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On the universal Gröbner bases of toric ideals of graphs

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ABSTRACT

The universal Gröbner basis of an ideal is a Gröbner basis with respect to all term orders simultaneously. We characterize in graph theoretical terms the elements of the universal Gröbner basis of the toric ideal of a graph. We also provide a new degree bound. Finally, we give examples of graphs for which the true degrees of their circuits are less than the degrees of some elements of the Graver basis.

1. Introduction

The universal Gröbner basis of an ideal I is the union of all reduced Gröbner bases $G_<$ of the ideal I as $<$ runs over all term orders. The universal Gröbner basis is a finite subset of I and it is a Gröbner basis for I with respect to all term orders simultaneously, see [8]. Universal Gröbner bases exist for every ideal in $\mathbb{K}[x_1,\dots,x_n]$. They were introduced by V. Weispfenning [11] and N. Schwartz [7].

Let $A=\{\mathbf{a}_1,\dots,\mathbf{a}_m\}\subseteq\mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A:=\{l_1\mathbf{a}_1+\cdots+l_m\mathbf{a}_m \mid l_i\in\mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1,\dots,x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i)=\mathbf{a}_i$ for $i=1,\dots,m$. For $\mathbf{u}=(u_1,\dots,u_m)\in\mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}}:=x_1^{u_1}\cdots x_m^{u_m}$ to be

$$\deg_A\left(\mathbf{x}^{\mathbf{u}}\right):=u_1\mathbf{a}_1+\cdots+u_m\mathbf{a}_m\in\mathbb{N}A.$$

The *toric ideal* I_A associated to A is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}})=\deg_A(\mathbf{x}^{\mathbf{v}})$, see [8]. For such binomials, we set $\deg_A(\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}):=\deg_A(\mathbf{x}^{\mathbf{u}})$. An irreducible binomial $\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}$ in I_A is called *primitive* if there exists no other binomial $\mathbf{x}^{\mathbf{w}}-\mathbf{x}^{\mathbf{z}}$ in I_A such that $\mathbf{x}^{\mathbf{w}}$ divides $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{z}}$ divides $\mathbf{x}^{\mathbf{v}}$. The set of primitive binomials forms the Graver basis of I_A and is denoted by Gr_A. An irreducible binomial is called a *circuit* if it has minimal support. The set of circuits

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is denoted by $\mathcal{C}_A$. The relation among the set of circuits, the Graver basis and the universal Gröbner basis, which is denoted by $\mathcal{U}_A$, for a toric ideal I_A is given by B. Sturmfels [8]:

Proposition 1.1. *For any toric ideal I_A we have $\mathcal{C}_A \subset \mathcal{U}_A \subset Gr_A$.*

Circuits of toric ideals of graphs were determined by R. Villarreal [10, Proposition 4.2]. The Graver basis of a toric ideal of a graph has first been studied by H. Ohsugi and T. Hibi [4, Lemma 2.1] and the form of its elements was determined by E. Reyes, Ch. Tatakis and A. Thoma [6]. In [2, Theorem 5.1] J. De Loera, B. Sturmfels and R. Thomas determined the universal Gröbner basis for toric ideals of graphs with less than nine vertices. The purpose of this article is to determine the universal Gröbner basis for the toric ideal of any graph. In particular in Section 2 we present some terminology, notations and results about the toric ideals of graphs. Section 3 contains the main result of the article which is a characterization of the binomials that belong to the universal Gröbner basis of a toric ideal of a graph. Section 4 provides a degree bound for the binomials in the universal Gröbner basis of the toric ideal of a graph. Section 5 answers a conjecture by B. Sturmfels [9, Conjecture 4.8] by giving examples of graphs for which elements of their Graver bases have degrees greater than the degrees of any of its circuits.

