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Intersection theorems under dimension constraints

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Abstract

In Ahlswede et al. [Discrete Math. 273(1–3) (2003) 9–21] we posed a series of extremal (set system) problems under dimension constraints. In the present paper, we study one of them: the intersection problem. The geometrical formulation of our problem is as follows. Given integers $0 \leqslant t, k \leqslant n$ determine or estimate the maximum number of (0, 1)-vectors in a k-dimensional subspace of the Euclidean n-space $\mathbb{R}^n$, such that the inner product ("intersection") of any two is at least t. Also we are interested in the restricted (or the uniform) case of the problem; namely, the problem considered for the (0, 1)-vectors of the same weight ω.

The paper consists of two parts, which concern similar questions but are essentially independent with respect to the methods used.

In Part I, we consider the unrestricted case of the problem. Surprisingly, in this case the problem can be reduced to a weighted version of the intersection problem for systems of finite sets. A general conjecture for this problem is proved for the cases mentioned in Ahlswede et al. [Discrete Math. 273(1–3) (2003) 9–21]. We also consider a diametric problem under dimension constraint.

In Part II, we study the restricted case and solve the problem for $t = 1$ and $k < 2\omega$, and also for any fixed $1 \leqslant t \leqslant \omega$ and k large.

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1. Introduction

$\mathbb{N}$ denotes the set of positive integers. For $i, j \in \mathbb{N}$, $i < j$ the set $\{i, i+1, \dots, j\}$ is denoted by $[i, j]$ and $[n]$ stands for $[1, n]$. For $w, n \in \mathbb{N}$, $w \leqslant n$ we set

$$2^{[n]} = \{F : F \subset [n]\}, \binom{[n]}{w} = \{F \in 2^{[n]} : |F| = w\}.$$

With each subset we associate its characteristic (0,1)-vector in $\mathbb{R}^n$. For the sets of (0,1)-vectors corresponding to $2^{[n]}$ and $\binom{[n]}{\omega}$ we use the notation

$$E(n) = \{0,1\}^n \subset \mathbb{R}^n \text{ and } E(n, w) = \{x^n \in E(n) : x^n \text{ has } w \text{ ones}\}.$$

For $A \subset E(n)$ we write $\dim(A) = k$ if the vector space spanned by A has dimension k.

The set theoretical extremal problems can be formulated in terms of vector spaces and vice versa. In particular, concepts like t-intersecting families of subsets and antichains of subsets translate in the language of (0,1)-vectors in a natural way.

A family $\mathcal{F} \subset 2^{[n]}$ is called t-intersecting if $|F_1 \cap F_2| \geqslant t$ holds for all $F_1, F_2 \in \mathcal{F}$.

Correspondingly $A \subset E(n)$ is called t-intersecting if any two vectors from A have at least t common ones.

Note that families of sets are denoted here by script letters.

We now ask for a maximum sized t-intersecting system $A \subset E(n)$, contained in a k-dimensional subspace of $\mathbb{R}^n$. Given $0 \leqslant t, k \leqslant n$ define

$J_t(n, k) = \max\{|A| : A \subset E(n) \text{ is a } t\text{-intersecting system with } \dim(A) = k\}$.

Notice that the case $k = n$ is the well known intersection problem solved by Katona [16]. Let us define the family

$$\mathcal{K}(n, t) = \left\{A \in 2^{[n]} : |A| \geqslant \frac{n+t}{2}\right\} = \bigcup_{i=\frac{n+t}{2}}^{n} \binom{[n]}{i} \quad \text{if } 2 \mid (n+t).$$

Theorem Ka (*Katona [16]*). *Suppose that $\mathcal{A} \subset 2^{[n]}$ is t-intersecting. Then*

$$|\mathcal{A}| \leqslant J_t(n, n) = \begin{cases} |\mathcal{K}(n, t)| & \text{if } 2 \mid (n+t), \\ 2|\mathcal{K}(n-1, t)| & \text{if } 2 \nmid (n+t). \end{cases} \tag{1.1}$$

The general case of our intersection problem under dimension constraint (called unrestricted case) is studied in Part I. We aim to prove the following conjecture, stated also in [3].

Conjecture 1. *For $t > n - k + 1$*

$$J_t(n, k) = \begin{cases} \sum\limits_{i=k-1-\frac{n-t}{2}}^{k-1} \binom{k-1}{i} + \sum\limits_{i=\frac{n+t}{2}}^{k-1} \binom{k-1}{i} & \text{if } 2 \mid (n+t), \\ 2 \sum\limits_{i=k-1-\frac{n-t+1}{2}}^{k-2} \binom{k-2}{i} + 2 \sum\limits_{i=\frac{n+t-1}{2}}^{k-2} \binom{k-2}{i} & \text{if } 2 \nmid (n+t). \end{cases} \tag{1.2}$$

We establish the conjecture for some range of parameters. Note that the case $t \leqslant n-k+1$ is simple as it is shown in Section 5. We also consider a diametric problem under dimension constraint.

In Part II our problem is considered for (0,1)-vectors of the same weight: the restricted case. Namely, given positive integers $t \leqslant \omega \leqslant n, k \leqslant n$, the problem is to determine or estimate

$$\begin{aligned} &J_t(n, k, \omega) \\ &\quad \triangleq \max\{|A| : A \subset E(n, \omega),\ A \text{ is a } t\text{-intersecting system with } \dim(A) \leqslant k\}. \end{aligned}$$

Here, we study the problem mainly for intersecting systems, that is for the case $t = 1$. For this case we use the notation $J(n, k, \omega)$. The general case of the problem seems to be more difficult.

We recall now the famous Erdős–Ko–Rado Theorem in our terminology.

Theorem EKR (*Erdos et al. [11]*).

(i) *For* $2\omega \leqslant n$

$$J(n, n, \omega) = \binom{n-1}{\omega-1}. \tag{1.3}$$

(ii) *For* $1 < t < \omega$ *and* $n \geqslant n_o(\omega, t)$

$$J_t(n, n, \omega) = \binom{n-t}{\omega-t}. \tag{1.4}$$

For sharpenings of Theorem EKR (with $t > 1$) see [9,14,18]. The complete solution of the problem is given in [6].

Note that for $2\omega < n$ the unique intersecting system $A \subset E(n, \omega)$ achieving bound (1.3) is a "star", that is all vectors in $E(n, \omega)$ with 1 in a fixed coordinate. For $2\omega = n$ there are many other choices for an optimal system. For the case (ii) the unique (up to obvious isomorphisms) optimal t-intersecting system is a "t-star", that is all vectors with ones in t fixed positions.

Observe that a t-star A has $\dim(A) = n - t$. Thus Theorem EKR also gives a solution for our intersection problem in the case $k = n - 1$.

Corollary EKR. *For* $2\omega \leqslant n$ *we have*

$$J(n, n-1, \omega) = \binom{n-1}{\omega-1}.$$

Note that the obvious restriction in Theorem EKR is just to avoid triviality, since in the case $2\omega > n$ $E(n, \omega)$ is "automatically" intersecting, and hence $J(n, n, \omega) = \binom{n}{\omega}$.

It is also clear that for $2\omega > n$:

$$J(n, k, \omega) = \max |U_k \cap E(n, \omega)|, \tag{1.5}$$

taken over all k-dimensional subspaces U_k of $\mathbb{R}^n$. For $1 \leqslant \omega \leqslant n$ let us denote by $M(n,k,\omega)$ the quantity in the RHS of (1.5).

In [2] $M(n,k,\omega)$ has been determined for all parameters.

Theorem AAK (*Ahlswede et al. [2]*). *Given $\omega, k, n \in \mathbb{N}$; $\omega, k \leqslant n$:*

(a) $M(n,k,\omega) = M(n,k,n-\omega)$
(b) *For $\omega \leqslant \frac{n}{2}$ we have*

$$M(n,k,\omega) = \begin{cases} \binom{k}{\omega} & \textit{if } \text{(i) } 2\omega \leqslant k, \\ \binom{2(k-\omega)}{k-\omega} 2^{2\omega-k} & \textit{if } \text{(ii) } k \leqslant 2\omega \leqslant 2(k-1), \\ 2^{k-1} & \textit{if } \text{(iii) } k-1 \leqslant \omega. \end{cases}$$

The key sets giving the maximal values in the three cases are:

(i) $S_1 = E(k,\omega) \times \{0\}^{n-k}$,
(ii) $S_2 = E(2k-2\omega, k-\omega) \times \{10, 01\}^{2\omega-k} \times \{0\}^{n-2\omega}$,
(iii) $S_3 = \{10, 01\}^{k-1} \times \{1\}^{\omega-k+1} \times \{0\}^{n-k-\omega+1}$.

We state now our conjecture in terms of $M(n,k,\omega)$.

Conjecture 2. *For $\omega \leqslant n/2$*

$$J(n,k,\omega) = M(n-1,k,\omega-1).$$

In Part II the conjecture is established for the case $k < 2\omega$.

We also determine $J_t(n,k,\omega)$ for any $1 \leqslant t \leqslant \omega$ and k sufficiently large.

Part I. The unrestricted case

The main results of this part concern Conjecture 1, and they are stated in Section 3. But we start with a key observation in Section 2, showing that the problem (for the unrestricted case) can be reduced to a weighted version of the t-intersection problem for systems of finite sets. In Section 5, we give proofs of the main results using auxiliary results from Section 4. Finally, in Section 6 we consider a diametric problem under dimension constraint which turns out to have a simple solution.

2. Reformulation of the problem

Given integers $1 \leqslant k \leqslant n$ we assign to each element $i \in [k]$ a weight $w_i \in \mathbb{N}$ such that $\sum_{i=1}^{k} w_i = n$. Then the sequence $(w_1, \ldots, w_k)$ is called a weight distribution on $[k]$.

For each $A \in 2^{[k]}$ define its weight $w(A) = \sum_{i \in A} w_i$.

Given a weight distribution w, we say that a set system $\mathcal{A} \subset 2^{[k]}$ is *t-weight intersecting* if $w(A \cap B) \geqslant t$ holds for all $A, B \in \mathcal{A}$.

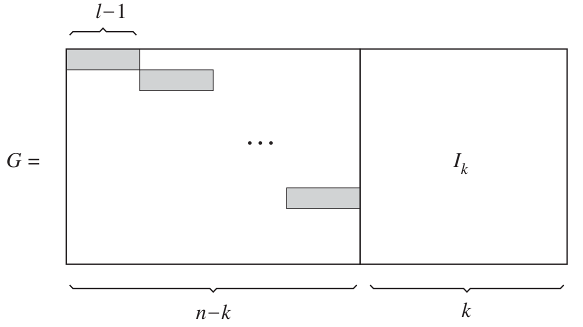

Fig. 1.

Our *weight–intersecting problem* is to determine

$$f(n,k,t) \triangleq \max_{w:\sum_{i=1}^{k} w_i=n} \{|\mathcal{A}| : \mathcal{A} \subset 2^{[k]} \text{ is } t\text{-weight intersecting with weight distribution } w\}.$$

Another problem is given t, k and a weight distribution $(w_1,\dots,w_k)$ determine

$$g(w_1,\dots,w_k;\ t) \triangleq \max\{|\mathcal{A}| : \mathcal{A} \subset 2^{[k]} \text{ is a } t\text{-weight intersecting system}\}.$$

The second problem seems to be more difficult than the first one. However, we will see below that for our purposes the first problem is more important.

Denote by $F(w_1,\dots,w_k)$ $(w_i \in \mathbb{N};\ i=1,\dots,k)$ the set of all k-tuples with entries 0 or w_i in the ith coordinate, i.e. $F(w_1,\dots,w_k)=\{0,w_1\}\times\cdots\times\{0,w_k\}$. This is another description of the set $2^{[k]}$ with the weight distribution $w:(w_1,\dots,w_k)$ on the ground set $[k]$.

We also need the following notion from [2].

An $r\times n$ real matrix M of $\operatorname{rank}(M)=r$ is said to have a *positive step form* if it has the form shown in Fig. 1, where each shade ("step") of size $\ell_i \geqslant 1$ $(i=1,\dots,r)$, with $\sum_{i=1}^{r}\ell_i=n$, depicts ℓ_i positive entries of the ith row and above the steps M has only zero entries.

Lemma 1.1 (*Ahlswede et al. [2]*). *A matrix M can be brought to a positive step form by elementary row operations or permutations of columns iff the space spanned by rows of M contains a positive vector* (*a vector with positive coordinates only*).

Remark 1.1. It follows from the proof of Lemma 1.1 [2] that M can be also brought to a positive step form with $\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_r$.

Lemma 1.2. *The quantities $J_t(n,k)$ and $f(n,k,t)$ are equal.*

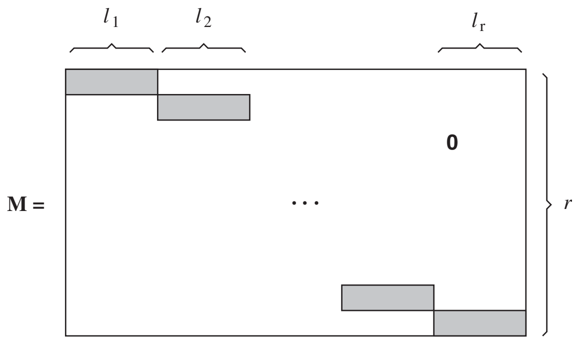

Fig. 2.

Proof. Let $A \subset E(n)$ be a t-intersecting system of vectors with $\dim(A) = k$. W.l.o.g. we can assume that there is no coordinate set $\{i_1, \dots, i_r\} \subset [n]$ in which all vectors of A have all-zeros. This is clear because otherwise we can replace the coordinates $i_1, \dots, i_r$ into 1's in each vector of $A \setminus A^*$, where A^* is a maximal subset of A with $\dim(A^*) = k - 1$. Obviously the new set A' is also t-intersecting and $\dim(A') = k$. Let G be a generator matrix for the k-dimensional subspace $U \triangleq \mathrm{span}(A) \subset \mathbb{R}^n$. U contains a positive vector, therefore (by Lemma 1.1) G can be transformed to a positive step form. In particular, w.l.o.g. we may assume that G has a form, shown in Fig. 2, where I_k is the $k \times k$ identity matrix.

Let u_1^n be the first row of G and suppose it has ℓ nonzero coordinates. Consider the following partition $A = A_1 \dot\cup A_0' \dot\cup A_0''$, where A_1 consists of the elements of A which are obtained by linear combinations (of the row vectors of G) involving u_1^n, i.e. the elements of A with 1 in the $(n-k+1)$-th coordinate, A_0' consists of the vectors from $A \setminus A_1$ with zeros in the first $\ell - 1$ coordinates, and $A_0'' = A \setminus (A_0 \cup A_0')$. Consider now the following transformation of A.

