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Extensions of Sperner and Tucker's lemma for manifolds [☆]



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ABSTRACT

The Sperner and Tucker lemmas are combinatorial analogs of the Brouwer and Borsuk–Ulam theorems with many useful applications. These classic lemmas are concerning labellings of triangulated discs and spheres. In this paper we show that discs and spheres can be substituted by large classes of manifolds with or without boundary.

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1. Introduction

1.1. Notation

Throughout this paper the symbol \mathbb{R}^d denotes the Euclidean space of dimension d . We denote by \mathbb{B}^d the d -dimensional ball and by \mathbb{S}^d the d -dimensional sphere. If we consider \mathbb{S}^d as the set of unit vectors x in \mathbb{R}^{d+1} , then points x and $-x$ are called *antipodal* and the symmetry given by the mapping $x \rightarrow -x$ is called the *antipodality* on \mathbb{S}^d .

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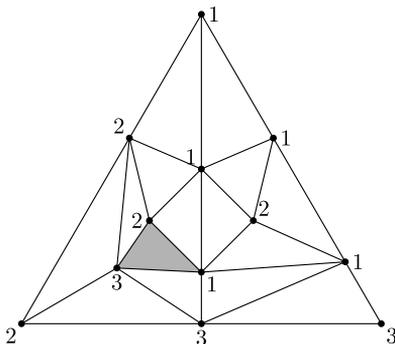


Fig. 1. A 2-dimensional illustration of Sperner's lemma.

1.2. Sperner's lemma

Sperner's lemma is a statement about labellings of triangulated simplices (d -balls). It is a discrete analog of the Brouwer fixed point theorem (see Fig. 1).

Let S be a d -dimensional simplex with vertices v_1, \dots, v_{d+1} . Let T be a triangulation of S . Suppose that each vertex of T is assigned a unique label from the set $\{1, 2, \dots, d+1\}$. A labelling L is called *Sperner's* if the vertices are labelled in such a way that a vertex of T belonging to the interior of a face F of S can only be labelled by k if v_k is on S .

Theorem 1.1 (*Sperner's lemma [13]*). *Every Sperner labelling of a triangulation of a d -dimensional simplex contains a cell labelled with a complete set of labels: $\{1, 2, \dots, d + 1\}$.*

There are several extensions of this lemma. One of the most interesting is the De Loera–Peterson–Su theorem. In the paper [4] they proved the Atanassov conjecture [1].

Theorem 1.2 (*Polytopal Sperner's lemma*). *Let P be a polytope in \mathbb{R}^d with vertices v_1, \dots, v_n . Let T be a triangulation of P . Let $L : V(T) \rightarrow \{1, 2, \dots, n\}$ be a Sperner labelling. Then there are at least $(n - d)$ fully-colored (i.e. with distinct labels) d -simplices of T .*

Meunier [8] extended this theorem:

Theorem 1.3. *Let P^d be a d -dimensional PL manifold embedded in \mathbb{R}^d that has boundary B . Suppose B has n vertices v_1, \dots, v_n . Let T be a triangulation of P . Let $L : V(T) \rightarrow \{1, 2, \dots, n\}$ be a Sperner labelling. Let d_i denote the number of edges of B which are connected to v_i . Then there are at least $n + \lceil \min_i \{d_i\} / d \rceil - d - 1$ fully-labelled d -simplices such that any pair of these fully-labelled simplices receives two different labellings.*

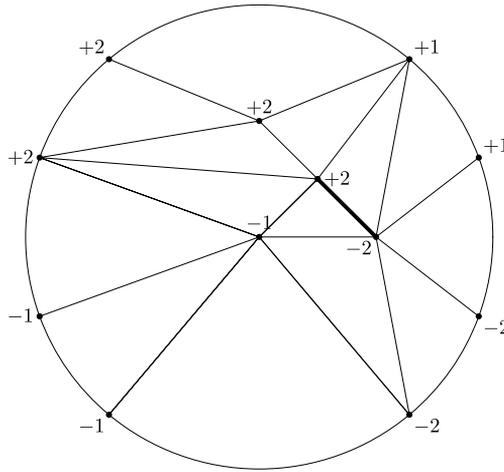


Fig. 2. A 2-dimensional illustration of Tucker's lemma.

1.3. Tucker's lemma

Let T be some triangulation of the d -dimensional ball \mathbb{B}^d . We call T *antipodally symmetric on the boundary* if the set of simplices of T contained in the boundary of $\mathbb{B}^d = \mathbb{S}^{d-1}$ is an antipodally symmetric triangulation of \mathbb{S}^{d-1} , that is if $s \subset \mathbb{S}^{d-1}$ is a simplex of T , then $-s$ is also a simplex of T (see Fig. 2).

Theorem 1.4 (Tucker's lemma [14]). *Let T be a triangulation of \mathbb{B}^d that antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be a labelling of the vertices of T that satisfies $L(-v) = -L(v)$ for every vertex v on the boundary. Then there exists an edge in T that is **complementary**, i.e. its two vertices are labelled by opposite numbers.*

Consider also the following version of Tucker's lemma:

Theorem 1.5. *Let T be a centrally symmetric triangulation of the sphere \mathbb{S}^d . Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

be an equivariant (or Tucker's) labelling, i.e. $L(-v) = -L(v)$. Then there exists a complementary edge.