2. Toric ideals of graphs

Let G be a finite simple connected graph with vertices $V(G) = \{v_1, \ldots, v_n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. Let $\mathbb{K}[e_1, \ldots, e_m]$ be the polynomial ring in the m variables $e_1, \ldots, e_m$ over a field $\mathbb{K}$. We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of G. With I_G we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

A *walk* connecting $v_{i_1} \in V(G)$ and $v_{i_{s+1}} \in V(G)$ is a finite sequence of the form

$$w = \big(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_s}, v_{i_{s+1}}\}\big)$$

with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$. The *length* of the walk w is the number s of edges of the walk. An even (respectively odd) walk is a walk of *even* (respectively odd) length. A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_s}, v_{i_{s+1}}\})$ is called *closed* if $v_{i_{s+1}} = v_{i_1}$. A *cycle* is a closed walk

$$\big(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_s}, v_{i_1}\}\big)$$

with $v_{i_k} \neq v_{i_j}$, for every $1 \leqslant k < j \leqslant s$. Note that, although the graph G has no multiple edges, the same edge e may appear more than once in a walk. In this case e is called a *multiple edge of the walk w*.

Given an even closed walk of the graph G

$$w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$$

write

$$E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}}, \qquad E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$$

and denote by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}.$$

It is easy to see that $B_w \in I_G$. Moreover, it is known that the toric ideal I_G is generated by binomials of this form, see [10]. For convenience, we denote by $\mathbf{w}$ the subgraph of G with vertices the vertices of the walk and edges the edges of the walk w. We call a walk $w' = (e_{j_1}, \ldots, e_{j_t})$ a *subwalk* of w if

$e_{j_1} \cdots e_{j_t} \mid e_{i_1} \cdots e_{i_{2q}}$. An even closed walk w is said to be primitive if there exists no even closed subwalk ξ of w of smaller length such that $E^+(\xi) \mid E^+(w)$ and $E^-(\xi) \mid E^-(w)$. The walk w is primitive if and only if the binomial B_w is primitive. Every even primitive walk $w = (e_{i_1}, \ldots, e_{i_{2k}})$ partitions the set of edges in the two sets $\mathbf{w}^+ = \{e_{i_j} \mid j \textit{ odd}\}, \mathbf{w}^- = \{e_{i_j} \mid j \textit{ even}\}$, otherwise the binomial B_w is not irreducible. By w^+ we denote the exponent vector of the monomial $E^+(w)$ and by w^- the exponent vector of the monomial $E^-(w)$. The edges of $\mathbf{w}^+$ are called odd edges of the walk and those of $\mathbf{w}^-$ even. Note that for an even closed walk whether an edge is even or odd depends on the edge that we start counting from. So it is not important to identify whether an edge is even or odd but to separate the edges in the two disjoint classes.

A *cut edge* (respectively *cut vertex*) is an edge (respectively vertex) of the graph whose removal increases the number of connected components of the remaining subgraph. A graph is called *biconnected* if it is connected and does not contain a cut vertex. A *block* is a maximal biconnected subgraph of a given graph G. A *sink* of a block B is a common vertex of two odd or two even edges of the walk w which belong to the block B. In particular if e is a cut edge of a primitive walk w then e appears at least twice in the walk and belongs either to $\mathbf{w}^+$ or $\mathbf{w}^-$. Therefore both vertices of e are sinks. Note that the definition of a sink depends only on the walk w and not on the underlying graph $\mathbf{w}$.

The following theorems determine the form of the circuits and the primitive binomials. R. Villarreal in [10, Proposition 4.2] gave a necessary and sufficient characterization of circuits:

Theorem 2.1. *Let G be a finite connected graph. The binomial $B \in I_G$ is circuit if and only if $B = B_w$ where*

(1) *w is an even cycle, or*
(2) *two odd cycles intersecting in exactly one vertex, or*
(3) *two vertex-disjoint odd cycles joined by a path.*

The next theorem by E. Reyes, Ch. Tatakis and A. Thoma [6] describes the form of the primitive binomials, i.e. the elements $B_w \in I_G$ that belong to the Graver basis.

