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The automorphism group of $M_{0,n}^{\text{trop}}$ and $\overline{M}_{0,n}^{\text{trop}}$ Alex Abreu, Marco Pacini¹*Universidade Federal Fluminense, Rua M. S. Braga, s/n, Valonguinho, 24020-005
Niterói (RJ), Brazil*

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ABSTRACT

In this paper we show that the automorphism groups of $M_{0,n}^{\text{trop}}$ and $\overline{M}_{0,n}^{\text{trop}}$ are isomorphic to the permutation group S_n for $n \geq 5$, while the automorphism groups of $M_{0,4}^{\text{trop}}$ and $\overline{M}_{0,4}^{\text{trop}}$ are isomorphic to the permutation group S_3 .

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1. Introduction

The study of the biregular and birational geometry of $\overline{M}_{g,n}$, the moduli space of Deligne–Mumford stable curves, has recently attracted a lot of interest. Some natural issues, such as, for example, the computation of the automorphism group of $\overline{M}_{g,n}$ have been answered only in the last few years. In a series of papers ([3] and [9]), Bruno, Massarenti and Mella proved that the automorphism group of $\overline{M}_{g,n}$ is the permutation group S_n , except in a few cases. This paper is devoted to the computation of the automorphism group of other moduli spaces which have an interesting geometric connection with $\overline{M}_{g,n}$.

In the last decade, many interesting parallels have been made between tropical and algebraic geometry. Some classical and new results in algebraic geometry have been proven

E-mail addresses: alexbra1@gmail.com (A. Abreu), pacini@impa.br, pacini.uff@gmail.com (M. Pacini).

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by means of tropical geometry, see for example [6], [7] and [8]. The tropical counterparts of $\overline{M}_{g,n}$ are the moduli space $M_{g,n}^{\text{trop}}$ of pointed tropical curves constructed in [10], [5] and [2], and its compactification $\overline{M}_{g,n}^{\text{trop}}$ constructed in [4] by means of extended tropical curves. In [1] the authors exhibited a geometrically meaningful connection between $\overline{M}_{g,n}$ and $\overline{M}_{g,n}^{\text{trop}}$.

This paper is motivated by the following questions:

- (1) what are the automorphisms of $M_{g,n}^{\text{trop}}$ and $\overline{M}_{g,n}^{\text{trop}}$?
- (2) what is the interplay between the automorphism group of $\overline{M}_{g,n}$ and the ones of $M_{g,n}^{\text{trop}}$ and $\overline{M}_{g,n}^{\text{trop}}$?

We prove that the two groups in Question (1) are equal for $g = 0$. Moreover for $n \geq 5$ the automorphism group of $M_{0,n}^{\text{trop}}$ is the symmetric group S_n , and the automorphism group of $M_{0,4}^{\text{trop}}$ is S_3 .

The techniques used in the paper belong to the theory of graphs. Indeed, since the graph underlying a tropical curve of genus 0 is a tree, the results of the paper follow from several combinatorial results about (legged) trees. Nevertheless, we do not see any trivial way to extend our techniques for higher g , because the underlying graph could be no longer a tree.

In [Example 2.6](#) we show that the automorphism group of M_2^{trop} is trivial and at the end of the paper we briefly discuss Question (2).

2. Preliminaries

A *tree* is a connected graph without cycles. For a tree Γ , we denote by $V(\Gamma)$ and $E(\Gamma)$ its sets of vertices and edges, respectively. For a vertex $v \in V(\Gamma)$ we denote by $E(v)$ the set of edges incident to v , and by $\text{val}(v)$ the cardinality of $E(v)$. An *isomorphism* g between trees Γ and Γ' is defined as the data of bijections $g_V: V(\Gamma) \rightarrow V(\Gamma')$ and $g_E: E(\Gamma) \rightarrow E(\Gamma')$ which are compatible with incidence.

