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Directed graphs and boron trees

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ABSTRACT

Let L_1 and L_2 be two disjoint relational signatures. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Ramsey classes of rigid relational structures in L_1 and L_2 respectively. Let $\mathcal{K}_1 * \mathcal{K}_2$ be the class of structures in $L_1 \cup L_2$ whose reducts to L_1 and L_2 belong to $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. We give a condition on $\mathcal{K}_1$ and $\mathcal{K}_2$ which implies that $\mathcal{K}_1 * \mathcal{K}_2$ is a Ramsey class. This is an extension of a result of M. Bodirsky.
In the second part of this paper we consider classes $\mathcal{OS}(2)$, $\mathcal{OS}(3)$, $\mathcal{OB}$ and $\mathcal{OH}$ which are obtained by expanding the class of finite dense local orders, the class of finite circular directed graphs, the class of finite boron tree structures, and the class of rooted trees respectively with linear orderings. We calculate Ramsey degrees for objects in these classes.

1. Introduction

In this paper we introduce some new Ramsey classes of finite relational structures. Examples of known Ramsey classes of finite structures include the class of ordered graphs and the class of ordered hypergraphs, see [1,15,16]; the class of ordered metric spaces, see [14]; the class of sets with two linear orderings (finite permutations), see [20]; the class of ordered incomparable chains in [21]; the class of ordered ultrametric spaces in which every open ball is an interval, see [17]; and the class of boron trees with ternary

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relation, see [10]. In [19], a method is given for combining two Ramsey classes into a new Ramsey class. This is given by using a cross construction, and a similar idea is used in [22] in order to describe a diagonal property. In this paper we consider a more general approach described as follows.

Let L_1 and L_2 be disjoint relational signatures. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Fraïssé classes in signature L_1 and L_2, respectively (see Section 3 for detailed definitions). Let $\mathcal{K}_1 * \mathcal{K}_2$ be the class of finite structures $\mathbb{A}$ in the signature $L_1 \cup L_2$ such that $\mathbb{A}|L_1 \in \mathcal{K}_1$ and $\mathbb{A}|L_2 \in \mathcal{K}_2$ where $\mathbb{A}|L_1$ and $\mathbb{A}|L_2$ are reducts of the structure $\mathbb{A}$ to the signatures L_1 and L_2 respectively. Then we have the following result:

Theorem 1. *Let L_1 and L_2 be two disjoint relational signatures. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Fraïssé classes in signatures L_1 and L_2 respectively. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are Ramsey classes of rigid structures, each satisfying the strong amalgamation property, then $\mathcal{K}_1 * \mathcal{K}_2$ is a Ramsey class.*

This result is an extension of Bodirsky's result from [3] who required that the signatures L_1 and L_2 be finite. Moreover, we extend this result further by removing the assumption that $\mathcal{K}_1$ and $\mathcal{K}_2$ are Fraïssé classes with the strong amalgamation property (see Theorem 4 below). The proof in [3] uses model theoretic concepts such as core models, model complete cores and ω-categorical structures. It also relies on the main result from [12], which connects topological dynamics and Ramsey theory. It is well-known that certain Ramsey type statements are not provable by finitistic methods. For example, one such a statement can be found in [11] involving Ramsey theorem for regressive functions. We use only elementary combinatorics to give a proof of Theorem 1 which in turn is motivated by the proof in [22]. We do not consider Ramsey numbers in this paper but our proof can give us some estimates on the corresponding Ramsey numbers; this is in contrast with the proof in [3] which cannot give us estimates on Ramsey numbers.

Many classical examples of Ramsey classes are obtained by adding arbitrary linear orderings to a given class of structures, such as ordered graphs or ordered metric spaces. Let $\mathcal{S}(2)$ be the class of dense local orders (see Section 5 for a precise definition). Let $\mathcal{S}(3)$ be the class of circular directed graphs (see Section 6 for definition). Let $\mathcal{B}$ be the class of boron trees (see Section 7 for a precise definition). By adding arbitrary linear orderings to structures in $\mathcal{S}(2)$, $\mathcal{S}(3)$ and $\mathcal{B}$ we obtain classes $\mathcal{OS}(2)$, $\mathcal{OS}(3)$ and $\mathcal{OB}$ respectively, none of which is a Ramsey class, see [18] and [10]. In this case, structural Ramsey theory asks for a measure of deviation of a given class from being a Ramsey class, i.e. for a Ramsey degree, see Section 4 for a definition. We calculate Ramsey degrees for structures in $\mathcal{OS}(2)$, $\mathcal{OS}(3)$ and $\mathcal{OB}$, see Theorem 9. In addition, we consider a certain class $\mathcal{H}$ of finite relational structures naturally related to $\mathcal{B}$. The class consists of finite sets with C-relations (see Section 8 for a precise definition). We refer the reader to [2] for a more detailed treatment of C-relations and boron tree structures. By adding arbitrary linear orderings to structures in $\mathcal{H}$ we obtain the class $\mathcal{OH}$. We calculate Ramsey degrees for structures in $\mathcal{OH}$, see Theorem 9.

In Section 2 we recall some preliminary definitions from model theory and we establish our notation, see [6,7,9]. In Section 3 we introduce lifting, thickening and diagonal thickening in order to facilitate the proofs in Section 4. We introduce the notion of an infinite sum of classes in order to obtain a Fraïssé class from a countable collection of Fraïssé classes. The main Ramsey statement in this paper is Theorem 4 and it is proved in Section 4. In order to calculate Ramsey degrees, we prove the expansion property (see Section 2 for definitions). This is done in Sections 5, 6, 7 and 8. The ordering property is a special case of the expansion property which is often proved as a corollary of the Ramsey property; this is done, for example, in the cases of ordered graphs and ordered metric spaces, see [14]. There are also cases in which one proves the Ramsey property by using the ordering property; see [20] where this is done in the case of finite posets with linear extensions. In Sections 5, 6, and 7 we use the Ramsey property to obtain the expansion property. In Section 8 we give two proofs of the ordering property, one based on the Ramsey property and another that is independent of it.

2. Preliminaries

For a non-empty set A, we denote by $lo(A)$, the collection of all linear orderings on the set A. The cardinality of a set A we denote by $|A|$. For a natural number n we denote by $[n]$ the set $\{1,\dots,n\}$. Let $\leq$ be a linear ordering on a set A, and let $B, C \subset A$ be such that $B \cap C = \emptyset$. If for all $b \in B$ and all $c \in C$ we have $b \leq c$ then we write $B \leq C$. If $\leq, \preceq \in lo(A)$ have the property that $a \leq b \Leftrightarrow b \preceq a$ for every $a, b \in A$ then we write $\preceq = op(\leq)$ or $\leq = op(\preceq)$, and we say that the linear orderings $\leq$ and $\preceq$ are opposite to each other. We say that a and b are *consecutive* in the linear ordering $\leq$ if for every c we have $a \leq c \leq b \Rightarrow (a = c$ or $a = b)$. For a linear ordering $\leq$ we denote its strict part by $<$. We denote by $\mathbb{Q}$ the set of rational numbers. We say that elements x and y in a given relational structure are *related* if there is a tuple of elements containing both x and y which belongs to some relation from the given structure. If the projection of a set $S \subseteq A \times B$ to each coordinate is a bijection then we say that S is a *diagonal set*.

If L is a relational signature and $\mathbb{A}$ and $\mathbb{B}$ are structures in L then we write $\mathbb{A} \hookrightarrow \mathbb{B}$ when $\mathbb{A}$ embeds into $\mathbb{B}$, $\mathbb{A} \leq \mathbb{B}$ when $\mathbb{A}$ is a substructure of $\mathbb{B}$ and $\mathbb{A} \cong \mathbb{B}$ when $\mathbb{A}$ and $\mathbb{B}$ are isomorphic. If S is a subset of the underlying set of a structure $\mathbb{A}$ then we write $\mathbb{A} \upharpoonright S$ for the substructure with the underlying set S. We say that a structure $\mathbb{A}$ is *rigid* if it has only one automorphism. For structures $\mathbb{A}$ and $\mathbb{B}$ we denote by $\binom{\mathbb{B}}{\mathbb{A}}$ the collection $\{\mathbb{A}' \leq \mathbb{B} : \mathbb{A}' \cong \mathbb{A}\}$. For a given structure $\mathbb{K}$ we denote by $Age(\mathbb{K})$ the class of all finite structures that can be embedded into $\mathbb{K}$.

We assume that every class of structures in this paper is closed under taking isomorphic images.

Let I and J be index sets such that $I \cap J = \emptyset$. Let $L = \{R_i\}_{i \in I}$ and $L' = L \cup \{R_j\}_{j \in J}$ be relational signatures such that for all $j \in J$ we have $R_j \notin L$. If $\mathbb{A}'$ is a structure in L' and $\mathbb{A}$ is a structure in L such that

$$\mathbb{A}' = \left(A, \left\{R_i^A\right\}_{i \in I \cup J}\right) \quad \text{and} \quad \mathbb{A} = \left(A, \left\{R_i^A\right\}_{i \in I}\right)$$

then we say that $\mathbb{A}$ is a *reduct* of $\mathbb{A}'$ or that $\mathbb{A}'$ is an *expansion* of $\mathbb{A}$. We denote this by $\mathbb{A} = \mathbb{A}'|L$ and we write $\mathbb{A}' = (\mathbb{A}, \{R_j^A\}_{j \in J})$. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures in L and L' respectively. If for every $\mathbb{A} \in \mathcal{K}$ there is some $\mathbb{B} \in \mathcal{K}$ such that for every $\mathbb{A}', \mathbb{B}' \in \mathcal{K}'$ with $\mathbb{A} = \mathbb{A}'|L$ and $\mathbb{B} = \mathbb{B}'|L$ there exists an embedding of $\mathbb{A}'$ into $\mathbb{B}'$ then we say that $\mathcal{K}'$ satisfies the *expansion property (EP)* with respect to $\mathcal{K}$. In this case we say that $\mathbb{B}$ *verifies EP* for $\mathbb{A}$. If $\mathbb{A}' \in \mathcal{K}'$ and $\mathbb{B} \in \mathcal{K}$ are such that for every $\mathbb{B}' \in \mathcal{K}'$ with $\mathbb{B}'|L = \mathbb{B}$ we have $\mathbb{A}' \hookrightarrow \mathbb{B}'$, then we say that $\mathbb{B}$ *verifies EP* for $\mathbb{A}'$. If $L' = \{\leq\}$, $\leq$ is interpreted in all structures in $\mathcal{K}'$ as a linear ordering and $\mathcal{K}'$ satisfies EP with respect to $\mathcal{K}$ then we say that $\mathcal{K}'$ satisfies the *ordering property (OP)* with respect to $\mathcal{K}$. Similarly, we say that $\mathbb{B}$ *verifies OP* for $\mathbb{A}$ or $\mathbb{A}'$. We say that $\mathcal{K}'$ is a *precompact expansion* of $\mathcal{K}$ if $|\{\mathbb{A}' \in \mathcal{K}' : \mathbb{A} = \mathbb{A}'|L\}|$ is a non-zero natural number for every $\mathbb{A} \in \mathcal{K}$. If $\mathcal{K}'$ is an expansion of the class $\mathcal{K}$ such that $\mathcal{K}'$ contains structures of the form $(\mathbb{A}, \leq^A)$ where $\mathbb{A} \in \mathcal{K}$ and $\leq^A$ is a linear ordering then we say that $\mathcal{K}'$ is an *ordered expansion* of $\mathcal{K}$.

Let $\mathcal{K}$ be a class of finite relational structures in a signature L. We say that $\mathcal{K}$ satisfies the following:

- *Hereditary property* (HP) if $\mathbb{A} \hookrightarrow \mathbb{B}$ and $\mathbb{B} \in \mathcal{K}$ then $\mathbb{A} \in \mathcal{K}$.
- *Joint embedding property* (JEP) if for all $\mathbb{A} \in \mathcal{K}$ and $\mathbb{B} \in \mathcal{K}$ there is some $\mathbb{C} \in \mathcal{K}$ such that $\mathbb{A} \hookrightarrow \mathbb{C}$ and $\mathbb{B} \hookrightarrow \mathbb{C}$.
- *Strong joint embedding property* (SJEP) if for all $\mathbb{A} \in \mathcal{K}$ and $\mathbb{B} \in \mathcal{K}$ there are $\mathbb{A}', \mathbb{B}', \mathbb{C} \in \mathcal{K}$ such that $\mathbb{A}' \leq \mathbb{C}$, $\mathbb{B}' \leq \mathbb{C}$, $\mathbb{A}' \cong \mathbb{A}$, $\mathbb{B}' \cong \mathbb{B}$, and the underlying sets of $\mathbb{A}'$ and $\mathbb{B}'$ are disjoint.
- *Amalgamation property* (AP) if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ and all embeddings $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ there are some $\mathbb{D} \in \mathcal{K}$ and embeddings $\bar{f} : \mathbb{B} \to \mathbb{D}$ and $\bar{g} : \mathbb{C} \to \mathbb{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$.
- *Strong amalgamation property* (SAP) if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ with the underlying sets A, B, C respectively and all embeddings $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ there are some $\mathbb{D} \in \mathcal{K}$ and embeddings $\bar{f} : \mathbb{B} \to \mathbb{D}$ and $\bar{g} : \mathbb{C} \to \mathbb{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$ and $\bar{f}(B) \cap \bar{g}(C) = \bar{f} \circ f(A) = \bar{g} \circ g(A)$.
- *Two point amalgamation property* (2AP) if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ with the underlying sets A, B, C respectively and all embeddings $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ with the property $|B \setminus f(A)| = 1$ and $|C \setminus g(A)| = 1$ there are $\mathbb{D} \in \mathcal{K}$ and embeddings $\bar{f} : \mathbb{B} \to \mathbb{D}$ and $\bar{g} : \mathbb{C} \to \mathbb{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$.

It should be clear that AP implies 2AP. We also have the converse under an additional assumption.

Lemma 1. *(See [6].) If a class $\mathcal{K}$ of finite relational structures satisfies HP and 2AP then it satisfies AP.*

A countable class $\mathcal{K}$ of finite structures, in a countable signature L which contains structures of arbitrary large finite cardinality, and satisfies HP, JEP and AP is called a *Fraïssé class*. We give the following lemma without a proof as a useful technical statement.

Lemma 2. *Let $\mathcal{K}$ be a class of structures in a relational signature L. Let $\mathcal{K}'$ be a class of structures in a relational signature L' such that $\mathcal{K}'$ is a precompact expansion of $\mathcal{K}$. If $\mathcal{K}'$ satisfies JEP and for every $\mathbb{A}' \in \mathcal{K}'$ there is a $\mathbb{B} \in \mathcal{K}$ that verifies EP for $\mathbb{A}'$ then $\mathcal{K}'$ has EP with respect to $\mathcal{K}$.*

3. Classes

Let L be a relational signature with arities $\{n_i\}_{i\in I}$. Let $\mathbb{A} = (A, \{R_i^A\}_{i\in I})$ and $\mathbb{B} = (B, \{R_i^B\}_{i\in I})$ be structures in L. If $A = B$ and $R_i^A \subset R_i^B$ for all $i \in I$ then we say that $\mathbb{B}$ is a *thickening* of $\mathbb{A}$. If $\mathcal{K}$ is a class of structures in L and $\mathbb{B} \in \mathcal{K}$ is a thickening of $\mathbb{A}$ then we say that $\mathbb{B}$ is a *thickening* of $\mathbb{A}$ in $\mathcal{K}$.

Let $\mathbb{A} = (A, \{R_i^A\}_{i\in I})$ be a structure in a relational signature $L = \{R_i\}_{i\in I}$, with arities $\{n_i\}_{i\in I}$. For a non-empty set B we define a relational L-structure $\mathbb{C} = (C, \{R_i^C\}_{i\in I})$ with $C = A\times B$. For $i \in I$ we define relation R_i^C as follows. Let $\bar{c} = (c^1, \dots, c^{n_i})$ be a sequence of points from C such that $c^s = (c_1^s, c_2^s)$, $s \in [n_i]$. Then, $R_i^C(\bar{c})$ iff for all $s, s' \in [n_i]$ we have:

- $R_i^A(c_1^1, c_1^2, \dots, c_1^{n_i})$,
- $c_1^s = c_1^{s'} \Rightarrow c^s = c^{s'}$.