Replace all nonzero entries of u_1^n by 1's: $u_1^n \to v_1^n$. Replace all other entries of the first $\ell - 1$ columns of G by 0's. Note that the same linear combinations as for G, (now for the new generating matrix) give a new set $B \subset E(n)$ with $|B| = |A|$. Denote the set of vectors (obtained after the described transformation) corresponding to A_1, A_0', A_0'' by B_1, B_0', B_0'' resp. That is we have $A_1 \to B_1$, $A_0' \to B_0'$, $A_0'' \to B_0''$. Turn now to the sets $B_1, B_0', B_0''+v_1^n$. Observe that these sets are disjoint and moreover their union $B^* = B_1 \dot\cup B_0' \dot\cup (B_0'' + v_1^n)$ is a t-intersecting system with $|B^*| = |A|$ and $\dim(B^*) = k$.

Applying this transformation to all other "steps" we can reduce G to a positive step form G' where all steps consist of 1's and all other entries in G' are 0's, i.e. all columns of G' are unit vectors.

Now the rows $v_1^n, v_2^n, \dots, v_k^n$ of G' can be associated with the elements of $\{1, \dots, k\}$ where each element $i \in \{1, \dots, k\}$ has weight $w_i \triangleq$ the number of $1'$s in v_i^n.

This completes the proof. $\square$

Given weight distribution $w = (w_1, \ldots, w_k)$ on $[k]$ we define the weighted Katona family

$$\mathcal{K}(k,t)_w = \left\{ F \in 2^{[k]} : w(F) \geqslant \frac{n+t}{2} \right\},$$

where $n = w_1 + \cdots + w_k$ and $2 \mid n + t$.

Note that $\mathcal{K}(k,t)_w$ is t-intersecting. Our Conjecture 1 can be explained now in terms of the weighted Katona family. It says that given n, k and $t > n - k + 1$, $2 \mid n + t$ an optimal t-weight intersecting family $\mathcal{A} \subset 2^{[k]}$ can be realized for the weight distribution $w = (n - k + 1, 1, \ldots, 1)$ and

$$\begin{aligned} J_t(n,k) = f(n,k,t) &= g(n-k+1, 1, \ldots, 1; t) \\ &= \begin{cases} |\mathcal{K}(k,t)_w| & \text{if } 2 \mid (n+t), \\ 2|\mathcal{K}(k-1,t)_w| & \text{if } 2 \nmid (n+t). \end{cases} \end{aligned} \tag{2.1}$$

For $2 \nmid n+t$ the bound in (1.2) is attained for $\mathcal{A} = \mathcal{K}(k,t+1) \cup \left\{ A \in 2^{[k-1]} : w(\mathcal{A}) = \frac{n+t-1}{2} \right\}$.

Remark 1.2. One may also expect that the family $\mathcal{K}(k,t)_w$ is optimal also for any weight distribution w. However, in general this is not the case. Note for instance that $g(3,3,3,1;6) \geqslant 4$, while the Katona family (consisting of the sets with weight $\geqslant 8$) contains only two elements: $\{1,2,3\}$ and $\{1,2,3,4\}$.

3. Main results

We now state our main results for the unrestricted case of our problem, considered in Part I. In the sequel we will use $f(n,k,t)$ rather than the identical (in view of Lemma 1.2) notation $J_t(n,k)$. Let us also denote the RHS of (1.2) by $m(n,k,t)$.

Theorem 1.1.

(i) $f(n,k,t) \leqslant 2^{k-1}$.
(ii) *For* $t \leqslant n - k + 1$ *we have* $f(n,k,t) = 2^{k-1}$.

The smallest t for which the problem is open is $t = n - k + 2$. The next theorem gives a partial solution for this case.

Theorem 1.2. *For* $n \geqslant \frac{3}{2}k - 1$, $t = n - k + 2$ *we have*

$$f(n,k,t) = m(n,k,t) = 2^{k-2}.$$

Theorem 1.3. *Given positive integers* t, k *and* $w_1 \geqslant \cdots \geqslant w_s \geqslant 2$, $s < k$, *such that* $t > w_1 + \cdots + w_s$. *Then*

$$g(w_1, \ldots, w_s, 1, \ldots, 1; t) = \begin{cases} |\mathcal{K}(k,t)_w| & \textit{if } 2 \mid (n+t), \\ 2|\mathcal{K}(k-1,t)_w| & \textit{if } 2 \nmid (n+t). \end{cases}$$

Corollary 1.1. *For positive integers k and $t > w_1$ ($n \triangleq w_1 + k - 1$) we have*

$$g(w_1, \underbrace{1, \ldots, 1}_{k-1}; t) = m(n, k, t)$$

$$= \begin{cases} \displaystyle\sum_{i=\frac{n+t}{2}-w_1}^{k-1} \binom{k-1}{i} + \sum_{i=\frac{n+t}{2}}^{k-1} \binom{k-1}{i} & 2 \mid (n+t), \\ \displaystyle 2 \sum_{i=\frac{n+t-1}{2}-w_1}^{k-2} \binom{k-2}{i} + 2 \sum_{i=\frac{n+t-1}{2}}^{k-2} \binom{k-2}{i} & 2 \nmid (n+t). \end{cases}$$

Theorem 1.4. *For $t \geqslant 2(n-k) - 1$ we have*

$$f(n, k, t) = m(n, k, t).$$

Corollary 1.2. *For $k \leqslant n \leqslant k + 3$ we have*

$$f(n, k, t) = m(n, k, t).$$

Theorem 1.5. *For $\frac{3}{2}(n-t) - 1 \leqslant k \leqslant \frac{n+t}{2}$ we have*

$$f(n, k, t) = m(n, k, t).$$

Theorem 1.6. *For $n > k\sqrt{2k}/2$, $t \geqslant n - k + 2$ we have*

$$f(n, k, t) = m(n, k, t).$$

4. Auxiliary tools and results

4.1. Distance properties

For $x^k, y^k \in F(w_1, \ldots, w_k)$ define the distance $\operatorname{dist}(x^k, y^k) = \sum_{i=1}^k |x_i - y_i|$.

Lemma 1.3. *Let $A, B \subset F(w_1, \ldots, w_k)$ be such that the nonzero distances occuring in A do not occur in B. Then*

$$|A||B| \leqslant 2^k.$$

Proof. Let us think of elements of $F(w_1, \ldots, w_k)$ as vectors in the k-dimensional vector space $GF(2)^k$. Then the statement follows from the observations:

(a) For $u^k, v^k \in F(w_1, \ldots, w_k)$ one has $\operatorname{dist}(u^k, v^k) = \operatorname{weight}(u^k + v^k)$.
(b) For $u_1^k, u_2^k \in A$ and for $v_1^k, v_2^k \in B$
$\operatorname{dist}(u_1^k, u_2^k) \neq \operatorname{dist}(v_1^k, v_2^k) \Rightarrow u_1^k + u_2^k \neq v_1^k + v_2^k \Rightarrow u_1^k + v_1^k \neq u_2^k + v_2^k$.
Hence

$$|A + B| \triangleq |\{u^k + v^k : u^k \in A, v^k \in B\}| = |A||B| \leqslant 2^k. \qquad \square$$

Lemma 1.4. *Given $a_1, a_2, \dots, a_k \in \mathbb{N}$, let $n = \sum_{i=1}^{k} a_i = \frac{3}{2}k - 1$, then there exist $I, J \subset [k]$ with $I \cap J = \emptyset$ such that*

(i) $\sum_{i \in I} a_i = \sum_{j \in J} a_j = \frac{k}{2}$, *if* $2 \mid k$,

(ii) $\sum_{i \in I} a_i = \sum_{j \in J} a_j = \frac{k-1}{2}$, *if* $2 \nmid k$.

Proof. We prove only case (i) (case (ii) is similar). W.l.o.g let $a_1 \geqslant \cdots \geqslant a_k$. Note that the number of 1's in $a_1, \dots, a_k$ is at least $\frac{k}{2} + 1$. That is we have

$$a_1 \geqslant \cdots \geqslant a_m \geqslant 2,\ a_{m+1} = \cdots = a_k = 1 \text{ with } m \leqslant \frac{k}{2} - 1.$$

Also note that if $a_1 + \cdots + a_m \leqslant \frac{k}{2}$, then we are done. Therefore, let $a_1 + \cdots + a_m > \frac{k}{2}$. Suppose now $a_1 + \cdots a_s \leqslant \frac{k}{2}$, for some $1 \leqslant s \leqslant m$. Such an s always exists since $a_1 \leqslant \frac{k}{2}$.

If $a_1 + \cdots + a_s = \frac{k}{2} - 1$ or $\frac{k}{2}$ then we are done, since we have at least $\frac{k}{2} + 1$ ones and one can find disjoint sets I and J satisfying condition (i). Thus let

$$a_1 + \cdots + a_s = \frac{k}{2} - y,\ \ y \geqslant 2 \text{ and } a_1 + \cdots + a_s + a_{s+1} \geqslant \frac{k}{2} + 1. \tag{4.1}$$

Now to complete the proof it suffices to justify the following

Claim. *The number of ones*, $k - m \geqslant \frac{k}{2} + y - 1$.

Proof. In view of (4.1) we have $a_{s+1} \geqslant y + 1$.

Therefore,

$$\sum_{i=1}^{s+1} a_i \geqslant (y+1)(s+1)$$

and hence

$2(m - s - 1) + k - m \leqslant a_{s+2} + \cdots + a_k \leqslant \frac{3}{2}k - 1 - (y+1)(s+1)$.

Finally, since $y \geqslant 2$ and $s \geqslant 1$, we get $k - m \geqslant \frac{k}{2} + (y-1)(s+1) - 1 \geqslant \frac{k}{2} + y$. $\square$

4.2. Shifting and multiexchange techniques

Recall the known operation in extremal set theory called shifting, which was introduced by Erdös et al. [11]. Given $\mathcal{B} \subset 2^{[k]}$, $B \in \mathcal{B}$ and integers $1 \leqslant i \leqslant j \leqslant k$

$$S_{ij}(B) \triangleq \begin{cases} \{i\} \cup (B \setminus \{j\}) & \text{if } i \notin B,\ j \in B,\ \{i\} \cup (B \setminus \{j\} \notin \mathcal{B}, \\ B & \text{otherwise,} \end{cases}$$

$$S_{ij}(\mathcal{B}) \triangleq \{S_{ij}(B) : B \in \mathcal{B}\}.$$

It is known (see e.g. [10]) that the following properties hold for any $\mathcal{B} \subset 2^{[k]}$.

S1. $|S_{ij}(\mathcal{B})| = |\mathcal{B}|$.
S2. After finitely many shifting operations $\mathcal{B}$ can be reduced to a shifted family, i.e. a family $\mathcal{B}'$ with $S_{ij}(\mathcal{B}') = \mathcal{B}'$ for all $1 \leqslant i < j \leqslant n$.
S3. If $\mathcal{B}$ is t-intersecting then so is $S_{ij}(\mathcal{B})$.

We extend now the shifting operation to any set system $\mathcal{B}$ over a ground set $[k]$ with a weight distribution $w : (w_1, \ldots, w_k)$, $w_1 \geqslant \cdots \geqslant w_k$. For any $B \in \mathcal{B} \subset 2^{[k]}$ define the weighted shifting operation $S_{i,J}$ where $i \in [k]$, $J = \{j_1, \ldots, j_m\} \subset [k]$ and $i < \min\{j_1, \ldots, j_m\}$.

$$S_{i,J}(B)_w \triangleq \begin{cases} \{i\} \cup (B \setminus J) & \text{if } i \notin B,\ J \subset B,\ \{i\} \cup (B \setminus J) \notin \mathcal{B} \text{ and} \\ & \quad w_{j_1} + \cdots + w_{j_m} \leqslant w_i, \\ B & \text{otherwise,} \end{cases}$$
$$S_{i,J}(\mathcal{B})_w \triangleq \{S_{i,J}(B)_w : B \in \mathcal{B}\}.$$

Note that for $|J| = 1$ we have standard shifting and properties S1–S3 are valid.

Suppose now $S_{i,j}(\mathcal{B}) = \mathcal{B}$ for all $1 \leqslant i < j \leqslant k$. Apply then $S_{i,J}$ for some admissible i and $J \in \binom{[k]}{2}$. It is easy to show that properties S1–S3 hold with respect to $S_{i,J}$. Thus $\mathcal{B}$ can be reduced to a shifted family, that is to a family $\mathcal{B}'$ with $S_{i,J}(\mathcal{B}')_w = \mathcal{B}'$ for all admissible i and $J \in \binom{[k]}{2}$. This procedure can be applied consecutively to all possible $J \subset [k]$ ($|J| = 2, 3, \ldots$) reducing $\mathcal{B}$ to a family $\mathcal{B}^*$, such that $S_{i,J}(\mathcal{B}^*)_w = \mathcal{B}^*$ for all admissible i and J. Such a family $\mathcal{B}^*$ is called then *w-shifted*.

More precisely, it is not difficult to prove the following.

Suppose $\mathcal{B} \subset 2^{[k]}$ is shifted with respect to all shifts $S_{i,J}$ with $J \subset \binom{[k]}{\ell}$. Then for any $J \subset \binom{[k]}{\ell+1}$ one has

(1) $|S_{i,J}(\mathcal{B})_w| = |\mathcal{B}|$.
(2) If $\mathcal{B}$ is t-intersecting then so is $S_{i,J}(\mathcal{B})_w$.

In the sequel, when we deal with families over weighted ground sets, by shiftedness we will always mean the w-shiftedness and t-intersecting will mean t-weight-intersecting.

Next define a *multiexchange operation* introduced in [8].

Given $\mathcal{A} \subset 2^{[k]}$ and disjoint sets $I, H \subset [k]$ define

$$\mathcal{A}_{I,H} = \{A \subset \mathcal{A} : I \subset A,\ H \cap A = \emptyset \text{ and } (A \setminus I) \cup H \notin \mathcal{A}\}.$$

Then the multiexchange operation $T_{I,H}$, called an $(|I|, |H|)$-exchange, is defined by

$$T_{I,H}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{I,H}) \cup \{(A \setminus I) \cup H : A \in \mathcal{A}_{I,H}\}.$$

Note that in case $|I| = |H| = 1$ we have the exchange operation in usual sense denoted here by T_{ij} ($i, j \in [k]$). We consider now only a special type of this operation when $|H| = |I| + 1$.

Given $\mathcal{A} \subset 2^{[k]}$ we apply first a (0,1)-exchange $T_{I,H}$. Repeatedly applying this operation for all $H \in \binom{[k]}{1}$ the family $\mathcal{A}$ can be brought to another family $\mathcal{B}$ which is stable with respect to each (0,1)-exchange, i.e. $T_{I,H}(\mathcal{B}) = \mathcal{B}$ for all $H \in \binom{[k]}{1}$. Clearly $|\mathcal{B}| = |\mathcal{A}|$ and if $\mathcal{A}$ is t-intersecting so is $\mathcal{B}$.

Next, we apply to $\mathcal{B}$ a (1,2)-exchange $T_{I,H}$.