Remark 2.1. If g_1 and g_2 are isomorphisms between trees Γ and Γ' with at least 3 vertices such that $g_{1,E} = g_{2,E}$, then $g_1 = g_2$.

A *legged tree* with legs indexed by the finite set L (the set of legs) is the data of a tree Γ and a map $\text{leg}_\Gamma: L \rightarrow V(\Gamma)$. Usually, we will still write Γ for a legged tree and denote by $L(\Gamma)$ its set of legs. Moreover we denote by $L(v)$ the set of legs incident to v , i.e., $L(v) := \text{leg}_\Gamma^{-1}(v)$ and by $\ell(v)$ the cardinality of $L(v)$. A *n-legged tree* is a legged tree Γ such that $L(\Gamma) = I_n := \{1, \dots, n\}$. A *n-legged tree* Γ is *stable* if $\text{val}(v) + \ell(v) \geq 3$ for every $v \in V(\Gamma)$. A *leaf* of a tree Γ is a vertex with $\text{val}(v) = 1$. A *chain* is a tree with only 2 leaves. A *path* in a tree Γ is a subtree of Γ that is a chain.

Given a subset $S \subset E(\Gamma)$, we define the legged tree Γ/S as the legged tree obtained by contracting all edges in S . We say that a legged tree Γ specializes to a legged tree Γ' if there exists $S \subset E(\Gamma)$ such that $\Gamma' = \Gamma/S$.

An *isomorphism* between n -legged trees Γ and Γ' is an isomorphism g between the underlying trees which is also compatible with incidence of legs, i.e., $L(g_V(v)) = L(v)$ for every $v \in V(\Gamma)$. We usually write $\Gamma \simeq \Gamma'$ if there exists an isomorphism between them.

Proposition 2.2. *A stable n -legged tree does not have nontrivial automorphisms.*

Proof. An automorphism of a stable n -legged tree fixes the legs. Therefore each leaf of the graph must be fixed by such an automorphism, because each leaf must have at least one leg attached to it. Consider the stable legged tree obtained by removing a leaf and making the only edge incident to that leaf a leg. Since by induction on the number of vertices this graph has no nontrivial automorphisms, the result follows. \square

Given a n -legged tree Γ and a permutation σ of I_n , we define the n -legged tree $\sigma(\Gamma)$ as the n -legged tree Γ' with same underlying tree, but with $\text{leg}_{\Gamma'} = \text{leg}_{\Gamma} \circ \sigma^{-1}$. Given two n -legged trees Γ and Γ' , and a permutation σ of I_n such that $\sigma(\Gamma) \simeq \Gamma'$, i.e., such that there exists an isomorphism g between $\sigma(\Gamma)$ and Γ' , (note that g is unique, by Proposition 2.2), we denote by $\sigma(v) = g_V(v)$ and $\sigma(e) = g_E(e)$ for every $v \in V(\Gamma)$ and $e \in E(\Gamma)$.

We note that there exists a permutation σ of I_n such that $\Gamma \simeq \sigma(\Gamma')$, if and only if Γ and Γ' are isomorphic as unlegged trees and the corresponding vertices have the same number of legs.

A *n -pointed tropical curve of genus 0* (respectively, an *extended n -pointed tropical curve of genus 0*) is the data of a n -legged tree Γ together with a length function $E(\Gamma) \rightarrow \mathbb{R}_{>0}$ (respectively, with a length function $E(\Gamma) \rightarrow \mathbb{R}_{>0} \cup \{\infty\}$). For a n -legged tree Γ , define the rational open polyhedral cone $C(\Gamma) := \mathbb{R}_{>0}^{|E(\Gamma)|}$ and the rational polyhedral cone (or simply cone) $\overline{C(\Gamma)} := \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$. We will identify $E(\Gamma)$ with the canonical basis of $\mathbb{R}^{|E(\Gamma)|}$.