In this case we say that $\mathbb{C}$ is a *lifting* of $\mathbb{A}$ by B. If $\mathbb{D}$ is a thickening of $\mathbb{C}$ such that for every diagonal $S \subset C$ we have $\mathbb{C} \restriction S = \mathbb{D} \restriction S$ then we say that $\mathbb{D}$ is a *diagonal thickening* of $\mathbb{C}$.

Let $\{\mathcal{K}_i\}_{i\in I}$ be a sequence of classes of finite relational structures in the relational signatures $\{L_i\}_{i\in I}$, respectively, where the signatures are pairwise disjoint. We define the *product of classes*, $\prod_{i\in I} \mathcal{K}_i$, to be the class of all finite relational structures $\mathbb{A}$ in the signature $L = \bigcup_{i\in I} L_i$ such that $\mathbb{A}|L_i \in \mathcal{K}_i$ for all $i \in I$. In particular, for $I = [n]$ we write $\prod_{i\in I} \mathcal{K}_i = \prod_{i=1}^n \mathcal{K}_i = \mathcal{K}_1 * \dots * \mathcal{K}_n$. Let us emphasize that even if $\mathcal{K}_1$ and $\mathcal{K}_2$ are non-empty it could be that $\mathcal{K}_1 * \mathcal{K}_2 = \emptyset$. For example, if $\mathcal{K}_1$ contains only structures of odd cardinality and $\mathcal{K}_2$ contains only structures of even cardinality then we have $\mathcal{K}_1 * \mathcal{K}_2 = \emptyset$. If I is finite and each of the classes $\mathcal{K}_i$ is countable then their product is at most countable. If I is infinite and the classes $\mathcal{K}_i$ are non-empty then their product can be uncountable. For example, if I is infinite and each $\mathcal{K}_i$ is the class of finite graphs then their product is uncountable. Let $\{\mathcal{K}_\lambda\}_{\lambda\in\Delta}$ be a sequence of classes of finite relational structures in relational signatures $\{L_\lambda\}_{\lambda\in\Delta}$, respectively, where the signatures are pairwise disjoint. We define the *sum of classes*, $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$, to be the class of all finite relational structures $\mathbb{A}$ in the signature $L = \bigcup_{\lambda\in\Delta} L_\lambda$ for which there is some finite $\Delta_0 \subseteq \Delta$ such that:

- $\mathbb{A}$ is a structure in $\bigcup_{\lambda\in\Delta_0} L_\lambda$ and
- for every $\lambda \in \Delta_0$ we have $\mathbb{A}|L_\lambda \in \mathcal{K}_\lambda$.

For finite $\Delta_1 = \{\lambda_1, \dots, \lambda_n\} \subseteq \Delta$ we have $\mathcal{K}_{\lambda_1} * \cdots * \mathcal{K}_{\lambda_n} \subseteq \sum_{\lambda\in\Delta} \mathcal{K}_\lambda$, but if $\Delta_0 = \{\mu_1, \dots, \mu_n\} \subseteq \Delta_1$ we do not have always $\mathcal{K}_{\mu_1} * \cdots * \mathcal{K}_{\mu_m} \subseteq \mathcal{K}_{\lambda_1} * \cdots * \mathcal{K}_{\lambda_n}$. For finite $\Delta_1 = \{\lambda_1, \dots, \lambda_n\}$ we also write $\mathcal{K}_{\lambda_1} + \cdots + \mathcal{K}_{\lambda_n}$ instead of $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$. If Δ is countable and $\mathcal{K}_\lambda$ is countable for each $\lambda \in \Delta$, then their sum is at most countable. In particular the class $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$ is the union of all classes $\mathcal{K}_{\lambda_1} * \cdots * \mathcal{K}_{\lambda_n}$ where $\{\lambda_1, \dots, \lambda_n\}$ ranges over all finite subsets of Δ. Note that for finite Δ of cardinality at least 2, typically, we have

$$\prod_{\lambda\in\Delta} \mathcal{K}_\lambda \neq \sum_{\lambda\in\Delta} \mathcal{K}_\lambda.$$

Example 1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of finite linearly ordered sets in two disjoint signatures $\{\leq\}$ and $\{\preceq\}$ respectively. Consider structures $\mathbb{A} = (A, \leq^A) \in \mathcal{K}_1$ and $\mathbb{B} = (B, \leq^B, \preceq^B) \in \mathcal{K}_1 * \mathcal{K}_2$. Suppose there is some $\mathbb{C} \in \mathcal{K}_1 + \mathcal{K}_2$ such that $\mathbb{A} \hookrightarrow \mathbb{C}$ and $\mathbb{B} \hookrightarrow \mathbb{C}$. Since $\mathbb{A} \hookrightarrow \mathbb{C}$ we have that $\mathbb{C}$ contains at most one linear ordering, and from $\mathbb{B} \hookrightarrow \mathbb{C}$ we have that it must contain two linear orderings. Therefore there is no such $\mathbb{C}$, and $\mathcal{K}_1 + \mathcal{K}_2$ does not satisfy JEP.

In order to avoid this obstacle we consider the following property. We say that the class $\mathcal{K}$ of finite structures in a relational signature $\{R_i\}_{i\in I}$ has the *spacing property* (SP) if it contains structures $(A, \{R_i^A\}_{i\in I})$ of arbitrary large finite cardinality such that for every $i \in I$ we have $R_i^A = \emptyset$. Note that the class of finite graphs has SP, but the class of finite linear orderings does not have SP.

In order to simplify the presentation of the proof of Lemma 3 and the proof of Lemma 4 we introduce 2AP modification and doubling as follows. Let L be a relational signature. Let $\mathcal{K}$ be a class of finite structures in the signature L which satisfies *2AP* and *HP*. Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be finite structures in L, with the underlying sets A, B and C respectively. Then we have the following:

- If $\mathbb{A} \in \mathcal{K}$, $\mathbb{B} \in \mathcal{K}$, $\mathbb{A} \cap \mathbb{B} \in \mathcal{K}$, $A \cap B = S$, $A = S \cup \{a\}$, $B = S \cup \{b\}$ and $C = S \cup \{a, b\}$ then there is some $\mathbb{D} \in \mathcal{K}$ with the underlying set C such that $\mathbb{A} \leq \mathbb{D}$ and $\mathbb{B} \leq \mathbb{D}$. We say that $\mathbb{D}$ is obtained by *2AP modification* of $\mathbb{C}$ from $\mathbb{A}$ and $\mathbb{B}$.
- If $\mathbb{A} \in \mathcal{K}$, $\mathbb{B} \in \mathcal{K}$, $B = A \cup \{b\}$, $b \notin A$ and $\mathbb{A} \leq \mathbb{B}$ then there is some $\mathbb{D} \in \mathcal{K}$ with the underlying set $C = B \cup \{c\}$, $c \notin B$, such that $\mathbb{B} \leq \mathbb{C}$ and $\mathbb{C}|(A \cup \{c\}) \cong \mathbb{B}$. We say that $\mathbb{C}$ is obtained by *doubling* $\mathbb{B}$ over $\mathbb{A}$.

Lemma 3. *Let L_1 and L_2 be disjoint signatures, and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of finite relational structures in L_1 and L_2 respectively. Let $\{\mathcal{K}_\lambda\}_{\lambda\in\Delta}$ be a sequence of classes of finite relational structures in the relational signatures $\{L_\lambda\}_{\lambda\in\Delta}$, respectively, where signatures are pairwise disjoint. Then we have:*

(i) [4] *If $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy HP and SAP then $\mathcal{K}_1 * \mathcal{K}_2$ satisfies HP and SAP.*
(ii) *If $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy HP and SJEP then $\mathcal{K}_1 * \mathcal{K}_2$ satisfies SJEP.*
(iii) *If $\mathcal{K}_1$ satisfies HP, JEP and SAP, and every one point structure in $\mathcal{K}_1$ can be embedded in a larger structure in $\mathcal{K}_1$ then $\mathcal{K}_1$ satisfies SJEP.*
(iv) *Let $\mathcal{K}_\lambda$, $\lambda \in \Delta$, satisfy HP, JEP, SAP and suppose that every one point structure in $\mathcal{K}_\lambda$ can be embedded in a larger structure in $\mathcal{K}_\lambda$. Then $\prod_{\lambda\in\Delta} \mathcal{K}_\lambda$ satisfies HP, SJEP and SAP. In particular if Δ is finite and each $\mathcal{K}_\lambda$ is a Fraïssé class which has SAP then $\prod_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Fraïssé class which has SJEP and SAP.*
(v) *Let $\mathcal{K}_\lambda$, $\lambda \in \Delta$, satisfy HP, JEP, SAP, SP and suppose that every one point structure in $\mathcal{K}_\lambda$ can be embedded in a larger structure in $\mathcal{K}_\lambda$. If Δ is at most countable then $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$ satisfies HP, SJEP and SAP. In particular if $\mathcal{K}_\lambda$, $\lambda \in \Delta$, is a Fraïssé class which has SAP and SP then $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Fraïssé class which has SJEP and SAP.*

Proof.

(i) This is Proposition 2.2 in [4].
(ii) Let $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathcal{K}_1 * \mathcal{K}_2$ with the underlying sets A and B respectively. Without loss of generality we may assume that $A \cap B = \emptyset$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy HP and SJEP there are $\mathbb{C}_1 \in \mathcal{K}_1$ and $\mathbb{C}_2 \in \mathcal{K}_2$ with the same underlying set $A \cup B$ such that $\mathbb{C}_1 \restriction A = \mathbb{A}|L_1$, $\mathbb{C}_1 \restriction B = \mathbb{B}|L_1$, $\mathbb{C}_2 \restriction A = \mathbb{A}|L_2$, $\mathbb{C}_2 \restriction B = \mathbb{B}|L_2$. Let $\mathbb{C} \in \mathcal{K}_1 * \mathcal{K}_2$ be such that $\mathbb{C}|L_1 = \mathbb{C}_1$ and $\mathbb{C}|L_2 = \mathbb{C}_2$. Therefore $\mathbb{C} \restriction A = \mathbb{A}$ and $\mathbb{C} \restriction B = \mathbb{B}$. Since $A \cap B = \emptyset$ this completes the verification of SJEP for the class $\mathcal{K}_1 * \mathcal{K}_2$.
(iii) Let $\mathbb{A}_0$, $\mathbb{A}_{00}$, $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathcal{K}_1$ with underlying sets A_0, A_{00}, A and B respectively. We assume that $B = A_0 \cup A_{00}$, $A = A_0 \cap A_{00}$ and that $\mathbb{A}_0$, $\mathbb{A}_{00}$ and $\mathbb{A}$ are substructures of $\mathbb{B}$ induced by the sets A_0, A_{00} and A respectively. Since every one point structure is contained in some larger structure, and since $\mathcal{K}_1$ satisfies HP we may assume $|A| \geq 2$. Let $A = \{a_1, \ldots, a_n\}$ for $n \geq 1$. Let $\mathbb{A}_1 \leq \mathbb{B}$ be induced by the set $B \setminus \{a_1\}$, and let $\mathbb{B}_1 \in \mathcal{K}_1$ be obtained by doubling $\mathbb{B}$ over $\mathbb{A}_1$. So $\mathbb{B}_1$ has the underlying set $B_1 = B \cup \{a'_1\}$ such that $a'_1 \notin B$. We define recursively structures $\mathbb{A}_i$ and $\mathbb{B}_i$ with underlying sets A_i and B_i, respectively, for $2 \leq i \leq n$. We take $A_i = B_{i-1} \setminus \{a_i\}$, and $\mathbb{B}_i$ is obtained by doubling $\mathbb{B}_{i-1}$ over $\mathbb{A}_i$. In particular we have $B_i = B_{i-1} \cup \{a'_i\}$ where $a'_i \notin B_{i-1}$. Now we take $A' = \{a'_1, \ldots, a'_n\}$, $A'_0 = (A_0 \setminus A) \cup A'$ and $\mathbb{A}'_0$ to be substructures of $\mathbb{B}_n$ induced by the sets A'_0. Then we have $A'_0 \cap A_{00} = \emptyset$ and $\mathbb{A}'_0 \cong \mathbb{A}_0$ so $\mathbb{B}_n$ verifies SJEP for $\mathbb{A}_0$ and $\mathbb{A}_{00}$.
(iv) This follows by the same arguments as (i), (ii) and (iii). We prove the claim for the class $\prod_{\lambda\in\Delta} \mathcal{K}_\lambda$ and then we prove the claim for the class $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$. It is enough to check the claim for finite Δ, and this follows from (i), (ii), (iii) and by induction on the size of Δ. We use SP in order to verify SJEP for the class $\mathcal{K} = \sum_{\lambda\in\Delta} \mathcal{K}_\lambda$. Let $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathcal{K}$ such that $\mathbb{A} \in \mathcal{K}_{\lambda_1} * \cdots * \mathcal{K}_{\lambda_n}$ and $\mathbb{B} \in \mathcal{K}_{\mu_1} * \cdots * \mathcal{K}_{\mu_m}$. Since $\mathcal{K}_\lambda$ satisfies SP for all $\lambda \in \Delta$, we may consider structures $\mathbb{A}$ and $\mathbb{B}$ as structures

in $\prod_{\lambda\in I}\mathcal{K}_\lambda$ where $I=\{\lambda_1,\dots,\lambda_n\}\cup\{\mu_1,\dots,\mu_m\}$. Since $\prod_{\lambda\in I}\mathcal{K}_\lambda$ satisfies SJEP, by the first part of this claim we have that $\sum_{\lambda\in\Delta}\mathcal{K}_\lambda$ satisfies SJEP.

(v) HP and SAP are verified in the same way as in (iv), but SJEP follows from SP. Let $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathcal{K}$ such that $\mathbb{A}\in\mathcal{K}_{\lambda_1}*\dots*\mathcal{K}_{\lambda_n}$ and $\mathbb{B}\in\mathcal{K}_{\mu_1}*\dots*\mathcal{K}_{\mu_m}$. Since $\mathcal{K}_\lambda$ satisfies SP for all $\lambda\in\Delta$, we may consider $\mathbb{A}$ and $\mathbb{B}$ also as structures in $\prod_{\lambda\in I}\mathcal{K}_\lambda$ where $I=\{\lambda_1,\dots,\lambda_n\}\cup\{\mu_1,\dots,\mu_m\}$. Since $\prod_{\lambda\in I}\mathcal{K}_\lambda$ satisfies SJEP, see (iv), we have that $\sum_{\lambda\in\Delta}\mathcal{K}_\lambda$ satisfies SJEP. □

4. Ramsey property

Let $\mathcal{K}$ be a class of finite structures in a given relational signature L. If for a natural number r and structures $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ from $\mathcal{K}$ we have that for every coloring $c:\binom{\mathbb{C}}{\mathbb{A}}\to[r]$, there is a $\mathbb{B}'\in\binom{\mathbb{C}}{\mathbb{B}}$ such that $c\restriction\binom{\mathbb{B}'}{\mathbb{A}}=const$, then we write $\mathbb{C}\to(\mathbb{B})^{\mathbb{A}}_r$. If for all $\mathbb{A}$ and $\mathbb{B}$ from $\mathcal{K}$ and all natural numbers r there is $\mathbb{C}\in\mathcal{K}$ such that $\mathbb{C}\to(\mathbb{B})^{\mathbb{A}}_r$, then we say that $\mathcal{K}$ is a *Ramsey class* or that $\mathcal{K}$ satisfies the *Ramsey property* (RP). In the case when a given class does not satisfy RP we measure its deviation from being Ramsey as follows. If for natural numbers r and t, and structures $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ from $\mathcal{K}$ we have that for every coloring $c:\binom{\mathbb{C}}{\mathbb{A}}\to[r]$, there is a $\mathbb{B}'\in\binom{\mathbb{C}}{\mathbb{B}}$ such that $\left|c\left(\binom{\mathbb{B}'}{\mathbb{A}}\right)\right|\le t$, then we write $\mathbb{C}\to(\mathbb{B})^{\mathbb{A}}_{r,t}$. If for $\mathbb{A}\in\mathcal{K}$ there is a natural number t_0 such that for any natural number r and any $\mathbb{B}\in\mathcal{K}$ there is a $\mathbb{C}\in\mathcal{K}$ such that $\mathbb{C}\to(\mathbb{B})^{\mathbb{A}}_{r,t_0}$, then we say that $\mathcal{K}$ has *finite Ramsey degree* in $\mathcal{K}$ and the smallest t_0 with this property is called the *Ramsey degree* of $\mathbb{A}$ in $\mathcal{K}$. The Ramsey degree of $\mathbb{A}$ in $\mathcal{K}$ is denoted by $t_{\mathcal{K}}(\mathbb{A})$. We calculate Ramsey degrees by using the following.