Tucker's lemma was extended by Ky Fan [5]:

Theorem 1.6. *Let T be a centrally symmetric triangulation of the sphere \mathbb{S}^d . Suppose that each vertex v of T is assigned a label $L(v)$ from $\{\pm 1, \pm 2, \dots, \pm n\}$ in such a way that $L(-v) = -L(v)$. Suppose this labelling does not have complementary edges. Then there are an odd number of d -simplices of T whose labels are of the form $\{k_0, -k_1, k_2, \dots, (-1)^d k_d\}$, where $1 \leq k_0 < k_1 < \dots < k_d \leq n$. In particular, $n \geq d + 1$.*

In this paper we consider extensions of the Sperner, De Loera–Peterson–Su, Tucker and Fan theorems for manifolds. We show that for all cases d -balls and spheres can be substituted by d -manifolds with or without boundary.

2. Preliminaries

Throughout this paper we consider manifolds that admit triangulations. The class of such manifolds is called piecewise linear (PL) manifolds. Note that a smooth manifold can be triangulated, therefore it is also a PL manifold. However, there are topological manifolds that do not admit a triangulation.

A *topological manifold* is a topological space that resembles Euclidean space near each point. More precisely, each point of a d -dimensional manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension d . A compact manifold without boundary is called *closed*. If a manifold contains its own boundary, it is called a *manifold with boundary*.

Smooth manifolds (also called *differentiable manifolds*) are manifolds for which overlapping charts “relate smoothly” to each other, meaning that the inverse of one followed by the other is an infinitely differentiable map from Euclidean space to itself.

M is called a *piecewise linear (PL) manifold* if it is a topological manifold together with a piecewise linear structure on it. Every PL manifold M admits a *triangulation*: that is, we can find a collection of simplices T of dimensions $0, 1, \dots, d$, such that (1) any face of a simplex belonging to T also belongs to T , (2) any nonempty intersection of any two simplices of T is a face of each, and (3) the union of the simplices of T is M . (See details in [3].)

Note that the circle is the only one-dimensional closed manifold. Closed manifolds in two dimensions are completely classified. (See details and proofs in [12].) An orientable two-manifold (surface) is the sphere or the connected sum of g tori, for $g \geq 1$. For any positive integer n , a distinct non-orientable surface can be produced by replacing n disks with Möbius bands. In particular, replacing one disk with a Möbius band produces the real projective plane and replacing two disks produces the Klein bottle. The sphere, the g -holed tori, and this sequence of non-orientable surfaces form a complete list of compact, boundaryless two-dimensional manifolds.

Example 2.1. The real projective plane, \mathbb{RP}^2 , can be viewed as the union of a Möbius band and a disc. The correspondent model of the Möbius band is shown in Fig. 3. Note that this model cannot be embedded to \mathbb{R}^3 .

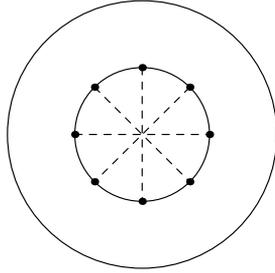


Fig. 3. Möbius band. Diametrically opposite points of the inner boundary circle are to be identified. The outer circle is the boundary of the Möbius band.

Let T be a triangulation of a PL manifold M . Then T is a simplicial complex. The vertex set of T , denoted by $V(T)$ is the union of the vertex sets of all simplices of T .

Given two triangulations T_1 and T_2 of two PL manifolds M_1 and M_2 . A *simplicial map* is a function $f : V(T_1) \rightarrow V(T_2)$ that maps the vertices of T_1 to the vertices of T_2 and that has the property that for any simplex (face) s of T_1 , the image set $f(s)$ is a face of T_2 .

Note that the original Brouwer proof of his fixed point theorem that is based on the concept of the degree of a continuous mapping. Let $f : M_1 \rightarrow M_2$ be a continuous map between two closed manifolds M_1 and M_2 of the same dimension. Intuitively, the degree is a number that represents the number of times that the domain manifold wraps around the range manifold under the mapping. Then $\deg_2(f)$ (the degree modulo 2) is 1 if this number is odd and 0 otherwise.

It is well known that the degree of a continuous map f of a closed manifold to a manifold is a topological invariant modulo 2 (see, for instance, [9] and [7, pp. 44–46]). Therefore, the degree of f is odd if any generic point in the range of the map has an odd number of preimages.

Now we define $\deg_2(f)$ more rigorously. Let T_1 be a triangulation of a closed d -dimensional PL manifold M_1 . Suppose that T_2 is a triangulation of a d -dimensional PL manifold M_2 . (We do not assume that M_2 is closed.) Let $f : V(T_1) \rightarrow V(T_2)$ be a simplicial map. Consider any d -simplex s of T_2 . Denote by m the number of preimages of s in T_1 . Then $\deg_2(f) = 1$ if m is odd and $\deg_2(f) = 0$ if m is even. Since the parity of m does not depend on s , the *degree of map modulo 2* is well defined. Thus, the degree of a continuous map of a closed manifold to a manifold is a topological invariant modulo 2.

Let $f : M_1 \rightarrow M_2$ be a continuous map between two manifolds M_i with $d_1 := \dim(M_1) \geq d_2 := \dim(M_2)$. Then for a point $y \in M_2$ the map f is called *transversal to y* (or *generic with respect to y*) if there are open sets $U_i \subset M_i$ such that U_2 contains y , $U_2 = f(U_1)$ and $U_1 = f^{-1}(U_2)$. In the case $M_2 = \mathbb{R}^{d_2}$ and $y = 0$, where 0 is the origin of \mathbb{R}^{d_2} , f is called *transversal to zero*.