The moduli space $M_{0,n}^{\text{trop}}$ of stable n -pointed tropical curves of genus 0 is the cone complex with cells $C(\Gamma)$, where Γ runs through all stable n -legged trees, with gluing conditions specified by specializations. More precisely if Γ specialized to Γ' , then $\overline{C(\Gamma')}$ is a face of $\overline{C(\Gamma)}$. The moduli space $\overline{M}_{0,n}^{\text{trop}}$ of stable extended n -pointed tropical curves of genus 0 is the extended cone complex

$$\overline{M}_{0,n}^{\text{trop}} = \coprod_{\Gamma} (\mathbb{R}_{>0} \cup \{\infty\})^{|E(\Gamma)|},$$

where Γ runs through all stable n -legged trees. The space $\overline{M}_{0,n}^{\text{trop}}$ compactifies $M_{0,n}^{\text{trop}}$. For more details about the terminology and the constructions of $M_{0,n}^{\text{trop}}$ and $\overline{M}_{0,n}^{\text{trop}}$, see [10, Section 2], [5, Section 3], [2, Sections 2.1 and 3.2] and [1, Sections 2 and 4].

A morphism of cones is a continuous map of topological spaces that is induced by an integral linear transformation. A morphism f of cone complexes is a continuous map of topological spaces such that for each cone σ in the source there exists a cone σ' in the target so that the restriction $f|_{\sigma}$ factors through a morphism of cones $\sigma \rightarrow \sigma'$ (see [1, Section 2]).

An automorphism f of $M_{0,n}^{\text{trop}}$ is a map of cone complexes $M_{0,n}^{\text{trop}} \rightarrow M_{0,n}^{\text{trop}}$ admitting an inverse which is also a map of cone complexes. Clearly an automorphism f induces a permutation of the set of cells of $M_{0,n}^{\text{trop}}$ that preserves the dimension of each cell.

Fix an automorphism f of $M_{0,n}^{\text{trop}}$. Assume that $f(C(\Gamma)) = C(\Gamma')$. By definition, $f|_{C(\Gamma)}$ is induced by an integral linear isomorphism $T: \mathbb{R}^{|E(\Gamma)|} \rightarrow \mathbb{R}^{|E(\Gamma')|}$. The extremal rays of $\overline{C(\Gamma)}$ and of $\overline{C(\Gamma')}$ can be respectively identified with $E(\Gamma)$ and $E(\Gamma')$. Since $T(C(\Gamma)) = C(\Gamma')$, we must have that T sends the extremal rays of $\overline{C(\Gamma)}$ into the extremal rays of $\overline{C(\Gamma')}$, i.e., for every $e \in E(\Gamma)$ there exists $e' \in E(\Gamma')$ such that $T(e) = \lambda e'$ with $\lambda \in \mathbb{R}_{>0}$. Since T is integral and primitive (because the inverse of $f|_{C(\Gamma)}$ must be integral as well), it follows that $\lambda = 1$ and then T is a permutation matrix, i.e., it is induced by a bijection between the sets $E(\Gamma)$ and $E(\Gamma')$. Abusing notation, we denote by $f: E(\Gamma) \rightarrow E(\Gamma')$ such a bijection. Note that given a subset $S \subset E(\Gamma)$, we have $f(C(\Gamma/S)) = C(\Gamma'/f(S))$, because $\overline{C(\Gamma/S)}$ is a face of $\overline{C(\Gamma)}$.