Again we have $|T_{I,H}(\mathcal{B})| = |\mathcal{B}|$ and it is easy to show that $T_{I,H}(\mathcal{B})$ preserves the intersection property of $\mathcal{A}$. This procedure we continue for all $I \in \binom{[k]}{1}$ and $H \in \binom{[k]}{2}$ reducing $\mathcal{B}$ to a stable family $\mathcal{C}$ such that $T_{I,H}(\mathcal{C}) = \mathcal{C}$.

Iteratively applying the described procedure of $(i-1,i)$-exchanges $(i = 1, 2, \ldots)$ $T_{I,H}$ for all $I \in \binom{[k]}{i-1}$ and $H \in \binom{[k]}{i}$ we reduce $\mathcal{A}$ to a *stable*, with respect to all multiexchange operations, family $\mathcal{A}^*$ (see [8]).

Now the following properties of $\mathcal{A}^*$ can be easily observed.

Let $\mathcal{A} \subset 2^{[k]}$ be a t-intersecting family. Then

T1. $|\mathcal{A}^*| = |\mathcal{A}|$.
T2. $\mathcal{A}^*$ is t-intersecting.
T3. If $A \in \mathcal{A}^*$ with $|A| = r$, then $\mathcal{A}^*$ contains all subsets $B \subset [k]$ with $|B| > r$.

Lemma 1.5. *Given weight distribution $w : (w_1, \ldots, w_s, 1, \ldots, 1)$ on $[k]$, with $2 \mid (n+t)$; $n = \sum_{i=1}^{k} w_i$, let $\mathcal{A} \subset 2^{[k]}$ be an optimal t-weight intersecting family. Then there exists a t-weight intersecting family $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|$ which is invariant on $[s+1,k]$. Namely if $A = (A_1 \cup A_2) \in \mathcal{A}^*$, where $A_1 = A \cap [1,s]$, $A_2 = A \cap [s+1,k]$, then $\mathcal{A}^*$ contains every subset $B = A_1 \cup E$ $(E \in 2^{[s+1,k]})$ with $|E| \geqslant |A_2|$.*

Proof. Apply successively multi-exchange operations $T_{I,H}(\mathcal{A})$ for all $I \in \binom{[s+1,k]}{i-1}$, $H \in \binom{[s+1,k]}{i}$ $i = 1, 2, \ldots$, reducing $\mathcal{A}$ to a stable, with respect to the described multiexchange operations, family. Thus w.l.o.g. we may assume that $\mathcal{A}$ is a shifted and stable (with respect to $(|I|, |H|)$-exchange) family. Note that, in view of property T_3, if $(A_1 \cup A_2) \in \mathcal{A}$ $(A_1 \subset [1,s],\ A_2 \subset [s+1,k])$, then $\mathcal{A}$ contains also all elements $(A_1 \cup F)$, $F \subset [s+1,k]$ with $|F| > |A_2|$.

Given $i, j \in [s+1,k]$, define now $\mathcal{A}_1 = \{A \in \mathcal{A}:\ i \in A, j \notin A\}$, $\mathcal{B} = \{B \in \mathcal{A}_1 : (B \setminus \{i\}) \cup \{j\} \notin \mathcal{A}_1\}$, $\mathcal{B}' = \{(B \setminus \{i\}) \cup \{j\} :\ B \in \mathcal{B}\}$. Thus $|\mathcal{B}| = |\mathcal{B}'|$, $\mathcal{B}' \cap \mathcal{A} = \emptyset$. Define also $\mathcal{F}_1 = \{B \in \mathcal{B} :\ w(B) \leqslant \frac{n+t}{2} - 1\}$, $\mathcal{F}_2 = \mathcal{B} \setminus \mathcal{F}_1$, $\mathcal{F}_1' = \{B' \in \mathcal{B}' :\ w(B') \leqslant \frac{n+t}{2} - 1\}$, $\mathcal{F}_2' = \mathcal{B}' \setminus \mathcal{F}_1'$. Consider now two new families

$$\mathcal{A}' = (\mathcal{A} \setminus \mathcal{F}_1) \cup \mathcal{F}_2', \quad \mathcal{A}'' = (\mathcal{A} \setminus \mathcal{F}_2) \cup \mathcal{F}_1'.$$

Claim. (i) *$\mathcal{A}'$ and $\mathcal{A}''$ are t-intersecting*; (ii) $|\mathcal{A}'| = |\mathcal{A}''| = |\mathcal{A}|$.

Proof. To prove that $\mathcal{A}'$ (resp. $\mathcal{A}''$) is intersecting it suffices to show that $\mathcal{F}_2 \cup \mathcal{F}_2'$ (resp. $\mathcal{F}_1 \cup \mathcal{F}_1'$) is intersecting. This follows from the definition of $\mathcal{A}_1$. Suppose now, for a contradiction, $E \in \mathcal{F}_1$, $F' \in \mathcal{F}'$ and $w(E \cap F') = t-1$. Since $w(E) + w(F') \leqslant n+t-2$, there exists $l \in [k]$ such that $l \notin (E \cup F')$. If $l \in [s]$, then (by the shiftedness of $\mathcal{A}$) $F \triangleq ((F' \setminus \{j\}) \cup \{l\}) \in \mathcal{A}$ and $w(E \cap F) = w(E \cap F') < t$, a contradiction. Suppose now $l \in [s+1,k]$. Then by the stability of $\mathcal{A}$ (defined above) $F \triangleq (F' \cup \{l\}) \in \mathcal{A}$ and again $w(E \cap F) < t$, a contradiction. Thus $\mathcal{F}_1 \cup \mathcal{F}_1'$ is t-intersecting. The family $\mathcal{F}_2 \cup \mathcal{F}_2'$ is t-intersecting because $w(F) \geqslant \frac{n+t}{2}$ for all $F \in (\mathcal{F}_2 \cup \mathcal{F}_2')$, completing the proof of (i).

To prove (ii) note that

$$|\mathcal{A}'| = |\mathcal{A}| + |\mathcal{F}_2'| - |\mathcal{F}_1|, \quad |\mathcal{A}''| = |\mathcal{A}| + |\mathcal{F}_1'| - |\mathcal{F}_2|. \tag{4.2}$$

Since $|\mathcal{F}_1| = |\mathcal{F}_1'|$, $|\mathcal{F}_2| = |\mathcal{F}_2'|$, we infer that $|\mathcal{F}_1| = |\mathcal{F}_2|$, otherwise by (4.2) $\max\{|\mathcal{A}'|, |\mathcal{A}''|\} > |\mathcal{A}|$, a contradiction with the optimality of $\mathcal{A}$. This with (4.2) completes the proof of the claim. $\square$

As a new intersecting family of the same size we take now $\mathcal{A}'$ for which $w(\mathcal{A}') \geqslant w(\mathcal{A})$ where $w(\mathcal{A}) \triangleq \sum_{A\in\mathcal{A}} w(A)$. We continue this procedure transforming $\mathcal{A}$ to a family $\mathcal{A}^*$ which is either stable with respect to all exchange operations T_{ij} with $i, j \in [s+1, k]$, or consists of all sets A with $w(A) \geqslant \frac{n+t}{2}$. This completes the proof of Lemma 1.5. $\square$

4.3. Properties of function $g(w_1, \ldots, w_k; t)$

Lemma 1.6. *For integers $w_1 \geqslant \cdots \geqslant w_k = 1$ with $2 \nmid \left(\sum_{i=1}^k w_i + t\right)$ we have*

$$g(w_1, \ldots, w_k; t) = 2g(w_1, \ldots, w_{k-1}; t).$$

Proof. Let $\mathcal{A} \subset 2^{[k]}$ be an optimal t-weight intersecting system for the weight distribution $w : (w_1, \ldots, w_k)$. Suppose also w.l.o.g. that $\mathcal{A}$ is shifted. Define the families

$$\mathcal{A}_0 = \{A \in \mathcal{A} : k \notin A\}, \ \mathcal{A}_1 = \mathcal{A} \setminus \mathcal{A}_0.$$

$\mathcal{A}_0 \subset 2^{[k-1]}$ is a t-intersecting family with the weight distribution $w : (w_1, \ldots, w_{k-1})$. Correspondingly the family $\mathcal{A}_0 \cup \mathcal{A}_0'$, where $\mathcal{A}_0' = \{A \cup \{k\} : A \in \mathcal{A}_0\}$, is also t-intersecting. Since $|\mathcal{A}_0'| = |\mathcal{A}_0|$ we have $|\mathcal{A}| \geqslant 2|\mathcal{A}_0|$ and hence, in view of optimality of $\mathcal{A}$, we have $|\mathcal{A}_1| \geqslant |\mathcal{A}_0|$. Note that if $|\mathcal{A}_1| = |\mathcal{A}_0|$, then we are done. Thus assume that $|\mathcal{A}_1| > |\mathcal{A}_0|$. The idea of the proof is to reduce $\mathcal{A}$ to another t-intersecting family $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|$ so that $|\mathcal{A}_0^*| = |\mathcal{A}_1^*|$ ($\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ are defined as $\mathcal{A}_0$ and $\mathcal{A}_1$ in $\mathcal{A}$).

Define the subclass $\mathcal{B} \subset \mathcal{A}_1$ and the family $\mathcal{B}'$ by

$$\mathcal{B} = \{B \in \mathcal{A}_1 : (B \setminus \{k\}) \notin \mathcal{A}\}, \ \mathcal{B}' = \{B \setminus \{k\} : B \in \mathcal{B}\}.$$

Next partition $\mathcal{B}$ into two sets $\mathcal{B} = \mathcal{F}_1 \cup \mathcal{F}_2$ with $\mathcal{F}_1 \triangleq \left\{B \in \mathcal{B} : w(B) < \frac{n+t}{2}\right\}$ and $\mathcal{F}_2 \triangleq \mathcal{B} \setminus \mathcal{F}_1$. Correspondingly for $\mathcal{B}'$ we get the induced partition $\mathcal{B}' = \mathcal{F}_1' \cup \mathcal{F}_2'$, where $\mathcal{F}_i' = \{F \setminus \{k\} : F \in \mathcal{F}_i\}$, $i = 1, 2$.

Define now the families $\mathcal{A}' = (\mathcal{A} \setminus \mathcal{F}_1) \cup \mathcal{F}_2'$, $\mathcal{A}'' = (\mathcal{A} \setminus \mathcal{F}_2) \cup \mathcal{F}_1'$.

Claim. (i) *$\mathcal{A}'$ and $\mathcal{A}''$ are t-intersecting families*; (ii) $|\mathcal{A}'| = |\mathcal{A}''| = |\mathcal{A}|$.

Proof. For (i) it suffices to show that both $\mathcal{F}_1 \cup \mathcal{F}_1'$ and $\mathcal{F}_2 \cup \mathcal{F}_2'$ are t-intersecting. Consequently, it suffices to show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(t+1)$-intersecting. Suppose $E, F \in \mathcal{F}_1$ and $w(E \cap F) = t$. Since $w(E) + w(F) < n + t$, the shiftedness of $\mathcal{A}$ implies that there exists $j \in [1, k-1]$ such that $F' \triangleq (F \setminus \{k\}) \cup \{j\} \in \mathcal{A}$. But $w(E \cap F') < t$, a contradiction. The family $\mathcal{F}_2$ is $(t+1)$-intersecting since $w(E) + w(F) \geqslant n + t + 1$ for all $E, F \in \mathcal{F}_2$.

Part (ii) can be proved by repeating the argument used in Lemma 1.5. Equivalently, it follows that $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}_1'| = |\mathcal{F}_2'|$. $\square$

To complete the proof of Lemma 1.6 note that we reduce $\mathcal{A}$ to a new t-intersecting family $\mathcal{A}^* = \mathcal{A}'$ or $\mathcal{A}''$ of the same size, so that $|\mathcal{A}^*| = 2|\mathcal{A}_0^*|$. $\square$

Lemma 1.7. *Let* $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_s \geqslant 2$, $w_{s+1} = \cdots = w_k = 1$ *and let* $\sum_{i=2}^{s} w_i < t$, *then*

$$g(w_1, \ldots, w_s, 1, \ldots, 1; t) \leqslant g(w_1 + 1, \ldots, w_s - 1, 1, \ldots, 1; t).$$

Proof. Let $\mathcal{A} \subset 2^{[k]}$ be an optimal t-intersecting system over $[k]$ with $w : (w_1, \ldots, w_s, 1, \ldots, 1)$, i.e. $|\mathcal{A}| = g(w_1, \ldots, w_s, 1, \ldots, 1; t)$, and $w_2 + \cdots + w_s < t$ (note that the case $s = k$ is trivial).

We start with the assumption that $\mathcal{A}$ is w-shifted. By Lemma 1.6 it suffices to prove the statement for $2 \mid (n+t)$ $(n = w_1 + \cdots + w_s + k - s)$. Then in view of Lemma 1.5 we may also assume that $\mathcal{A}$ is invariant in $[s+1, \ldots, k]$.

Next partition $\mathcal{A}$ into four subfamilies $\mathcal{A} = \mathcal{A}_{00} \cup \mathcal{A}_{01} \cup \mathcal{A}_{10} \cup \mathcal{A}_{11}$,

$$\mathcal{A}_{00} = \{A \in \mathcal{A} : 1 \notin A, s \notin A\}, \mathcal{A}_{10} = \{A \in \mathcal{A} : 1 \in A, s \notin A\},$$
$$\mathcal{A}_{01} = \{A \in \mathcal{A} : 1 \notin A, s \in A\}, \mathcal{A}_{11} = \{A \in \mathcal{A} : 1 \in A, s \in A\}.$$

Define now the set of minimal elements $\mathcal{M} \subset \mathcal{A} : \mathcal{M} = \{M \in \mathcal{A} : E \subset M \Rightarrow E \notin \mathcal{A}\}$. Define also $\mathcal{M}_{01} = \mathcal{M} \cap \mathcal{A}_{01}$ and $\mathcal{M}_{10} = \mathcal{M} \cap \mathcal{A}_{10}$.
Represent each element $A \in \mathcal{A}$ by the pair (X, Y), where $X = A \cap [1, s]$, $Y = A \cap [s+1, k]$.

The following properties of $\mathcal{A}$ will be used below:

(a) For $A_1 \in \mathcal{A}_{10}$ and $A_2 \in \mathcal{A}_{00}$ we have $w(A_1 \cap A_2) \geqslant t + w_s$.
(b) For $E \in \mathcal{A}_{01}$ and $F \in \mathcal{A}_{10}$ we have $w(E) + w(F) \geqslant n + t$ $(n = w_1 + \cdots + w_s + k - s)$.
(c) Suppose $(X, V) \in \mathcal{M}_{10}$ and $((X \setminus \{1\}) \cup \{s\}, U) \in \mathcal{M}_{01}$. Then $|U| \geqslant |V| + w_1 - w_s$.
(d) If $(X, W) \in \mathcal{M}$ then there are exactly $\binom{k-s}{|W|}$ elements $(X, Y) \in \mathcal{M}$ with $|Y| = |W|$.