Let M be a closed PL-manifold. A simplicial map $A : M \rightarrow M$ is called a *free involution* if $A(A(x)) = x$ and $A(x) \neq x$ for all $x \in M$. A triangulation T of M is called

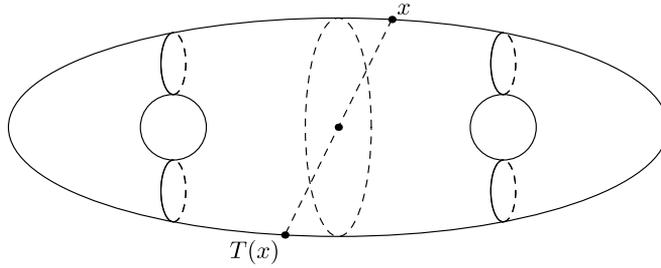


Fig. 4. The double torus that is centrally symmetric embedded to \mathbb{R}^3 .

antipodal or equivariant if $A : T \rightarrow T$ is a simplicial map. Let us call a pair (M, A) , where A is a free simplicial involution as \mathbb{Z}_2 -manifold.

Example 2.2. It is clear that (\mathbb{S}^d, A) with $A(x) = -x$ is a \mathbb{Z}_2 -manifold. Suppose that M can be represented as a connected sum $N \# N$, where N is a closed PL manifold. Then M admits a free involution. Indeed, M can be “centrally symmetric” embedded to \mathbb{R}^k with some k and the antipodal symmetry $x \rightarrow -x$ in \mathbb{R}^k implies a free involution $T : M \rightarrow M$ [11, Corollary 1]. For instance, orientable two-dimensional manifolds M_g^2 with even genus g and non-orientable manifolds P_m^2 with even m , where m is the number of Möbius bands, are \mathbb{Z}_2 -manifolds.

Suppose that M admits a free simplicial involution A . We say that a map $f : M \rightarrow \mathbb{R}^d$ is antipodal (or equivariant) if $f(A(x)) = -f(x)$. (See Fig. 4.)

3. Extensions of Sperner’s lemma for manifolds

A d -simplex S where each corner is labelled between 1 and $d + 1$ such that all labels are used exactly once is called fully labelled. Suppose that points are added in S , then it may be triangulated, i.e. subdivided into smaller d -simplices such that none of the smaller simplices contain any points: all the points are corners of smaller simplices. This subdivision may be done in many ways.

Now label all the interior points according to the following rule: an interior point that is on a facet of the simplex must be given one of the labels of one of the corners of that facet. The result is called a Sperner labelling.

Note that in this definition an interior point of S can be labelled by any label. So Sperner’s constraint is only for the boundary of S that is homeomorphic to \mathbb{S}^{d-1} . Let us extend this definition to any closed manifold.

Definition 3.1. Let K be a closed m -dimensional PL manifold with vertices $V = \{v_1, \dots, v_n\}$ and faces $\{F_i\}$ of dimension from 1 to m . Let T be a triangulation of K such that for any face (that is a simplex) F_i it is a triangulation of F_i . Suppose that the vertices of T have a labelling satisfying the following conditions: each vertex v_k of V

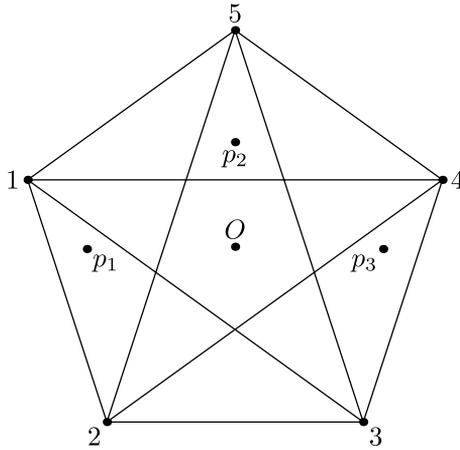


Fig. 5. Pebbles and $\text{cov}(x)$ for a pentagon.

is assigned a unique label from $\{1, 2, \dots, n\}$, and each other vertex v of T belonging to a face F_i with vertices $V(F_i) := \{v_{i_1}, \dots, v_{i_\ell}\}$ from V is assigned a label of one of the vertices of $V(F_i)$. Such a labelling is called a *Sperner labelling* of T .

Definition 3.2. We say that a d -simplex is a *fully labelled cell* or simply a *full cell* if all its labels are distinct.

Let T be a triangulation of a d -dimensional PL manifold. Let $L : V(T) \rightarrow \{1, \dots, n\}$ be a labelling of T . Let Q be a set of $(d+1)$ -subsets of $\{1, \dots, n\}$. We denote by $\text{fc}(L, T, Q)$ the number of fully labelled cells that are labelled as labels in Q . In the case $n = d + 1$ we denote $\text{fc}(L, T) := \text{fc}(L, T, \{1, \dots, d + 1\})$.

Let P be a set of n points p_1, \dots, p_n in \mathbb{R}^d . Denote by $S(P)$ the collection of all simplices spanned by vertices $\{p_{i_1}, \dots, p_{i_k}\}$ with $1 \leq k \leq d + 1$. Consider a point $x \in \mathbb{R}^d$ and the set $S_x(P)$ of all simplices from $S(P)$ which cover x . If no such simplices exist, we write $S_x(P) = \emptyset$. Denote this set of simplices by $\text{cov}_P(x)$ or just by $\text{cov}(x)$.