We now give the definition of morphism of extended cones for the cases of interest of this paper, namely the extended cones $(\mathbb{R}_{\geq 0} \cup \{\infty\})^n$. In this case, a morphism of extended cones

$$f: (\mathbb{R}_{\geq 0} \cup \{\infty\})^n \rightarrow (\mathbb{R}_{\geq 0} \cup \{\infty\})^m$$

is a continuous map of topological spaces that factors as

$$(\mathbb{R}_{\geq 0} \cup \{\infty\})^n \rightarrow (\mathbb{R}_{\geq 0} \cup \{\infty\})^r \rightarrow (\mathbb{R}_{\geq 0} \cup \{\infty\})^m$$

where the first map is induced by a morphism of cones $\mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^r$ and the second map is the inclusion $(\mathbb{R}_{\geq 0} \cup \{\infty\})^r \times \{\infty\}^{m-r} \subset (\mathbb{R}_{\geq 0} \cup \{\infty\})^m$ (up to coordinate permutation). For a more general definition of a morphism of extended cones see [1, Section 2].

A morphism f of extended cone complexes is a continuous map of topological spaces such that for each extended cone σ in the source there exists an extended cone σ' in the target such that the restriction $f|_{\sigma}$ factors through a morphism of extended cones $\sigma \rightarrow \sigma'$.

An automorphism \bar{f} of $\overline{M}_{0,n}^{\text{trop}}$ is a morphism of extended cone complexes $\overline{M}_{0,n}^{\text{trop}} \rightarrow \overline{M}_{0,n}^{\text{trop}}$ admitting an inverse which is also a morphism of extended cone complexes.

Proposition 2.3. *There is a canonical isomorphism between the automorphism groups of $\overline{M}_{0,n}^{\text{trop}}$ and $M_{0,n}^{\text{trop}}$.*

Proof. We note that each automorphism of $M_{0,n}^{\text{trop}}$ extends to an automorphism of $\overline{M}_{0,n}^{\text{trop}}$ by linearity. Let \bar{f} be an automorphism of $\overline{M}_{0,n}^{\text{trop}}$. By the definition of morphism of

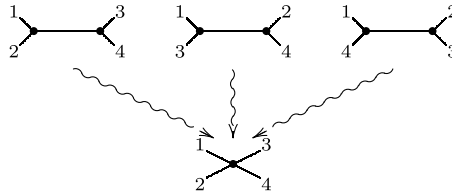


Fig. 1. Stable 4-legged genus-0 graphs.

extended cone complex, for each extended cone $\bar{\sigma}$ in $\overline{M}_{0,n}^{\text{trop}}$, there exists an extended cone $\bar{\sigma}'$ in $\overline{M}_{0,n}^{\text{trop}}$ such that $\bar{f}|_{\bar{\sigma}}$ factors through a morphism of extended cones $\bar{f}|_{\bar{\sigma}}: \bar{\sigma} \rightarrow \bar{\sigma}'$. If $\bar{\sigma}$ is a maximal cone, then, since \bar{f} is an automorphism, so must be $\bar{\sigma}'$. By definition of a morphism of extended cones, we have that $\bar{f}|_{\bar{\sigma}}$ is a morphism of cones $\bar{f}|_{\bar{\sigma}}: \sigma \rightarrow \tau$ where τ is a face (possibly at infinity) of $\bar{\sigma}'$. Hence, since $\bar{f}|_{\bar{\sigma}}$ is injective (because \bar{f} is an automorphism) and $\dim(\sigma) = \dim(\sigma')$, we must have that $\tau = \sigma'$. The same holds true for every maximal extended cone of $\overline{M}_{0,n}^{\text{trop}}$. Hence $\bar{f}(M_{0,n}^{\text{trop}}) = M_{0,n}^{\text{trop}}$ because $M_{0,n}^{\text{trop}}$ is the union of its maximal cones. So the restriction of \bar{f} to $M_{0,n}^{\text{trop}}$ is an automorphism of $M_{0,n}^{\text{trop}}$. \square

We finish this section with a description of $M_{0,4}^{\text{trop}}$ and $M_{0,5}^{\text{trop}}$, and their automorphisms.

Example 2.4. In Fig. 1 above we draw all possible stable 4-legged genus-0 graphs and their specializations.

