitions of $[n]$ we say that $\mu \leq \nu$ if every block of μ is contained in some block of ν . The resulting poset NC_n , which is a subposet of the lattice Π_n of partitions of $[n]$, is itself a lattice and has been studied extensively. More information on NC_n can be found in Simion's survey article [13] and the references given there.

Π_{n+1} was shown to be supersolvable in [16] and so can be given a snelling λ as in Example 2.6. We can choose the M-chain to be the maximal chain consisting of the bottom element and those partitions of $[n + 1]$ whose only non-singleton block is $[i]$ where $2 \leq i \leq n + 1$. In the literature, λ is often seen in the following form, which can be shown to be equivalent. If ν covers μ in Π_{n+1} , then ν is obtained from μ by merging two blocks B and B' of μ . We set

$$\lambda(\mu, \nu) = \max\{\min B, \min B'\} - 1.$$

It was observed by Björner and Edelman [2] that λ restricts to NC_{n+1} to give an EL-labeling for NC_{n+1} . In fact, it is readily checked that we get a snelling for NC_{n+1} . Theorem 1 now gives a new proof of the supersolvability of NC_{n+1} . Fig. 6

Fig. 6. NC_4 with snelling.

shows NC_4 with $L_m = Q_m$ highlighted for when m is the maximal chain $\hat{0} < 24-1-3 < 234-1 < \hat{1}$. In this case, P_{ω_m} is just 3 incomparable elements and so $J(P_{\omega_m}) = B_3 \cong Q_m$.

6. Lattice snellings and good $\mathcal{H}_n(0)$ actions

Our main aim for this section is to prove Theorem 2.

Recall that P denotes a finite graded bowtie-free poset of rank n with $\hat{0}$ and $\hat{1}$. We suppose that P has a good $\mathcal{H}_n(0)$ action and we let χ_P denote the character of the defining representation of this action on the space $\mathbb{C}\mathcal{M}(P)$. In other words, we suppose that there exist functions $U_1, U_2, \dots, U_{n-1} : \mathcal{M}(P) \rightarrow \mathcal{M}(P)$ satisfying the following properties:

1. The action of U_1, U_2, \dots, U_{n-1} is local.
2. $U_i^2 = U_i$ for $i = 1, 2, \dots, n-1$.
3. $U_i U_j = U_j U_i$ if $|i - j| \geq 2$.
4. $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$ for $i = 1, 2, \dots, n-2$.
5. $\omega F_P(x) = \text{ch}(\chi_P)$.

As we have previously suggested, given any maximal chain m of P , we define the descent set of m to be the set

$$\{i \in [n-1] : U_i(m) \neq m\}.$$

We wish to show that P is snellable. The following approach was suggested by Stanley. Suppose P has a unique maximal chain m_0 with empty descent set. Given a

maximal chain m of P , suppose we can find $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ with r minimal such that

$$U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0. \quad (6.5)$$

Then to m we associate the permutation $\omega_m = s_{i_1} s_{i_2} \cdots s_{i_r}$ and we label the edges of m by $\omega_m(1), \omega_m(2), \dots, \omega_m(n)$ from bottom to top. Our proof of the validity of this approach divides into four main parts. The first task is to show that m_0 exists and is unique. The next is to show that, given m , we can always find $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ satisfying (6.5). The third task is to show that ω_m is well-defined. Finally, we must show that this gives a snelling for P .

Definition 6.1. Given maximal chains m and m' of P , we say that the expression $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m'$ is *restless* if $U_{i_r}(m) \neq m$ and if

$$U_{i_j} U_{i_{j+1}} \cdots U_{i_r}(m) \neq U_{i_{j+1}} \cdots U_{i_r}(m) \quad \text{for } j = 1, 2, \dots, r-1.$$

We say that two sequences $U_{i_1} U_{i_2} \cdots U_{i_r}$ and $U_{j_1} U_{j_2} \cdots U_{j_r}$ are in the same *braid class* if we can get from one to the other by applying properties 3 and 4 repeatedly. It can be readily checked that if $U_{i_1} U_{i_2} \cdots U_{i_r}$ and $U_{j_1} U_{j_2} \cdots U_{j_r}$ are in the same braid class and if $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m'$ is restless, then $U_{j_1} U_{j_2} \cdots U_{j_r}(m) = m'$ is restless. Here we use the bowtie-free property of P .

To every sequence i_1, i_2, \dots, i_r such that $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m'$, we can associate a counting vector of length $n-1$ where the j th coordinate equals the number of times that i_j appears in the sequence i_1, i_2, \dots, i_r . We say that the expression $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m'$ is *lexicographically minimal* (or *lex. minimal* for short) if no sequence $U_{j_1} U_{j_2} \cdots U_{j_r}$ in the braid class of $U_{i_1} U_{i_2} \cdots U_{i_r}$ and satisfying $U_{j_1} U_{j_2} \cdots U_{j_r}(m) = m'$ has a lexicographically less counting vector.