Theorem 2. *(See [12,18].) Let $\mathcal{K}$ be a relational Fraïssé class in the signature L and let $\mathcal{K}'$ be a relational Fraïssé class in a signature L' such that $L\cap(L'\setminus L)=\emptyset$. Suppose that $\mathcal{K}'$ is a precompact expansion of $\mathcal{K}$. If $\mathcal{K}'$ satisfies RP and EP with respect to $\mathcal{K}$ then*

$$t_{\mathcal{K}}(\mathbb{A})=\frac{|\{\mathbb{A}'\in\mathcal{K}':\mathbb{A}'|L=\mathbb{A}\}|}{|Aut(\mathbb{A})|}.$$

In particular, if $\mathbb{A}$ is a rigid structure then $t_{\mathcal{K}}(\mathbb{A})=|\{\mathbb{A}'\in\mathcal{K}':\mathbb{A}'|L=\mathbb{A}\}|$.

In this Section we will show how to transfer the Ramsey property to the product of classes. Our approach is motivated by the approach in [22], so we will use some notation from [22]. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of structures. Let $\vec{\mathbb{A}}=(\mathbb{A}_1,\mathbb{A}_2)$ and $\vec{\mathbb{B}}=(\mathbb{B}_1,\mathbb{B}_2)$ be from $\mathcal{K}_1\times\mathcal{K}_2$, and let A_1 and A_2 be the underlying sets for the structures $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively. Then we define:

(1) $\vec{\mathbb{A}}\le\vec{\mathbb{B}}$ iff $\mathbb{A}_i\le\mathbb{B}_i$ for all $i\in[2]$,
(2) $\vec{\mathbb{A}}\cong\vec{\mathbb{B}}$ iff $\mathbb{A}_i\cong\mathbb{B}_i$ for all $i\in[2]$,
(3) $\binom{\vec{\mathbb{B}}}{\vec{\mathbb{A}}}=\{\vec{\mathbb{C}}\in\mathcal{K}_1\times\mathcal{K}_2:\vec{\mathbb{C}}\le\vec{\mathbb{B}}\text{ and }\vec{\mathbb{C}}\cong\vec{\mathbb{A}}\}$.

We refer to the following result as the product Ramsey theorem for classes, see [21].

Theorem 3. *Let r be a natural number, and let $\mathcal{K}_1$, $\mathcal{K}_2$ be Ramsey classes. Then for all $\vec{\mathbb{A}}, \vec{\mathbb{B}} \in \mathcal{K}_1 \times \mathcal{K}_2$ with $\binom{\vec{\mathbb{B}}}{\vec{\mathbb{A}}} \neq \emptyset$ there is a $\vec{\mathbb{C}} \in \mathcal{K}_1 \times \mathcal{K}_2$ such that for every coloring $\chi : \binom{\vec{\mathbb{C}}}{\vec{\mathbb{A}}} \to [r]$ there is a $\vec{\mathbb{B}'} \in \mathcal{K}_1 \times \mathcal{K}_2$ such that $\vec{\mathbb{B}'} \cong \vec{\mathbb{B}}$ and $\chi \restriction \binom{\vec{\mathbb{B}'}}{\vec{\mathbb{A}}} = const$.*

If $\vec{\mathbb{A}}, \vec{\mathbb{B}}, \vec{\mathbb{C}} \in \mathcal{K}_1 \times \mathcal{K}_2$ satisfy the statement of the previous theorem then we write $\vec{\mathbb{C}} \to (\vec{\mathbb{B}})^{\vec{\mathbb{A}}}_r$. If in the previous theorem we take $\mathcal{K}_1$ and $\mathcal{K}_2$ to be the classes of finite linearly ordered sets then we obtain the classical product Ramsey theorem, see page 97 in [8]. In the following we obtain the Ramsey statement for the product of two Ramsey classes.

Theorem 4. *Let $L_1 = \{R_i\}_{i \in I}$ and $L_2 = \{R_j\}_{j \in J}$ be disjoint relational signatures with arities $\{n_i\}_{i \in I}$ and $\{n_j\}_{j \in J}$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Ramsey classes of finite rigid structures in L_1 and L_2, respectively. If for every lifting of a structure in $\mathcal{K}_1$ ($\mathcal{K}_2$) there is a diagonal thickening in $\mathcal{K}_1$ ($\mathcal{K}_2$) then $\mathcal{K} = \mathcal{K}_1 * \mathcal{K}_2$ is a Ramsey class.*

Proof. Let r be a given natural number. Let $\mathbb{A} = (A, \{R_i^A\}_{i \in I}, \{R_j^A\}_{j \in J})$ and $\mathbb{B} = (B, \{R_i^B\}_{i \in I}, \{R_j^B\}_{j \in J})$ be structures from $\mathcal{K}$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$. We consider structures:

$$\mathbb{A}_1 = \left(A, \left\{R_i^A\right\}_{i \in I}\right) = \mathbb{A}|L_I, \qquad \mathbb{A}_2 = \left(A, \left\{R_j^A\right\}\right) = \mathbb{A}|L_J,$$
$$\mathbb{B}_1 = \left(B, \left\{R_i^B\right\}_{i \in I}\right) = \mathbb{B}|L_I, \qquad \mathbb{B}_2 = \left(B, \left\{R_j^B\right\}\right) = \mathbb{B}|L_J.$$

Note that $\mathbb{A}_1, \mathbb{B}_1 \in \mathcal{K}_1$ and $\mathbb{A}_2, \mathbb{B}_2 \in \mathcal{K}_2$, and that $\binom{\mathbb{B}_1}{\mathbb{A}_1} \neq \emptyset$ and $\binom{\mathbb{B}_2}{\mathbb{A}_2} \neq \emptyset$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are Ramsey classes, by Theorem 3, there are structures $\mathbb{C}_1 = (C_1, \{R_i^{C_1}\}_{i \in I}) \in \mathcal{K}_1$ and $\mathbb{C}_2 = (C_2, \{R_i^{C_2}\}_{i \in I}) \in \mathcal{K}_2$ such that

$$(\mathbb{C}_1, \mathbb{C}_2) \to (\mathbb{B}_1, \mathbb{B}_2)^{(\mathbb{A}_1, \mathbb{A}_2)}_r.$$

We define a structure $\mathbb{C} = (C, \{R_i^C\}_{i \in I}, \{R_j^C\})$ with the underlying set $C = C_1 \times C_2$ in signature $L_1 \cup L_2$ such that:

- $\mathbb{C}|L_1$ is a lifting of $\mathbb{C}_1$ by C_2,
- $\mathbb{C}|L_2$ is a lifting of $\mathbb{C}_2$ by C_1.

We denote by $\pi_i : C \to C_i$ the projection $\pi_i(c_1, c_2) = c_i$, $i \in [2]$. For diagonals S and S' in C, $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ we have the following four facts:

Fact 1. $\mathbb{C} \restriction \mathbb{S} \in \mathcal{K}$ iff $\mathbb{C}_i \restriction \pi_i(S) \in \mathcal{K}_i$ for $i \in [2]$.

Fact 2. If $\mathbb{C} \restriction S \cong \mathbb{C} \restriction S' \in \mathcal{K}$ then $\mathbb{C}_i \restriction \pi_i(S) \cong \mathbb{C}_i \restriction \pi_i(S')$ for $i \in [2]$ and there are unique isomorphisms which verify this (because the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ contain only rigid structures).

Fact 3. If $\mathbb{C} \restriction \mathbb{S} \in \mathcal{K}$, $\mathbb{C}_1 \restriction A_1 \cong \mathbb{C}_1 \restriction \pi_1(S)$ and $\mathbb{C}_2 \restriction A_2 \cong \mathbb{C}_2 \restriction \pi_2(S)$ then there is a unique diagonal $S' \subset A_1 \times A_2$ such that $\mathbb{C} \restriction S \cong \mathbb{C} \restriction S'$.

We check Fact 3. From Fact 1 we have $\mathbb{C}_i \restriction \pi_i(S) \in \mathcal{K}_i$ for $i \in [2]$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of rigid structures there are unique isomorphisms $\varphi_1 : \pi_1(S) \to A_1$ and $\varphi_1 : \pi_2(S) \to A_2$. Consider the map $f : S \to A_1 \times A_2$ given by

$$f(x) = \big(\varphi_1 \circ \pi_1(x), \varphi_2 \circ \pi_2(x)\big).$$

Clearly, f is a bijection from S to $S' = f(S)$ such that $\mathbb{C} \restriction S \cong \mathbb{C} \restriction S'$. Uniqueness of S' follows from the fact that $\mathbb{C}_1 \restriction \pi_1(S')$ and $\mathbb{C}_2 \restriction \pi_2(S')$ are rigid structures and $\pi_1(S') = A_1$ and $\pi_2(S') = A_2$.

Fact 4. For each $\mathbb{K} \in \mathcal{K}$, if $\mathbb{C}_1 \restriction A_1 \cong \mathbb{K}|L_I$ and $\mathbb{C}_2 \restriction A_2 \cong \mathbb{K}|L_J$ then there is a unique diagonal $S \subset A_1 \times A_2$ such that $\mathbb{C} \restriction S \cong \mathbb{K}$.

The proof is similar to that of Fact 3. There are unique isomorphisms $\varphi_1 : \mathbb{K}|L_I \to \mathbb{C}_1 \restriction A_1$ and $\varphi_2 : \mathbb{K}|L_J \to \mathbb{C}_2 \restriction A_2$. If K is the underlying set for $\mathbb{K}$, then the unique embedding $f : K \to A_1 \times A_2$ is given by $f(x) = (\varphi_1(x), \varphi_2(x))$.

Since liftings of structures in $\mathcal{K}_1$ and $\mathcal{K}_2$ have diagonal thickenings in $\mathcal{K}_1$ and $\mathcal{K}_2$ there are $\mathbb{D}_1 \in \mathcal{K}_1$ and $\mathbb{D}_2 \in \mathcal{K}_2$ which are thickening of $\mathbb{C}|L_1$ and $\mathbb{C}|L_2$ respectively. Moreover, there is $\mathbb{D} = (D, \{R_i^C\}_{i \in I}, \{R_j^C\}) \in \mathcal{K}$ which is a thickening of $\mathbb{C}$ such that $\mathbb{D}|L_1 = \mathbb{D}_1$ and $\mathbb{D}|L_2 = \mathbb{D}_2$ and for every diagonal $S \subset C$ we have $\mathbb{D} \restriction S = \mathbb{C} \restriction S$.

We claim that $\mathbb{D} \to (\mathbb{B})^{\mathbb{A}}_r$. Let $p : \binom{\mathbb{D}}{\mathbb{A}} \to [r]$ be a given coloring. There is an induced coloring:

$$\bar{p} : \binom{(\mathbb{C}_1, \mathbb{C}_2)}{(\mathbb{A}_1, \mathbb{A}_2)} \to [r],$$
$$\bar{p}\big((\mathbb{A}'_1, \mathbb{A}'_2)\big) = p\big(\mathbb{A}'\big),$$

where $\mathbb{A}'$, with underlying set A', is the structure given by the unique diagonal such that:

- A'_1 is the underlying set for $\mathbb{A}'_1$,
- A'_2 is the underlying set for $\mathbb{A}'_2$,
- $\mathbb{C} \restriction A' \cong \mathbb{A}$.

The coloring $\bar{p}$ is well-defined by Fact 3 and since $\mathbb{D} \restriction S = \mathbb{C} \restriction S$ for a given diagonal $S \subseteq C$. According to the choice of the sequence $(\mathbb{C}_1, \mathbb{C}_2)$ there is a $\mathbb{B}'_1 \in \binom{\mathbb{C}_1}{\mathbb{B}_1}$, with underlying set B'_1, and a $\mathbb{B}'_2 \in \binom{\mathbb{C}_2}{\mathbb{B}_2}$, with underlying set B'_2, such that $\bar{p} \restriction \binom{(\mathbb{B}'_1, \mathbb{B}'_2)}{(\mathbb{A}_1, \mathbb{A}_2)} = const$. There is a unique diagonal set $B' \subset B'_1 \times B'_2$ such that $\mathbb{C} \restriction B' = \mathbb{D} \restriction B' \cong \mathbb{B}$, by Fact 4. Therefore $p \restriction \binom{\mathbb{D} \restriction B'}{\mathbb{A}} = const$. $\square$

Corollary 1. *Let $\{L_\lambda\}_{\lambda\in\Delta}$ be a list of mutually disjoint relational signatures. Let $\{\mathcal{K}_\lambda\}_{\lambda\in\Delta}$ be a list of classes of finite relational structures in the signatures $\{L_\lambda\}_{\lambda\in\Delta}$, respectively. For each $\lambda \in \Delta$, let $\mathcal{K}_\lambda$ be a Ramsey class and suppose that for every lifting of a structure in $\mathcal{K}_\lambda$ there is a diagonal thickening in $\mathcal{K}_\lambda$. Then $\mathcal{S} = \sum_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Ramsey class and if Δ is finite then $\mathcal{K} = \prod_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Ramsey class.*

Proof. Note that from the definition of the classes $\mathcal{K}$ and $\mathcal{S}$ it is enough to consider only the case in which Δ is finite. For finite Δ we prove this by induction on $|\Delta|$. This follows from Theorem 4 and the fact that if liftings of structures in $\mathcal{K}_1$ and $\mathcal{K}_2$ have diagonal thickenings then liftings of structures in $\mathcal{K}_1 * \mathcal{K}_2$ have diagonal thickenings. $\square$

Lemma 4. *Let $\mathcal{K}$ be a relational Fraïssé class with SAP. Let $\mathbb{A}$ be a structure from $\mathcal{K}$, and let $\mathbb{C}$ be a lifting of $\mathbb{A}$ by B where B is a non-empty set. Then there is a $\mathbb{D} \in \mathcal{K}$ such that $\mathbb{D}$ is a* diagonal thickening *of $\mathbb{C}$.*

Proof. Let $A = \{a_1, a_2, \dots, a_n\}$ be the underlying set of $\mathbb{A}$, let $B = \{b_1, b_2, \dots, b_m\}$, and let $C = A \times B$ be the underlying set of $\mathbb{C}$. We modify the structure $\mathbb{C}$ to the structure $\mathbb{D} \in \mathcal{K}$ in finitely many recursive steps by considering sets

$$P_{i,j} = \big\{(a_r, b_s) : \big(s < j \text{ and } r \in [n]\big) \text{ or } \big(s = j \text{ and } r \in [i]\big)\big\}$$

for $i \in [n]$ and $j \in [m]$. Our recursion is based on the lexicographical ordering $<$ of the set $[n] \times [m]$ such that $(i, j) < (i', j')$ iff $(j < j')$ or $(j = j',\ i < i')$. We start our construction from $(n, 1)$ instead of $(1, 1)$. During our construction we define structures $\mathbb{P}_{i,j} \in \mathcal{K}$ for $(n, 1) \le (i, j) \le (n, m)$. The main steps in our construction are:

Step $(n, 1)$: Note that $\mathbb{C} \restriction P_{n,1} \in \mathcal{K}$ by definition of lifting. We take $\mathbb{P}_{n,1} = \mathbb{C} \restriction P_{n,1}$.

Step $(1, 2)$: By definition of the lifting we have $\mathbb{C} \restriction P_{n,1} \cong \mathbb{C} \restriction (P_{1,2} \setminus \{(a_1, b_1)\}) \in \mathcal{K}$. Then we take $\mathbb{P}_{1,2}$ to be a structure in $\mathcal{K}$ obtained by 2AP modification of $\mathbb{C} \restriction P_{1,2}$ from $\mathbb{C} \restriction P_{n,1}$ and $\mathbb{C} \restriction (P_{1,2} \setminus \{(a_1, b_1)\})$. So we have $\mathbb{P}_{n,1} \le \mathbb{P}_{1,2}$ and $\mathbb{P}_{1,2}$ is a diagonal thickening of $\mathbb{C} \restriction P_{1,2}$. In particular we assume that the underlying set of $\mathbb{P}_{1,2}$ is $P_{1,2}$.

Step $(i, j),\ i > 1,\ j > 2$: The structure $\mathbb{P}_{i-1,j} \in \mathcal{K}$ was previously defined as a diagonal thickening of $\mathbb{C} \restriction P_{i-1,j}$. The structure $\mathbb{P}_{i,j} \in \mathcal{K}$ is obtained by doubling

$$\mathbb{P}_{i-1,j} \text{ over } \mathbb{P}_{i-1,j} \restriction \big(P_{i-1,j} \setminus \big\{(a_i, b_{j-1})\big\}\big).$$

Without loss of generality we may assume that the underlying set of $\mathbb{P}_{i,j}$ is $P_{i,j}$, so we also have $\mathbb{P}_{i-1,j} \le \mathbb{P}_{i,j}$ and that $\mathbb{P}_{i,j}$ is a diagonal thickening of $\mathbb{C} \restriction P_{i,j}$.