Consequently, $M_{0,4}^{\text{trop}}$ has 3 maximal cones which have dimension 1 and they intersect along the 0-dimensional cone. Therefore it is clear that its group of automorphisms is isomorphic to S_3 . Note that there is a map $S_4 \rightarrow \text{Aut}(M_{0,4}^{\text{trop}}) \simeq S_3$ induced by the natural action of S_4 on the set of legs. This map has kernel equal to the normal Klein subgroup $\{(), (12)(34), (13)(24), (14)(23)\}$.

Example 2.5. In Fig. 2 below we draw a stable 5-legged genus-0 graph with 2 edges together with its specializations. Note that every 5-legged genus-0 graph is obtained from one in the Figure by permuting the legs. Consequently, $M_{0,5}^{\text{trop}}$ has 15 maximal cones of dimension 2, 10 cones of dimension 1, and a cone of dimension 0.

We have a natural map $S_5 \rightarrow \text{Aut}(M_{0,5}^{\text{trop}})$ induced by the natural action of S_5 on the set of legs. Indeed, given a permutation $\sigma \in S_5$, we can define the map $f: M_{0,5}^{\text{trop}} \rightarrow M_{0,5}^{\text{trop}}$ such that $f(\overline{C(\Gamma)}) = \overline{C(\sigma(\Gamma))}$ and $f|_{\overline{C(\Gamma)}}$ is the restriction to $\overline{C(\Gamma)}$ of the map

$$\begin{aligned} \mathbb{R}^{|E(\Gamma)|} &\rightarrow \mathbb{R}^{|E(\sigma(\Gamma))|} \\ \sum_{e \in E(\Gamma)} a_e e &\mapsto \sum_{e \in E(\Gamma)} a_e \sigma(e). \end{aligned}$$

In Theorem 3.9 we will prove that this map is in fact an isomorphism.

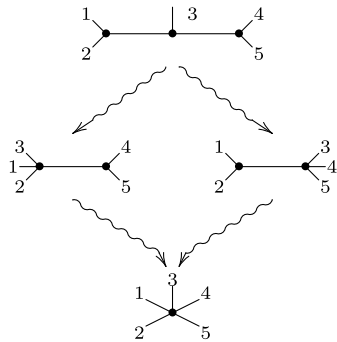


Fig. 2. Stable 5-legged genus-0 graphs.

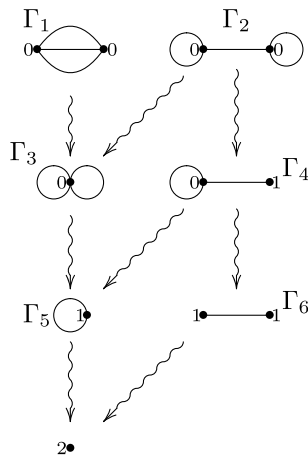


Fig. 3. Genus-2 graphs.

Example 2.6. The notion of automorphism of $M_{0,n}^{\text{trop}}$ naturally extends to $M_{g,n}^{\text{trop}}$. However, a series of complications arise. For instance, Proposition 2.2 does not hold for general graphs. Hence the typical cell of $M_{g,n}^{\text{trop}}$ is of type $C(\Gamma) = \mathbb{R}_{\geq 0}^{|E(\Gamma)|} / \text{Aut}(\Gamma)$. By the definition of morphism of generalized cone complexes (see [1, Section 2]), if f is an automorphism of $M_{g,n}^{\text{trop}}$ such that $f(\overline{C(\Gamma)}) = \overline{C(\Gamma')}$, then $f|_{\overline{C(\Gamma)}}$ is induced by a bijection $f: E(\Gamma) \rightarrow E(\Gamma')$ that is compatible with $\text{Aut}(\Gamma)$ and $\text{Aut}(\Gamma')$, i.e., the maps $E(\Gamma) \rightarrow E(\Gamma') / \text{Aut}(\Gamma')$ and $E(\Gamma') \rightarrow E(\Gamma) / \text{Aut}(\Gamma)$ induced by f and f^{-1} are $\text{Aut}(\Gamma)$ -equivariant and $\text{Aut}(\Gamma')$ -equivariant, respectively. Moreover, we do not see how to extend Proposition 3.1 for higher genus.