The following result will help us to complete our first two tasks.

Lemma 6.2. Let m' be any maximal chain of P . Suppose $U_i(m') \neq m'$. Then there do not exist i_1, i_2, \dots, i_r satisfying $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(m') = m'$.

Proof. Suppose there exists a sequence i_1, i_2, \dots, i_r satisfying $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(m') = m'$. It suffices to consider the case when $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(m') = m'$ is restless and lex. minimal. Let $l \in [n-1]$ denote the minimum element of the sequence i_1, i_2, \dots, i_r, i . Since our equation is restless U_l must occur at least twice in the sequence.

Take any pair of U_l appearances with no U_l between them. If we had no U_{l+1} between them, we could apply property 3 until we had an appearance of $U_l U_l$, contradicting the restless property since $U_l^2 = U_l$. If there is just one U_{l+1} between them, we can apply property 3 to get $U_l U_{l+1} U_l$ appearing and then apply property 4 to get $U_{l+1} U_l U_{l+1}$, contradicting the lex. minimal property. We conclude that, between the two appearances of U_l , there are at least two appearances of U_{l+1} . Choose any two of these appearances of U_{l+1} that do not have another U_{l+1} between them and apply the same argument to show that there must be at least two appearances of U_{l+2} between them. Repeating this process, we eventually get $U_i U_i$ appearing, yielding a contradiction. \square

More generally, we can apply the same argument to prove the following statement:

Lemma 6.3. *Suppose $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}_0$ is restless and lex. minimal. Let l denote the minimum element of the sequence i_1, i_2, \dots, i_r . Then U_l appears exactly once and for $l < i \leq n-1$, there must be an appearance of U_{i-1} between any two appearances of U_i .*

The following result is essentially a rephrasing of property 5 of our good $\mathcal{H}_n(0)$ action into more amenable terms.

Proposition 6.4. *For all $S \subseteq [n-1]$, $\alpha_P(S)$ equals the number of maximal chains of P with descent set contained in S .*

Proof. We know that

$$\chi_P = \sum_{S \subseteq [n-1]} b_{P,S} \chi_S$$

for some set of coefficients $\{b_{P,S}\}_{S \subseteq [n-1]}$ and hence

$$\text{ch}(\chi_P) = \sum_{S \subseteq [n-1]} b_{P,S} L_{S,n}.$$

By (4.2) and property 5, we see that $b_{P,S} = \beta_P(S^c)$.

Now let $J = \{i_1, i_2, \dots, i_k\} \subseteq [n-1]$. Then

$$\begin{aligned} \sum_{S \supseteq J} \beta_P(S^c) &= \sum_{S \subseteq [n-1]} \beta_P(S^c) \chi_S(U_{i_1} U_{i_2} \cdots U_{i_k}) \\ &= \chi_P(U_{i_1} U_{i_2} \cdots U_{i_k}) \\ &= \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has no descents in } J\} \end{aligned}$$

by Lemma 6.2. Therefore,

$$\sum_{S \supseteq J} \beta_P(S^c) = \sum_{S \supseteq J} \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set } S^c\}.$$

Since this holds for all $J \subseteq [n-1]$, we get that

$$\beta_P(S) = \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set } S\}$$

for all $S \subseteq [n-1]$. By Inclusion-Exclusion, this is equivalent to

$$\alpha_P(S) = \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set contained in } S\}. \quad \square$$

In particular, setting $S = \emptyset$, we see that P has exactly one maximal chain, which we denote by \mathfrak{m}_0 , with no descents. Also, given a maximal chain \mathfrak{m} of P , by Lemma 6.2 and the finiteness of P , we can find $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ with r minimal such that $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}_0$. This completes our first two tasks.

Given any maximal chain \mathfrak{m} of P , we consider the braid classes of the set of sequences $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ such that $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}_0$ is restless. Our next task is to show that there is only one such braid class. Every braid class contains at least one

element $U_{i_1}, U_{i_2}, \dots, U_{i_r}$ such that $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$ is restless and lex. minimal. For such an element, the minimum, l , of i_1, i_2, \dots, i_r is the lowest rank for which $m \neq m_0$, by Lemma 6.3. It follows that l is the same for all the braid classes. It suffices to consider the case when $l = 1$.

The following result is central to our proof that there is just one braid class.