Example 3.3. Let P be a pentagon, see Fig. 5. Then

$$\begin{aligned} \text{cov}(p_1) &= (123) \cup (124) \cup (125); & \text{cov}(p_2) &= (135) \cup (145) \cup (235) \cup (245); \\ \text{cov}(p_3) &= (134) \cup (234) \cup (345); & \text{cov}(O) &= (124) \cup (134) \cup (135) \cup (235) \cup (245). \end{aligned}$$

Definition 3.4. Let $P := \{p_1, \dots, p_n\}$ be points in \mathbb{R}^d . Let T be a triangulation of a closed PL manifold M of dimension m . Let L be an n -labelling of T , i.e. a labelling (map) $L : V(T) \rightarrow \{1, 2, \dots, n\}$. If for $v \in V(T)$ we have $L(v) = i$, then set $f_{L,P}(v) := p_i$. Therefore, $f_{L,P}$ is defined for all vertices of T , and it uniquely defines a simplicial (piecewise linear) map $f_{L,P} : M \rightarrow \mathbb{R}^d$.

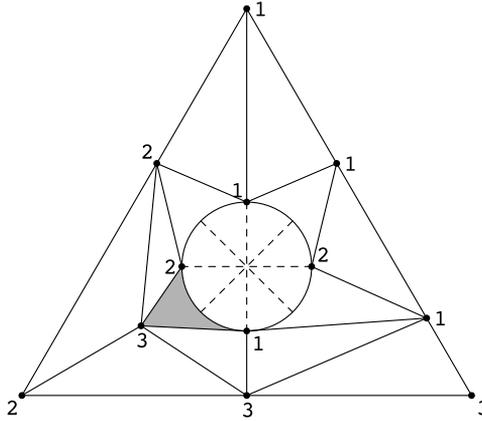


Fig. 6. Sperner’s lemma for the Möbius band.

Theorem 3.5. Let $P := \{p_1, \dots, p_n\} \subset \mathbb{R}^d$. Suppose $x \in \mathbb{R}^d$ is such that $\text{cov}_P(x)$ consists of d -simplices. Let M be a closed PL d -dimensional manifold. Then any n -labelling L of a triangulation T of M must contain an even number of full cells which are labelled as simplices in $\text{cov}_P(x)$.

Proof. Consider $f_{L,P} : T \rightarrow \mathbb{R}^d$. It is easy to see that $\deg_2(f_{L,P}) = 0$. Indeed, if $y \in \mathbb{R}^d$ lies outside of the convex hull of P in \mathbb{R}^d , then $f_{L,P}^{-1}(y) = \emptyset$. Therefore, for any point x in \mathbb{R}^d which is a regular value of $f_{L,P}$, we have $|f_{L,P}^{-1}(x)| \equiv |f_{L,P}^{-1}(y)| \equiv 0 \pmod{2}$. Thus, the number of full cells which are labelled as simplices in $\text{cov}_P(x)$ is even. \square

For the classical case $n = d + 1$ we have the following result (also see [6]):

Corollary 3.1. Let T be a triangulation of a closed PL manifold M^d . Any $(d + 1)$ -labelling of T must contain an even number of full cells.

Corollary 3.2. Let M be a d -dimensional compact PL manifold with boundary B . Let B be PL homeomorphic to the boundary of a d -simplex (i.e. $B \cong \mathbb{S}^{d-1}$) with vertices v_1, \dots, v_{d+1} . Then any $(d + 1)$ -labelling L of a triangulation T of M such that $L(v_i) = i$ and L is a Sperner labelling on the boundary B must contain an odd number of full cells.

Proof. We prove this corollary by induction on d . It is clear for $d = 1$. Let S denote a d -simplex. Two manifolds M and S can be glued together along B . We denote the new manifold by N . Then N is a closed manifold. Corollary 3.1 implies that any $(d + 1)$ -labelling of any triangulation of N has an even number of full cells. (See Fig. 6.)

Let us add to the vertices of T one more vertex q that is an internal point of S . Let $C = \text{cone}(T|_B)$ be the cone triangulation of S with vertex q . (Here $T|_B$ denote the triangulation T on B .) Actually, C consists of simplices formed by the union of all segments connecting the points of B with q and the boundary triangulation $T|_B$. Then we obtain the triangulation $\tilde{T} := T \cup C$ of N .

Consider the following labelling \tilde{L} on \tilde{T} . Let $\tilde{L}(v) := L(v)$ for all $v \in V(T)$ and $\tilde{L}(q) := 1$. Since \tilde{T} is a triangulation of N , we have that the number of full cells $\text{fc}(\tilde{L}, \tilde{T})$ is even.

By induction the face $F = v_2 \dots v_{d+1}$ of B has an odd number of full cells. Then $\text{fc}(\tilde{L}, C) = \text{fc}(L, T|_F, \{2, \dots, d + 1\})$ is odd. Note that

$$\text{fc}(\tilde{L}, \tilde{T}) = \text{fc}(L, T) + \text{fc}(\tilde{L}, C).$$

Thus T must contain an odd number of full cells. \square

Note that for the case when M is a d -simplex [Corollary 3.2](#) is Sperner’s lemma.

Now we show how the De Loera–Peterson–Su theorem follows from [Theorem 3.5](#).