We prove that the automorphism group of M_2^{trop} is trivial. In the sequel, we will follow the notation in the Fig. 3, where we draw all possible stable genus-2 graphs and their specializations. Consequently, M_2^{trop} has maximal cones $\overline{C(\Gamma_1)}$ and $\overline{C(\Gamma_2)}$ of dimension 3, cones $\overline{C(\Gamma_3)}$ and $\overline{C(\Gamma_4)}$ of dimension 2, cones $\overline{C(\Gamma_5)}$ and $\overline{C(\Gamma_6)}$ of dimension 1, and a single cone of dimension 0.

Let f be an automorphism of M_2^{trop} . Since Γ_1 only specializes to Γ_3 , and Γ_2 specializes to Γ_3 and Γ_4 , we see that $f(\overline{C(\Gamma_1)}) = \overline{C(\Gamma_1)}$ and $f(\overline{C(\Gamma_2)}) = \overline{C(\Gamma_2)}$ (the argument is essentially the one in [Corollary 3.2](#)). Since $\text{Aut}(\Gamma_1) = S_3$, every bijection $E(\Gamma_1) \rightarrow E(\Gamma_1)$ is compatible with the action of $\text{Aut}(\Gamma_1) = S_3$ and it will induce the identity map

$$\text{Id}: \overline{C(\Gamma_1)} = \mathbb{R}_{\geq 0}^3 / S_3 \rightarrow \mathbb{R}_{\geq 0}^3 / S_3 = \overline{C(\Gamma_1)}$$

Hence $f|_{\overline{C(\Gamma_1)}}$ is the identity map. Analogously, $f|_{\overline{C(\Gamma_2)}}$ is induced by a bijection $E(\Gamma_2) \rightarrow E(\Gamma_2)$ which is compatible with the action of $\text{Aut}(\Gamma_2) = S_2$, and any such bijection must preserve the unique edge of Γ_2 that is not a loop. Hence $f|_{\overline{C(\Gamma_2)}}$ is the identity map by the same argument above. Since f restricts to the identity over the maximal cones, it must be the identity.

3. The result

Throughout the section f will be a fixed automorphism of $M_{0,n}^{\text{trop}}$.

Proposition 3.1. *Let Γ be a stable n -legged tree with m edges. Then the number of $(m+1)$ -dimensional cones in $M_{0,n}^{\text{trop}}$ whose closure contain $C(\Gamma)$ is*

$$\sum_{v \in V(\Gamma)} \left(2^{\ell(v) + \text{val}(v) - 1} - (\ell(v) + \text{val}(v) + 1) \right).$$

Proof. The number of $(m+1)$ -dimensional cones whose closure contain $C(\Gamma)$ is precisely the number of stable n -legged trees with $m+1$ edges that specialize to Γ . To construct a stable n -legged tree Γ' with $m+1$ edges that specializes to Γ , it is equivalent to replace a vertex v of Γ by two vertices v_1 and v_2 , connected by an edge e , with $L(v) = L(v_1) \amalg L(v_2)$ and $E(v) = (E(v_1) \setminus \{e\}) \amalg (E(v_2) \setminus \{e\})$.

Since $\ell(v_i) + \text{val}(v_i) \geq 3$ for $i = 1, 2$, we have to make a partition of $L(v) \cup E(v)$ into two subsets such that each one has at least 2 elements. Clearly, the number of ways to do this is $2^{\ell(v) + \text{val}(v)} - 2(\ell(v) + \text{val}(v) + 1)$. By symmetry, we have to divide by 2, and we obtain the result. \square

For the next several results, recall that there exists a permutation σ of I_n such that $\Gamma \simeq \sigma(\Gamma')$, if and only if Γ and Γ' are isomorphic as unlegged trees and the corresponding vertices have the same number of legs.