Theorem 2.2. *Let G be a graph and w an even closed walk of G. The binomial B_w is primitive if and only if*

(1) *every block of* $\mathbf{w}$ *is a cycle or a cut edge,*
(2) *every multiple edge of the walk w is a double edge of the walk and a cut edge of* $\mathbf{w}$,
(3) *every cut vertex of* $\mathbf{w}$ *belongs to exactly two blocks and it is a sink of both.*

Fig. 1 shows a graph $\mathbf{w}$ of a primitive walk and a block B with four sinks.

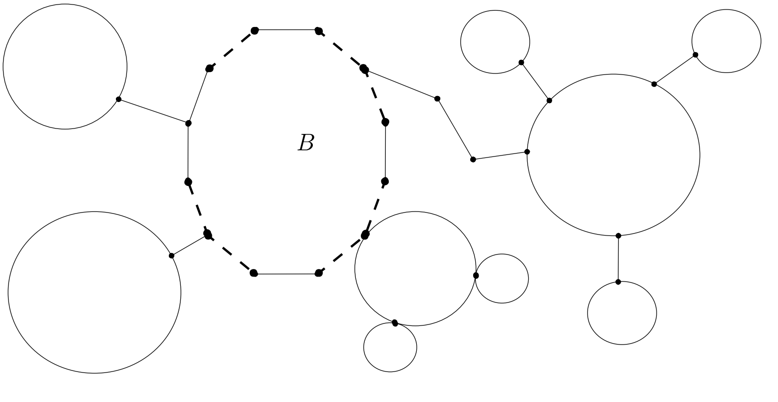

Fig. 1.

3. Universal Gröbner bases

In this section we will characterize the elements of the universal Gröbner basis of the toric ideal of a graph. The elements B_w of the universal Gröbner basis belong to the Graver basis, therefore their form is determined by Theorem 2.2. Let $w=(e_{i_1},e_{i_2},\dots,e_{i_{2q}})$ be a primitive walk. Then, the blocks of the graph **w** are cycles or they are cut edges. The simplest example of a walk w such that B_w is in the Graver basis but not in the universal Gröbner basis is the one with degree 6 whose graph is in Fig. 2.

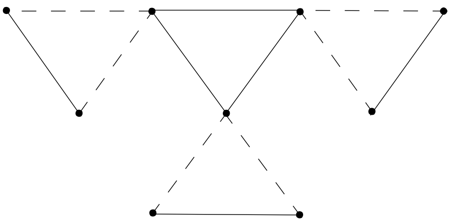

Fig. 2.

The existence of this walk implies for $n \geqslant 9$ that $\mathcal{U}_{K_n} \neq Gr_{K_n}$, where K_n is the complete graph on n vertices. Note that in [2] J. De Loera, B. Sturmfels and R. Thomas prove that $\mathcal{C}_{K_n}=\mathcal{U}_{K_n}=Gr_{K_n}$ for $n \leqslant 7$ and $\mathcal{C}_{K_8} \neq \mathcal{U}_{K_8}=Gr_{K_8}$. This walk is not in the universal Gröbner basis because there exists a *pure* cyclic block, the one in the center, all of whose edges are either in $\mathbf{w}^+$ or in $\mathbf{w}^-$. In Proposition 3.2 we will see that whenever a primitive walk w has a pure cyclic block, then the binomial B_w is not in the universal Gröbner basis. In Theorem 3.4 we will see the converse: whenever an element B_w is in the Graver basis but not in the universal Gröbner basis then w has a pure cyclic block.

Definition 3.1. A cyclic block B of a primitive walk w is called pure if all edges of B are either in $\mathbf{w}^+$ or in $\mathbf{w}^-$.

Proposition 3.2. *Let w be an even primitive walk that has a pure cyclic block. Then B_w does not belong to the universal Gröbner basis of I_G.*

Proof. Suppose that w has a pure cyclic block B with edges $\epsilon_1,\dots,\epsilon_s$ which we can assume belong to $\mathbf{w}^-$. Then the walk w can be written in the form $(w_1,\epsilon_1,\dots,w_s,\epsilon_s)$, where w_i are subwalks of w of odd length, see Fig. 3.