Step $(1, j),\ j > 2$: The structure $\mathbb{P}_{n,j-1} \in \mathcal{K}$ was previously defined as a diagonal thickening of $\mathbb{C} \restriction P_{n,j-1}$. The structure $\mathbb{P}_{1,j} \in \mathcal{K}$ is obtained by doubling

$$\mathbb{P}_{n,j-1} \text{ over } \mathbb{P}_{n,j-1} \restriction \big(P_{n,j-1} \setminus \big\{(a_1, b_{j-1})\big\}\big).$$

Without loss of generality we may assume that $P_{1,j}$ is the underlying set of $\mathbb{P}_{i,j}$. Then we have $\mathbb{P}_{n,j-1} \leq \mathbb{P}_{1,j}$ and that $\mathbb{P}_{1,j}$ is a diagonal thickening of $\mathbb{C} \restriction P_{1,j}$.

In the end we take $\mathbb{D} = \mathbb{P}_{n,m}$. $\square$

Corollary 1 and Lemma 4 imply the following.

Corollary 2. *Let $\{L_\lambda\}_{\lambda\in\Delta}$ be a list of pairwise disjoint relational signatures. Let $\{\mathcal{K}_\lambda\}_{\lambda\in\Delta}$ be a list of Fraïssé classes in signatures $\{L_\lambda\}_{\lambda\in\Delta}$, respectively. Suppose that for every $\lambda \in \Delta$, the class $\mathcal{K}_\lambda$ has RP and SAP. Then $\sum_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Ramsey class and if Δ is finite then $\prod_{\lambda\in\Delta} \mathcal{K}_\lambda$ is a Ramsey class.*

5. Dense local order

Let T denote the unit circle in the complex plane. We define an oriented graph structure $\mathbb{T} = (T, \longrightarrow^T)$ in a relational binary signature $\{\longrightarrow\}$. For $x, y \in T$ we define $x \longrightarrow^T y$ iff $0 < \arg(\frac{y}{x}) < \pi$. The *dense local order* is the structure $\mathbb{S}(2) = (S(2), \longrightarrow^{S(2)})$ which is the substructure of $\mathbb{T}$ determined by the set $S(2) = \{t \in T : \arg(t) \in \mathbb{Q}\}$. We consider $\mathcal{S}(2) = Age(\mathbb{S}(2))$, *the class of finite dense local orders*. It is a Fraïssé class which satisfies SAP. Let L and R be unary relational symbols. Let $\mathcal{US}(2) = (S(2), \longrightarrow^{S(2)}, L^{S(2)}, R^{S(2)})$ be the structure such that $\mathcal{US}(2)|\{\longrightarrow\} = \mathbb{S}(2)$ and for $x \in S(2)$ we have $L^{S(2)}(x)$ iff x is in the left half plane, and $R^{S(2)}(x)$ iff x is in the right half plane. We consider $\mathcal{US}(2) = Age(\mathcal{US}(2))$, a Fraïssé class which satisfies SAP. If a point from a structure in $\mathcal{US}(2)$ is indicated by L then we say that it lies in the *left half* of the structure, and otherwise we say it lies in the *right half* of the structure. For more on the structure $\mathbb{S}(2)$ we refer the reader to [6] and [18], and for more on the structure $\mathcal{US}(2)$ we refer the reader to [18].

Theorem 5. *(See [18].) For dense local order we have the following:*

(i) *The class $\mathcal{US}(2)$ has RP and EP with respect to $\mathcal{S}(2)$.*
(ii) *No order expansion of $\mathcal{S}(2)$ has RP and EP with respect to $\mathcal{S}(2)$.*

We consider the class

$$\mathcal{OS}(2) = \{(A, \longrightarrow^A, \leq^A) : (A, \longrightarrow^A) \in \mathcal{S}(2) \text{ and } \leq^A \in lo(A)\}.$$

Since $\mathcal{S}(2)$ satisfies SAP, $\mathcal{OS}(2)$ also satisfies SAP, see Proposition 5.3 in [12]. We consider $\mathcal{UOS}(2)$, a Fraïssé class, given by

$$\mathcal{UOS}(2) = \{(\mathbb{A}, \leq^A, L^A, R^A) : \mathbb{A} \in \mathcal{S}(2) \text{ and } (\mathbb{A}, \leq^A) \in \mathcal{OS}(2) \text{ and } (\mathbb{A}, L^A, R^A) \in \mathcal{US}(2)\}.$$

We say that *ordering and arrow agree* on a structure $\mathbb{A} = (A, \longrightarrow^A, \leq^A, L^A, R^A)$ if for all $a, b \in A$ we have $a \longrightarrow^A b$ and $a \leq^A b$. Otherwise we say that they *disagree*.

Corollary 3. *$\mathcal{UOS}(2)$ is a Ramsey class.*

Proof. We have $\mathcal{UOS}(2) = \mathcal{US}(2) * \mathcal{L}$ where $\mathcal{L}$ is the class of finite linearly ordered sets. Since $\mathcal{US}(2)$ and $\mathcal{L}$ are both Fraïssé classes of rigid structures with SAP and RP, it follows from Corollary 2 that $\mathcal{UOS}(2)$ is a Ramsey class. □

We use RP in order to obtain EP. More precisely, we compare two expansions of a structure from $\mathcal{OS}(2)$ to structures from $\mathcal{UOS}(2)$.

For the purpose of Proposition 1 and Proposition 2 we consider the following structures:

- $\mathbb{L}_1 = (L_1, \longrightarrow^{L_1}, \leq^{L_1}, L^{L_1}, R^{L_1})$, $L_1 = \{l_{1,1}, l_{1,2}\}$, $l_{1,1} \longrightarrow^{L_1} l_{1,2}$, $l_{1,1} \leq^{L_1} l_{1,2}$, $L^{L_1}(l_{1,1})$, $L^{L_1}(l_{1,2})$.
- $\mathbb{L}_2 = (L_2, \longrightarrow^{L_2}, \leq^{L_2}, L^{L_2}, R^{L_2})$, $L_2 = \{l_{2,1}, l_{2,2}\}$, $l_{2,1} \longrightarrow^{L_2} l_{2,2}$, $l_{2,2} \leq^{L_2} l_{2,1}$, $L^{L_2}(l_{2,1})$, $L^{L_2}(l_{2,2})$.
- $\mathbb{R}_1 = (R_1, \longrightarrow^{R_1}, \leq^{R_1}, L^{R_1}, R^{R_1})$, $R_1 = \{r_{1,1}, r_{1,2}\}$, $r_{1,1} \longrightarrow^{R_1} r_{1,2}$, $r_{1,1} \leq^{R_1} r_{1,2}$, $R^{R_1}(r_{1,1})$, $R^{R_1}(r_{1,2})$.
- $\mathbb{R}_2 = (R_2, \longrightarrow^{R_2}, \leq^{R_2}, L^{R_2}, R^{R_2})$, $R_2 = \{r_{2,1}, r_{2,2}\}$, $r_{2,1} \longrightarrow^{R_2} r_{2,2}$, $r_{2,2} \leq^{R_2} r_{2,1}$, $R^{R_2}(r_{2,1})$, $R^{R_2}(r_{2,2})$.
- $\mathbb{X}_1 = (X_1, \longrightarrow^{X_1}, \leq^{X_1}, L^{X_1}, R^{X_1})$, $X_1 = \{x_{1,1}, x_{1,2}, x_{1,3}\}$, $x_{1,1} \longrightarrow^{X_1} x_{1,2}$, $x_{1,1} \longrightarrow^{X_1} x_{1,3}$, $x_{1,2} \longrightarrow^{X_1} x_{1,3}$, $x_{1,1} \leq^{X_1} x_{1,2} \leq^{X_1} x_{1,3}$, $L^{X_1}(x_{1,1})$, $L^{X_1}(x_{1,2})$, $L^{X_1}(x_{1,3})$.
- $\mathbb{X}_2 = (X_2, \longrightarrow^{X_2}, \leq^{X_2}, L^{X_2}, R^{X_2})$, $X_2 = \{x_{2,1}, x_{2,2}, x_{2,3}\}$, $x_{2,1} \longrightarrow^{X_2} x_{2,2}$, $x_{2,1} \longrightarrow^{X_2} x_{2,3}$, $x_{2,2} \longrightarrow^{X_2} x_{2,3}$, $x_{2,3} \leq^{X_2} x_{2,2} \leq^{X_2} x_{2,1}$, $L^{X_2}(x_{2,1})$, $L^{X_2}(x_{2,2})$, $L^{X_2}(x_{2,3})$.
- $\mathbb{Y}_1 = (Y_1, \longrightarrow^{Y_1}, \leq^{Y_1}, L^{Y_1}, R^{Y_1})$, $Y_1 = \{y_{1,1}, y_{1,2}, y_{1,3}\}$, $y_{1,1} \longrightarrow^{Y_1} y_{1,2}$, $y_{1,1} \longrightarrow^{Y_1} y_{1,3}$, $y_{1,2} \longrightarrow^{Y_1} y_{1,3}$, $y_{1,1} \leq^{Y_1} y_{1,2} \leq^{Y_1} y_{1,3}$, $R^{Y_1}(y_{1,1})$, $R^{Y_1}(y_{1,2})$, $R^{Y_1}(y_{1,3})$.
- $\mathbb{Y}_2 = (Y_2, \longrightarrow^{Y_2}, \leq^{Y_2}, L^{Y_2}, R^{Y_2})$, $Y_2 = \{y_{2,1}, y_{2,2}, y_{2,3}\}$, $y_{2,1} \longrightarrow^{Y_2} y_{2,2}$, $y_{2,1} \longrightarrow^{Y_2} y_{2,3}$, $y_{2,2} \longrightarrow^{Y_2} y_{2,3}$, $y_{2,3} \leq^{Y_2} y_{2,2} \leq^{Y_2} y_{2,1}$, $R^{Y_2}(y_{2,1})$, $R^{Y_2}(y_{2,2})$, $R^{Y_2}(y_{2,3})$.
- $\mathbb{Z}_1 = (Z_1, \longrightarrow^{Z_1}, \leq^{Z_1}, L^{Z_1}, R^{Z_1})$, $Z_1 = \{z_{1,1}, z_{1,2}, z_{1,3}\}$, $z_{1,1} \longrightarrow^{Z_1} z_{1,2}$, $z_{1,1} \longrightarrow^{Z_1} z_{1,3}$, $z_{1,2} \longrightarrow^{Z_1} z_{1,3}$, $z_{1,3} \leq^{Z_1} z_{1,1} \leq^{Z_1} z_{1,2}$, $L^{Z_1}(z_{1,1})$, $L^{Z_1}(z_{1,2})$, $L^{Z_1}(z_{1,3})$.
- $\mathbb{Z}_2 = (Z_2, \longrightarrow^{Z_2}, \leq^{Z_2}, L^{Z_2}, R^{Z_2})$, $Z_2 = \{z_{2,1}, z_{2,2}, z_{2,3}\}$, $z_{2,1} \longrightarrow^{Z_2} z_{2,2}$, $z_{2,1} \longrightarrow^{Z_2} z_{2,3}$, $z_{2,2} \longrightarrow^{Z_2} z_{2,3}$, $z_{2,3} \leq^{Z_2} z_{2,1} \leq^{Z_2} z_{2,2}$, $R^{Z_2}(z_{2,1})$, $R^{Z_2}(z_{2,2})$, $R^{Z_2}(z_{2,3})$.
- $\mathbb{W} = (W, \longrightarrow^{W}, \leq^{W}, L^{W}, R^{W})$, $W = \{w_1, w_2, w_3\}$, $w_1 \longrightarrow^W w_2$, $w_2 \longrightarrow^W w_3$, $w_3 \longrightarrow^W w_1$, $w_1 \leq^W w_2 \leq^W w_3$, $R^W(w_1)$, $L^W(w_2)$, $R^W(w_3)$.

Each of the structures $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{R}_1$, $\mathbb{R}_2$, $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{Y}_1$, $\mathbb{Y}_2$, $\mathbb{Z}_1$ and $\mathbb{Z}_2$ has only one half. Structure $\mathbb{W}$ has two halves, but arrows in this structure make a circle, so any expansion of the structure $\mathbb{W}|\{\longrightarrow, \leq\}$ to a structure in $\mathcal{UOS}(2)$ must have points in both halves. Ordering and arrow agree on $\mathbb{L}_1$, $\mathbb{R}_1$, $\mathbb{X}_1$, $\mathbb{Y}_1$; while they disagree on $\mathbb{L}_2$, $\mathbb{R}_2$, $\mathbb{X}_2$, $\mathbb{Y}_2$.

Proposition 1. *$\mathcal{UOS}(2)$ satisfies EP with respect to $\mathcal{OS}(2)$.*

Proof. By Lemma 2, it is enough to fix $\mathbb{A} = (A, \longrightarrow^A, \leq^A, L^A, R^A) \in \mathcal{UOS}(2)$ and find $\mathbb{E} = (E, \longrightarrow^E, \leq^E) \in \mathcal{OS}(2)$ that verifies EP for $\mathbb{A}$. Since the class $\mathcal{UOS}(2)$ satisfies JEP there exists a $\mathbb{B} = (B, \longrightarrow^B, \leq^B, L^B, R^B) \in \mathcal{UOS}(2)$ containing $\mathbb{A}$, $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{R}_1$, $\mathbb{R}_2$, $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{Y}_1$, $\mathbb{Y}_2$, $\mathbb{Z}_1$, $\mathbb{Z}_2$ and $\mathbb{W}$ as substructures. There is a structure $\mathbb{C} \in \mathcal{UOS}(2)$ with the same underlying set as $\mathbb{B}$ such that for all $x, y \in B$ we have:

- $\mathbb{C} = (B, \longrightarrow^C, \leq^C, L^C, R^C)$,
- $x \longrightarrow^B y \Leftrightarrow x \longrightarrow^C y$,
- $x \leq^B y \Leftrightarrow x \leq^C y$,
- $L^B(x) \Leftrightarrow R^C(x)$ and
- $R^B(x) \Leftrightarrow L^C(x)$.

The structures $\mathbb{B}$ and $\mathbb{C}$ agree on arrow relations, agree on orderings and they are opposite with respect to L and R. Again, using JEP for the class $\mathcal{UOS}(2)$ there is some $\mathbb{D} = (D, \longrightarrow^D, \leq^D, L^D, R^D) \in \mathcal{UOS}(2)$ such that $\mathbb{B} \hookrightarrow \mathbb{D}$ and $\mathbb{C} \hookrightarrow \mathbb{D}$.