Let $A_1 = (X_1, Y_1)$, $A_2 = (X_2, Y_2)$ and $w(A_1 \cap A_2) = l$. Since $t > w_2 + \cdots + w_s$ we have $|Y_1 \cap Y_2| > w_s$. By the shiftedness $\mathcal{A}$ contains an element $B = (X_2 \cup \{s\}, Y_2 \setminus Z)$ with $Z \subset (Y_1 \cap Y_2)$ and $|Z| = w_s$. Hence $w(A_1 \cap B) \geqslant l - w_s \geqslant t$ concluding property (a).

To prove (b) suppose the converse. $\mathcal{A}$ is w-shifted and invariant in $[s+1, n]$. This with $t > w_2 + \cdots + w_s$ implies that there exist $E \in \mathcal{M}_{10}$ and $F \in \mathcal{M}_{01}$ with $E \cup F = [k]$. Observe now that the assumption $w(E) + w(F) < n + t$ is contradictory with the t-weight intersection property of $\mathcal{A}$.

Property (c) directly follows from the shiftedness of $\mathcal{A}$. Since $\mathcal{A}$ is invariant in $[s+1, n]$, property (d) follows as well.

Define then $\mathcal{M}_{10}^* = \{(X, Y \setminus \{i\}) : (X, Y) \in \mathcal{M}_{10}, i \in Y\}$ and consider a new family $\mathcal{A}^*$,

$$\mathcal{A}^* \triangleq (\mathcal{A}_{00} \cup \mathcal{A}_{11}) \cup (\mathcal{A}_{01} \setminus \mathcal{M}_{01}) \cup (\mathcal{A}_{10} \cup \mathcal{M}_{10}^*).$$

Claim. (i) $\mathcal{A}^*$ *is a t-weight intersecting family for the weight distribution* $w^* : (w_1 + 1, \dots, w_s - 1, \dots, 1)$ *of the ground set* $[k]$.
(ii) $|\mathcal{A}^*| \geqslant |\mathcal{A}|$.

Proof. First note that $\mathcal{M}_{10}^*$ is t-intersecting. Further in view of property (a), $\mathcal{M}_{10}^* \cup \mathcal{A}_{10} \cup \mathcal{A}_{00}$ is t-intersecting for weight distribution w^*. Note also that $\mathcal{M}_{10}^* \cup \mathcal{A}_{11}$ is t-intersecting for w^* (since it is $(t-1)$-intersecting for w). By property (b), for $E \in \mathcal{M}_{10}$ and $B \in \mathcal{A}_{01} \setminus \mathcal{M}_{01}$ holds $w(E)+w(B) \geqslant n+t+1$. This implies that for all $M \in \mathcal{M}_{10}^*$ holds $w(M)+w(B) \geqslant n+t$ and hence $\mathcal{M}_{10}^* \cup \mathcal{A}_{01}$ is also t-intersecting, completing the proof of (i).

Let us show now that for every $E = (X, U) \in \mathcal{M}_{01}$ there exists an element $F = \big((X \setminus \{s\}) \cup \{1\}, V\big) \in \mathcal{M}_{10}$. This is true because otherwise, in view of the shiftedness of $\mathcal{A}$, there exists $F' \in \mathcal{M}_{01}$, $F' \subset F$ such that $F' \cap [s+1, k] = \emptyset$. But this is a contradiction since $w_2 + \dots + w_s < t$ and hence $w(F' \cap E) < t$. In fact, since $w_s \geqslant 2$ we have $|V| \geqslant 3$.

To prove (ii) (that is $|\mathcal{M}_{01}| \leqslant |\mathcal{M}_{10}^*|$) it suffices, in view of property (d), to show that

$$\binom{k-s}{|U|} \leqslant \binom{k-s}{|V|-1}. \tag{4.3}$$

First observe that property (b) implies

$$|U| + |V| \geqslant k - s + t - (w_2 + \dots + w_{s-1}). \tag{4.4}$$

This with property (c) and the condition $t > w_2 + \dots + w_{s-1}$ implies that $|U| > \frac{k-s}{2}$. Then (4.4) gives $k - s - |U| < |V| \leqslant |U|$, which implies (4.3) and consequently (ii).

This completes the proof of Lemma 1.7. □

Remark 1.3. Note that Lemma 1.7 is not true in general. For example, observe that $g(2, 2, 2, 2, 2, 2; 8) = 7$ while $g(3, 2, 2, 2, 2, 1; 8) = 6$.

Lemma 1.8. *Given* $n, a_1, \dots, a_k, \alpha \in \mathbb{N}$ *with* $a_1 + \dots + a_k = n$, $\alpha \geqslant \frac{n+1}{2}$. *Let X be the set of solutions* $(x_1, \dots, x_k) \in E(k)$ *of the inequality*

$$\sum_{i=1}^{k} a_i x_i \geqslant \alpha. \tag{4.5}$$

Then

$$G(n, k, \alpha) \triangleq \max_{\sum_{i=1}^{k} a_i = n} |X| = \sum_{i=\alpha-(n-k+1)}^{k-1} \binom{k-1}{i} + \sum_{i=\alpha}^{k-1} \binom{k-1}{i} \tag{4.6}$$

and the maximum is assumed for $a_1 = n - k + 1$, $a_2 = \dots = a_k = 1$.

Proof. Assume w.l.o.g. $a_1 \geqslant \dots \geqslant a_k$. We proceed by induction on α, n, k.

Claim. *For* $\frac{n+1}{2} \leqslant \alpha \leqslant n - k + 1$

$$G(n, k, \alpha) = 2^{k-1}.$$

Proof. Note first that $G(n,k,\alpha)\leqslant 2^{k-1}$. Indeed, since $\alpha\geqslant\frac{n+1}{2}$ then $(x_1,\dots,x_k)\in X$ implies that $(1-x_1,\dots,1-x_k)\notin X$.

On the other hand, setting $a_1=n-k+1$, $a_2=\dots=a_k=1$ we get

$$G(n,k,\alpha)\geqslant|X|=\mathrm{RHS}(4.6)=2^{k-1}. \qquad \square$$

Suppose now that $\alpha>\frac{n+1}{2}$ and $\alpha>n-k+1$. Given n and k suppose also that (4.6) holds for all smaller values of n and k. Separating the solutions $(x_1,\dots,x_k)\in X$ of (4.5) with $x_k=0$ and with $x_k=1$, we may apply the induction hypothesis since $\alpha-a_k\geqslant\frac{n-a_k+1}{2}$. Then we get

$$\begin{aligned}|X| &\leqslant G(n-a_k,k-1,\alpha)+G(n-a_k,k-1,\alpha-a_k)\\ &=\sum_{i=\alpha-(n-a_k-k+2)}^{k-2}\binom{k-2}{i}+\sum_{i=\alpha}^{k-2}\binom{k-2}{i}\\ &\quad+\sum_{i=\alpha-(n-k+2)}^{k-2}\binom{k-2}{i}+\sum_{i=\alpha-a_k}^{k-2}\binom{k-2}{i}.\end{aligned} \tag{4.7}$$

To complete the proof of the lemma it remains to verify that RHS(4.7) $\leqslant$ RHS(4.6). $\square$

Remark 1.4. We note that later on Lemma 1.8 will not be used. However, it shows that given n, k and t the "biggest" Katona family $\mathcal{K}(k,t)_w$ is assumed for the weight distribution $w:(n-k+1,1,\dots,1)$. This fact could be helpful for comparisons of $\mathcal{K}(k,t)_w$ with other "competitor" t-weight intersecting families.

5. Proofs of Theorems 1.1–1.6

Proof of Theorem 1.1. In view of Lemma 1.2 the statement (i) follows.

Let now $t\leqslant n-k+1$. Then for the weight distribution $w:(n-k+1,1,\dots,1)$ we take the t-weight intersecting family $\mathcal{A}=\{A\in 2^{[k]}:\{1\}\in A\}$. $\square$

Proof of Theorem 1.2. It is more convenient to proceed here with $F(w_1,\dots,w_k)$ defined in Section 2.

First, we prove the theorem for

Case $n=\frac{3k}{2}-1;\ 2\mid k$: Let $B\subset F(w_1,\dots,w_k)$ be a t-weight intersecting system. In view of Lemma 1.4 there exist subsets $I,J\subset[1,k]$; $I\cap J=\emptyset$ so that

$$\sum_{i\in I}w_i=\sum_{j\in J}w_j=\frac{k}{2}. \tag{5.1}$$

Consider the following subsets of $[1,k]$:

$$S_1=\emptyset,\ S_2=[1,k]\setminus I,\ S_3=[1,k]\setminus J,\ S_4=I\cup J.$$

For any subset $S \subset [k]$ we define now its characteristic vector by

$$\mathcal{X}(S) = (x_1, \dots, x_k) \in F(w_1, \dots, w_k), \quad \text{where } x_j = \begin{cases} w_j & \text{if } j \in S, \\ 0 & \text{if } j \notin S. \end{cases}$$

Thus let $\mathcal{X}(S_i) = v_i^k$ $(i = 1, 2, 3, 4)$, and let us denote $C = \{v_1^k, v_2^k, v_3^k, v_4^k\}$. In view of (5.1) the minimum distance of C is $k-1$, that is $\operatorname{dist}(v_i^k, v_j^k) \geqslant k-1$ for any distinct $v_i^k, v_j^k \in C$. Since B is t-intersecting, it is also clear that for any $x^k, y^k \in B$ $\operatorname{dist}(x^k, y^k) \leqslant n - t = k - 2$.

Applying now Lemma 1.3 we get $|B||C| = |B|4 \leqslant 2^k$. □

Case $n > \frac{3k}{2} - 1;\ 2 \mid k$: Given $w_1 \geqslant \cdots \geqslant w_k$, with $\sum_{i=1}^{k} w_i = n$ clearly there exist $w_1' \geqslant \cdots \geqslant w_k'$ so that $w_i \geqslant w_i'$, $\sum_{i=1}^{k} w_i' = \frac{3k}{2} - 1$. Then by the previous case there exists a set

$C' = \{v_1^{k\prime}, v_2^{k\prime}, v_3^{k\prime}, v_4^{k\prime}\} \subset F(w_1', \dots, w_k')$ with minimum distance $k-1$.

Construct now the set $C = \{v_1^k, v_2^k, v_3^k, v_4^k\} \subset F(w_1, \dots, w_k)$, where each vector v_i^k $(i = 1, 2, 3, 4)$ is obtained from $v_i^{k\prime}$ by replacing every nonzero coordinate w_j' $(j \in [k])$ by w_j.

Since $w_j \geqslant w_j'$ the new set C has minimum distance at least $k-1$. The rest of the proof is the same as for the case $n = \frac{3}{2}k - 1$.

The proof of case $2 \nmid k$ is similar. □

Proof of Theorem 1.3. Let $\mathcal{A} \subset 2^{[k]}$ be an optimal t-weight intersecting family with $w : (w_1, \dots, w_k)$. In view of Lemma 1.6, we consider only the case $2 \mid n + t$. Suppose $A \in \mathcal{A}$. We assume w.l.o.g. that $\mathcal{A}$ is shifted and invariant in $[s+1, n]$. These properties of $\mathcal{A}$ with $t > w_1 + \cdots + w_s$ imply that $\mathcal{A}$ contains also an element B, with $w(B) = w(A)$, so that $A \cup B = [k]$. Since $\mathcal{A}$ is t-weight intersecting we infer that $w(A) \geqslant \frac{n+t}{2}$. □

Proof of Theorem 1.4. Suppose $\mathcal{A} \subset 2^{[k]}$ is an optimal t-weight intersecting family for a weight distribution $w : (w_1, \dots, w_k)$, $\sum_{i=1}^{k} w_i = n$. Assume $w_1 \geqslant \cdots \geqslant w_s \geqslant 2$, $w_{s+1} = \cdots = w_k = 1$, $1 \leqslant s \leqslant k$. Then $w_2 + \cdots + w_s = n - k + s - w_1 < 2(n-k) - 1 \leqslant t$.

Now using Lemma 1.7 and then Corollary 1.1 we get

$$|\mathcal{A}| \leqslant g(n-k+1, 1, \dots, 1;\ t) = m(n, k, t). \qquad \square$$

Proof of Theorem 1.5. We need the following simple

Fact. For positive integers $w_1, \dots, w_k$ with $w_i \geqslant 2$, $i \in [k]$ we have

$$g(w_1, \dots, w_k;\ t) \leqslant g(w_1, \dots, w_i - 1, \dots, w_k;\ t - 1). \tag{5.2}$$

This is clear because any t-weight intersecting family $\mathcal{A} \subset 2^{[k]}$ with $w : (w_1, \dots, w_k)$ is also $(t-1)$-weight intersecting for $w : (w_1, \dots, w_i - 1, \dots, w_k)$.

Note also that (5.2) implies

$$f(n+1,k,t+1) \leqslant f(n,k,t). \tag{5.3}$$

Note first that the condition of the theorem can be written as $n - \frac{4k+1}{3} \leqslant \frac{n+t}{2} - k$, $\frac{n+t}{2} \geqslant k$. Suppose first that $n \leqslant \frac{4k+1}{3}$. This together with $\frac{n+t}{2} \geqslant k$ implies that $t \geqslant 2(n-k)-1$. Then by Theorem 1.4 we have $f(n,k,t) = m(n,k,t)$.

Suppose now $n > \frac{4k+1}{3}$. Then in the case $2|(n+t)$ there exists an integer $\alpha \geqslant 1$ such that $n - \frac{4k+1}{3} \leqslant \alpha \leqslant \frac{n+t}{2} - k$. Put further $n' = n - \alpha$, $t' = t - \alpha$ and note that $n' \leqslant \frac{4k+1}{3}$, $n' + t' \geqslant 2k$. Note also that for $2 \nmid (n+t)$ there exists an integer $1 \leqslant \alpha < \frac{n+t}{2} - k$ such that $n' = n - \alpha = \lceil \frac{4k+1}{3} \rceil$ and $t' + n' \geqslant 2k+1$ ($t' = t - \alpha$). In both cases this implies that $t' \geqslant 2(n'-k) - 1$ and hence (by Theorem 1.4) $f(n',k,t') = m(n',k,t')$. Now by (5.3) we get $f(n,k,t) \leqslant m(n',k,t')$. On the other hand, since $\frac{n'+t'}{2} = \frac{n+t}{2} - \alpha \geqslant k$, for $2|(n+t)$ (and similarly for $2 \nmid (n+t)$) we have

$$\begin{aligned} f(n,k,t) \geqslant m(n,k,t) &= \sum_{i=k-1-\frac{n-t}{2}}^{k-1} \binom{k-1}{i} + \sum_{i=\frac{n+t}{2}}^{k-1} \binom{k-1}{i} \\ &= \sum_{i=k-1-\frac{n-t}{2}}^{k-1} \binom{k-1}{i} = m(n',k,t'). \qquad \square \end{aligned}$$

Proof of Theorem 1.6. We proceed by induction on t and n.

Case $t = n - k + 2$, $n \geqslant \frac{k\sqrt{2k}}{2}$: Since $\frac{k\sqrt{2k}}{2} \geqslant \frac{3k}{2} - 1$, Theorem 1.2 gives the result.