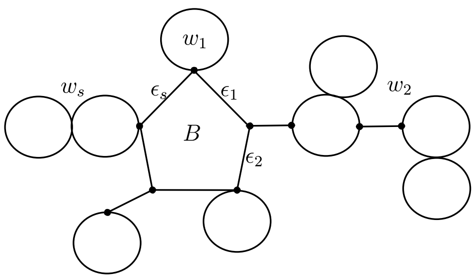

Fig. 3.

For a subwalk w_i we denote by

$$E^+(w_i)=\prod_{e_{i_{2k-1}}\in w_i} e_{i_{2k-1}}, \qquad E^-(w_i)=\prod_{e_{i_{2k}}\in w_i} e_{i_{2k}}.$$

Then $B_w = E^+(w_1)E^+(w_2)\cdots E^+(w_s) - \epsilon_1\epsilon_2\cdots\epsilon_s E^-(w_1)E^-(w_2)\cdots E^-(w_s)$. Look at the even walks $(w_i, \epsilon_i, w_{i+1}, \epsilon_i)$ and the corresponding binomials $F_i = E^+(w_i)E^+(w_{i+1}) - \epsilon_i^2 E^-(w_i)E^-(w_{i+1}) \in I_G$, where $1 \leqslant i \leqslant s-1$ and $F_s = E^+(w_s)E^+(w_1) - \epsilon_s^2 E^-(w_s)E^-(w_1) \in I_G$.

Suppose that B_w belongs to a reduced Gröbner basis for I_G with respect to a term order $<$. There are two cases.

First case: $E^+(w_1)E^+(w_2)\cdots E^+(w_s) > \epsilon_1\epsilon_2\cdots\epsilon_s E^-(w_1)E^-(w_2)\cdots E^-(w_s)$. Then necessarily $E^+(w_i)E^+(w_{i+1}) < \epsilon_i^2 E^-(w_i)E^-(w_{i+1})$ for every i, since $E^+(w_i)E^+(w_{i+1})$ divides $E^+(w_1)E^+(w_2)\cdots E^+(w_s)$ and $F_i \in I_G$.

Multiplying all these inequalities for different i's we get

$$\left(E^+(w_1)E^+(w_2)\cdots E^+(w_s)\right)^2 < \left(\epsilon_1\epsilon_2\cdots\epsilon_s E^-(w_1)E^-(w_2)\cdots E^-(w_s)\right)^2,$$

which is a contradiction.

Second case: $E^+(w_1)E^+(w_2)\cdots E^+(w_s) < \epsilon_1\epsilon_2\cdots\epsilon_s E^-(w_1)E^-(w_2)\cdots E^-(w_s)$. In the case that $s = 2k$ the binomial $G = \epsilon_1\epsilon_3\cdots\epsilon_{2k-1} - \epsilon_2\epsilon_4\cdots\epsilon_{2k}$ is in I_G and both monomials of G divide $\epsilon_1\epsilon_2\cdots\epsilon_{2k}E^-(w_1)E^-(w_2)\cdots E^-(w_s)$, a contradiction to the fact that B_w belongs to the reduced Gröbner basis.

In the case that $s = 2k+1$ the binomials $G_i = E^+(w_i)\epsilon_{i+1}\epsilon_{i+3}\cdots\epsilon_{i+2k-1} - E^-(w_i)\epsilon_i\epsilon_{i+2}\cdots\epsilon_{i+2k}$ are in I_G, where $\epsilon_j = \epsilon_l$ if $j \equiv l \bmod (2k+1)$. Therefore $E^+(w_i)\epsilon_{i+1}\epsilon_{i+3}\cdots\epsilon_{i+2k-1} > E^-(w_i)\epsilon_i\epsilon_{i+2}\cdots\epsilon_{i+2k}$, since $E^-(w_i)\epsilon_i\epsilon_{i+2}\cdots\epsilon_{i+2k}$ divides $\epsilon_1\epsilon_2\cdots\epsilon_{2k}E^-(w_1)E^-(w_2)\cdots E^-(w_s)$. Multiplying them all and canceling common factors we get

$$E^+(w_1)E^+(w_2)\cdots E^+(w_s) > \epsilon_1\epsilon_2\cdots\epsilon_s E^-(w_1)E^-(w_2)\cdots E^-(w_s),$$

a contradiction. Therefore B_w does not belong to any reduced Gröbner basis of I_G and neither to the minimal universal Gröbner basis of I_G. $\square$

Definition 3.3. A primitive walk w is called mixed if no cyclic block of w is pure.