Using RP for the class $\mathcal{UOS}(2)$ we recursively define structures $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathbb{E}_4 \in \mathcal{UOS}(2)$ such that:

$$\mathbb{E}_1 \to (\mathbb{D})_4^{\mathbb{L}_1}, \qquad \mathbb{E}_2 \to (\mathbb{E}_1)_4^{\mathbb{L}_2}, \qquad \mathbb{E}_3 \to (\mathbb{E}_2)_4^{\mathbb{R}_1}, \qquad \mathbb{E}_4 \to (\mathbb{E}_3)_4^{\mathbb{R}_2}.$$

Let $\mathbb{E}_4 = (E, \longrightarrow^E, \leq^E, L^E, R^E)$ and take $\mathbb{E} = (E, \longrightarrow^E, \leq^E) \in \mathcal{OS}(2)$. We claim that $\mathbb{E}$ verifies EP for $\mathbb{A}$. Let $\Gamma = \{(L, L), (L, R), (R, L), (R, R)\}$ be a set of colors and let $\Gamma_0 = \{(L, L), (R, R)\}$. In order to check our claim we need to consider an arbitrary expansion of the structure $\mathbb{E}$ to a structure in $\mathcal{UOS}(2)$, say $(\mathbb{E}, L^0, R^0)$. We define colorings

$$\chi_1 : \binom{\mathbb{E}_4}{\mathbb{L}_1} \to \Gamma, \qquad \chi_2 : \binom{\mathbb{E}_4}{\mathbb{L}_2} \to \Gamma, \qquad \chi_3 : \binom{\mathbb{E}_4}{\mathbb{R}_1} \to \Gamma, \qquad \chi_4 : \binom{\mathbb{E}_4}{\mathbb{R}_2} \to \Gamma,$$

such that for each $i \in [4]$ we have $\chi_i(\mathbb{U}) = (\gamma_1, \gamma_2)$ where $\mathbb{U}$ has the underlying set $U = \{x, y\}$ with $x \longrightarrow^E y$ and

$$\gamma_1 = \begin{cases} L; & L^0(x) \\ R; & R^0(x) \end{cases}; \quad \text{and} \quad \gamma_2 = \begin{cases} L; & L^0(y) \\ R; & R^0(y) \end{cases}.$$

According to the construction of the sequence $(\mathbb{E}_i)_{i=1}^4$ there is a sequence $(\mathbb{E}'_i)_{i=0}^3$ such that:

$$\mathbb{E}'_0 \leq \mathbb{E}'_1 \leq \mathbb{E}'_2 \leq \mathbb{E}'_3 \leq \mathbb{E}_4, \qquad \mathbb{E}'_0 \cong \mathbb{D}, \qquad \mathbb{E}'_1 \cong \mathbb{E}_1, \qquad \mathbb{E}'_2 \cong \mathbb{E}_2, \qquad \mathbb{E}'_3 \cong \mathbb{E}_3,$$

$$\chi_4 \restriction \binom{\mathbb{E}'_3}{\mathbb{R}_2} = (\gamma_{41}, \gamma_{42}), \qquad \chi_3 \restriction \binom{\mathbb{E}'_2}{\mathbb{R}_1} = (\gamma_{31}, \gamma_{32}),$$
$$\chi_2 \restriction \binom{\mathbb{E}'_1}{\mathbb{L}_2} = (\gamma_{21}, \gamma_{22}), \qquad \chi_1 \restriction \binom{\mathbb{E}'_0}{\mathbb{R}_1} = (\gamma_{11}, \gamma_{12}).$$

In particular we have that

$$\chi_4 \restriction \binom{\mathbb{E}'_0}{\mathbb{R}_2} = (\gamma_{41}, \gamma_{42}), \qquad \chi_3 \restriction \binom{\mathbb{E}'_0}{\mathbb{R}_1} = (\gamma_{31}, \gamma_{32}),$$
$$\chi_2 \restriction \binom{\mathbb{E}'_0}{\mathbb{L}_2} = (\gamma_{21}, \gamma_{22}), \qquad \chi_1 \restriction \binom{\mathbb{E}'_0}{\mathbb{R}_1} = (\gamma_{11}, \gamma_{12}).$$

Since $\mathbb{X}_1$ and $\mathbb{X}_2$ can be embedded into $\mathbb{B}$ and $\mathbb{B} \hookrightarrow \mathbb{E}'_0$, we must have

$$(\gamma_{11}, \gamma_{12}),\ (\gamma_{21}, \gamma_{22}) \in \Gamma_0.$$

We use here the fact that every two substructures of $\mathbb{X}_1$ ($\mathbb{X}_2$) isomorphic with $\mathbb{L}_1$ ($\mathbb{L}_2$) must have a point in common. For the similar reason, since $\mathbb{Y}_1$ and $\mathbb{Y}_2$ can be embedded into $\mathbb{B}$ and $\mathbb{B} \hookrightarrow \mathbb{E}'_0$, we must have

$$(\gamma_{31}, \gamma_{32}),\ (\gamma_{41}, \gamma_{42}) \in \Gamma_0.$$

Moreover, $\mathbb{Z}_1 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$ and $\mathbb{Z}_2 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$, so we have

$$(\gamma_{11}, \gamma_{12}) = (\gamma_{21}, \gamma_{22}) \in \big\{(L, L), (R, R)\big\},$$
$$(\gamma_{31}, \gamma_{32}) = (\gamma_{41}, \gamma_{42}) \in \big\{(L, L), (R, R)\big\}.$$

This follows from the fact that $\mathbb{Z}_1$ contains two copies of $\mathbb{L}_1$ and $\mathbb{L}_2$ which have a point in common, and similarly for $\mathbb{Z}_2$ and $\mathbb{R}_1$ and $\mathbb{R}_2$. We recall the earlier observation that in every expansion of the structure $(W, \longrightarrow^W, \leq^W)$ (which is in $\mathcal{OS}(2)$) to a structure in $\mathcal{UOS}(2)$, there are points indicated by L as well as points indicated by R, i.e. each expansion contains both halves. Since $\mathbb{W} \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$, we have

$$(\gamma_{11}, \gamma_{12}) = (\gamma_{21}, \gamma_{22}) \neq (\gamma_{31}, \gamma_{32}) = (\gamma_{41}, \gamma_{42}).$$

Now we have two cases:

(1) $(L, L) = (\gamma_{11}, \gamma_{12}) = (\gamma_{21}, \gamma_{22}) \neq (\gamma_{31}, \gamma_{32}) = (\gamma_{41}, \gamma_{42}) = (R, R)$,
(2) $(R, R) = (\gamma_{11}, \gamma_{12}) = (\gamma_{21}, \gamma_{22}) \neq (\gamma_{31}, \gamma_{32}) = (\gamma_{41}, \gamma_{42}) = (L, L)$.

In the first case, this means that the relations L^0 and L^E agree on the underlying set of $\mathbb{E}'_0$. The same holds for R^0 and R^E. Furthermore, $\mathbb{A} \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$, so $\mathbb{A} \hookrightarrow (\mathbb{E}, L^0, R^0)$. In the second case, we have that L^0 and R^E agree on the underlying set of $\mathbb{E}'_0$; and the same for R^0 and L^E. Since $\mathbb{C} \hookrightarrow \mathbb{E}'_0$, there is $\mathbb{C}' \leq \mathbb{E}'_0$ with the underlying set C' such that $\mathbb{C}' \cong \mathbb{C}$. By the construction of the structure $\mathbb{C}$, we get $\mathbb{B} \cong (C', \longrightarrow^E \restriction C', \leq^E \restriction C', L^0 \restriction C', R^0 \restriction C')$, and in particular that $\mathbb{A} \hookrightarrow (\mathbb{E}, L^0, R^0)$. This completes the verification of EP for $\mathbb{A}$ by $\mathbb{E}$. $\square$

6. Circular directed graph

Let J denote the unit circle in the complex plane and let $\longrightarrow$ be a binary relational symbol. We define an oriented graph structure $\mathbb{J} = (J, \longrightarrow^J)$ such that for $x, y \in J$ we have $x \longrightarrow^J y$ iff $0 < \arg(\frac{y}{x}) < \frac{2\pi}{3}$. The *circular directed graph* is the structure $\mathbb{S}(3) = (S(3), \longrightarrow^{S(3)})$ which is a substructure of $\mathbb{J}$ such that $S(3) = \{t \in J : \arg(t) \in \mathbb{Q}\}$. Note that the structures $\mathbb{S}(2)$ and $\mathbb{S}(3)$ have the same underlying set but the graph relation is defined differently. We consider $\mathcal{S}(3) = Age(\mathbb{S}(3))$, *the class of finite circular directed graphs*. It is a Fraïssé class which satisfies SAP. Let L, R and D be unary relational symbols. Let $\mathcal{US}(3) = (S(3), \longrightarrow^{S(3)}, L^{S(3)}, R^{S(3)}, D^{S(3)})$ be the structure such that $\mathcal{US}(3)|\{\longrightarrow\} = \mathbb{S}(3)$ and for $x \in S(2)$ we have

$$\begin{aligned} L^{S(3)}(x) \quad &\Leftrightarrow \quad \frac{\pi}{2} < \arg(x) < \frac{7\pi}{6}, \\ R^{S(3)}(x) \quad &\Leftrightarrow \quad -\frac{\pi}{6} < \arg(x) < \frac{\pi}{2}, \\ D^{S(3)}(x) \quad &\Leftrightarrow \quad \frac{7\pi}{6} < \arg(x) < \frac{11\pi}{6}. \end{aligned}$$

We consider $\mathcal{US}(3) = Age(\mathcal{US}(3))$, a Fraïssé class which has SAP. Structures in $\mathcal{S}(2)$ are partitioned into three parts which we call the *left, right and down part* and they are indicated by L, R and D respectively. More details about the classes $\mathcal{S}(2)$ and $\mathcal{S}(3)$ can be found in [6], and more details about the class $\mathcal{US}(3)$ in [18].

Theorem 6. *(See [18].) For the circular directed graph we have the following:*

(i) *The class $\mathcal{US}(3)$ satisfies RP and EP with respect to $\mathcal{S}(3)$.*
(ii) *No order expansion of $\mathcal{S}(3)$ has RP and EP with respect to $\mathcal{S}(3)$.*

We consider the class

$$\mathcal{OS}(3) = \{(A, \longrightarrow^A, \leq^A) : (A, \longrightarrow^A) \in \mathcal{S}(3) \text{ and } \leq^A \in lo(A)\}.$$

Since $\mathcal{S}(3)$ satisfies SAP, $\mathcal{OS}(3)$ satisfies SAP. We consider $\mathcal{UOS}(3)$, a Fraïssé class, given by

$$\begin{aligned} \mathcal{UOS}(3) = \{(\mathbb{A}, \leq^A, L^A, R^A, D^A) : \mathbb{A} \in \mathcal{S}(3) \text{ and} \\ (\mathbb{A}, \leq^A) \in \mathcal{OS}(3) \text{ and } (\mathbb{A}, L^A, R^A, D^A) \in \mathcal{US}(3)\}. \end{aligned}$$

Similar to dense local orderings, we say that *ordering and arrow agree on* $\mathbb{A} = (A, \longrightarrow^A, \leq^A, L^A, R^A, D^R)$ if for all $a, b \in A$ we have $a \longrightarrow^A b$ and $a \leq^A b$. Otherwise, we say that they *disagree*.

Corollary 4. *$\mathcal{UOS}(3)$ is a Ramsey class.*

Proof. We have $\mathcal{UOS}(3) = \mathcal{US}(3) * \mathcal{L}$ where $\mathcal{L}$ is the class of finite linearly ordered sets. Since $\mathcal{US}(3)$ and $\mathcal{L}$ are Fraïssé classes of rigid structures with SAP and RP, Corollary 2 applies, and we obtain RP for the class $\mathcal{UOS}(3)$. □

For the purpose of Proposition 2 we consider structures $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{R}_1$, $\mathbb{R}_2$, $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{Y}_1$, $\mathbb{Y}_2$, $\mathbb{Z}_1$, and $\mathbb{Z}_2$, introduced in Section 5, as structures in the class $\mathcal{UOS}(3)$. In addition to those structures we need to consider the following structures:

- $\mathbb{D}_1 = (D_1, \longrightarrow^{D_1}, \leq^{D_1}, L^{D_1}, R^{D_1}, D^{D_1})$, $D_1 = \{d_{1,1}, d_{1,2}\}$, $d_{1,1} \longrightarrow^{D_1} d_{1,2}$, $d_{1,1} \leq^{D_1} d_{1,2}$, $D^{D_1}(d_{1,1})$, $D^{D_1}(d_{1,2})$.
- $\mathbb{D}_2 = (D_2, \longrightarrow^{D_2}, \leq^{D_2}, L^{D_2}, R^{D_2}, D^{D_2})$, $D_2 = \{d_{2,1}, d_{2,2}\}$, $d_{2,1} \longrightarrow^{D_2} d_{2,2}$, $d_{2,2} \leq^{D_2} d_{2,1}$, $D^{D_2}(d_{2,1})$, $D^{D_2}(d_{2,2})$.
- $\mathbb{P}_1 = (P_1, \longrightarrow^{P_1}, \leq^{P_1}, L^{P_1}, R^{P_1}, D^{P_1})$, $P_1 = \{p_{1,1}, p_{1,2}, p_{1,3}\}$, $p_{1,1} \longrightarrow^{P_1} p_{1,2}$, $p_{1,1} \longrightarrow^{P_1} p_{1,3}$, $p_{1,2} \longrightarrow^{P_1} p_{1,3}$, $p_{1,1} \leq^{P_1} p_{1,2} \leq^{P_1} p_{1,3}$, $D^{P_1}(p_{1,1})$, $D^{P_1}(p_{1,2})$, $D^{P_1}(p_{1,3})$.
- $\mathbb{P}_2 = (P_2, \longrightarrow^{P_2}, \leq^{P_2}, L^{P_2}, R^{P_2}, D^{P_2})$, $P_2 = \{p_{2,1}, p_{2,2}, p_{2,3}\}$, $p_{2,1} \longrightarrow^{P_2} p_{2,2}$, $p_{2,1} \longrightarrow^{P_2} p_{2,3}$, $p_{2,2} \longrightarrow^{P_2} p_{2,3}$, $p_{2,3} \leq^{P_2} p_{2,2} \leq^{P_2} p_{2,1}$, $D^{P_2}(p_{2,1})$, $D^{P_2}(p_{2,2})$, $D^{P_2}(p_{2,3})$.
- $\mathbb{Z}_3 = (Z_3, \longrightarrow^{Z_3}, \leq^{Z_3}, L^{Z_3}, R^{Z_3}, D^{Z_3})$, $Z_3 = \{z_{3,1}, z_{3,2}, z_{3,3}\}$, $z_{3,1} \longrightarrow^{Z_3} z_{3,2}$, $z_{3,1} \longrightarrow^{Z_3} z_{3,3}$, $z_{3,2} \longrightarrow^{Z_3} z_{3,3}$, $z_{3,3} \leq^{Z_3} z_{3,1} \leq^{Z_3} z_{3,2}$, $D^{Z_3}(z_{3,1})$, $D^{Z_3}(z_{3,2})$, $D^{Z_3}(z_{3,3})$.
- $\mathbb{W}_{rl} = (W_{rl}, \longrightarrow^{W_{rl}}, \leq^{W_{rl}}, L^{W_{rl}}, R^{W_{rl}}, D^{W_{rl}})$, $W_{rl} = \{w_{rl,1}, w_{rl,2}, w_{rl,3}\}$, $w_{rl,1} \longrightarrow^{W_{rl}} w_{rl,2}$, $w_{rl,2} \longrightarrow^{W_{rl}} w_{rl,3}$, $w_{rl,1} \leq^{W_{rl}} w_{rl,2} \leq^{W_{rl}} w_{rl,3}$, $R^{W_{rl}}(w_{rl,1})$, $R^{W_{rl}}(w_{rl,2})$, $L^{W_{rl}}(w_{rl,3})$.
- $\mathbb{W}_{ld} = (W_{ld}, \longrightarrow^{W_{ld}}, \leq^{W_{ld}}, L^{W_{ld}}, R^{W_{ld}}, D^{W_{ld}})$, $W_{ld} = \{w_{ld,1}, w_{ld,2}, w_{ld,3}\}$, $w_{ld,1} \longrightarrow^{W_{ld}} w_{ld,2}$, $w_{ld,2} \longrightarrow^{W_{ld}} w_{ld,3}$, $w_{ld,1} \leq^{W_{ld}} w_{ld,2} \leq^{W_{ld}} w_{ld,3}$, $L^{W_{ld}}(w_{ld,1})$, $L^{W_{ld}}(w_{ld,2})$, $D^{W_{ld}}(w_{ld,3})$.
- $\mathbb{W}_{dr} = (W_{dr}, \longrightarrow^{W_{dr}}, \leq^{W_{dr}}, L^{W_{dr}}, R^{W_{dr}}, D^{W_{dr}})$, $W_{dr} = \{w_{dr,1}, w_{dr,2}, w_{dr,3}\}$, $w_{dr,1} \longrightarrow^{W_{dr}} w_{dr,2}$, $w_{dr,2} \longrightarrow^{W_{dr}} w_{dr,3}$, $w_{dr,1} \leq^{W_{dr}} w_{dr,2} \leq^{W_{dr}} w_{dr,3}$, $D^{W_{dr}}(w_{dr,1})$, $D^{W_{dr}}(w_{dr,2})$, $R^{W_{dr}}(w_{dr,3})$.

Structures $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{Z}_3$ are obtained from $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{Z}_1$ by changing the left part into the down part. Each of the structures $\mathbb{W}_{rl}$, $\mathbb{W}_{ld}$ and $\mathbb{W}_{dr}$ contains points from exactly two parts and contains points which are not connected by an edge.