Case $n = \left\lceil \frac{k\sqrt{2k}}{2} \right\rceil$, $t \geqslant n - k + 2$ (for convenience let $\frac{k\sqrt{2k}}{2}$ be an integer): Suppose $\mathcal{A} \subset 2^{[k]}$ is a t-weight intersecting family with weight distribution $w : (w_1, \ldots, w_k)$ of the ground set and $w_1 \geqslant \cdots \geqslant w_k$. By property (5.2)

$$g(w_1, \ldots, w_k;\ t) \leqslant g(w_1, \ldots, w_s, 1, \ldots, 1;\ t')$$

for some $1 \leqslant s < k$ and $t' = t - (w_{s+1} + \cdots + w_k) + (k - s)$. Let us denote $\alpha = (w_{s+1} + \cdots + w_k) - (k - s)$, $n' = w_1 + \cdots + w_s + (k - s)$. The family $\mathcal{A}$ considered for the weight distribution $(w_1, \ldots, w_s, 1, \ldots, 1)$ is a t'-weight intersecting family, with $t' = t - \alpha$; $n' = n - \alpha$.

We have $\frac{n+t}{2} \geqslant k$ and we aim to choose now an α, such that

$$\frac{n'+t'}{2} = \frac{n+t}{2} - \alpha \geqslant k. \tag{5.4}$$

Observe then that $\alpha \leqslant (k-s)(\sqrt{2k}/2 - 1) \leqslant n - k - s(\sqrt{2k}/2 - 1)$.

Consequently,

$$n' \geqslant k + s\left(\frac{\sqrt{2k}}{2} - 1\right), \quad t' \geqslant n' - k + 2 \geqslant s\left(\frac{\sqrt{2k}}{2} - 1\right),$$

$$\frac{n'+t'}{2} \geqslant s\left(\frac{\sqrt{2k}}{2} - 1\right) + 1 + \frac{k}{2}. \tag{5.5}$$

Hence to guarantee (5.4) it is sufficient to take $s = \frac{\sqrt{2k}}{2} + 1$. Thus, given $t' = t - \alpha$ there exists $s \leqslant \frac{\sqrt{2k}}{2} + 1$ so that (5.4) holds.

Observe now that $w_1 \geqslant \frac{\sqrt{2k}}{2} \geqslant s - 2$. This together with $w_2 + \cdots + w_s = n' - k + s - w_1$ and $t' > n' - k + 2$ implies that $t' > w_2 + \cdots + w_s$.

Finally applying Lemma 1.7 and Corollary 1.1 we get

$$|\mathcal{A}| \leqslant g(w_1, \ldots, w_s, 1, \ldots, 1;\ t') \leqslant m(n', k, t') = \sum_{i=k-1-\frac{n-t}{2}}^{k-1} \binom{k-1}{i}.$$

We are now prepared to apply induction.

Let $n \geqslant \frac{k\sqrt{2k}}{2} + 1$ and $t > n - k + 2$. Then the value $\alpha = 1$ satisfies inequality (5.4). Since $|\mathcal{A}| \leqslant f(n, k, t) \leqslant f(n-1, k, t-1)$, the induction hypothesis gives

$$|\mathcal{A}| \leqslant f(n-1, k, t-1) = m(n-1, k, t-1) = m(n, k, t).$$

This completes the proof of Theorem 1.6. □

6. Diametric problems

The Hamming distance between two vectors $x^n = (x_1, \ldots, x_n)$, $y^n = (y_1, \ldots, y_n) \in E(n)$ is defined by $d_H(x^n, y^n) = |\{i \in [n] : x_i \neq y_i\}|$. The diameter of a set $A \subset E(n)$ is defined by $\operatorname{diam}(A) = \max_{x^n, y^n \in A} d_H(x^n, y^n)$.

Kleitman proved the following

Theorem Kl (*Kleitman [17]*). *For a set $A \subset E(n)$ with* $\operatorname{diam}(A) = \delta < n$ *one has*

$$\max_{A \subset E(n)} |A| = \begin{cases} \sum_{i=0}^{\delta/2} \binom{n}{i} & \text{if } 2 \mid \delta, \\ 2 \sum_{i=0}^{(\delta-1)/2} \binom{n-1}{i} & \text{if } 2 \nmid \delta. \end{cases} \tag{6.1}$$

The diametric problem for n-sequences over any q-ary alphabet is solved in [7].

In [4] it was shown that the intersection and diametric problems are equivalent, that is Theorems Ka and Kl can be reduced to each other.

Consider now the *diametric problem under dimension constraint*.

Define

$$D_\delta(n, k) = \max\{|A| : A \subset E(n), \operatorname{diam}(A) = \delta, \dim(A) = k < n\}.$$

For $n = k$ by Theorem Kl we readily have $D_\delta(n, n) = \text{RHS (6.1)}$. A simple observation shows that Theorem Kl gives the answer for any $n \geqslant k$.

Theorem 1.7.

$$D_\delta(n, k) = \begin{cases} 2^k & \text{if } k \leqslant \delta \leqslant n, \\ D_\delta(k, k) & \text{if } \delta < k. \end{cases} \tag{6.2}$$

Proof. Let $A \subset E(n)$ with $\operatorname{diam}(A) = \delta$ and $\dim(A) = k$. One needs only to note that there exist $n-k$ coordinates $i_1, \dots, i_{n-k} \in [n]$, such that deleting them in all vectors of A we get a new set of vectors $A' \subset E(k)$ with $\dim(A') = k$, $|A'| = |A|$ and $\operatorname{diam}(A') \leqslant \delta$. $\square$

Thus the intersection and diametric problems under dimension constraint are not equivalent!

Let us consider also the following *weight diametric problem* in $F(w_1, \dots, w_k)$.

The diameter δ of a set $B \subset F(w_1, \dots, w_k)$ is defined by $\delta(B) = \max\limits_{x^k, y^k \in B} \operatorname{dist}(x^k, y^k)$.

Given n, k, δ define

$$f^*(n,k,\delta) = \max_{\substack{F(w_1,\dots,w_k)\\ \sum_{i=1}^k w_i = n}} \left\{|B| : \delta(B) = \delta,\ B \subset F(w_1, \dots, w_k)\right\}.$$

Given k, δ and $F(w_1, \dots, w_k)$ define also the function

$$g^*(w_1, \dots, w_k; \delta) = \max\left\{|B| : \delta(B) = \delta,\ B \subset F(w_1, \dots, w_k)\right\}.$$

Then we have the following

Lemma 1.9.

(i) $f(n,k,t) = f^*(n,k,n-t)$.

(ii) $g(w_1, \dots, w_k; t) = g^*(w_1, \dots, w_k; n-t)$ $\left(n \triangleq \sum_{i=1}^k w_i\right)$.

Proof. We mentioned above that the case $n = k$ was proved in [4]. The idea of the proof works also in our case and the reader can prove the statement repeating all the steps. $\square$

Part II. The restricted case

Our main result in this part is

Theorem 2.1.

$$J(n,k,\omega) = \begin{cases} M(n-1,k,\omega-1) & \\ \quad = \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2} & \textit{if } \text{(i) } k < 2\omega \leqslant 2(k-1);\ \omega \leqslant \frac{n}{2}, \\ M(n-1,k,\omega-1) = 2^{k-1} & \textit{if } \text{(ii) } k \leqslant \omega \leqslant \frac{n}{2}, \\ M(n,k,\omega) & \textit{if } \text{(iii) } \omega > \frac{n}{2}. \end{cases}$$

Note that Theorem 2.1 does not cover the case $k \geqslant 2\omega$.

Besides the cases in Theorem 2.1 we establish Conjecture 1 for k sufficiently large. In this case we have a more general result for t-intersecting systems.

Theorem 2.2. *For* $1 \leqslant t \leqslant \omega$ *and* $k \geqslant k_o(\omega, t)$

$$J_t(n,k,\omega) = \begin{cases} M(n-t,k,\omega-t) = \binom{k}{\omega-t} & \text{if } k \leqslant n-t, \\ \binom{n-t}{\omega-t} & \text{if } k > n-t. \end{cases}$$

To prove these theorems we use several auxiliary results derived in Sections 7 and 8 and also results and tools from [2]. However, the main auxiliary result is a LYM-type inequality proved in Section 7. It should be also noted that the shifting technique used in Part 1 does not seem to work here.

7. Main auxiliary result

Recall the notion of a chain and antichain for set systems, (translated into the language of (0, 1)-vectors). $A \subset E(n)$ is called a chain of length $|A|$ if $a^n \geqslant b^n$ or $a^n \leqslant b^n$ holds for all $a^n, b^n \in A$ (here we mean the componentwise inequality, which corresponds to an inclusion for the corresponding sets). A chain of length $n+1$ is called maximal. Also $A \subset E(n)$ is an antichain if it contains no chain of length two.

Given $a_1, \dots, a_n, \lambda \in \mathbb{R}^+$ let $X \subset E(n)$ be the (0, 1)-solutions of the equation

$$\sum_{i=1}^{n} a_i x_i = \lambda. \tag{7.1}$$

Clearly for any such Eq. (7.1) the set of solutions X corresponds to some antichain (whereas the opposite is not true).

Recall now the well known LYM inequality (see e.g. [10]), which says that for any antichain $A \subset E(n)$ (in particular for X)

$$\sum_{v^n \in A} \frac{1}{\binom{n}{\|v^n\|}} \leqslant 1 \qquad \text{(LYM inequality)}, \tag{7.2}$$

where $\|v^n\|$ denotes the number of 1's in v^n.

Equality in (7.2) holds iff $A = E(n,i)$ for some $i \in [n]$. For the solutions of (7.1) this means that $a_1 = a_2 = \cdots = a_n$. What can we say in the case when not all a_i's are equal? Can we improve (7.2) in this case?

Define $\alpha_i \triangleq |\{x^n \in X : \|x^n\| = i\}|$, that is $\alpha_i = |X \cap E(n,i)|$.

Lemma 2.1 (*LYM-type inequality for equations*). *Assume in* (7.1) $a_i \neq a_j$ *for some* $i, j \in [n]$, *and* $\sum_{i=1}^n a_i \neq \lambda$. *Then*

$$\sum_{x^n \in X} \frac{1}{\binom{n}{\|x^n\|}} \leqslant \frac{n-1}{n} \tag{7.3}$$

or equivalently

$$\sum_{i=1}^{n} \frac{\alpha_i}{\binom{n}{i}} \leqslant \frac{n-1}{n}.$$

Proof. W.l.o.g. let $a_1 > a_2$. Let $\mathcal{C}_n$ be the set of maximal chains in $E(n)$, and let $\mathcal{C}_n^*$ be the set of maximal chains, which do not meet any member of X, that is the elements (maximal chains) of $\mathcal{C}_n$ which do not contain a solution of (7.1).

We claim that

$$|\mathcal{C}_n^*| \geqslant (n-1)! \tag{7.4}$$

and proceed by induction on $n \geqslant 2$.

Induction beginning: $n = 2$. We have $a_1 \neq a_2$ and $a_1 + a_2 \neq \lambda$. Clearly there exists at most one solution of (7.1): 10 or 01, since 00 and 11 are not solutions. Hence $|\mathcal{C}_2^*| \geqslant 1$ (since either $\{00, 10, 11\}$ or $\{00, 01, 11\} \in \mathcal{C}_2^*$.

Induction step: $n \to n+1$. Partition $\mathcal{C}_{n+1}$ into $(n-1)!$ "equivalent" classes $\mathcal{S}_1, \ldots, \mathcal{S}_{(n-1)!}$, with $|\mathcal{S}_i| = n(n+1)$; $(i = 1, \ldots, (n-1)!)$ in the following way. Let A be a maximal chain in $E(n+1)$, i.e. $|A| = n+2$.

Denote by A_0 the set of all vectors obtained from A by deletion of the first two coordinates. Clearly $|A_0| = n$; moreover A_0 is a maximal chain in $E(n-1)$. We call A_0 the kernel of A. Consider now the set of all maximal chains in $E(n+1)$, which have a given kernel. There are $n(n+1)$ such maximal chains which we join into one class of maximal chains $\mathcal{S}_i$. There are $(n-1)!$ distinct kernels, so we get a partition of $\mathcal{C}_{n+1}$ into $(n-1)!$ classes $\mathcal{S}_1, \ldots, \mathcal{S}_{(n-1)!}$. We call them equivalent because the property of a class we are going to prove does not depend on the choice of a class. Note that to prove claim (7.4) it suffices to show that each class $\mathcal{S}_i$ contains at least n "forbidden" chains, i.e. chains from $\mathcal{C}_{n+1}^*$.

This was shown to be true for $n = 2$ where we have only one class $\mathcal{S}_1$ consisting of two maximal chains and the kernel $A_0 = \varnothing$. Thus we proceed assuming that this property holds for the partition of $\mathcal{C}_n$ into $(n-2)!$ equivalent classes. For convenience here we represent each maximal chain $A = \{v_1^n, \ldots, v_{n+1}^n\} \subset E(n)$, with $\|v_i^n\| = i - 1$ $(i = 1, \ldots, n+1)$, by the $(n+1) \times n$ array with the vector v_i^n as its ith row. W.l.o.g. let $\mathcal{S}_1$ be the class with the following kernel:

$$\left.\begin{array}{ccccccc}
x_3 & x_4 & & \cdots & & x_{n+1} \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 \\
. & . & . & \cdots & . & . \\
1 & 1 & 1 & \cdots & 1 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1
\end{array}\right\} (n-1)\text{-chain.}$$

Define the subclass $\mathcal{S}_1' \subset \mathcal{S}_1$ by $\mathcal{S}_1' = \{A \in \mathcal{S}_1 : x_{n+1} = 0\ \forall (x_1, \ldots, x_{n+1}) \in (A \setminus \{1^{n+1}\})\}$, where 1^{n+1} is the all one vector. Note that $|\mathcal{S}_1'| = n(n-1)$. Clearly deleting the last row and the last column in any member of $\mathcal{S}_1'$ we get a maximal chain from $\mathcal{C}_n$.

We distinguish between two cases.

Case 1: $\sum_{i=1}^n a_i \neq \lambda$, i.e. $(11\ldots10) \notin X$. In this case, we can apply the induction hypothesis to $\mathcal{S}_1'$ (more precisely to the restriction of $\mathcal{S}_1'$ on coordinates $x_1, \ldots, x_n$) considered as a subclass of $\mathcal{C}_n$. By induction hypothesis $\mathcal{S}_1'$ contains at least $n-1$ elements from $\mathcal{C}_{n+1}^*$. Let us show that $\mathcal{S}_1$ contains one more forbidden chain $B \in \mathcal{S}_1 \setminus \mathcal{S}_1'$ $(B \in \mathcal{C}_{n+1}^*)$.

Subcase (a): For some $1 \leqslant t \leqslant n$ there exists
$x^n = (x_1, x_2, \dots, x_{t+1}, \dots, x_n) = (0\underbrace{1\dots1}_{t}0\dots0) \in X$.