The next theorem is the main result of this article and describes the elements of the universal Gröbner basis of I_G, for a general graph G. For any mixed primitive walk w we construct a term order $<_w$ that depends on w to prove that B_w belongs to the reduced Gröbner basis with respect to $<_w$. To prove it we will show that whenever one monomial of a binomial B in I_G divides one of $E^+(w)$, $E^-(w)$, then the other monomial of B is greater with respect to $<_w$ and does not divide either $E^+(w)$ or $E^-(w)$.

Let w be a mixed primitive walk. We define a term order $<_w$ on $\mathbb{K}[e_1,\dots,e_n]$, as an elimination order with the variables that do not belong to $\mathbf{w}$ larger than the variables in $\mathbf{w}$. We order the first set of variables by any term order, and the second set of variables as follows: Let $B_1,\dots,B_{s_0}$ be any enumeration of all cyclic blocks of $\mathbf{w}$. Let t_i^+ denotes the number of edges in $\mathbf{w}^+ \cap B_i$ and t_i^- denotes the number of edges in $\mathbf{w}^- \cap B_i$. Let $W = (w_{ij})$ be the $(s_0) \times m$ matrix

$$w_{ij} = \begin{cases} 0, & \text{if } e_j \notin B_i, \\ t_i^-, & \text{if } e_j \in B_i \cap \mathbf{w}^+, \\ t_i^+, & \text{if } e_j \in B_i \cap \mathbf{w}^- \end{cases}$$

where m is the number of edges of $\mathbf{w}$.

Note that each column has at most one nonzero entry since each edge belongs to exactly one block of $\mathbf{w}$. Denote by $[u]$ the vector u written as a column vector. We say that $e^u <_w e^v$ if and only if the first nonzero coordinate of $W[u-v]$ is negative, otherwise, if $W[u-v] = \mathbf{0}$, order them lexicographically. Note that for the walk w we have $W[w^+ - w^-] = \mathbf{0}$. Fig. 4 shows a mixed primitive walk with degrees w_{ij}.

Theorem 3.4. *Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.*

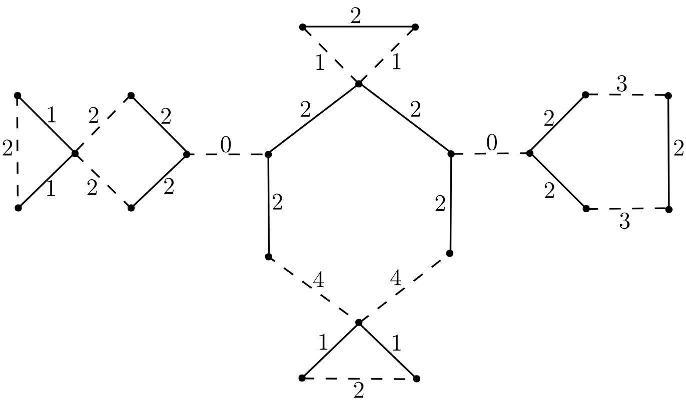

Fig. 4.

Proof. If w is not mixed then it has a pure cyclic block and the result follows from Proposition 3.2.

Let w be a mixed primitive walk. We will prove that B_w belongs to the reduced Gröbner basis of I_G with respect to the term order $<_w$. It is enough to prove that whenever there exists a primitive binomial B_z such that $E^+(z) \mid E^+(w)$ then $E^-(z) >_w E^+(z)$. Note that $E^-(z) \nmid E^-(w)$ since w is primitive and $E^-(z) \nmid E^+(w)$ since w is mixed. We remark that if $\mathbf{z} \not\subseteq \mathbf{w}$, since $\mathbf{z}^+ \subset \mathbf{w}^+$, there is an edge of $\mathbf{z}^-$ which is not an edge of $\mathbf{w}$. But then $E^-(z) >_w E^+(z)$ since $<_w$ is an elimination order. Thus we can suppose that $\mathbf{z} \subset \mathbf{w}$, see also [8, Proposition 4.13].