Lemma 6.5. *Suppose the expressions $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$ and $U_{j_1} U_{j_2} \cdots U_{j_s}(m) = m_0$ are both restless. Then there exists an element of the braid class of $U_{i_1} U_{i_2} \cdots U_{i_r}$ and an element of the braid class of $U_{j_1} U_{j_2} \cdots U_{j_s}$ both ending on the right with the same U_i .*

Proof. Suppose $U_{i_1} U_{i_2} \cdots U_{i_r}$ and $U_{j_1} U_{j_2} \cdots U_{j_s}$ are in different braid classes. Without loss of generality, we take them both to be lex. minimal. If U_1 can be moved to the right-hand end in both by applying property 3, then there is nothing to prove. Suppose, by applying property 3, that U_1 can be brought to the right end in one sequence but not in the other. Then P must have the edges shown in Fig. 7, where m and m_0 are the maximal chains on the left and right, respectively. We see that we get a contradiction with the bowtie-free property unless $a = b$. In this case, U_2 appears at least twice in the latter sequence to the right of the unique appearance of U_1 , contradicting Lemma 6.3. We conclude that U_1 cannot be brought to the right end in either sequence. Now we consider that portion of each sequence to the right of the unique U_1 . By the same logic, the maximal chains we get when we apply these portions to m must have the same element at rank 2.

Consider the unique U_2 in each of these portions. By a similar argument, we conclude that either we have nothing to prove or else U_2 cannot be brought to the right of either sequence by applying property 3. In the latter case, we consider the portion of each sequence to the right of the unique U_2 . The maximal chains we get

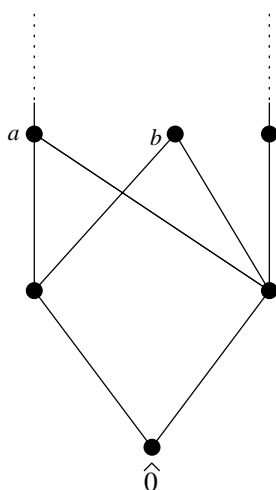


Fig. 7. A portion of P .

when we apply these portions to m must have the same element at rank 3. Repeating the same argument, we are eventually reduced to the case where U_i is the element at the right end of both sequences, for some i . \square

Proposition 6.6. *If the expressions $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$ and $U_{j_1} U_{j_2} \cdots U_{j_s}(m) = m_0$ are both restless then $s_{i_1} s_{i_2} \cdots s_{i_r} = s_{j_1} s_{j_2} \cdots s_{j_s}$.*

Proof. It suffices to prove the result in the case when r is as small as possible. We prove the result by induction on r , the result being trivially true when $r = 0$.

For $r > 0$, by the previous lemma, there exists an element $U_{i_1} U_{i_2} \cdots U_{i_{r-1}} U_i$ of the braid class of $U_{i_1} U_{i_2} \cdots U_{i_r}$ and an element $U_{j_1} U_{j_2} \cdots U_{j_{s-1}} U_i$ of the braid class of $U_{j_1} U_{j_2} \cdots U_{j_s}$. Consider $U_i(m)$. By the induction hypothesis,

$$s_{i_1} s_{i_2} \cdots s_{i_{r-1}} = s_{j_1} s_{j_2} \cdots s_{j_{s-1}}.$$

Therefore, since permutations are invariant under braid moves,

$$s_{i_1} s_{i_2} \cdots s_{i_r} = s_{i_1} s_{i_2} \cdots s_{i_{r-1}} s_i = s_{j_1} s_{j_2} \cdots s_{j_{s-1}} s_i = s_{j_1} s_{j_2} \cdots s_{j_s}. \quad \square$$

Finally, we can make the following definition:

Definition 6.7. If $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$ is restless then we define

$$\omega_m = s_{i_1} s_{i_2} \cdots s_{i_r}.$$

For every maximal chain m of P , we label the edges of m from bottom to top by $\omega_m(1), \omega_m(2), \dots, \omega_m(n)$. Our final task is to show that this gives an edge-labeling, and in particular a snelling, for P . We divide the proof into a number of small steps.

Step 1: If $U_{i_1} U_{i_2} \cdots U_{i_r}(m) = m_0$ is restless then $\omega_m = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression. Furthermore, if $\omega_m = s_{j_1} s_{j_2} \cdots s_{j_r}$ is another reduced expression, then $U_{j_1} U_{j_2} \cdots U_{j_r}(m) = m_0$ is restless.

The first assertion follows from the fact that if $\omega_m = s_{i_1} s_{i_2} \cdots s_{i_r}$ is not reduced then we can apply a sequence of braid moves to get $s_i s_i$ appearing. This contradicts the restless property. The second assertion follows from Tits' Word Theorem.