Since $a_1 > a_2$, it is not hard to see that the following chain $B \in \mathcal{S}_1 \setminus \mathcal{S}_1'$:

$$B = \begin{array}{cc|cccccccc}
x_1 & x_2 & x_3 & & \cdots & x_t & & \cdots & x_{n+1} \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
. & . & . & . & \cdots & . & . & \cdots & . \\
0 & 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
. & . & . & . & \cdots & . & . & \cdots & . \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1
\end{array}$$

is from $\mathcal{C}^*_{n+1}$.

Subcase (b): For every $1 \leqslant t \leqslant n$

$$x^{n+1} = (x_1, x_2, \dots, x_{t+1}, \dots, x_{n+1}) = (0\underbrace{1\dots1}_{t}0\dots0) \notin X.$$

Then the following chain $B \in \mathcal{S}_1 \setminus \mathcal{S}_1'$:

$$B = \begin{array}{cc|cccc}
x_1 & x_2 & x_3 & & \cdots & x_{n+1} \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
. & . & . & . & \cdots & \\
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & 1
\end{array}$$

does the work, i.e. $B \in \mathcal{C}^*_{n+1}$.

Case 2: $\sum_{i=1}^n a_i = \lambda$, i.e. $(11\dots10) \in X$. In this case, we cannot use the induction hypothesis, but now we will describe a direct construction of at least n forbidden maximal chains.

Consider the following vectors:

$$\begin{array}{cc|ccc}
x_1 & x_2 & & \cdots & x_{n+1} \\
\hline
0 & 0 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1.
\end{array} \tag{7.5}$$

Clearly we can have at most one vector out of these three as a solution of our Eq. (7.1), since $a_1 \neq a_2$.

Subcase (a): None of the vectors from (7.5) is in X.

Consider then the maximal chains $A_1, \dots, A_n \in \mathcal{S}_1 \setminus \mathcal{S}_1'$ shown below. Note that in each A_i $(i = 1, \dots, n)$ the first coordinate x_1 gets the value $x_1 = 1$ first time in the $(n+2-i)$th member (row) of A_i.

$$A_1 = \begin{array}{cc|cccc} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{array} \qquad A_2 = \begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array} \quad \cdots \quad A_n = \begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & & \cdot \\ 1 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array}$$

Consider also the maximal chains $B_1, \dots, B_n \in \mathcal{S}_1 \setminus \mathcal{S}_1'$, where B_i $(i = 1, \dots, n)$ is obtained from A_i by transposition of the first two coordinates.

$$B_1 = \begin{array}{cc|cccc} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{array} \qquad B_2 = \begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array} \quad \cdots \quad B_n = \begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array}$$

Observe now that all $2n$ maximal chains defined above are from $\mathcal{C}_{n+1}^*$. This is clear since all vectors contained in $A_1, \dots, A_n, B_1, \dots, B_n$, except of those which are from chain (7.5), are "covered" by the vector $(1 \dots 10)$.

Subcase (b): $(101 \dots 1) \in X$.

Then the chains $A_1, \dots, A_n$ are forbidden.

Symmetrically if $(011 \dots 1) \in X$, then $B_1, \dots, B_n$ are forbidden.

Subcase (c): $(001 \dots 1) \in X$.

Then except for the A_1 and B_1 all $2(n-1)$ remaining maximal chains are forbidden.

Thus we have proved that in $\mathcal{S}_1$ there are at least n maximal chains from $\mathcal{C}_{n+1}^*$. Note also that all our arguments in this proof did not depend on the choice of an equivalent class $\mathcal{S}_i$, $i = 1, \dots, (n-1)!$.

This means that for given n the total number of forbidden chains $|\mathcal{C}_n^*| \geqslant (n-1)(n-2)!$, completing the proof of claim (7.4).

Since $|\mathcal{C}_n| = n!$, the number of maximal chains containing elements from X (solutions of Eq. (7.1))

$$|\mathcal{C}_n| - |\mathcal{C}_n^*| \leqslant n! - (n-1)! = (n-1)(n-1)!$$

On the other hand, there are exactly $i!(n-i)!$ maximal chains containing a given vector x^n with $\|x^n\| = i$, and each maximal chain contains at most one element from X. Therefore, we have

$$\sum_{i=1}^{n} \alpha_i \cdot i(n-i)! \leqslant (n-1)(n-1)!,$$

or equivalently

$$\sum_{i=1}^{n} \frac{\alpha_i}{\binom{n}{i}} \leqslant \frac{n-1}{n}.$$

Clearly Lemma 2.1 implies. □

Corollary 2.1. *Under the hypothesis of Lemma* 1

$$|X| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{n-1}{n}. \tag{7.6}$$

A refined version of Corollary 2.1 is the following.

Lemma 2.1*. *Assume w.l.o.g. that* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n = 1$, *and let* $N(n)$ *be the maximum number of* $(0, 1)$*-solutions of the equation*

$$a_1 x_1 + \cdots + a_n x_n = \lambda \tag{7.7}$$

among all choices of $a_1, \ldots, a_n, \lambda \in \mathbb{R}^+$ *with* $a_i \neq a_j$ *for some* $i, j \in [n]$. *Then*

(i) $$N(n) = \begin{cases} \binom{n}{\frac{n-2}{2}} & \text{if } 2 \mid n, \\ 2\binom{n-1}{\frac{n-3}{2}} & \text{if } 2 \nmid n, n \geqslant 3. \end{cases} \tag{7.8}$$

(ii) *Bound* (7.8) *is attained if and only if*

$a_1 = 2, a_2 = \cdots = a_n = 1;\ \lambda = \begin{cases} \frac{n}{2} \text{ or } \frac{n+2}{2} & \text{if } 2 \mid n, \\ \frac{n+1}{2} & \text{if } 2 \nmid n, \end{cases}$

or

$n \in \{3, 4\}, a_1 = a_2 > 1, a_3 = a_n = 1, \lambda = a + 1.$

Proof. Taking $a_1 = 2, a_2 = \cdots = a_n = 1, \lambda = \lceil \frac{n+1}{2} \rceil$ we see that the RHS of (7.8) is a lower bound for $N(n)$.

Let $X \subset E(n)$ be the set of solutions of (7.7), so $|X| = N(n)$.

Case $2 \mid n$: Setting $n = 2\ell$, by Lemma 2.1 we have

$$\sum_{i=0}^{2\ell} \frac{\alpha_i}{\binom{2\ell}{i}} \leqslant \frac{2\ell - 1}{2\ell}. \tag{7.9}$$

Let us first estimate α_ℓ, that is the size of the set of solutions $X' \subset X$ of equations

$$\sum_{i=1}^{n} a_i x_i = \lambda,$$
$$\sum_{i=1}^{n} x_i = \ell. \tag{7.10}$$

Claim. $\alpha_\ell \leqslant \binom{2\ell-1}{\ell-1}$, *for* $\ell > 1$.

Proof. Since $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n = 1$, Eq. (7.10) can be transformed as following:

$$\begin{aligned} b_1x_1 + \cdots + b_mx_m &= \lambda', \\ x_1 + \cdots + x_n &= \ell, \end{aligned} \tag{7.11}$$

where $b_i = a_i - 1$, $i = 1, \ldots, m$, $\lambda' = \lambda - \ell$, $1 \leqslant m \leqslant n-1$. Suppose first, there are two distinct elements among $\{b_1, \ldots, b_m\}$. Then by Corollary 2.2 the number of solutions of the first equation of (7.11) is upper bounded by $\binom{m}{\lfloor \frac{m}{2} \rfloor}\frac{m-1}{m}$. This clearly implies that

$$|X'| \leqslant \frac{m-1}{m}\binom{m}{\lfloor \frac{m}{2} \rfloor}\binom{n-m}{\lfloor \frac{n-m}{2} \rfloor}.$$

It can be verified that $\frac{m-1}{m}\binom{m}{\lfloor \frac{m}{2} \rfloor}\binom{n-m}{\lfloor \frac{n-m}{2} \rfloor} \leqslant \max_{1 \leqslant m \leqslant n-1} \frac{m-1}{m}\binom{m}{\lfloor \frac{m}{2} \rfloor}\binom{n-m}{\lfloor \frac{n-m}{2} \rfloor} = \frac{2\ell-3}{2\ell-2}\binom{2\ell-2}{\ell-1}\binom{2}{1}$ $< \binom{2\ell-1}{\ell-1}$.

Let now $b_1 = \cdots = b_m$, which means that $a_1 = \cdots = a_m \triangleq a > 1$, $a_{m+1} = \cdots = a_n = 1$.

To complete the proof of the claim we need the following simple facts.

Fact 1. Given integers $1 \leqslant m \leqslant n-1$ we have

(a) $$\binom{m}{\lfloor \frac{m}{2} \rfloor}\binom{n-m}{\lfloor \frac{n-m}{2} \rfloor} > \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}, \tag{7.12}$$

if and only if $2 \mid n$, $m = 2$ $(m = n-2)$, or $n = 2m = 8$.

(b) $$\binom{m}{\lfloor \frac{n}{2} \rfloor}\binom{n-m}{\lfloor \frac{n-m}{2} \rfloor} = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}, \tag{7.13}$$

if and only if $2 \nmid n$, $m = 2$ $(m = n-2)$, or $m = 1$ $(m = n-1)$.

Fact 2. Given integer $\ell \geqslant 2$ we have

$$2\binom{2\ell-2}{\ell-1} + 2\binom{2\ell-2}{\ell-3} \begin{cases} < \binom{2\ell}{\ell-1} & \text{if } \ell > 2, \\ = \binom{2\ell}{\ell-1} & \text{if } \ell = 2. \end{cases}$$

Clearly (7.12) and (7.13) imply that to prove the claim it suffices to consider the cases $m = 2$, $m = n-2$ and $n = 2m = 8$.

Suppose $m = 2$. Then (7.11) can be written as

$$\begin{aligned} (a-1)x_1 + (a-1)x_2 &= \lambda - \ell, \\ x_1 + \cdots + x_n &= \ell. \end{aligned} \tag{7.14}$$

Note that in order that $|X'| = \alpha_\ell > \binom{2\ell-1}{\ell-1}$ holds, the first equation of (7.14) must have two solutions (otherwise $|X'| \leqslant \binom{2\ell-2}{\ell-1}$).

Hence if $|X'| > \binom{2\ell-1}{\ell-1}$, then we must have $\lambda - \ell = a - 1$. This means that Eq. (7.7) is of the form

$$ax_1 + ax_2 + x_3 + \cdots + x_n = \ell + a - 1.$$

Now an easy calculation shows that

$$|X| \leqslant 2\binom{2\ell-2}{\ell-1} + \binom{2\ell-2}{\ell-1+a} + \binom{2\ell-2}{\ell-1-a}$$
$$= 2\left\{\binom{2\ell-2}{\ell-1} + \binom{2\ell-2}{\ell-1-a}\right\} \leqslant 2\left\{\binom{2\ell-2}{\ell-1} + \binom{2\ell-2}{\ell-3}\right\}.$$

In view of Fact 2, we now conclude that for $\ell > 2$

$$|X| < \binom{2\ell}{\ell-1} \leqslant N(n),$$

a contradiction with the assumption $|X| = N(n)$.

In the case $\ell = 2$ we have $|X| = \binom{2\ell}{\ell-1} = 4$, moreover this can be achieved for any positive $a \neq 1$ and $\lambda = a+1$. Similarly observe that for the case $m = n-2$ we have

$$|X| \leqslant 2\binom{2\ell-2}{\ell-1} < \binom{2\ell}{\ell-1}, \text{ if } \ell > 2.$$

The same can be shown for $n = 2m = 8$. This completes the proof of the claim. □

Let us rewrite (7.9) as

$$\sum_{i=1}^{\ell-1} \frac{\alpha_i}{\binom{2\ell}{i}} + \sum_{j=\ell+1}^{2\ell} \frac{\alpha_j}{\binom{2\ell}{j}} \leqslant \frac{2\ell-1}{2\ell} - \frac{\alpha_\ell}{\binom{2\ell}{\ell}}.$$

This clearly implies

$$\sum_{i=1}^{\ell-1} \alpha_i + \sum_{j=\ell+1}^{2\ell} \alpha_j \leqslant \frac{2\ell-1}{2\ell}\binom{2\ell}{\ell-1} - \alpha_\ell \frac{\binom{2\ell}{\ell-1}}{\binom{2\ell}{\ell}}.$$

Hence

$$\sum_{i=1}^{2\ell} \alpha_i \leqslant \frac{2\ell-1}{2\ell}\binom{2\ell}{\ell-1} + \alpha_\ell - \alpha_\ell \frac{\binom{2\ell}{\ell-1}}{\binom{2\ell}{\ell}} = \frac{2\ell-1}{2\ell}\binom{2\ell}{\ell-1} + \frac{1}{\ell+1}\alpha_\ell.$$

Since $\alpha_\ell \leqslant \binom{2\ell-1}{\ell-1}$ (for $\ell > 2$) we get

$$\sum_{i=1}^{2\ell} \alpha_i \leqslant \frac{2\ell-1}{2\ell}\binom{2\ell}{\ell-1} + \frac{1}{\ell+1}\binom{2\ell-1}{\ell-1} = \binom{2\ell}{\ell-1} = \binom{n}{\frac{n-2}{2}}. \tag{7.15}$$

Note that (7.15) implies that $|X| < \binom{2\ell}{\ell-1}$ if $\alpha_\ell < \binom{2\ell-1}{\ell-1}$. The latter, in view of the claim, means that except for the case $\ell = 2$ one has $|X| = \binom{2\ell}{\ell-1}$ only if $\alpha_\ell = \binom{2\ell-1}{\ell-1}$. The observation in the proof of the claim shows that $\alpha_\ell = \binom{2\ell-1}{\ell-1}$ if and only if $m = 1$; $a_1 > 1$, $a_2 = \cdots = a_n = 1$, or $m = n-1$; $a_1 = \cdots = a_{n-1} > 1$, $a_n = 1$.

Observe further that the case $m = n - 1$ is excluded, since otherwise $|X| \leqslant \binom{2\ell-1}{\ell-1}$, a contradiction.

Finally, observe that the equation $ax_1 + x_2 + \cdots + x_n = \lambda$ has $\binom{2\ell}{\ell-1}$ solutions from $E(n)$ if and only if $a = 2$, $\lambda = \ell + 1$. This completes the proof of the case $2 \mid n$.

Case $2 \nmid n$: The upper bound (7.8) directly follows from (7.6).

The part (ii) for this case can be easily derived, proceeding along the same lines as for the even case. □

Remark 2.1. In fact equality (7.8) gives the second biggest size for the (0, 1)-solutions of equation (7.1). We emphasize that this is not true for antichains in general, i.e. the second biggest size of an antichain can exceed the RHS of (7.8).

8. Further preparations

Consider a system of $n - k$ independent equations

$$\langle v_i^n, x^n \rangle = 0; \ i = 1, \ldots, n-k \tag{8.1}$$

where $v_1^n, \ldots, v_{n-k}^n \in \mathbb{R}^n$ ($\langle \cdot, \cdot \rangle$ means the standard inner product).