Proposition 2. *$\mathcal{UOS}(3)$ satisfies EP with respect to $\mathcal{OS}(3)$.*

Proof. It is enough to fix $\mathbb{A} = (A, \longrightarrow^A, \leq^A, L^A, R^A, D^A) \in \mathcal{UOS}(3)$ and find an $\mathbb{F} \in \mathcal{OS}(3)$ that verifies EP for $\mathbb{A}$. Since the class $\mathcal{UOS}(3)$ satisfies JEP there exists a $\mathbb{B} = (B, \longrightarrow^B, \leq^B, L^B, R^B, D^B) \in \mathcal{UOS}(3)$ that contains $\mathbb{A}$, $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{R}_1$, $\mathbb{R}_2$, $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{Y}_1$, $\mathbb{Y}_2$, $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{W}_{rl}$, $\mathbb{W}_{ld}$ and $\mathbb{W}_{dr}$ as substructures. We define structures $\mathbb{B}_1$ and $\mathbb{B}_2$ in $\mathcal{UOS}(3)$ which have the same underlying set as $\mathbb{B}$ such that for all $x, y \in B$ we have:

- $\mathbb{B}_1 = (B, \longrightarrow^{B_1}, \leq^{B_1}, L^{B_1}, R^{B_1}, D^{B_1})$,
- $\mathbb{B}_2 = (B, \longrightarrow^{B_2}, \leq^{B_2}, L^{B_2}, R^{B_2}, D^{B_2})$,
- $x \longrightarrow^B y \Leftrightarrow x \longrightarrow^{B_1} y \Leftrightarrow x \longrightarrow^{B_2} y$,
- $x \leq^B y \Leftrightarrow x \leq^{B_1} y \Leftrightarrow x \leq^{B_2} y$,
- $L^B(x) \Leftrightarrow R^{B_1}(x) \Leftrightarrow D^{B_2}(x)$,
- $R^B(x) \Leftrightarrow D^{B_1}(x) \Leftrightarrow L^{B_2}(x)$ and
- $D^B(x) \Leftrightarrow L^{B_1}(x) \Leftrightarrow R^{B_2}(x)$.

The structures $\mathbb{B}_1$ and $\mathbb{B}_2$ are obtained from the structure $\mathbb{B}$ by rotating the labeling counterclockwise for one or two places. JEP for the class $\mathcal{UOS}(3)$ implies that there is an $\mathbb{E}_0 \in \mathcal{UOS}(3)$ such that $\mathbb{B} \hookrightarrow \mathbb{E}_0$, $\mathbb{B}_1 \hookrightarrow \mathbb{E}_0$ and $\mathbb{B}_2 \hookrightarrow \mathbb{E}_0$. Since $\mathcal{UOS}(2)$ is a Ramsey class, we can recursively define a sequence $(\mathbb{E}_i)_{i=0}^6$ of structures in $\mathcal{UOS}(3)$ such that:

$$\begin{aligned} &\mathbb{E}_1 \to (\mathbb{E}_0)_6^{\mathbb{L}_1}, \qquad &&\mathbb{E}_2 \to (\mathbb{E}_1)_6^{\mathbb{L}_2}, \\ &\mathbb{E}_3 \to (\mathbb{E}_2)_6^{\mathbb{R}_1}, \qquad &&\mathbb{E}_4 \to (\mathbb{E}_3)_6^{\mathbb{R}_2}, \\ &\mathbb{E}_5 \to (\mathbb{E}_4)_6^{\mathbb{D}_1}, \qquad &&\mathbb{E}_6 \to (\mathbb{E}_5)_6^{\mathbb{D}_2}. \end{aligned}$$

Let $\mathbb{E}_6 = (F, \longrightarrow^F, \leq^F, L^F, R^F, D^F)$, and let $\mathbb{F} = (F, \longrightarrow^F, \leq^F) \in \mathcal{OS}(3)$. We claim that $\mathbb{F}$ verifies EP for $\mathbb{A}$. Let $\Lambda = \{(\lambda_1, \lambda_2) : \lambda_i \in \{R, L, D\}$ for $i \in [2]\}$ be a set of colors and let $\Lambda_0 = \{(L, L), (R, R), (D, D)\}$. In order to check our claim we need to consider an arbitrary expansion of the structure $\mathbb{F}$ to a structure in $\mathcal{UOS}(3)$, say $(\mathbb{F}, L^0, R^0, D^0)$. We define colorings

$$\chi_1 : \binom{\mathbb{E}_6}{\mathbb{L}_1} \to \Lambda, \qquad \chi_2 : \binom{\mathbb{E}_6}{\mathbb{L}_2} \to \Lambda, \qquad \chi_3 : \binom{\mathbb{E}_6}{\mathbb{R}_1} \to \Lambda,$$
$$\chi_4 : \binom{\mathbb{E}_6}{\mathbb{R}_2} \to \Lambda, \qquad \chi_5 : \binom{\mathbb{E}_6}{\mathbb{P}_1} \to \Lambda, \qquad \chi_6 : \binom{\mathbb{E}_6}{\mathbb{P}_2} \to \Lambda,$$

such that for each $i \in [6]$ we have $\chi_i(\mathbb{U}) = (\gamma_1, \gamma_2)$ where $U = \{x, y\}$ is the underlying set of the structure $\mathbb{U}$ with $x \longrightarrow^F y$ and

$$\gamma_1 = \begin{cases} R; & R^0(x) \\ L; & L^0(x) \\ D; & D^0(x) \end{cases}; \quad \text{and} \quad \gamma_2 = \begin{cases} R; & R^0(y) \\ L; & L^0(y) \\ D; & D^0(y) \end{cases}.$$

The construction of the sequence $(\mathbb{E}_i)_{i=0}^6$ implies the existence of a sequence $(\mathbb{E}'_i)_{i=0}^5$ such that for each $0 \le i \le 5$ we have

$$\mathbb{E}'_i \le \mathbb{E}'_{i+1}, \qquad \mathbb{E}'_5 \le \mathbb{E}_6, \qquad \mathbb{E}'_i \cong \mathbb{E}_i,$$
$$\chi_6 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{P}_2} = (\gamma_{61}, \gamma_{62}), \qquad \chi_5 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{P}_1} = (\gamma_{51}, \gamma_{52}),$$
$$\chi_4 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{R}_2} = (\gamma_{41}, \gamma_{42}), \qquad \chi_3 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{R}_1} = (\gamma_{31}, \gamma_{32}),$$
$$\chi_2 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{L}_2} = (\gamma_{21}, \gamma_{22}), \qquad \chi_1 \upharpoonright \binom{\mathbb{E}'_0}{\mathbb{L}_1} = (\gamma_{11}, \gamma_{12}).$$

Since $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{Y}_1$, $\mathbb{Y}_2$, $\mathbb{P}_1$ and $\mathbb{P}_2$ can be embedded into $\mathbb{B}$ and $\mathbb{B} \hookrightarrow \mathbb{E}'_0$, the same reason as in the proof of Proposition 1 implies that

$$(\gamma_{11}, \gamma_{12}),\ (\gamma_{21}, \gamma_{22}),\ (\gamma_{31}, \gamma_{32}),\ (\gamma_{41}, \gamma_{42}),\ (\gamma_{51}, \gamma_{52}),\ (\gamma_{61}, \gamma_{62}) \in \Lambda_0.$$

Moreover, $\mathbb{Z}_1 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$, $\mathbb{Z}_2 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$ and $\mathbb{Z}_3 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$ so we have

$$(\gamma_{11}, \gamma_{12}) = (\gamma_{21}, \gamma_{22}) \in \Lambda_0, \qquad (\gamma_{31}, \gamma_{32}) = (\gamma_{41}, \gamma_{42}) \in \Lambda_0,$$
$$(\gamma_{51}, \gamma_{52}) = (\gamma_{61}, \gamma_{62}) \in \Lambda_0.$$

Since $\mathbb{W}_{rl} \hookrightarrow \mathbb{B}$, $\mathbb{W}_{ld} \hookrightarrow \mathbb{B}$, $\mathbb{W}_{dr} \hookrightarrow \mathbb{B}$ and $\mathbb{B} \hookrightarrow \mathbb{E}'_0$, we have

$$(\gamma_{11}, \gamma_{12}) \ne (\gamma_{31}, \gamma_{32}), \qquad (\gamma_{11}, \gamma_{12}) \ne (\gamma_{51}, \gamma_{52}), \qquad (\gamma_{51}, \gamma_{52}) \ne (\gamma_{31}, \gamma_{32}).$$

This follows from the fact that points $w_{rl,1}$ and $w_{rl,3}$ are not in the same part. Similarly points $w_{ld,1}$ and $w_{rl,3}$, and points $w_{dr,1}$ and $w_{rl,3}$ are in different parts. Therefore, we are left with three cases:

(1) $(\gamma_{11}, \gamma_{12}) = (L, L)$, $(\gamma_{31}, \gamma_{32}) = (R, R)$, $(\gamma_{51}, \gamma_{52}) = (D, D)$. This means that the relations L^0 and L^E agree on the underlying set of $\mathbb{E}'_0$; and the same for R^0 and R^E, and D^0 and D^E. Furthermore, $\mathbb{A} \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{E}'_0$, so $\mathbb{A} \hookrightarrow (\mathbb{F}, L^0, R^0, D^0)$.
(2) $(\gamma_{11}, \gamma_{12}) = (R, R)$, $(\gamma_{31}, \gamma_{32}) = (D, D)$, $(\gamma_{51}, \gamma_{52}) = (L, L)$. This means that the relations L^0, R^0, D^0 agree with the relations D^E, L^E, R^E respectively on the underlying set of $\mathbb{E}'_0$. Since $\mathbb{B}_1 \hookrightarrow \mathbb{E}_0$, there is an embedding $\mathbb{A} \hookrightarrow (\mathbb{F}, L^0, R^0, D^0)$.
(3) $(\gamma_{11}, \gamma_{12}) = (D, D)$, $(\gamma_{31}, \gamma_{32}) = (L, L)$, $(\gamma_{51}, \gamma_{52}) = (R, R)$. In this case we have agreement of the relations L^0, R^0, D^0 with the relations R^E, D^E, L^E respectively on the underlying set of $\mathbb{E}'_0$. Since $\mathbb{B}_2 \hookrightarrow \mathbb{E}_0$, there is an embedding $\mathbb{A} \hookrightarrow (\mathbb{F}, L^0, R^0, D^0)$.

We point out that the case where $(\gamma_{11}, \gamma_{12}) = (L, L)$ and $(\gamma_{31}, \gamma_{32}) = (D, D)$ is impossible since there is no arrow from the down part to the left part. This completes verification of EP for $\mathbb{A}$ by $\mathbb{F}$. $\square$

7. Boron trees

Let $\mathbb{A} = (A, E^A)$ be a graph. A *path* in the graph $\mathbb{A}$ is a sequence of points $x_1, x_2, \ldots, x_n$, $n \geq 2$, such that for all $1 \leq i < n$ we have $x_i E^A x_{i+1}$; the path is said to connect the vertices x_1 and x_n. If $x_1 = x_n$ then we call such a path a *circle*. A graph is *connected* if for every two points there is a path that connects two vertices. A connected graph without circles is called a *graph theoretic tree*. In every graph theoretic tree there is a unique shortest path connecting distinct vertices x and y which we denote by xy. If for distinct vertices x, y, z, w, we have that the paths xy and zw do not intersect then we write $xy|zw$. If x is a vertex in a graph $\mathbb{A}$ then we denote by $val(x)$, the *valence of the vertex* x, the cardinality of the set $|\{y : xE^A y\}|$. If $val(x) = 1$ then we say that x is a *terminal node* of the graph. Let $\mathcal{T}$ denote the class of finite graph theoretic trees all of whose vertices have valence 1 or 3. Every structure from the class $\mathcal{T}$ is of the form $\mathbb{T} = (T, E^T)$ for some non-empty set T with graph relation E^T. We consider a relational symbol R with arity 4 and to each $\mathbb{T} \in \mathcal{T}$ we assign the structure $B(\mathbb{T}) = (B(T), R^{B(T)})$ where

- $B(T) = \{x \in T : val(x) = 1\}$,
- $R^{B(T)}(x, y, z, w) \Leftrightarrow (xy|zw \text{ and } |\{x, y, z, w\}| = 4)$ for $x, y, z, w \in B(T)$.

The class of all structures of the form $B(\mathbb{T})$ where $\mathbb{T} \in \mathcal{T}$, together with the one point structure, is denoted by $\mathcal{B}$ and is called the *class of boron tree structures*. The class $\mathcal{B}$ is a Fraïssé class with SAP, see [4]. By adding arbitrary linear orderings to structures from $\mathcal{B}$ we obtain the class

$$\mathcal{OB} = \big\{\big(A, R^A, \leq^A\big) : \big(A, R^A\big) \in \mathcal{B} \text{ and } \leq^A \in lo(A)\big\}.$$

It is straightforward to see that $\mathcal{OB}$ is also a Fraïssé class with SAP.

We denote by $\mathbb{T}_n = (T_n, E^{T_n}) \in \mathcal{T}$, $n \geq 1$, where:

- $T_n = \{(x_1, \ldots, x_k) : k \in [n] \text{ and } x_i \in \{0, 1\} \text{ for all } i \in [k]\}$.
- For $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_l)$ in T_n we have $xE^{T_n}y$ iff ($l = k+1$ and $x = (y_1, \ldots, y_k)$) or ($k = l+1$ and $y = (x_1, \ldots, x_l)$) or ($k = l = 1$ and $x_1 \neq y_1$).

Let $\leq^n$ be a lexicographic ordering on the set $B(T_n)$ such that for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $B(T_n)$ we have

$$x \leq^n y \quad \Leftrightarrow \quad \big((x = y) \text{ or } \big(x \neq y \text{ and } x_k \leq y_k \text{ where } k = \min\{j : x_j \neq y_j\}\big)\big).$$

Let $\mathbb{B}_n = B(\mathbb{T}_n) = (B_n, R^{B_n})$ and let $\mathbb{B}_0 = (B_0, R^{B_0})$ where $|B_0| = 1$ and $R^{B_0} = \emptyset$.

We consider S as a relational symbol of arity 3. Let $\mathbb{A} = (A, R^A) \in \mathcal{B}$ and let $\phi : \mathbb{A} \to \mathbb{B}_n$ be an embedding. We define a structure $\mathbb{A}_\phi = (A, R^A, S^A)$ by taking

$$S^A(x,y,z) \quad \Leftrightarrow \quad \begin{cases} \phi(x), \phi(y) \leq^n \phi(z) \quad \text{and} \\ \min\{k : \phi(x)_{k+1} \neq \phi(y)_{k+1}\} > \min\{k : \phi(y)_{k+1} \neq \phi(z)_{k+1}\} \end{cases}$$

for all $x, y, z \in A$. We denote by $\mathcal{SB}$ the class of all structures obtained from structures in $\mathcal{B}$ via the previous embeddings. Recall that a rooted tree is a graph theoretic tree with distinguished point. So far we have not used rooted trees, but the expansion $\mathcal{SB}$ indirectly selects a root. A vertex in our structure is not chosen to be a root, rather an edge is selected to have an imaginary root at its midpoint. Every structure in $\mathcal{SB}$ comes with a unique graph theoretic tree, which induces a boron tree structure, and with a ternary relation that selects an edge to carry a root.

We consider the following class

$$\mathcal{SOB} = \{(A, R^A, \leq^A, S^A) : (A, R^A, \leq^A) \in \mathcal{OB} \text{ and } (A, R^A, S^A) \in \mathcal{SB}\}.$$

More details about boron tree structures can be found in [5] and [4], and more details about the class $\mathcal{SB}$ can be found in [10].

Theorem 7. *(See [10].) For the boron tree structures we have the following:*

(i) *The class $\mathcal{SB}$ has RP and EP with respect to $\mathcal{B}$.*
(ii) *The class $\mathcal{OB}$ is not a Ramsey class.*

It is proved that $\mathcal{SB}$ satisfies RP also in [23] by a different method than in [10].

Corollary 5. *$\mathcal{SOB}$ is a Ramsey class.*

Proof. We have $\mathcal{SOB} = \mathcal{SB} * \mathcal{L}$ where $\mathcal{L}$ is the class of linearly ordered sets. Since $\mathcal{SB}$ and $\mathcal{L}$ are Fraïssé classes of rigid structures which satisfy RP and SAP, Corollary 2 applies, and the class $\mathcal{SOB}$ satisfies RP. □

In the proofs of Proposition 1 and Proposition 2 we used RP to obtain EP, but in the following we obtain EP without using RP.