Corollary 3.3. *Let P be a convex polytope in \mathbb{R}^d with vertices p_1, \dots, p_n . Let M be a compact d -dimensional PL manifold with boundary B . Let B be piecewise linearly homeomorphic to the boundary of P . Suppose $x \in \mathbb{R}^d$ is such that $\text{cov}_P(x)$ consists of d -simplices. Then any n -labelling L of a triangulation T of M that is a Sperner labelling on the boundary must contain an odd number of full cells which are labelled as simplices in $\text{cov}_P(x)$. In other words, $\text{fc}(L, T, \text{cov}(x))$ is odd.*

Proof. This corollary can be proved by similar arguments as [Corollary 3.2](#). Indeed, two manifolds M and P can be glued together along B . We denote the new manifold by N . Then N is a closed manifold.

Let $C := \text{cone}(T|_B)$ be the cone triangulation of P with vertex q , where q is an internal point of P . Then we have the triangulation $\tilde{T} := T \cup C$ of N .

Consider the following labelling \tilde{L} on \tilde{T} . Let $\tilde{L}(v) := L(v)$ for all $v \in V(T)$ and $\tilde{L}(q) := 1$. Now we show that $\text{fc}(\tilde{L}, C, \text{cov}(x))$ is odd.

Consider the line in \mathbb{R}^d passes through points p_1 and x . By assumptions, this line intersects the boundary B of the polytope P in two points p_1 and y , where y is an internal point of some $(d - 1)$ -simplex of $T|_B$ with distinct labels ℓ_1, \dots, ℓ_d . Therefore, y lies on the face $F = v_{\ell_1} \dots v_{\ell_d}$. By induction $\text{fc}(L, T|_F, \{\ell_1, \dots, \ell_d\})$ is odd. Note that \tilde{L} on C contains only one labelling $(1\ell_1 \dots \ell_d)$ from $\text{cov}(x)$. Then

$$\begin{aligned} \text{fc}(\tilde{L}, C, \text{cov}(x)) &= \text{fc}(\tilde{L}, C, \{1, \ell_1, \dots, \ell_d\}) \\ &= \text{fc}(L, T|_F, \{\ell_1, \dots, \ell_d\}) = 1 \pmod{2}. \end{aligned}$$

We have

$$\text{fc}(\tilde{L}, \tilde{T}, \text{cov}(x)) = \text{fc}(L, T, \text{cov}(x)) + \text{fc}(\tilde{L}, C, \text{cov}(x)),$$

where $\text{fc}(\tilde{L}, \tilde{T}, \text{cov}(x))$ is even and $\text{fc}(\tilde{L}, C, \text{cov}(x))$ is odd. Thus $\text{fc}(L, T, \text{cov}(x))$ is odd. \square

Corollary 3.4. *Let P be a convex polytope in \mathbb{R}^d with vertices p_1, \dots, p_n . Let M be a compact d -dimensional PL manifold with boundary B . Let B be PL homeomorphic to the boundary of P . Then any n -labelling of a triangulation of M that is a Sperner labelling on the boundary contains at least $n - d$ full cells.*

Proof. A proof of this corollary follows from another theorem from [4, Th. 4]: *Any convex polytope P in \mathbb{R}^d with n vertices contains a pebble set of size $n - d$.* (A finite set of points (pebbles) in P is called a pebble set if each d -simplex of P contains at most one pebble interior to chambers.)

Consider a pebble set $\{p_i\}$ of size $n - d$. Then for $i \neq j$ we have $\text{cov}(p_i) \cap \text{cov}(p_j) = \emptyset$. Thus Corollary 3.3 guarantees that there are at least $n - d$ full cells. \square

Remark. (i) For the case $M = P$ this statement is the polytopal Sperner lemma [4, Th. 1].

(ii) In fact, Meunier’s proof of his extension of the polytopal Sperner lemma (De Loera–Peterson–Su’s theorem) is not based on the “pebbles set theorem.” It is an interesting problem to find an extension of Meunier’s theorem for manifolds.

Example 3.6. Consider the case when P is a pentagon. In P there are three pebbles p_1, p_2, p_3 , see Fig. 5. So the polytopal Sperner’s lemma (Theorem 1.1) and Corollary 3.4 implies that there are at least three fully labelled triangles. Actually, this statement can be improved.

Note that there are 10 5-labellings for triangles. Five of them are consecutive: (123), (234), (345), (451), (512) and five are non-consecutive. In fact, $\text{cov}(O)$ consists of non-consecutive labellings, see Example 3.3. Then Corollary 3.3 implies the following statement:

Any Sperner 5-labelling of a triangulation T of a pentagon P must contain at least three full cells. Moreover, at least one of them is not consecutive labelled.

4. Extensions of Tucker’s lemma for manifolds

Definition 4.1. Let M be a closed PL d -dimensional manifold with a free simplicial involution $A : M \rightarrow M$. We say that a pair (M, A) is a BUT (Borsuk–Ulam Type) manifold if for any continuous $g : M \rightarrow \mathbb{R}^d$ there is a point $x \in M$ such that $g(A(x)) = g(x)$. Equivalently, if a continuous map $f : M \rightarrow \mathbb{R}^d$ is antipodal, then the zeros set $Z_f := f^{-1}(0)$ is not empty.

In [11], we found several equivalent necessary and sufficient conditions for manifolds to be BUT. For instance, M is a BUT manifold if and only if M admits an antipodal continuous transversal to zeros map $h : M \rightarrow \mathbb{R}^d$ with $|Z_h| = 2 \pmod{4}$.

Let T be any equivariant triangulation of M . We say that

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

is an *equivariant* (or *Tucker’s*) labelling if $L(A(v)) = -L(v)$.

An edge e in T is called *complementary* if its two ends are labelled by opposite numbers, i.e. if $e = uv$, then $L(v) = -L(u)$.