Consider only the solutions of (8.1) which are in $E(n, \omega)$. That is consider the set X of all solutions of the system

$$\begin{cases} \langle v_i^n, x^n \rangle = 0; \ i = 1, \ldots, n-k \\ \langle 1^n, x^n \rangle = \omega. \end{cases} \tag{8.2}$$

where $x^n \in \{0, 1\}^n$ and 1^n is the all-one vector.

In view of Lemma 1.1, system (8.2) can be brought to a form

$$\langle u_i^n, x^n \rangle = c_i; \ i = 1, \ldots, n-k+1, \tag{8.3}$$

where the matrix of coefficients has a positive step form with the step sizes $\ell_1 \geqslant \cdots \geqslant \ell_{n-k+1} \geqslant 1$.

Lemma 2.2 (*Ahlswede et al. [2]*). *For the set X of solutions of* (8.2) *we have*

$$|X| \leqslant \max_{\Sigma \omega_i = \omega} \prod_{i=1}^{n-k+1} \binom{\ell_i}{\omega_i} \leqslant M(n, k, \omega). \tag{8.4}$$

Lemma 2.3. *Let $2 \leqslant \ell_1 \leqslant 2k - 2\omega - 2$ and $\omega < k < 2\omega \leqslant n$. Then for the set X of solutions of* (8.2) *we have*

$$|X| \leqslant \begin{cases} \binom{2k-2\omega-2}{k-\omega-2} 2^{2\omega-k+2} & \text{if } k > \omega + 3, \\ \binom{4}{2}^2 2^{\omega-4} & \text{if } k = \omega + 3, \\ 2^{\omega} & \text{if } k = \omega + 2 \end{cases} \tag{8.5}$$

and equality holds if and only if $\ell_1 = 2k - 2\omega - 2$ or $\ell_1 = 2k - 2\omega - 3$.

Proof. The proof is rather elementary although somewhat tedious and requires a step-by-step verification of several inequalities.

First, we proof that the maximum is attained when $\ell_1 = 2k - 2\omega - 2$ or $2k - 2\omega - 3$. The proof is based on the following inequalities which can be easily verified.

(i) For $\ell > 2\omega$; $\omega \geqslant 1$, $r > 1$

$$\binom{\ell}{\omega}\binom{2r}{r} < \binom{\ell + 2r - 1}{\omega + r}. \tag{8.6}$$

(ii) For $\ell \geqslant s \geqslant 2$

$$\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{s}{\lfloor \frac{s}{2} \rfloor} < \binom{\ell + 2}{\lfloor \frac{\ell}{2} \rfloor + 1}\binom{s - 2}{\lfloor \frac{s}{2} \rfloor - 1}. \tag{8.7}$$

(iii) For $\ell > 2\omega + 1$

$$\binom{\ell}{\omega} 2 < \binom{\ell + 1}{\omega + 1}. \tag{8.8}$$

In view of Lemma 2.2 for given ℓ_1

$$|X| \leqslant \binom{\ell_1}{\omega_1} M(n - \ell_1, k - \ell_1 + 1, \omega - \omega_1) \text{ for some } 1 \leqslant \omega_1 \leqslant \frac{\ell_1}{2}.$$

Now notice that the expression for $M(n, k, \omega)$ in Theorem AAK is always of the form $\binom{2r}{r}2^t$ for suitable parameters. Therefore, we can write now

$$|X| \leqslant \binom{\ell_1}{\omega_1}\binom{2r}{r}2^t, \tag{8.9}$$

where $\ell_1 + 2r + t - 1 = k$ and $\omega_1 + r + t \leqslant \omega$.

Suppose now X has maximum cardinality, then in view of (8.6) we have $\ell_1 + 2r \geqslant 2k - 2\omega$.

Suppose then that $\ell_1 < 2k - 2\omega - 3$. Then we can see from Theorem AAK that $t \geqslant 1$. This and (8.8) yield $\omega_1 = \lfloor \frac{\ell}{2} \rfloor$.

But in this case we get a contradiction with (8.7) and the assumption that X is a maximal set. Thus $\ell_1 = 2k - 2\omega - 2$ or $\ell_1 = 2k - 2\omega - 3$. Denote the RHS of (8.9) by $f(\ell_1)$. Note further that if $\ell_1 = 2k - 2\omega - 3$ then necessarily $\omega_1 = k - \omega - 1$ by maximality of X and (8.8) and therefore $f(2k - 2\omega - 3) \leqslant f(2k - 2\omega - 2)$. Thus, we have

$$|X| = \binom{2k - 2\omega - 2}{\omega_1} M(n', k', \omega'),$$

where $n' = n - 2k + 2\omega + 2$, $k' = 2\omega - k + 3$, $\omega' = \omega - \omega_1$.

Observe also that $n' \geqslant 2\omega'$ and $k' \leqslant 2\omega'$. Moreover, since $\omega_1 \leqslant k - \omega - 1$ we have $\omega' \geqslant 2\omega - k + 1$. We can now apply Theorem AAK to determine $M(n', k', \omega')$.

We distinguish between two cases:

(a) $\omega' = 2\omega - k + 1$.
Then $M(n', k', \omega') = \binom{4}{2} 2^{2\omega-k-1}$. Furthermore clearly $\omega_1 = k - \omega - 1$ and hence

$$|X| = \binom{2k-2\omega-2}{k-\omega-1}\binom{4}{2} 2^{2\omega-k-1}. \tag{8.10}$$

(b) $\omega' \geqslant 2\omega - k + 2 \geqslant k' - 1$.
Then $M(n', k', \omega') = 2^{2\omega-k+2}$, which implies that $\omega_1 = k - \omega - 2$ and hence

$$|X| = \binom{2k-2\omega-2}{k-\omega-2} 2^{2\omega-k+2}. \tag{8.11}$$

Comparing now the RHS of (8.10) with the RHS of (8.11) we get:

(1) For $k > \omega + 4$ RHS (8.10) < RHS (8.11)
(2) For $\omega < k < \omega + 4$ RHS (8.10) > RHS (8.11)
(3) For $k = \omega + 4$ RHS (8.10) = RHS (8.11)

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $\ell_1 \geqslant 2k - 2\omega + 1$.*
Then for the set X of solutions of (8.2) *we have*

$$|X| \leqslant \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2} \tag{8.12}$$

and equality holds iff $\ell_1 = 2k - 2\omega + 1$ or $2k - 2\omega + 2$.

Proof. For $\ell_1 \geqslant 2k - 2\omega + 2$ clearly we have

$$|X| \leqslant \max_{0 \leqslant \omega_1 \leqslant \frac{\ell_1}{2}} \binom{\ell_1}{\omega_1} M(n - \ell_1, k - \ell_1 + 1, \omega - \omega_1).$$

Since $M(n - \ell_1, k - \ell_1 + 1, \omega - \omega_1) \leqslant 2^{k-\ell_1}$, we get

$$|X| \leqslant \max_{0 \leqslant \omega_1 \leqslant \frac{\ell_1}{2}} \binom{\ell_1}{\omega_1} 2^{k-\ell_1} = \binom{\ell_1}{\lfloor \frac{\ell_1}{2} \rfloor} 2^{k-\ell_1}.$$

Suppose $2 \mid \ell_1$, then we have

$$|X| \leqslant \max_{1 \leqslant i \leqslant \omega-k} \binom{2k-2\omega+2i}{k-\omega+i} 2^{2\omega-k-2i}. \tag{8.13}$$

But the function in RHS of (8.13) is strictly decreasing with respect to i. This simple fact together with the identity $\binom{\ell_1}{\frac{\ell_1}{2}} 2^{k-\ell_1} = \binom{\ell_1-1}{\frac{\ell_1}{2}-1} 2^{k-\ell_1+1}$ implies that $|X|$ is bounded from above by the RHS of (8.12). On the other hand, one can observe that this bound is attainable if (and by the statement above only if) $\ell_1 = 2k - 2\omega + 1$ or $2k - 2\omega + 2$. This completes the proof. □

Our next lemma combines the two previous ones.

Lemma 2.5. *Let $\omega < k < 2\omega \leqslant n$ and let $\ell_1 \neq 2k - 2\omega, 2k - 2\omega - 1$. Then we have*

(i) $|X| \leqslant \binom{4}{2}^2 2^{\omega-4}$ *if $k = \omega + 3$; $\omega \geqslant 4$*
and equality holds iff $\ell_1 = 4$ or 3

(ii) $|X| \leqslant \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2}$
and equality holds iff $\ell_1 = 2k - 2\omega + 1$ or $2k - 2\omega + 2$.

Proof. One has only to compare the bounds in Lemmas 2.3 and 2.4. □

For our purposes we also need the following sharpening of Lemma 2.2 [2] in a special case.

Lemma 2.6. *Let $X \subset E(n)$ be the set of solutions of Eq.* (8.3) *given in a positive step form with step sizes $\ell_1, \dots, \ell_{n-k+1} \geqslant 1$. Let also the rth step have two distinct entries. Then*

$$|X| \leqslant \frac{\ell_r - 1}{\ell_r} \max_{\Sigma \omega_i = \omega} \prod_{i=1}^{n-k+1} \binom{\ell_i}{\omega_i} \leqslant \frac{\ell_r - 1}{\ell_r} M(n, k, \omega). \tag{8.14}$$

Consider the partition of the coordinate set $[n] = N_1 \cup \dots \cup N_{n-k+1}$, with $N_i = \left[\sum_{j=1}^{i-1} \ell_j + 1, \ell_i\right]$, and let us write each vector $x^n \in X$ as $x^n = (x^{\ell_1}, x^{\ell_2}, \dots, x^{\ell_{n-k+1}})$, where $x^{\ell_i} \in E(\ell_i)$ is the restriction of x^n on coordinate subset $N_i \subset [n]$, $i = 1, \dots, n - k + 1$.

To prove Lemma 2.6 we use the following (more general) version of Lemma 2.2.

Lemma 2.2′ [2]**.** *For the set X of solutions of* (8.3) *we have*

$$\sum_{(x^{\ell_1}, \dots, x^{\ell_{n-k+1}}) \in X} \frac{1}{\prod_{i=1}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}} \leqslant 1. \tag{8.15}$$

Proof of Lemma 2.6. Define $X_{i_1,\dots,i_s}$ to be the restriction of the vectors of X on the subset $N_{i_1} \cup \dots \cup N_{i_s}$ of the coordinate set $[n]$.

For $(a^{\ell_1}, \dots, a^{\ell_s}) \in X_{1,\dots,s}$ also define

$$X(a^{\ell_1}, \dots, a^{\ell_s}) = \{x^n = (x^{\ell_1}, \dots, x^{\ell_{n-k+1}}) : x^{\ell_i} = a^{\ell_i}, i = 1, \dots, s\}.$$

Let us first consider the case $r = 1$, that is suppose the first step has two distinct entries.

By Lemma 2.1, for the first equation of (8.3) we can write

$$\sum_{x^{\ell_1} \in X_1} \frac{1}{\binom{\ell_1}{\|x^{\ell_1}\|}} \leqslant \frac{\ell_1 - 1}{\ell_1}. \tag{8.16}$$

Also in view of (8.15), for each $a^{\ell_1} \in X_1$ we have

$$\sum_{(x^{\ell_1},\dots,x^{\ell_{n-k+1}})\in X(a^{\ell_1})} \frac{1}{\prod_{i=2}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}} \leqslant 1. \tag{8.17}$$

Combining (8.16) and (8.17) (namely multiplying each summand in LHS (8.16) by its corresponding sum written in LHS (8.17)) we get

$$\begin{aligned}\frac{\ell_1-1}{\ell_1} &\geqslant \sum_{x^{\ell_1}\in X_1} \sum_{(x^{\ell_1},\dots,x^{\ell_{n-k+1}})\in X(x^{\ell_1})} \frac{1}{\binom{\ell_1}{\|x^{\ell_1}\|}} \cdot \frac{1}{\prod_{i=2}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}} \\ &= \sum_{x^n\in X} \frac{1}{\prod_{i=1}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}}.\end{aligned} \tag{8.18}$$

Suppose now $r \geqslant 2$. Then in view of (8.18), for each $\underline{b} \in X_{1,\dots,r-1}$ we have

$$\sum_{(x^{\ell_1},\dots,x^{\ell_{n-k+1}})\in X(\underline{b})} \frac{1}{\prod_{i=r}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}} \leqslant \frac{\ell_r-1}{\ell_r}. \tag{8.19}$$

For the first $r-1$ equations we also have (by Lemma 2.2')

$$\sum_{(x^{\ell_1},\dots,x^{\ell_{r-1}})\in X_{1,\dots,r-1}} \frac{1}{\prod_{j=1}^{r-1} \binom{\ell_j}{\|x^{\ell_j}\|}} \leqslant 1. \tag{8.20}$$

Finally (8.19) and (8.20) imply

$$\begin{aligned}\frac{\ell_r-1}{\ell_r} &\geqslant \sum_{(x^{\ell_1},\dots,x^{\ell_{r-1}})\in X_{1,\dots,r-1}} \sum_{x^n\in X(x^{\ell_1},\dots,x^{\ell_{r-1}})} \frac{1}{\prod_{i=1}^{r-1} \binom{\ell_i}{\|x^{\ell_i}\|}} \cdot \frac{1}{\prod_{j=r}^{n-k+1} \binom{\ell_j}{\|x^{\ell_j}\|}} \\ &= \sum_{x^n\in X} \frac{1}{\prod_{i=1}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}} \geqslant \frac{|X|}{\max_{x^n\in X} \prod_{i=1}^{n-k+1} \binom{\ell_i}{\|x^{\ell_i}\|}}.\end{aligned} \tag{8.21}$$

In particular for $X \subset E(n,\omega)$ (8.21) implies

$$\frac{\ell_r-1}{\ell_r} \geqslant \frac{|X|}{\max_{\Sigma\omega_i=\omega} \prod_{i=1}^{n-k+1} \binom{\ell_i}{\omega_i}} \geqslant \frac{|X|}{M(n,k,\omega)}. \qquad \square$$

9. Proof of Theorem 2.1.

Case (iii): $\omega > \frac{n}{2}$.

This case is trivial, because $E(n,\omega)$ is intersecting.

Case (ii): $k \leqslant \omega \leqslant \frac{n}{2}$.

This case is also evident since $J(n,k,\omega) \leqslant M(n,k,\omega)$ and (by Theorem AAK) $M(n,k,\omega) = M(n-1,k,\omega-1) = 2^{k-1}$. Moreover, the family S_3 (with $|S_3| = M(n,k,\omega)$) in Theorem AAK is $(\omega - k + 1)$-intersecting.

Thus it remains to prove

Case (i): $k < 2\omega \leqslant 2(k-1)$.

Let $A \subset E(n,\omega)$ be an optimal intersecting family, that is $|A| = J(n,k,\omega)$.

The proof consists of two parts.