Proposition 3. *$\mathcal{SOB}$ satisfies EP with respect to $\mathcal{OB}$.*

Proof. By Lemma 2 it is enough to find a structure in $\mathcal{OB}$ that verifies EP for a given $\mathbb{A} = (A, R^A, \leq^A, S^A) \in \mathcal{SOB}$. Since the class $\mathcal{SB}$ has EP with respect to $\mathcal{B}$ there is a structure $\mathbb{B}' = (B', R^{B'}) \in \mathcal{B}$ which verifies EP for $\mathbb{A}' = (A, R^A)$ (see Theorem 7). There is a tree $\mathbb{T}' = (T', E^{T'}) \in \mathcal{T}$ such that $B(\mathbb{T}') = \mathbb{B}'$. Let $(\mathbb{A}'_i)_{i=1}^n$ be the list of all substructures of $\mathbb{B}'$ isomorphic with $\mathbb{A}'$. Let k be the number of all expansions of the structure $\mathbb{A}'$ to structure $(\mathbb{A}', S^{A'}) \in \mathcal{SB}$ isomorphic to (A, R^A, S^A). For each $\mathbb{A}'_i$ we have a list $(\mathbb{A}'_{i,j})_{j=1}^k$ of all such expansions. Let l be a natural number such that $2^l > nk$. We construct a tree $\mathbb{T} = (T, E^T) \in \mathcal{T}$ by attaching a copy of $\mathbb{T}_l$ to $\mathbb{T}'$ over each $b \in B'$ as follows:

- $T = T' \sqcup (\bigsqcup_{b\in B'} T_{l,b})$ where $T_{l,b}$ is a copy of T_l for $b \in B'$.
- There is a sequence of bijections $(\phi_b : T_l \to T_{l,b})_{b\in B'}$ such that for each $b \in B'$ we have a structure $\mathbb{T}_{l,b} = (T_{l,b}, E^{T_{l,b}}) \in \mathcal{T}$ which is isomorphic to $\mathbb{T}_l$ by ϕ_b. Let $b_0 = \phi_b((0))$ and let $b_1 = \phi_b((1))$.
- $E^T = E^{T'} \sqcup (\bigsqcup_{b\in B'} E^{T_{l,b}} \setminus \{(b_0, b_1), (b_1, b_0)\}) \sqcup (\bigsqcup_{b\in B'} \{(b, b_0), (b_0, b), (b, b_1), (b_1, b)\})$. The edge between "the lowest" vertices in each tree $\mathbb{T}_{l,b}$ is removed and replaced by adding two edges between these vertices and the corresponding vertex in the tree $\mathbb{T}'$.

Terminal nodes of the tree $\mathbb{T}$ are the underlying set of the structure $B(\mathbb{T}) = (B(T), R^{B(T)})$. Before we introduce a linear ordering $\le^{B(T)}$ on the set $B(T)$ we emphasize the following fact.

Fact. Let $\mathbb{A}'_i$ be one of the structures from our list with the set of nodes $A'_i \subseteq B'$. Let $A''_i \subseteq B$ be the set obtained when we replace each node $b' \in A'_i$ with a node $b \in B$ which is a terminal node of the tree $\mathbb{T}_{l,b'}$. Let $\mathbb{A}''_i$ be the substructure of $B(\mathbb{T})$ induced on the set A''_i. Then we have the following:

(1) $\mathbb{A}'_i \cong \mathbb{A}''_i$.
(2) Every expansion of the structure $\mathbb{A}''_i$ to a structure in $\mathcal{SB}$ is an expansion of the structure $\mathbb{A}'_i$ to a structure in $\mathcal{SB}$.

For $i \in [n]$ and $j \in [k]$ we define a set $A''_{i,j} \subseteq B(T)$ such that:

- $(i, j) \ne (i', j') \Rightarrow A''_{i,j} \cap A''_{i',j'} = \emptyset$.
- If $b' \in A_{i,j}$ where $A_{i,j}$ is the underlying set of $\mathbb{A}'_{i,j}$ then $A''_{i,j}$ has exactly one point among terminal nodes of $\mathbb{T}_{l,b'}$.
- If $b' \notin A_{i,j}$ where $A_{i,j}$ is the underlying set of $\mathbb{A}'_{i,j}$ then $A''_{i,j}$ has no points among terminal nodes of $\mathbb{T}_{l,b'}$.

Also, every set $A''_{i,j}$ induces a substructure $\mathbb{A}''_{i,j} \le B(\mathbb{T})$ such that $\mathbb{A}''_{i,j} \cong \mathbb{A}'_i$. Moreover, each structure $\mathbb{A}'_{i,j}$ naturally gives $\mathbb{A}'''_{i,j}$, an expansion $\mathbb{A}'''_{i,j}$ of the structure $\mathbb{A}''_{i,j}$ to a structure in $\mathcal{SB}$, such that $\mathbb{A}'''_{i,j} \cong \mathbb{A}'_{i,j}$. On each set $A_{i,j}$ we define a linear ordering $\le^{i,j}$ such that

$$\left(\mathbb{A}'''_{i,j}, \le^{i,j}\right) \cong \mathbb{A}.$$

We take $\le^{B(T)}$ to be a linear ordering on the set $B(T)$ such that for all i, j we have:

$$\le^{B(T)} \upharpoonright \left(A''_{i,j}\right)^2 = \le^{i,j}.$$

We claim that the structure $(B(\mathbb{T}), \le^{B(T)}) = (B(T), R^{B(T)}, \le^{B(T)})$ verifies EP for $\mathbb{A}$. Let S^0 be a relation defined on the set $B(T)$ such that $(B(T), R^{B(T)}, \le^{B(T)}, S^0) \in \mathcal{SOB}$. We have the following fact.

Fact. Let b'_1, b'_2, b'_3 be distinct points in B'. Let a_1, a_2, a_3 and a'_1, a'_2, a'_3 be points from $B(T)$ such that for each i points a_i and a'_i are terminal nodes in $\mathbb{T}_{l,b'_i}$. Then we have

$$S^0(a_1,a_2,a_3) \quad\Leftrightarrow\quad S^0\big(a'_1,a'_2,a'_3\big),$$
$$S^0(a_1,a_1,a_2) \quad\Leftrightarrow\quad S^0\big(a'_1,a'_1,a'_2\big).$$

In other words, the expansion is independent of the attached trees $\mathbb{T}_{l,b}$.

By the last fact, there is an induced relation S^1 on the set B' such that $(B', R^{B'}, S^1) \in \mathcal{SB}$. For distinct b'_1, b'_2, b'_3 in B we consider $a_1, a_2, a_3 \in B(T)$ where each a_i is a terminal node in $\mathbb{T}_{l,b'_i}$ and we consider S^1 defined by:

$$S^1\big(b'_1,b'_2,b'_3\big) \quad\Leftrightarrow\quad S^0(a_1,a_2,a_3),$$
$$S^1\big(b'_1,b'_1,b'_2\big) \quad\Leftrightarrow\quad S^0(a_1,a_1,a_2).$$

Since $\mathbb{B}'$ verifies EP for $\mathbb{A}'$, there is an $\mathbb{A}'_i$ with underlying set A'_i such that $(\mathbb{A}'_i, S^1 \upharpoonright (A'_i)^3) \cong (A, R^A, S^A)$ and there is some j such that $(A, R^A, S^A) = \mathbb{A}'_{i,j}$. Furthermore, we have $\mathbb{A}'''_{i,j} \cong \mathbb{A}'_{i,j}$ and by the choice of the linear ordering $\leq^{B(T)}$ we have that $(\mathbb{A}'''_{i,j}, \leq^{i,j}) \cong \mathbb{A}$. Therefore there is an embedding of $\mathbb{A}$ into $(B(T), R^{B(T)}, \leq^{B(T)}, S^0)$. $\square$

8. Rooted trees

In this section we consider C as a ternary relational symbol. We recall that the class $\mathcal{T}$ was introduced in Section 7. For a structure $\mathbb{T} = (T, E^T) \in \mathcal{T}$ and a terminal node ∞^T in $\mathbb{T}$ we denote by $(\mathbb{T}, \infty^T) = (T, E^T, \infty^T)$ the tree $\mathbb{T}$ with the root ∞^T. The *height of the tree* $(\mathbb{T}, \infty^T)$ is the length of the longest path $\infty^T x$ in $\mathbb{T}$. The height of a vertex in a tree is the length of the shortest path between the root and the vertex. Let $\mathcal{T}_\infty$ denote the class of all rooted trees $(\mathbb{T}, \infty^T)$ where $\mathbb{T} \in \mathcal{T}$. To every $(\mathbb{T}, \infty^T) \in \mathcal{T}_\infty$ we assign a structure $H(\mathbb{T}, \infty^T) = H(T, E^T, \infty^T) = (H(T), C^{H(T)}) \in \mathcal{H}$ such that $H(T) = B(T) \setminus \{\infty^T\}$ and for $x, y, z \in H(T)$ we have

$$C^{H(T)}(x,y,z) \quad\Leftrightarrow\quad R^{B(T)}(\infty,x,y,z) \quad\Leftrightarrow\quad \infty^T x|yz.$$

We consider $\mathcal{H} = \{H(\mathbb{T}, \infty^T) : (\mathbb{T}, \infty^T) \in \mathcal{T}_\infty\}$, a Fraïssé class with SAP.

Let $\mathbb{A} = (H(A), C^{H(A)}) = H(A, E^A, \infty^A)$ and $\mathbb{B} = (H(B), C^{H(B)}) = H(B, E^B, \infty^B)$ be structures from $\mathcal{H}$ such that $\mathbb{A} \leq \mathbb{B}$. Let V be the set of vertices in the tree (B, E^B) which are on the paths of the form xy for $x, y \in H(A)$. Let v be the vertex with the smallest height in V and let $val_V(x)$ denote the valence of a given vertex x in the tree $(V, E^B \upharpoonright V^2)$. We consider set $W = \{w \in V : val_V(w) \in \{1,3\}\}$ and a binary relation E^W on W such that for $w_1, w_2 \in W$ we have

$$w_1 E^W w_2 \quad\Leftrightarrow\quad \big(\{v \in w_1w_2 : val_V(v) \in \{1,3\}\} = \{w_1, w_2\}\big).$$

Clearly, $(W, E^W, v) \in \mathcal{T}_\infty$ and $H(W, E^W, v) = \mathbb{A}$. For a natural number n we denote by $\mathbb{T}_{n,\infty} = (T_{n,\infty}, E^{T_{n,\infty}}, \infty^n) \in \mathcal{T}_\infty$, the structure with the property that all terminal nodes $x \neq \infty^n$ in $\mathbb{T}_{n,\infty}$ have height n. In particular if the height of a tree $(\mathbb{T}, \infty^T) \in \mathcal{T}_\infty$ is n, then the structure $H(\mathbb{T}, \infty^T)$ can be embedded in any structure $H(\mathbb{T}_{N,\infty})$ for $N \geq n$.

Let $(\mathbb{T}, \infty^T) = (T, E^T, \infty^T) \in \mathcal{T}_\infty$. A function $\phi : T \setminus \{\infty^T\} \to \{0, 1\}$ is a *valuation* on $(\mathbb{T}, \infty^T)$ if it has the property that for every non-terminal vertex $v \in T$ and vertices $v_1, v_2 \in T$ such that $vE^T v_1$, $vE^T v_2$ and v_1 and v_2 have height greater than v we have $\phi(v_1) \neq \phi(v_2)$. Let $Val(\mathbb{T}, \infty^T)$ denote the set of all valuations on the rooted tree $(\mathbb{T}, \infty^T)$. Then every valuation ϕ gives, for each terminal node $x \in T \setminus \{\infty^T\}$, a sequence $\phi(x) = (v_1, \ldots, v_n)$ such that $\infty^T x = \infty^T, x_1, \ldots, x_n = x$ and $v_i = \phi(x_i)$. For a given valuation ϕ we consider the lexicographic ordering on all terminal nodes of $(\mathbb{T}, \infty^T)$, excluding ∞, such that for x, y we have

$$x <^\phi y \quad \Leftrightarrow \quad \phi(x)_k < \phi(y)_k,$$

where $k = \min\{l : \phi(x)_l \neq \phi(y)_l\}$. Consequently, we have a linear ordering on the set $H(T)$ and an ordered structure $(H(T), C^{H(T)}, \leq^\phi)$. We consider $\mathcal{CH}$, the class of the structures of the form $(A, C^A, \leq^A)$ with the property that: $(A, C^A) \in \mathcal{H}$, there are $(T, E^T, \infty^T) \in \mathcal{T}_\infty$ and $\phi \in Val(T, E^T, \infty^T)$ such that $H(T, E^T, \infty^T) = (A, C^A)$ and $\leq^A = \leq^\phi$. We also consider

$$\mathcal{OH} = \{(A, C^A, \leq^A) : (A, C^A) \in \mathcal{H} \text{ and } \leq^A \in lo(A)\}.$$

It is straightforward to see that $\mathcal{CH}$ and $\mathcal{OH}$ are Fraïssé classes with SAP. More details about the class $\mathcal{H}$ can be found in [2] and [3]. We point out a difference in definition of the relation C in this paper and in [2]. In our consideration we assume that only three distinct points can be related in C. On the other hand for distinct $a \neq b$ we note that $C(a, b, b)$ is allowed in [2]. Since we are considering only embedding, this can be done without loss of generality.

Theorem 8. *(See [13].) $\mathcal{CH}$ is a Ramsey class.*

We also have the following.

Lemma 5. *$\mathcal{CH}$ satisfies OP with respect to $\mathcal{H}$.*

Proof. Let $\mathbb{A} = (A, C^A) \in \mathcal{H}$. There is a structure $(T, E^T, \infty^T) \in \mathcal{T}_\infty$ such that $H(T, E^T, \infty^T) = (A, C^A)$. Let n be the height of the tree (T, E^T, ∞^T). From the definition of the classes $\mathcal{CH}$ and $\mathcal{H}$ it follows that $H(\mathbb{T}_{n,\infty})$ verifies OP for the structure $\mathbb{A}$. □

We combine classes $\mathcal{CH}$ and $\mathcal{OH}$ into the class

$$\mathcal{COH} = \{(A, C^A, \leq^A, \preceq^A) : (A, C^A, \leq^A) \in \mathcal{OH} \text{ and } (A, C^A, \preceq^A) \in \mathcal{CH}\}.$$

Corollary 6. *$\mathcal{COH}$ is a Ramsey class.*

Proof. This follows from Corollary 2 and the fact that $\mathcal{COH} = \mathcal{CH} * \mathcal{L}$ where $\mathcal{CH}$ and $\mathcal{L}$ are Fraïssé classes of rigid structures with RP and SAP. $\square$

At this point we compare proofs from the previous three sections. It is not clear how to obtain EP for the classes $\mathcal{UOS}(2)$ and $\mathcal{UOS}(3)$ with respect to $\mathcal{OS}(2)$ and $\mathcal{OS}(3)$ without using RP. On the other hand, it is not clear how to obtain EP for the class $\mathcal{SOB}$ with respect to $\mathcal{OB}$ from RP. In the following we show that $\mathcal{COH}$ satisfies EP with respect to $\mathcal{OH}$, in this case EP is OP. Moreover we give two proofs, the first is obtained without using RP and the second is based on RP. The first proof is similar to the proof of Proposition 3, and the second is similar to the proofs of Proposition 1 and Proposition 2.

Proposition 4 *(1st proof). $\mathcal{COH}$ satisfies OP with respect to $\mathcal{OH}$.*

Proof. By Lemma 2 it is enough to find a structure in $\mathcal{OH}$ that verifies EP for a given $\mathbb{A} = (A, C^A, \leq^A, \preceq^A) \in \mathcal{COH}$. Since the class $\mathcal{CH}$ has OP with respect to $\mathcal{H}$ there is a structure $\mathbb{B}' = (B', C^{B'}) \in \mathcal{H}$ which verifies OP for $\mathbb{A}' = (A, C^A)$. There is a tree $(\mathbb{T}', \infty^{T'}) = (T', E^{T'}, \infty^{T'}) \in \mathcal{T}_\infty$ such that $H(\mathbb{T}', \infty^{T'}) = \mathbb{B}'$. Let $(\mathbb{A}'_i)_{i=1}^n$ be the list of all substructures of $\mathbb{B}'$ isomorphic with $\mathbb{A}'$. Let k be the number of all ordered expansions of the structure $\mathbb{A}'$ to a structure $(\mathbb{A}', \preceq^{A'}) \in \mathcal{CH}$ which are isomorphic to $(A, \leq^A, \preceq^A)$. In particular for each $\mathbb{A}'_i$ we have a list $(\mathbb{A}'_{i,j})_{j=1}^k$ of all such expansions. We choose a natural number l such that $2^l > nk$. We construct a tree $(\mathbb{T}, \infty^T) = (T, E^T, \infty^T) \in \mathcal{T}_\infty$ using $(\mathbb{T}', \infty^{T'})$ and copies of $\mathbb{T}_{l,\infty} = (T_{l,\infty}, E^{T_{l,\infty}}, \infty^l) \in \mathcal{T}_\infty$ such that:

- $T = T' \sqcup (\bigsqcup_{b \in B'} T_{l,b})$ and $|T_{l,b}| = |T_{l,\infty}| - 1$.
- There is a sequence of bijections $(\phi_b : T_{l,\infty} \to T_{l,b} \cup \{b\})_{b \in B'}$ such that for each $b \in B'$ we have $\phi_b(\infty^l) = b$. Each ϕ_b gives naturally a graph structure E^b on the set $T_{l,b} \cup \{b\}$ such that ϕ_b is also isomorphism between structure $(T_{l,\infty}, E^{T_{l,\infty}})$ and structure $(T_{l,b} \cup \{b\}, E^b)$.
- $E^T = E^{T'} \sqcup (\bigsqcup_{b \in B'} E^b)$.
- The root ∞^T in the new tree is the same as the root $\infty^{T'}$.