Theorem 4.2. *A closed PL d -dimensional manifold M with a free simplicial involution A is BUT if and only if for any equivariant labelling $L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$ of any equivariant triangulation T of M there exists a complementary edge.*

For the case $M = \mathbb{S}^d$ this is Tucker’s lemma.

Proof. Let e_1, \dots, e_d be an orthonormal basis of \mathbb{R}^d . Any equivariant labelling L of a triangulation T of M defines a simplicial map $f_L : T \rightarrow C^d$, where C^d is the crosspolytope in \mathbb{R}^d with vertex set $\{e_1, -e_1, e_2, -e_2, \dots, e_d, -e_d\}$, where for $v \in V(T)$, $f_L(v) = e_i$ if $L(v) = i$ and $f_L(v) = -e_i$ if $L(v) = -i$. (See details in [7, Sec. 2.3].) In other words, $f_L = f_{L, C^d}$ (see Section 3).

Note that any fully labelled simplex contains a complementary edge. Therefore, if L has no complementary edges, then $f_L : T \rightarrow \mathbb{R}^d$ has no zeros.

The reverse implication can be proved by the same arguments as equivalence of the Borsuk–Ulam theorem and Tucker’s lemma in [7, 2.3.2], i.e. if there is continuous antipodal map $f : M \rightarrow \mathbb{S}^{d-1}$ (i.e. $Z_f = \emptyset$) then T and L can be constructed with no complementary edges. (See also Theorem 4.3.) \square

Theorem 4.2 and [11, Theorem 2] immediately imply:

Corollary 4.1. *Let M be a closed PL manifold with a free involution A . Then M is a BUT manifold if and only if there exist an equivariant triangulation Λ of M and an equivariant labelling of $V(\Lambda)$ such that $f_L : \Lambda \rightarrow \mathbb{R}^d$ is transversal to zeros and the number of complementary edges is $4k + 2$, where k is integer.*

Corollary 4.2. *Let T be a triangulation of a PL-compact d -dimensional manifold M with boundary B that is homeomorphic to \mathbb{S}^{d-1} . Assume T is antipodally symmetric on the boundary. Let $L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$ be a labelling of the vertices of T which satisfies $L(-v) = -L(v)$ for all vertices v in B . Then there is a complementary edge in T .*

Proof. Consider two copies of M : M_+ and M_- , where for M_+ we take a given labelling L and for M_- we take a labelling $\bar{L} = (-L)$, i.e. $\bar{L}(v) = -L(v)$. Since L is antipodal on the boundary $B = \mathbb{S}^{d-1}$ the connected sum $N := M \# M$ with a free involution $I : N \rightarrow N$, where $I(M_+) = M_-$, is well defined. [11, Corollary 1] implies that N is BUT. Thus, from Theorem 4.2 follows that there is a complementary edge. \square

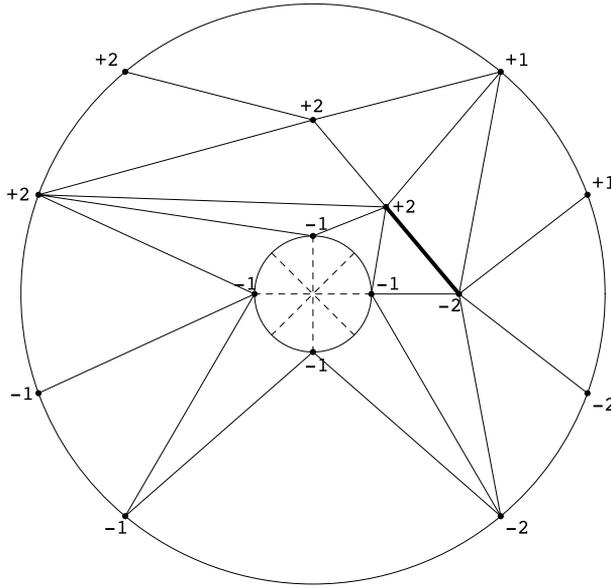


Fig. 7. Tucker's lemma for the Möbius band.

Now we extend [Theorem 4.2](#) for n -labellings. (See [Fig. 7.](#))

Theorem 4.3. *Let $P = \{p_1, -p_1, \dots, p_n, -p_n\}$ be a centrally symmetric set of $2n$ points in \mathbb{R}^d . Let points in P be equivariantly labelled by $\{+1, -1, +2, -2, \dots, +n, -n\}$. Let M be a closed PL d -dimensional manifold with a free involution. Then M is a BUT manifold if and only if for any equivariant triangulation T of M and for any equivariant labelling $L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$ there exists a simplex s in T such that $0 \in f_{L,P}(s)$.*

Proof. If M is BUT, then $f_{L,P}$ has zeros, so there is a simplex s as required.

Suppose M is not BUT. Then there is a continuous antipodal $h : M \rightarrow \mathbb{S}^{d-1}$. Let T be an equivariant triangulation of M . Let Q denote the boundary of the convex hull of P in \mathbb{R}^d . Without loss of generality we may assume that $h : M \rightarrow Q$ and for any vertex $v \in V(T)$ the image $f(v)$ has only one closest vertex p in Q . Then set $L(v) := L(p)$. This labelling implies that $f_{L,P}$ is an antipodal simplicial map from T to Q . Thus, 0 in \mathbb{R}^d is not covered by $f_{L,P}$, a contradiction. \square

5. Radon partitions and Ky Fan's lemma for manifolds

In this section we show that Ky Fan's lemma follows from [Theorem 4.3](#).