We claim that there exists at least one i, such that $B_i \cap \mathbf{z} \neq \emptyset$ and $B_i \cap \mathbf{z}^+ \subsetneqq B_i \cap \mathbf{w}^+$. Suppose not, then for every i, either $B_i \cap \mathbf{z} = \emptyset$ or $B_i \cap \mathbf{z}^+ = B_i \cap \mathbf{w}^+$, since the containment holds because $E^+(z) \mid E^+(w)$. Let B_i be a cyclic block such that $B_i \cap \mathbf{z}^+ = B_i \cap \mathbf{w}^+$. Then $B_i \cap \mathbf{z}^- = B_i \cap \mathbf{w}^-$. If not, then B_i is not a block of $\mathbf{z}$ which implies that every edge e in $B_i \cap \mathbf{z}^+$ is a cut edge of $\mathbf{z}$ and therefore e is a double edge of z. The B_i is a cyclic block of $\mathbf{w}$ which means that every edge of B_i is a single edge of w. Then $e^2 \mid E^+(z)$ and $e^2 \nmid E^+(w)$ which is impossible since $E^+(z) \mid E^+(w)$. Therefore $B_i \cap \mathbf{z} = B_i$ or $B_i \cap \mathbf{z} = \emptyset$. This is true also for blocks which are cut edges. There exists at least one block of $\mathbf{w}$ such that $B_i \cap \mathbf{z} = \emptyset$ and at least one such that $B_i \cap \mathbf{z} = B_i$, since $z \neq w$. The graph $\mathbf{w}$ is connected, as a graph of a walk. Therefore two adjacent blocks B_j and B_i exist such that $B_j \cap \mathbf{z} = \emptyset$ and $B_i \cap \mathbf{z} = B_i$. Let v be the common cut vertex of B_j and B_i. Then, $2v$ appears in exactly one of the degrees $\deg_A(E^+(z))$, $\deg_A(E^-(z))$. Therefore $B_z \notin I_G$, a contradiction.

Let i be the smallest integer such that $B_i \cap \mathbf{z} \neq \emptyset$ and $B_i \cap \mathbf{z}^+ \subsetneqq B_i \cap \mathbf{w}^+$. We claim that the first $i-1$ coordinates of $W[z^+ - z^-]$ are zero. Denote by w_j the j-row of W. In the case $B_j \cap \mathbf{z} = \emptyset$ then $w_j[z^+] = 0 = w_j[z^-]$. In the other case $B_j \cap \mathbf{z}^+ = B_j \cap \mathbf{w}^+$, from the argument in the previous paragraph we have also $B_j \cap \mathbf{z}^- = B_j \cap \mathbf{w}^-$ and then $w_j[z^+] = t_j^- t_j^+ = w_j[z^-]$.

For the block B_i we have two cases: either $B_i \cap \mathbf{z} \neq B_i$ or $B_i \cap \mathbf{z} = B_i$.

First case: let $e \in B_i \cap \mathbf{z}$. Then e is a cut edge and $e \in \mathbf{z}^-$, otherwise e is a double edge of z and a simple of w, contradicting the fact that $E^+(z) \mid E^+(w)$. So every edge of $B_i \cap \mathbf{z}$ is in $\mathbf{z}^-$ and therefore $w_i[z^+] = 0$ and $w_i[z^-] > 0$. Thus $E^-(z) >_w E^+(z)$.

Second case: $B_i \cap \mathbf{z} = B_i$ implies $B_i \cap \mathbf{z}^- = B_i \setminus (B_i \cap \mathbf{z}^+)$ and since $B_i \cap \mathbf{z}^+ \subsetneqq B_i \cap \mathbf{w}^+$, we have $w_i[z^+] < t_i^- t_i^+ < w_i[z^-]$. Therefore $E^-(z) >_w E^+(z)$.