Corollary 3.2. *Let Γ and Γ' be two stable n -legged trees with 2 vertices such that $f(C(\Gamma)) = C(\Gamma')$. Then there exists a permutation σ of I_n such that $\sigma(\Gamma) \simeq \Gamma'$.*

Proof. Let v_1, v_2 be the vertices of Γ and v'_1, v'_2 the ones of Γ' . Since f is an automorphism, the numbers of 2-dimensional cones whose closure contain $C(\Gamma)$ and $C(\Gamma')$ are equal. Then we have

$$2^{\ell(v_1)} + 2^{\ell(v_2)} - (\ell(v_1) + \ell(v_2) + 4) = 2^{\ell(v'_1)} + 2^{\ell(v'_2)} - (\ell(v'_1) + \ell(v'_2) + 4),$$

and hence

$$2^{\ell(v_1)} + 2^{\ell(v_2)} = 2^{\ell(v'_1)} + 2^{\ell(v'_2)}$$

from which we get $\ell(v_1) = \ell(v'_1)$ or $\ell(v_1) = \ell(v'_2)$ and the result follows. \square

Lemma 3.3. *Let Γ and Γ' be two stable n -legged trees such that $f(C(\Gamma)) = C(\Gamma')$. Then there exist leaves v, v' of Γ, Γ' such that $\ell(v) = \ell(v')$ and $f(E(v)) = E(v')$.*

Proof. Let v be a leaf of Γ and e be the only edge attached to v . Contracting all edges of Γ except e , we get a stable n -legged tree with 2 vertices v and \bar{v} . Contracting all edges of Γ' except $f(e)$, by Corollary 3.2 we must get a stable n -legged tree with 2-vertices w' and \bar{w}' , such that $\ell(v) = \ell(w')$ (up to switching w' and \bar{w}'). If the vertex \tilde{v}' of $f(e)$ that contracts to w' is (respectively, is not) a leaf, then there exists a leaf v' of Γ' with $\ell(v') = \ell(w')$ and $v' = \tilde{v}'$ (respectively, $\ell(v') < \ell(w')$ and $v' \neq \tilde{v}'$). In particular, by the same argument applied to f^{-1} , for any given leaf v' of Γ' , there exists a leaf w in Γ such that $\ell(w) \leq \ell(v')$.

In the setting above, if we choose v to be a leaf of Γ with the minimum number of legs attached to it, then it follows that the vertex \tilde{v}' attached to $f(e)$ that contracts to w' is also a leaf, otherwise there would be a leaf w of Γ such that $\ell(w) < \ell(v)$, a contradiction. Then, for the chosen v we have that $v' = \tilde{v}'$ and $\ell(v) = \ell(v')$. \square

Proposition 3.4. *Let Γ and Γ' be two stable n -legged trees such that $f(C(\Gamma)) = C(\Gamma')$. Then there exists a permutation σ of I_n such that $\sigma(\Gamma) \simeq \Gamma'$ and $f(e) = \sigma(e)$ for every $e \in E(\Gamma)$.*

Proof. The proof is by induction on the number of edges of Γ . If Γ and Γ' have each one exactly one edge, then the result is Corollary 3.2.

Step 1. *The trees Γ and Γ' have exactly 2 edges.*

Let v_1, v_2, v_3 be the vertices of Γ and v'_1, v'_2, v'_3 the ones of Γ' , where v_2 and v'_2 are not leaves. Let e_i be the edge between v_i and v_{i+1} and e'_i be the one between v'_i and v'_{i+1} , for $i = 1, 2$. By Lemma 3.3, up to relabeling the vertices, we have that $\ell(v_1) = \ell(v'_1)$ and $f(e_1) = e'_1$. This implies that $f(e_2) = e'_2$.