Now we have a structure $H(\mathbb{T}, \infty^T) = (H(T), C^{H(T)}) \in \mathcal{H}$. Before we define $\leq^{H(T)}$, a linear ordering on the set $H(T)$, we have the following fact.

Fact. Let $\mathbb{A}'_i$ be one of the structures from our list with the set of nodes $A'_i \subseteq B'$. Let $A''_i \subseteq B$ be the set obtained when we replace each node $b' \in A'_i$ with a node $b \in B$ which is a terminal node of the tree $\mathbb{T}_{l,b'}$. Let $\mathbb{A}''_i$ be the substructure of $B(\mathbb{T})$ induced on the set A''_i. Then we have the following:

(1) $\mathbb{A}'_i \cong \mathbb{A}''_i$.

(2) Every expansion of the structure $\mathbb{A}_i''$ to a structure in $\mathcal{CH}$ is an expansion of the structure $\mathbb{A}_i'$ to a structure in $\mathcal{CH}$.

For $i \in [n]$ and $j \in [k]$ we consider a set $A_{i,j}'' \subseteq H(T)$ such that:

- $(i,j) \neq (i',j') \Rightarrow A_{i,j}'' \cap A_{i',j'}'' = \emptyset$.
- If $b' \in A_{i,j}$ where $A_{i,j}$ is the underlying set of $\mathbb{A}_{i,j}'$ then $A_{i,j}''$ has exactly one point among terminal nodes of $\mathbb{T}_{l,b'}$.
- If $b' \notin A_{i,j}$ where $A_{i,j}$ is the underlying set of $\mathbb{A}_{i,j}'$ then $A_{i,j}''$ has no points among terminal nodes of $\mathbb{T}_{l,b'}$.

Also, every set $A_{i,j}''$ induces a substructure $\mathbb{A}_{i,j}'' \leq H(\mathbb{T}, \infty^T)$ such that $\mathbb{A}_{i,j}'' \cong \mathbb{A}_i'$. Moreover, each structure $\mathbb{A}_{i,j}'$ naturally gives an expansion $\mathbb{A}_{i,j}'''$ of the structure $\mathbb{A}_{i,j}''$ to a structure in $\mathcal{CH}$ such that $\mathbb{A}_{i,j}''' \cong \mathbb{A}_{i,j}'$. On each set $A_{i,j}$ we define linear ordering $\leq^{i,j}$ such that

$$\left(\mathbb{A}_{i,j}''', \leq^{i,j}\right) \cong \mathbb{A}.$$

Finally, we have only one requirement for the linear ordering $\leq^{H(T)}$ on the set $H(T)$. For all i, j it must be:

$$\leq^{H(T)} \restriction (A_{i,j})^2 = \leq^{i,j}.$$

We claim that the structure $(H(\mathbb{T}), \leq^{H(T)}) = (H(T), C^{H(T)}, \leq^{H(T)})$ verifies OP for $\mathbb{A}$. Let $\preceq^0$ be a linear ordering on the set $H(T)$ such that $(H(T), R^{H(T)}, \leq^{H(T)}, S^0) \in \mathcal{COH}$. We have the following fact.

Fact. Let b_1' and b_2' be distinct points in B'. Let a_1, a_2 and a_1', a_2' be points from $H(T)$ such that for each i points a_i and a_i' are terminal nodes in $\mathbb{T}_{l,b_i'}$. Then we have

$$a_1 \preceq^0 a_2 \quad \Leftrightarrow \quad a_1' \preceq^0 a_2''.$$

By this fact, there is an induced relation $\preceq^1$ on the set B' such that $(B', C^{B'}, \preceq^1) \in \mathcal{CH}$. For distinct b_1', b_2' points in B' we take points $a_1, a_2 \in H(T)$ such that each a_i is a terminal node in $\mathbb{T}_{l,b_i'}$. Then take

$$b_1' \preceq^1 b_2' \quad \Leftrightarrow \quad a_1 \preceq^0 a_2.$$

Since $\mathbb{B}'$ verifies EP for $\mathbb{A}'$, there is an $\mathbb{A}_i'$ with the underlying set A_i' such that $(\mathbb{A}_i', \preceq^1 \restriction (A_i')^2) \cong (A, C^A, \preceq^A)$ and there is j such that $(A, C^A, \preceq^A) = \mathbb{A}_{i,j}'$. Furthermore, we have $\mathbb{A}_{i,j}''' \cong \mathbb{A}_{i,j}'$ and by the choice of the linear ordering $\leq^{H(T)}$ we have that $(\mathbb{A}_{i,j}''', \leq^{i,j}) \cong \mathbb{A}$. Therefore there is an embedding of $\mathbb{A}$ into $(H(T), R^{H(T)}, \leq^{H(T)}, S^0)$. $\square$

Proposition 5 *(2nd proof). $\mathcal{COH}$ satisfies OP with respect to $\mathcal{OH}$.*

Proof. By Lemma 2 it is enough to find a structure in $\mathcal{OH}$ that verifies EP for a given $\mathbb{A} = (A, C^A, \leq^A, \preceq^A) \in \mathcal{COH}$. We consider structures $\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$ in $\mathcal{COH}$ given as follows:

- $\mathbb{I} = (I, C^I, \leq^I, \preceq^I)$, $I = \{i_1, i_2\}$, $C^I = \emptyset$, $i_1 \leq^I i_2$, $i_1 \preceq^I i_2$.
- $\mathbb{J} = (J, C^J, \leq^J, \preceq^J)$, $J = \{j_1, j_2\}$, $C^J = \emptyset$, $j_1 \leq^J j_2$, $j_2 \preceq^J j_1$.
- $\mathbb{K} = (K, C^K, \leq^K, \preceq^K)$, $(K, C^K, \leq^K) \notin \mathcal{CH}$, $(K, C^K, op(\leq^K)) \notin \mathcal{CH}$.

By JEP for the class $\mathcal{COH}$ there is $\mathbb{B} = (B, C^B, \leq^B, \preceq^B) \in \mathcal{COH}$ such that $\mathbb{A} \hookrightarrow \mathbb{B}$, $\mathbb{I} \hookrightarrow \mathbb{B}$, $\mathbb{J} \hookrightarrow \mathbb{B}$, $\mathbb{K} \hookrightarrow \mathbb{B}$. Moreover we have $\mathbb{B}^{op} = (B, C^B, \leq^B, op(\preceq^B)) \in \mathcal{COH}$. Using JEP for the class $\mathcal{COH}$ there is $\mathbb{D} = (D, C^D, \leq^D, \preceq^D) \in \mathcal{COH}$ such that $\mathbb{B} \hookrightarrow \mathbb{D}$ and $\mathbb{B}^{op} \hookrightarrow \mathbb{D}$. By RP for the class $\mathcal{COH}$ there are structures $\mathbb{E}$ and $\mathbb{F}$ in $\mathcal{COH}$ such that:

$$\mathbb{E} \to (\mathbb{D})^{\mathbb{I}}_2 \quad \text{and} \quad \mathbb{F} \to (\mathbb{E})^{\mathbb{J}}_2.$$

Let $\mathbb{F} = (F, C^F, \leq^F, \preceq^F)$. We claim that $(F, C^F, \leq^F)$ verifies OP for $\mathbb{A}$. Let $\preceq^0$ be a linear ordering on the set F such that $(F, C^F, \leq^F, \preceq^0) \in \mathcal{COH}$. We have induced colorings:

$$\chi_I : \binom{\mathbb{F}}{\mathbb{I}} \to \{0, 1\} \quad \text{and} \quad \chi_J : \binom{\mathbb{F}}{\mathbb{J}} \to \{0, 1\},$$

$$\chi_I(\mathbb{I}') = \begin{cases} 1 & \text{if } \preceq^0 \restriction I' = \preceq^F \restriction I' \\ 0 & \text{if otherwise} \end{cases} \quad \text{and} \quad \chi_J(\mathbb{J}') = \begin{cases} 1 & \text{if } \preceq^0 \restriction J' = \preceq^F \restriction J' \\ 0 & \text{if otherwise} \end{cases}$$

where I' and J' are the underlying sets of the structures $\mathbb{I}'$ and $\mathbb{J}'$ respectively. By the choice of the structures $\mathbb{E}$ and $\mathbb{F}$ there are structures $\mathbb{D}'$ and $\mathbb{E}'$ such that $\mathbb{D}' \leq \mathbb{E}' \leq \mathbb{F}$, $\mathbb{D}' \cong \mathbb{D}$, $\mathbb{E}' \cong \mathbb{E}$, and numbers $p_I, p_J \in \{0, 1\}$ such that:

$$\chi_I \restriction \binom{\mathbb{D}'}{\mathbb{I}} = p_I \quad \text{and} \quad \chi_J \restriction \binom{\mathbb{D}'}{\mathbb{J}} = p_J.$$

Let D' be the underlying set of the structure $\mathbb{D}'$. We complete the verification by checking four cases depending on the value of the pair (p_I, p_J):

(1) $(p_I, p_J) = (0, 0)$ – In this case we have $\preceq^0 \restriction D' = op(\preceq^F \restriction D')$, and also we have $\mathbb{B}^{op} \hookrightarrow \mathbb{D}'$. So there is $\mathbb{B}' \leq \mathbb{D}'$ such that $\mathbb{B}' \cong \mathbb{B}^{op}$, where $\mathbb{B}'$ has the underlying set B'. Therefore the substructure of $(F, C^F, \leq^F, \preceq^0)$ induced by the set B' is isomorphic to $\mathbb{B}$, and it contains substructures isomorphic with $\mathbb{A}$.

(2) $(p_I, p_J) = (0, 1)$ – Now we have $\preceq^0 \restriction D' = \leq^F \restriction D'$. This is in contradiction with the fact that $\mathbb{K} \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{D}'$ and with the definition of the linear ordering $\leq^K$. Therefore this case is impossible.

(3) $(p_I, p_J) = (1, 0)$ – Same as the previous case.

(4) $(p_I, p_J) = (1, 1)$ – In this case we have $\preceq^0 \restriction D' = \preceq^F \restriction D'$, so there is an embedding of $\mathbb{A}$ into $(F, C^F, \leq^F, \preceq^0)$. $\square$

9. Conclusion

In the following we give a list of Ramsey degrees.

Theorem 9. *Let $\mathcal{K} \in \{\mathcal{OS}(2), \mathcal{OS}(3), \mathcal{OB}, \mathcal{H}, \mathcal{OH}\}$ and let $\mathbb{A} \in \mathcal{K}$. Then we have that $t_{\mathcal{K}}(\mathbb{A})$ is equal to:*

(i)
$$\begin{aligned}&\left|\left\{\mathbb{A}' \in \mathcal{UOS}(2) : \mathbb{A}'|\{\longrightarrow, \leq\} = \mathbb{A}\right\}\right| \\ &\quad = \left|\left\{\mathbb{A}' \in \mathcal{US}(2) : \mathbb{A}'|\{\longrightarrow\} = \mathbb{A}|\{\longrightarrow\}\right\}\right| = 2|\mathbb{A}|\end{aligned}$$

for $\mathcal{K} = \mathcal{OS}(2)$.

(ii)
$$\begin{aligned}&\left|\left\{\mathbb{A}' \in \mathcal{UOS}(3) : \mathbb{A}'|\{\longrightarrow, \leq\} = \mathbb{A}\right\}\right| \\ &\quad = \left|\left\{\mathbb{A}' \in \mathcal{US}(3) : \mathbb{A}'|\{\longrightarrow\} = \mathbb{A}|\{\longrightarrow\}\right\}\right| = 3|\mathbb{A}|\end{aligned}$$

for $\mathcal{K} = \mathcal{OS}(3)$.

(iii)
$$\begin{aligned}&\left|\left\{\mathbb{A}' \in \mathcal{SOB} : \mathbb{A}'|\{R, \leq\} = \mathbb{A}\right\}\right| \\ &\quad = \left|\left\{\mathbb{A}' \in \mathcal{SB} : \mathbb{A}'|\{R\} = \mathbb{A}|\{R\}\right\}\right|\end{aligned}$$

for $\mathcal{K} = \mathcal{OB}$.

(iv)
$$\left|\left\{\mathbb{A}' \in \mathcal{CH} : \mathbb{A}'|\{C\} = \mathbb{A}\right\}\right| \cdot \left|Aut(\mathbb{A})\right|^{-1}$$

for $\mathcal{K} = \mathcal{H}$.

(v)
$$\begin{aligned}&\left|\left\{\mathbb{A}' \in \mathcal{COH} : \mathbb{A}'|\{C, \leq\} = \mathbb{A}\right\}\right| \\ &\quad = \left|\left\{\mathbb{A}' \in \mathcal{CH} : \mathbb{A}'|\{C\} = \mathbb{A}|\{C\}\right\}\right|\end{aligned}$$

for $\mathcal{K} = \mathcal{OH}$.

Proof. This follows from Theorem 2 and the following results:

(i) Corollary 3, Proposition 1 and from the calculation of the Ramsey degree in the class $\mathcal{S}(2)$, see [18].
(ii) Corollary 4, Proposition 2 and from the calculation of the Ramsey degree in the class $\mathcal{S}(3)$, see [18].
(iii) Corollary 5 and Proposition 3.
(iv) Theorem 8 and Lemma 5.
(v) Corollary 6 and Proposition 4 or Proposition 5. $\square$

Since calculation of the Ramsey degree for objects in $\mathcal{B}$ is not simple, see [10], we avoid going into more details. Since the only Ramsey objects in $\mathcal{B}$ are one, two and three point structures we have the same for the class $\mathcal{OB}$. Even $\mathcal{OS}(2)$ and $\mathcal{OS}(3)$ are classes of rigid structures they have no Ramsey objects. A structure $\mathbb{A} \in \mathcal{H}$ is a Ramsey object iff there is some n such that $\mathbb{A} = H(\mathbb{T}_{n,\infty})$. The only Ramsey object in $\mathcal{OH}$ is the one point structure.

Let L_1 and L_2 be disjoint relational signatures, and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of finite relational structures in L_1 and L_2 respectively. Let L_1^* and L_2^* be disjoint relational signatures such that $L_1 \subset L_1^*$ and $L_2 \subset L_2^*$. Suppose that $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are expansions of the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. If $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are Fraïssé classes such that $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are also Ramsey classes of rigid structures then we know that $\mathcal{K}_1^* * \mathcal{K}_2^*$ is a Ramsey class by the results from Section 4. Moreover we know that $\mathcal{K}_1^* * \mathcal{K}_2^*$ is a Ramsey expansion of the class $\mathcal{K}_1 * \mathcal{K}_2$. Furthermore, we may suppose that $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ satisfy EP with respect to $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Now we would like to know whether or not $\mathcal{K}_1^* * \mathcal{K}_2^*$ satisfies EP with respect to $\mathcal{K}_1 * \mathcal{K}_2$. According to Theorem 10.7. in [12] we may expect that there is a class $\mathcal{K} \subseteq \mathcal{K}_1 * \mathcal{K}_2$ which is a Ramsey class and satisfies EP with respect to $\mathcal{K}_1 * \mathcal{K}_2$. Since there is no characterization of the class $\mathcal{K}$ we ask the following question.

Problem 1. Find a characterization of the class $\mathcal{K}$ in terms of the classes $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$.

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