(1) First we show that

$$|A| \geqslant m \triangleq 2^{2\omega-k-2}\binom{2k-2\omega+2}{k-\omega+1}. \tag{9.1}$$

Consider the following three sets:

$$\begin{aligned}
A_1 &= E(2k-2\omega+2, k-\omega+1) \times \{01, 10\}^{2\omega-k-2} \times \{1\} \times \{0^{n-2\omega+1}\},\\
A_2 &= E(2k-2\omega+1, k-\omega) \times \{01, 10\}^{2\omega-k-1} \times \{1\} \times \{0^{n-2\omega}\},\\
A_3 &= E(2k-2\omega+1, k-\omega+1) \times \{01, 10\}^{2\omega-k-1} \times \{0^{n-2\omega+1}\}.
\end{aligned}$$

Observe now that

(a) $\dim(A_1) = \dim(A_2) = \dim(A_3) = k$
(b) $|A_1| = |A_2| = |A_3| = m$
(c) A_1, A_2 and A_3 are intersecting.

This clearly implies (9.1).

(2) Let us show now that $|A| \leqslant m$.

As we mentioned above, A can be viewed to be a subset of X, the set of solutions of a system of Eqs. (8.3). Rewrite now system (8.3) in the matrix form

$$H(x_1,\ldots,x_n)^T = (c_1,\ldots,c_{n-k+1})^T, \tag{9.2}$$

so that H has a positive step form with step sizes $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n-k+1}$.

Our aim is now to show that $\ell_1 = 2k-2\omega+1$ or $2k-2\omega+2$.

The proof consists of several observations on the structure of matrix H.

Claim 1. *Each step of H consists of equal elements.*

Proof. Suppose the *j*th step has two distinct elements. Then by Lemma 2.6 we get

$$\begin{aligned}
|X| &\leqslant \max_{\Sigma\omega_i=\omega} \prod_{i=1}^{n-k+1}\binom{\ell_i}{\omega_i}\frac{\ell_j-1}{\ell_j}\\
&\leqslant M(n,k,\omega)\frac{\ell_j-1}{\ell_j} < M(n,k,\omega)\frac{2k-2\omega+1}{2k-2\omega+2}\\
&= 2^{2\omega-k}\binom{2k-2\omega}{k-\omega}\frac{2k-2\omega+1}{2k-2\omega+2} = m,
\end{aligned}$$

a contradiction with $|A| \geqslant m$. $\square$

Thus w.l.o.g. we can assume that the entries of all "steps" consist of only ones.

Suppose now, for a contradiction, $|A| > m$.

Then Lemma 2.5 implies that the only possible values for ℓ_1 are 3 or 4, if $k = \omega + 3$ and $2k - 2\omega$ or $2k - 2\omega - 1$, if $k \neq \omega + 3$. Let us consider the case $k \neq \omega + 3$, $\ell_1 = 2k - 2\omega$.

In view of Claim 1 the set X of all solutions of system (8.3) is a subset of a direct product

$$E(\ell_1, \omega_1) \times E(n - \ell_1, \omega - \omega_1)$$

for some $0 \leqslant \omega_1 = c_1 \leqslant \omega$ determined from the first equation of (8.3): $x_1 + \cdots + x_{\ell_1} = c_1$.

This with Theorem AAK implies

$$|X| \leqslant \binom{2k - 2\omega}{\omega_1} M(n - 2k + 2\omega, 2\omega - k + 1, \omega - \omega_1) = \binom{2k - 2\omega}{\omega_1} 2^{2\omega - k}.$$

Simple calculations show that for $\omega_1 \neq k - \omega$ we have

$$\binom{2k - 2\omega}{\omega_1} 2^{2\omega - k} < \binom{2k - 2\omega + 2}{k - \omega + 1} 2^{2\omega - k - 2} = m.$$

Thus, we conclude that $\omega_1 = k - \omega$. Similarly one can show that $\omega_1 = \left\lfloor \frac{\ell_1}{2} \right\rfloor$ for other cases.

Next let us show that providing $|X| \geqslant m$ we must have $\ell_2 = \cdots = \ell_{2\omega - k + 1} = 2$, $\ell_{2\omega - k + 2} = \cdots = \ell_{n - k + 1} = 1$. Suppose $\ell_2 \geqslant 3$. Then using Lemma 2.2 (with some direct calculations) we can verify that

$$\begin{aligned} |X| &\leqslant \binom{2k - 2\omega}{k - \omega} \max_{\Sigma \omega_i = 2\omega - k} \prod_{i=2}^{n-k+1} \binom{\ell_i}{\omega_i} \\ &\leqslant \binom{2k - 2\omega}{k - \omega} \binom{\ell_2}{\left\lfloor \frac{\ell_2}{2} \right\rfloor} 2^{2\omega - k - \left\lfloor \frac{\ell_2}{2} \right\rfloor} \\ &< \binom{2k - 2\omega + 2}{k - \omega + 1} 2^{2\omega - k - 2} = m \end{aligned}$$

a contradiction.

The same fact can be shown for the case $\ell_1 = 2k - 2\omega - 1$.

Let H' be the submatrix of H formed by the first $2\omega - k + 1$ rows and the first 2ω columns of H. By our observations above H' has a positive step form with $\ell_1 = 2k - 2\omega$, $\ell_2 = \cdots = \ell_{2\omega - k + 1} = 2$, moreover the entries of all steps are ones.

Our last discovery is

Claim 2. *The columns of H' corresponding to each step are equal. In other words H' can be transformed to the positive step form (with $\ell_1 = 2k - 2\omega$, $\ell_2 = \cdots = \ell_{2\omega - k + 1} = 2$) where all steps consist of ones and all other entries of H consist of zeros.*

Proof. First we prove the claim for the steps of size two. Let $r_1, \ldots, r_{2\omega - k + 1}$ be the rows of H' and let $h_1, \ldots, h_{2\omega}$ be the columns of H'. Let also $\begin{pmatrix} 1 & 1 & 0 & 0 \\ a & b & 1 & 1 \end{pmatrix}$ be the submatrix formed by the rows r_2, r_3 and the columns $h_{2k-2\omega+1}, \dots, h_{2k-2\omega+4}$ (i.e. columns corresponding to the second and third steps).

If $a < b$ then the latter submatrix can be transformed (by linear combinations of rows r_2 and r_3) to $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & b-a & 1 & 1 \end{pmatrix}$.

Exchanging now the second and third rows (of the transformed matrix) we get a contradiction with the assertion that the size of each step, except possibly for the first one, must have size not greater than two. This clearly implies that $a = b$ and the same fact holds for all other steps of size two.

Let now $T = \begin{pmatrix} 1 & \dots & 1 & 0 & 0 \\ a_1 & \dots & a_{2k-2\omega} & 1 & 1 \end{pmatrix}$ be the submatrix formed by the first two rows and the first two steps. W.l.o.g. we may assume that $0 = a_1 \leqslant \dots \leqslant a_{2k-2\omega}$.

Let also ℓ_1' be the number of nonzero entries in the second row of T. Then exchanging the first two rows of H we obtain a new matrix of positive step form with the first two steps of sizes ℓ_1' and $\ell_2' = 2k - 2\omega + 2 - \ell_1'$, respectively.

In view of Claim 1 the first step must consist of ones. Suppose now $\ell_1' \notin \{2, 2k - 2\omega\}$. Then by (7.12) and (7.13)

$$|X| \leqslant \binom{\ell_1'}{\lfloor \frac{\ell_1}{2} \rfloor} \binom{\ell_2'}{\lfloor \frac{\ell_2}{2} \rfloor} 2^{2\omega-k-1} \leqslant \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2},$$

a contradiction with $|X| > m$.

Next observe the case $\ell_1' = 2k - 2\omega$, that is $a_1 = a_2 = 0$, $a_3 = \dots = a_{2k-2\omega} = 1$. Consider the first two equations of our system (8.3). In view of our observations above it has the form

$$\begin{aligned} x_1 + \dots + x_{2k-2\omega} &= c_1, \\ x_3 + \dots + x_{2k-2\omega+2} &= c_2. \end{aligned} \tag{9.3}$$

We observed before that $|X| > m$ holds only if $c_1 = k - \omega$. Therefore by symmetry (exchanging the first two rows) $c_2 = k - \omega$ as well. Let $Y \subset E(n)$ be the set of solutions of (9.2). Observe that

$$|Y| = 4\binom{2k-2\omega-2}{k-\omega-1} + 2\binom{2k-2\omega-2}{k-\omega} = \binom{2k-2\omega}{k-\omega} + 2\binom{2k-2\omega-2}{k-\omega-1}.$$

Hence

$$|X| \leqslant |Y| \cdot 2^{2\omega-k-1} = \left\{ \binom{2k-2\omega}{k-\omega} + 2\binom{2k-2\omega-2}{k-\omega-1} \right\} 2^{2\omega-k-1}. \tag{9.4}$$

It is not hard to verify now that the

$$\text{RHS (9.3)} < \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2} = m,$$

a contradiction.

Hence, we conclude that $\ell_1' = 2$, that is $a_1 = \cdots = a_{2k-2\omega} = 0$.
Clearly the same can be shown for all other rows of H'.
This completes the proof of Claim 2. $\square$

Observe now that Claim 2 implies that $|X| > m$ only if Eq. (9.1) has the form

$$H(x_1, \ldots, x_n)^T = (k-\omega, 1, \ldots, 1)^T. \tag{9.5}$$

The latter clearly means that for $\ell_1 = 2k - 2\omega$ one has $|X| > m$ only if

$$X \subset E(2k-2\omega, k-\omega) \times E(2,1)^{2\omega-k} \times \{0^{n-2\omega}\}. \tag{9.6}$$

In fact we can show (by a counting argument) that we have equality in (9.6), however this is not necessary here.

Similarly (repeating all the steps) one can easily show that for $\ell_1 = 2k-2\omega-1$, one has $|X| > m$ only if

$$X \subset E(2k-2\omega-1, k-\omega-1) \times E(2,1)^{2\omega-k+1} \times \{0^{n-2\omega-1}\}$$

and for the other possible cases , $\ell_1 = 4$ or 3 (with $k = \omega + 3$), one has $|X| \geqslant m$ only if X is in one of the following configurations:

$$\begin{aligned}
&E(4,2)^2 \times E(2,1)^{\omega-4} \times \{0^{n-2\omega}\}, \text{ or}\\
&E(4,2) \times E(3,1) \times E(2,1)^{\omega-3} \times \{0^{n-2\omega-1}\}, \text{ or}\\
&E(3,1)^2 \times E(2,1)^{\omega-2} \times \{0^{n-2\omega-2}\}.
\end{aligned}$$

In other words, we have proved that $|X| \geqslant m$ only if X is a direct product with the specified parameters. It is easy to show that for an intersecting system A in a direct product $X = E(\ell_1, \omega_1) \times \cdots \times E(\ell_r, \omega_r)$ with $2\omega_i \leqslant \ell_i (i = 1, \ldots, r)$ we have

$$|A| \leqslant \tfrac{1}{2}|X|. \tag{9.7}$$

This is also a special case of a result in [13], where the maximum size of an intersecting family A is determined for direct products (for the complete solution of the t-intersection problem for direct products see [1]).

We now turn to our intersecting system $A \subset X$. One can easily verify that for all possible configurations X (with $|X| \geqslant m$) described above we have $\frac{1}{2}|X| < m$.

Hence by (9.7) for a corresponding intersecting system A we have

$$|A| < m = \binom{2k-2\omega+2}{k-\omega+1} 2^{2\omega-k-2},$$

a contradiction. Thus the only configuration which can achieve this bound must have $\ell_1 = 2k-2\omega+1$ or $\ell_1 = 2k-2\omega+2$. This completes the proof of Theorem 2.1.

Remark 2.2. Using the same approach as in the proof of Theorem 2.1 it is not difficult to show that there are no other optimal intersecting systems except for the systems A_1, A_2, A_3 described above.

10. Proof of Theorem 2.2.

We need some new definitions and notation. A t-intersecting family $\mathcal{F} \subset \binom{[n]}{\omega}$ is called nontrivial if

$$\Big|\bigcap_{F \in \mathcal{F}} F\Big| < t.$$

Define the following set systems:

$$\begin{aligned} \nu_t(n,\omega) &= \left\{F \in \binom{[n]}{\omega} : |[1,t+2] \cap F| \geqslant t+1\right\}, \\ \mu_t(n,\omega) &= \left\{F \in \binom{[n]}{\omega} : [1,t] \subset F,\ F \cap [1+t,\omega+1] \neq \emptyset\right\} \\ &\quad \cup \{[1,\omega+1] \setminus \{i\} : i \in [1,t]\}. \end{aligned}$$

Denote also by $J_t(n,\omega)$ the maximum possible size of a nontrivial t-intersecting family $\mathcal{F} \subset \binom{[n]}{\omega}$. Hilton and Milner [15] determined $J_1(n,\omega)$, Frankl [12] extended the result to $J_t(n,\omega)$ when n is big enough and finally a solution for all n was given in [5]. We use here

Theorem F (*Frankl [12]*). *For* $1 \leqslant t \leqslant \omega \leqslant n$, $n > n_1(\omega,t)$ (*suitable*) *we have*

(a) *for* $t+1 \leqslant \omega \leqslant 2t+1$

$$J_t(n,\omega) = |\nu_t(n,\omega)|,$$

(b) *for* $\omega > 2t+1$

$$J_t(n,\omega) = |\mu_t(n,\omega)|.$$

Let us turn now to the language of (0,1)-vectors.

Let $A \subset E(n,\omega)$ be an optimal t-intersecting system with $\dim(A) = k$. Consider first the case $k \leqslant n-t$. Note that if A is a t-star then $|A| = M(n-t,k,\omega-t)$. Therefore, let A be an optimal nontrivial t-intersecting system. Observe now that $n \leqslant k\omega$. This is clear, because otherwise $\dim(A) > k$.

We have

$$\begin{aligned} |\nu_t(n,\omega)| &= \binom{t+2}{t+1}\binom{n-t-2}{\omega-t-1} + \binom{n-t-2}{\omega-t-2}, \\ |\mu_t(n,\omega)| &= \binom{n-t}{\omega-t} - \binom{n-\omega-1}{\omega-t} + t. \end{aligned} \tag{10.1}$$

Then given ω and t clearly $|\nu_t(n,\omega)|$ and $|\mu_t(n,\omega)| = O(n^{\omega-t-1})$. Consequently $|\nu_t(n,\omega)|$ and $|\mu_t(n,\omega)| = O(k^{\omega-t-1})$, since $n \leqslant k\omega$. Thus by the assumption $|A| = O(k^{\omega-t-1})$. However, $M(n-t,k,\omega-t) = \binom{k}{\omega-t} \sim ck^{\omega-t}$ (for some constant c), a contradiction with the optimality of A. This means that for k large an optimal t-intersecting system $A \subset E(n,\omega)$

with $\dim(A) = k$ forms a t-star. Hence for $k \geqslant k_o(\omega, t)$ and $k \leqslant n - t$ we have

$$J_t(n, k, \omega) = M(n - t, k, \omega - t) = \binom{k}{\omega - t}.$$

The case $k > n - t$ directly follows from Theorem EKR.

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