Radon's theorem on convex sets states that *any set S of $d + 2$ points in \mathbb{R}^d can be partitioned into two (disjoint) sets A and B whose convex hulls intersect. Moreover, if $\text{rank}(S) = d$, then this partition is unique.*

The partition $S = A \cup B$ is called the *Radon partition* of S .

Breen [2] proved that if S is a $(d + 2)$ -subset of the moment curve C_d in \mathbb{R}^d , then $S = A \cup B$ is the Radon partition if and only if A and B alternate along C_d . Actually, Breen’s theorem can be extended for convex curves in \mathbb{R}^d .

We say that a curve K in \mathbb{R}^d is *convex* if for every hyperplane K intersects it at no more than d points. It is well known that the moment curve C_d is convex. In [10, Sec. 3] we considered several other examples of convex curves.

Definition 5.1. Let $K = \{x(t) = (x_1(t), \dots, x_d(t)) : t \in [a, b]\}$ be a curve in \mathbb{R}^d . Let $S = \{x(t_1), \dots, x(t_{d+2})\}$, where $a < t_1 < t_2 < \dots < t_{d+2} < b$. We say that A and B *alternate along K* if $S = A \cup B$, where $A = \{x(t_1), x(t_3), \dots\}$ and $B = \{x(t_2), x(t_4), \dots\}$.

Theorem 5.2. *A curve K in \mathbb{R}^d is convex if and only if for any $(d + 2)$ -subset S of K its Radon partition sets A and B alternate along K .*

Proof. Let K be convex and $S = \{x(t_1), \dots, x(t_{d+2})\}$ be a $(d + 2)$ -subset of K . Let $A \cap B$ be the Radon partition of S . If A and B do not alternate along K , there are at most d points $P = \{x(\tau_i)\}$ which separate A and B on K . If $r = |P| < d$, we add to P $d - r$ points $x(\tau)$ with $\tau \in (a, t_1)$. Then P defines a hyperplane H which passes through the points in P . Clearly, H separates A and B in \mathbb{R}^d . Thus, $A \cap B$ cannot be the Radon partition of S , a contradiction.

Suppose that for any $(d + 2)$ -subset S of K , its Radon partition sets A and B alternate along K . If K is not convex, then there is a hyperplane H which intersects K at $r \geq d + 1$ points. Therefore, H separates K into $r + 1$ connected components C_1, \dots, C_{r+1} . Let $S = \{x(t_1), \dots, x(t_{d+2})\}$, where $x(t_i) \in C_i$. Since A and B which alternate along K are separated by H , the partition $S = A \cup B$ is not Radon’s — a contradiction. \square

Definition 5.3. Let P be a convex polytope in \mathbb{R}^d with $2n$ centrally symmetric vertices $\{p_1, -p_1, \dots, p_n, -p_n\}$. We say that P is *ACS (Alternating Centrally Symmetric) (n, d) -polytope* if the set of all simplices in $\text{cov}_P(0)$, that contain the origin 0 of \mathbb{R}^d inside, consists of edges $(p_i, -p_i)$ and d -simplices with vertices $\{p_{k_0}, -p_{k_1}, \dots, (-1)^d p_{k_d}\}$ and $\{-p_{k_0}, p_{k_1}, \dots, (-1)^{d+1} p_{k_d}\}$, where $1 \leq k_0 < k_1 < \dots < k_d \leq n$.

Theorem 5.4. *For any integer $d \geq 2$ and $n \geq d$ there exists ACS (n, d) -polytope.*

Proof. Let q_1, \dots, q_n be points on a convex curve K in \mathbb{R}^{d-1} . Let $p_i = (q_i, 1) \in \mathbb{R}^d$. Denote by $P(n, d)$ a convex polytope with vertices $\{p_1, -p_1, \dots, p_n, -p_n\}$. Clearly, $0 \in (-p_i, p_i)$. Let Δ be a simplex spanned by vertices of $P(n, d)$. Let $V(\Delta) = A \cup (-B)$, where A and B are vertices with $x_d = 1$. It is easy to see that $0 \in \Delta$ if and only if $\text{conv}(A) \cap \text{conv}(B) = \emptyset$, i.e. $S = A \cup B$ is the Radon partition of S . Then Theorem 5.2 implies that A and B alternate along K . Thus, $P(n, d)$ is an ACS (n, d) -polytope. \square

Let P be an ACS (n, d) -polytope. If we apply [Theorem 4.3](#) for P , then we obtain the following theorem.

Theorem 5.5. *Let M be a BUT d -dimensional manifold with a free involution A . Let T be any equivariant triangulation of M . Let $L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$ be an equivariant labelling. Suppose that there are no complementary edges in T . Then there are an odd number of d -simplices with labels in the form $\{k_0, -k_1, k_2, \dots, (-1)^d k_d\}$, where $1 \leq k_0 < k_1 < \dots < k_d \leq n$.*

For the case $M = \mathbb{S}^d$ this theorem is Ky Fan’s combinatorial lemma [\[5\]](#). Actually, it is a new proof of this lemma.

6. Sperner and Tucker’s type lemmas for the case $m \geq d$

Now we consider extensions of the polytopal Sperner and Tucker lemmas for the case when $d \leq \dim M = m$. In this case, the set of fully-colored d -simplices defines certain $(m - d)$ -submanifold S of M . A natural extension of [Theorem 3.5](#) is that S is cobordant to zero. We also consider an extension of the Tucker lemma.