We conclude that B_w is in the reduced Gröbner basis with respect to the term order $<_w$ and thus it belongs to the universal Gröbner basis of I_G. $\square$

4. Degree bounds

The number of elements in the universal Gröbner basis is usually very large, for example in [2] J. De Loera, B. Sturmfels and R. Thomas computed that the number of the elements in the universal Gröbner basis of I_{K_8} is 45 570, where K_n is the complete graph on n vertices. An estimate for the size of a universal Gröbner basis can be a bound for the degrees of the elements in the universal Gröbner basis. Let d_n be the largest degree of a binomial in the universal Gröbner basis for I_{K_n}. In [2] J. De Loera, B. Sturmfels and R. Thomas proved that d_n satisfies $n-2 \leqslant d_n \leqslant \binom{n}{2}$.

We will improve this result by proving that d_n takes always the value $n-2$.

Proposition 4.1. *The largest degree d_n of any binomial in the Graver basis (and in the universal Gröbner basis) for I_{K_n} is $d_n = n-2$, for $n \geqslant 4$.*

Proof. We will prove that the largest degree d_n of a binomial in the Graver basis for I_{K_n} is $d_n = n-2$ and it is attained by a circuit, see also [2]. Theorem 2.2 implies that the graph of a primitive walk consists of blocks which are cut edges and cyclic blocks. The cyclic blocks are at least two, except in the case that the walk is a cycle. Let w be a primitive walk and suppose that w has s_0 cyclic blocks and s_1 cut edges. Thus $s = s_0 + s_1$ is the total number of blocks. From Theorem 2.2 we know that there are exactly $s-1$ cut vertices and each one belongs to exactly two blocks. Let $B_1, \dots, B_{s_0}$ be the cyclic blocks and t_i denotes the number of edges (vertices) of the cyclic block B_i. Then the total number of vertices of **w** is

$$t_1 + \cdots + t_{s_0} + 2s_1 - (s-1) \leqslant n,$$

since the cut vertices are counted twice, see Theorem 3.1 of [6]. Two times the degree of B_w is the sum of edges of the cyclic blocks $t_1 + \cdots + t_{s_0}$ plus two times the number of cut edges s_1, since cut edges are double edges of the walk w and edges of cycles are always single. Therefore

$$2\deg(B_w) = t_1 + \cdots + t_{s_0} + 2s_1 \leqslant n + s - 1.$$

So the largest degree is attained when the number of blocks of **w** is the largest possible and equality is achieved only if the walk w passes through all n vertices. But from $t_1 + \cdots + t_{s_0} + 2s_1 \leqslant n + s - 1$ we get $s + (t_1 - 2) + \cdots + (t_{s_0} - 2) \leqslant n - 1$. Note that $(t_1 - 2) + \cdots + (t_{s_0} - 2) \geqslant 2$ since cyclic blocks have at least three vertices and the walk has at least two cyclic blocks, except if w is a cycle. In the latter case there is just one block and the degree is small, $\deg(B_w) \leqslant n/2$. Therefore $s \leqslant n-3$. Note that $s = n-3$ is attained by a circuit with $n-5$ cut edges plus 2 cyclic blocks of three vertices each. This has the maximal possible degree $(n + (n-3) - 1)/2 = n-2$. □

Since any graph G with m vertices is a subgraph of the complete graph K_m we have the following corollary.

Corollary 4.2. *Let G be a graph with m vertices, $m \geqslant 4$. The largest degree d of any binomial in the Graver basis (and in the universal Gröbner basis) for I_G is $d \leqslant m-2$.*

The bound $m-2$ is sharp for the toric ideals of the complete graphs K_m. Note that from the proof of Proposition 4.1 the maximum degree $m-2$ is attained only by a circuit with $m-5$ cut edges plus 2 cyclic blocks of three vertices each. This remark gives a characterization of all graphs with $m > 4$ vertices for which the bound $m-2$ is sharp: the largest degree of any binomial in the Graver basis (and in the universal Gröbner basis) for I_G is $m-2$ if and only if G contains a circuit with $m-5$ cut edges plus 2 cyclic blocks of three vertices each.