Step 2: The permutation ω_m has a descent at i if and only if $U_i(m) \neq m$. In this case, $\omega_{U_i(m)}$ is the same as ω_m except that the i th and $(i+1)$ st elements have been switched, removing the descent.

$$U_i(m) \neq m$$

$$\Leftrightarrow U_{i_1} U_{i_2} \cdots U_{i_r} U_i(m) = m_0 \text{ is restless for some } i_1, i_2, \dots, i_r$$

$$\Leftrightarrow s_{i_1} s_{i_2} \cdots s_{i_r} s_i = \omega_m \text{ is a reduced expression for some } i_1, i_2, \dots, i_r$$

$$\Leftrightarrow \omega_m s_i \text{ has one less inversion than } \omega_m$$

$$\Leftrightarrow \omega_m \text{ has a descent at } i.$$

When $U_i(m) \neq m$ and $\omega_m = s_{i_1}s_{i_2} \cdots s_{i_r}s_i$ is reduced we see that $\omega_{U_i(m)} = s_{i_1}s_{i_2} \cdots s_{i_r}$, yielding the second statement.

Step 3: Let $S \subseteq [n-1]$. Then every chain in P with rank set equal to S has exactly one extension to a maximal chain of P with descent set contained in S .

Given any chain c with rank set S , let m be any extension of c to a maximal chain in P . Apply U_i for $i \notin S$ repeatedly to m . By Step 2, this will eventually yield an extension of c which is a maximal chain with descent set contained in S . Therefore, every chain with rank set S has at least one such extension. We get

$$\alpha_P(S) \leq \#\{m \in \mathcal{M}(P) : m \text{ has descent set contained in } S\}.$$

However, by Proposition 6.4,

$$\alpha_P(S) = \#\{m \in \mathcal{M}(P) : m \text{ has descent set contained in } S\}.$$

Thus c has exactly one extension to a maximal chain of P with descent set contained in S .

Step 4: For every maximal chain m of P , labeling the edges of m from bottom to top by $\omega_m(1), \omega_m(2), \dots, \omega_m(n)$ gives an edge-labeling for P .

Let $x, y \in P$ be such that y covers x and let m and m' be maximal chains of P containing both x and y . Define $S = [\text{rk}(x), \text{rk}(y)]$ and let $m_{(x,y)}$ denote the unique extension of $x < y$ to a maximal chain with descent set contained in S . By applying U_i for $i \notin S$ repeatedly to m , we can reach $m_{(x,y)}$. By Step 2, m and $m_{(x,y)}$ give the same label to the edge (x, y) . Similarly, m' and $m_{(x,y)}$ give the same label to the edge (x, y) . Therefore, m and m' give the same label to the edge (x, y) and so we have an edge labeling for P .

Step 5: This edge-labeling is a snelling for P .

Let $x, y \in P$ be such that $x < y$. Let

$$S = [n-1] - \{\text{rk}(x) + 1, \text{rk}(x) + 2, \dots, \text{rk}(y) - 1\}$$

in Step 3. The fact that the interval $[x, y]$ has exactly one increasing maximal chain follows from Step 3 and the fact that we now have an edge-labeling. Every maximal chain is labeled by a permutation by definition. Therefore, P is snellable, proving Theorem 2. \square

Remark 6.8. Theorem 2 does indeed contain information not contained in Corollary 1, in that there exist finite graded bowtie-free posets with $\hat{0}$ and $\hat{1}$ that are snellable but are not lattices. For example, take the lattice B_4 with a snelling as described in Example 2.3. Now delete the edge $(\{3, 4\}, \{2, 3, 4\})$ in the Hasse diagram of B_4 to form the Hasse diagram of a new poset. We can check that the new poset has the desired properties.

It seems that we have fully answered the question of finite graded posets with $\hat{0}$ and $\hat{1}$ in the bowtie-free case. What can we say about such posets that are not bowtie-free? In Example 4.2 we saw a poset with a bowtie that has a good $\mathcal{H}_n(0)$

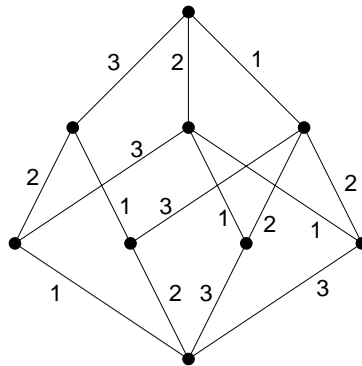


Fig. 8.

action but which is not snellable. On the other hand, Fig. 8 shows a finite graded poset with $\hat{0}$ and $\hat{1}$ that has a bowtie but which is still snellable.

This suggests the following question.

Question. Let \mathcal{C} denote the class of finite graded posets with $\hat{0}$, $\hat{1}$ and a good $\mathcal{H}_n(0)$ action. Is there some “nice” characterization of \mathcal{C} , possibly in terms of edge-labelings?

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