An m -dimensional manifold M is called *null-cobordant* (or *cobordant to zero*) if there is a cobordism between M and the empty manifold; in other words, if M is the entire boundary of some $(m + 1)$ -manifold. Equivalently, its cobordism class is trivial.

Theorem 6.1. *Let $P = \{p_1, \dots, p_n\}$ be a set of points in \mathbb{R}^d . Suppose $y \in \mathbb{R}^d$ is such that $\text{cov}_P(y)$ consists of d -simplices. Let M be a closed PL m -dimensional manifold with $m \geq d$. Then for any n -labelling L of a triangulation T of M the set $S := f_{L,P}^{-1}(y)$ is a null-cobordant manifold of dimension $d - m$.*

Note that for $d = m$ this theorem yields [Theorem 3.5](#). In this case S consists of even number of points.

Proof. Let

$$W := M \times [0, 1], \quad M_0 := M \times \{0\} \quad \text{and} \quad M_1 := M \times \{1\}.$$

Let $f_0 := f_{L,P} : M_0 \rightarrow \mathbb{R}^d$. Let us fix a point $q \neq y$ in \mathbb{R}^d and set $f_1(x) = q$ for all $x \in M_1$.

Note that f_0 is transversal to y and $f_1^{-1}(y)$ is empty. Let

$$F(x, t) := (1 - t)f_0(x) + tf_1(x).$$

Then $F : W \rightarrow \mathbb{R}^d$ is transversal to y with $F|_{M_0} = f_0$ and $F|_{M_1} = f_1$. Therefore, $Z_F := F^{-1}(y)$ is a manifold of dimension $(m + 1 - d)$.

Denote $Z_i := Z_F \cap M_i = f_i^{-1}(y)$, $i = 0, 1$. It is clear that $Z_0 = S$ and Z_1 is empty. Thus, Z_0 is the boundary of Z_F and so it is a null-cobordant $(m-d)$ -dimensional manifold. \square

Now we extend the class of BUT manifolds.

Definition 6.2. We say that a closed PL-free m -dimensional \mathbb{Z}_2 -manifold (M, A) is a $\text{BUT}_{m,d}$ if for any continuous $g : M \rightarrow \mathbb{R}^d$ there is a point $x \in M$ such that $g(A(x)) = g(x)$. Equivalently, if a continuous map $f : M \rightarrow \mathbb{R}^d$ is antipodal, then the zeros set $Z_f := f^{-1}(0)$ is not empty.

We obviously have

$$\text{BUT} = \text{BUT}_{m,m} \subset \text{BUT}_{m,m-1} \subset \cdots \subset \text{BUT}_{m,1}.$$

Note that in our paper [11] we found a sufficient condition for (M, A) to be a $\text{BUT}_{m,d}$, see [11, Corollary 3].

Let T be an antipodal triangulation of M . Any equivariant labelling $L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$ defines a simplicial map $f_L : T \rightarrow C^d$, where C^d is the crosspolytope in \mathbb{R}^d (see Section 4). It is easy to see that if L has no complementary edges, then $f_L : T \rightarrow \mathbb{R}^d$ has no zeros. It implies the following theorem.

Theorem 6.3. *Let $m \geq d$. Let T be any equivariant triangulation of a $\text{BUT}_{m,d}$ manifold (M, A) . Let $L : V(A) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$ be any equivariant labelling of T . Then there exists a complementary edge in T .*

This theorem is an extension of Theorem 4.2. When $m \geq d$, it is not hard to extend other theorems and corollaries from Sections 4 and 5.

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References

- [1] K.T. Atanassov, On Sperner's lemma, *Studia Sci. Math. Hungar.* 32 (1996) 71–74.
- [2] M. Breen, Primitive Radon partitions for cyclic polytopes, *Israel J. Math.* 15 (1973) 156–157.
- [3] J.L. Bryant, Piecewise linear topology, in: *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, pp. 219–259.
- [4] J.A. De Loera, E. Peterson, F.E. Su, A polytopal generalization of Sperner's lemma, *J. Combin. Theory Ser. A* 100 (2002) 1–26.
- [5] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications, *Ann. of Math.* 56 (1952) 431–437.
- [6] K. Fan, Simplicial maps from an orientable n -pseudomanifold into \mathbf{S}^m with the octahedral triangulation, *J. Combin. Theory* 2 (1967) 588–602.
- [7] J. Matoušek, *Using the Borsuk–Ulam Theorem*, Springer-Verlag, Berlin, 2003.

- [8] F. Meunier, Sperner labellings: a combinatorial approach, *J. Combin. Theory Ser. A* 113 (2006) 1462–1475.
- [9] J.W. Milnor, *Topology from the differentiable viewpoint*, The University Press of Virginia, Charlottesville, Virginia, 1969.
- [10] O.R. Musin, Chebyshev systems and zeros of a function on a convex curve, *Proc. Steklov Inst. Math.* 221 (1998) 236–246.
- [11] O.R. Musin, Borsuk–Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* 140 (2012) 2551–2560.
- [12] H. Seifert, W. Threlfall, *Lehrbuch der Topologie*, Teubner, 1934, Chelsea, 1947; translated as *A Textbook of Topology*, Academic Press, 1980.
- [13] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Semin. Univ. Hambg.* 6 (1928) 265–272.
- [14] A.W. Tucker, Some topological properties of the disk and sphere, in: *Proc. of the First Canadian Math. Congress, Montreal, 1945*, pp. 285–309.