5. The true circuit conjecture

The knowledge of the form of the circuits [10, Proposition 4.2], the elements of the Graver basis [6], the minimal systems of generators [6] and the elements of the universal Gröbner basis of the toric ideal of a graph G, Theorem 3.4, allow us to produce examples of toric ideals having specific properties. For example, one can easily construct graphs such that the universal Gröbner basis is equal to the Graver basis, either by avoiding creating pure blocks in the elements of the Graver basis, or making subdivisions in some of the edges of pure blocks. For other toric ideals that have this property see the recent work [1] of T. Bogart, R. Hemmecke and S. Petrović.

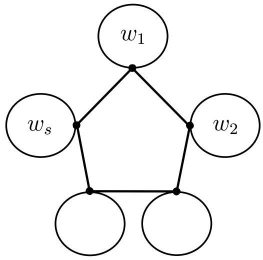

Fig. 5.

B. Sturmfels in his lecture at Santa Cruz (July 1995, see [9]), made the conjecture that circuits always have the maximal degree among the elements of the Graver basis. S. Hosten and R. Thomas gave a counterexample of a toric ideal such that the maximal degree of the elements of the Graver basis was 16 while the maximal degree of the circuits was 15, see [3,9]. This example led B. Sturmfels to alter the conjecture to the following: the degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$ [9, Conjecture 4.8]. Following [9] we define the true degree of a circuit as follows: Consider any circuit $C \in \mathcal{C}_A$ and regard its support $\operatorname{supp}(C)$ as a subset of A. The lattice $\mathbb{Z}(\operatorname{supp}(C))$ has finite index in the lattice $R(\operatorname{supp}(C)) \cap \mathbb{Z}A$, which is called the index of the circuit C and denoted by $\operatorname{index}(C)$. The *true degree* of the circuit C is the product $\deg(C) \cdot \operatorname{index}(C)$.

There are several examples of families of toric ideals where circuits do attain the maximum degree, see for example [5]. This is also true for families of toric ideals of graphs, for example in Proposition 4.1 the binomial that has the maximal degree in I_{K_n} is a circuit. But this is not true in the general case.

Next we give examples of circuits for which their true degrees are less than the degrees of some elements of the Graver basis. Let us consider a graph G consisting of a cycle of length s and s odd cycles of length l, each one attached to a vertex of the initial cycle, see Fig. 5.

Let w be the walk that passes through every edge of the graph G. The length of the walk w is $ls + s = s(l+1)$, which is even. Thus, B_w is an element of the Graver basis of I_G, see [6], and has degree $s(l+1)/2$. In the graph G there are a lot of circuits. The longest one consists of two odd cycles joined by a path of length $s-1$. Its degree is $(2l + 2(s-1))/2 = l + s - 1$. Note that s, l are greater than two, as lengths of cycles, which implies that $s(l+1)/2 > l + s - 1$. So there exists an element B_w in the Graver basis that has larger degree than any of the circuits. The difference of the degrees can be made as large as one wishes, by choosing large values for l and s. Note that an easy, but lengthy, computation of the true degree of this circuit shows that the true degree is equal to the usual degree. Therefore this family provides infinitely many counterexamples to the true circuit conjecture [9, Conjecture 4.8].

In the previous examples B_w is in the Graver basis but it is not in the universal Gröbner basis, since it has a pure block, see Theorem 3.4. A small alteration of w can provide a family of examples of graphs that have an element in the universal Gröbner basis that has larger degree than any of the circuits. Let us consider a graph G consisting of a cycle of length s and $s-2$ odd cycles of length l each one attached to a vertex of the initial cycle. Let w' be the walk that passes through every edge of the graph G. Then w' is mixed. $B_{w'}$ is in the universal Gröbner basis and the degree of $B_{w'}$ is larger than any of the degrees of circuits, for large l and s.

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