By the same argument in the proof of Corollary 3.2 we get that

$$2^{\ell(v_1)} + 2^{\ell(v_2)+1} + 2^{\ell(v_3)} = 2^{\ell(v'_1)} + 2^{\ell(v'_2)+1} + 2^{\ell(v'_3)}.$$

Since $\ell(v_1) = \ell(v'_1)$, we have that either $\ell(v_i) = \ell(v'_i)$ for $i = 2, 3$, and in this case we are done, or

$$\ell(v_3) = \ell(v'_2) + 1, \quad \ell(v_2) = \ell(v'_3) - 1. \quad (1)$$

If Equation (1) holds, we argue as follows. Contracting e_1 and $f(e_1) = e'_1$, we get that $f(C(\Gamma/\{e_1\})) = C(\Gamma'/\{e'_1\})$. Note that $\Gamma/\{e_1\}$ has two vertices, one with $\ell(v_1) + \ell(v_2)$ legs attached to it and the other with $\ell(v_3)$ legs attached to it. A similar property holds for $\Gamma'/\{e'_1\}$. By Corollary 3.2 we get that either $\ell(v_3) = \ell(v'_3)$ or $\ell(v_3) = \ell(v'_1) + \ell(v'_2)$. If $\ell(v_3) = \ell(v'_3)$, then Equation (1) implies that $\ell(v_2) = \ell(v'_2)$ and the result follows. If $\ell(v_3) = \ell(v'_1) + \ell(v'_2)$, then Equation (1) implies that $\ell(v'_1) = 1$ which is a contradiction because Γ' is stable and v'_1 is a leaf.

Step 2. *The trees Γ and Γ' are chains with exactly 3 edges.*

Assume that v_i (respectively, v'_i) are the vertices of Γ (respectively, Γ') for $i = 1, 2, 3, 4$, and e_i (respectively, e'_i) the edge connecting v_i and v_{i+1} (respectively, v'_i and v'_{i+1}), for $i = 1, 2, 3$. By Lemma 3.3 we can assume, without loss of generality, that $\ell(v_1) = \ell(v'_1)$ and $f(e_1) = e'_1$. Then we have two cases.

In the first case $f(e_2) = e'_2$ and $f(e_3) = e'_3$, hence contracting e_1 and $f(e_1) = e'_1$, and applying Step 1, we get that $\ell(v_3) = \ell(v'_3)$ and $\ell(v_4) = \ell(v'_4)$ hence $\ell(v_2) = \ell(v'_2)$ and the result follows.

In the second case $f(e_2) = e'_3$ and $f(e_3) = e'_2$. Contracting e_1 and $f(e_1) = e'_1$, and applying Step 1, we get that $\ell(v_3) = \ell(v'_3)$, $\ell(v_4) = \ell(v'_1) + \ell(v'_2)$ and $\ell(v'_4) = \ell(v_1) + \ell(v_2)$. These equalities translates to $\ell(v'_1) = \ell(v_1)$, $\ell(v'_2) = \ell(v_4) - \ell(v_1)$, $\ell(v'_3) = \ell(v_3)$ and $\ell(v'_4) = \ell(v_1) + \ell(v_2)$. Now contracting e_2 and $f(e_2) = e'_3$, we get that $\ell(v_2) + \ell(v_3) = \ell(v'_2)$, which implies $\ell(v_2) + \ell(v_3) = \ell(v_4) - \ell(v_1)$. Contracting e_3 and $f(e_3) = e'_2$, we get that $\ell(v_2) = \ell(v'_2) + \ell(v'_3)$, which implies that $\ell(v_2) = \ell(v_4) - \ell(v_1) + \ell(v_3)$. Clearly this yields $\ell(v_3) = 0$, contradiction.

Note that we have proved that, if Γ and Γ' are chains with 3 edges, then f takes the edge of Γ attached to no leaf to the edge of Γ' attached to no leaf.

Step 3. *The trees Γ and Γ' have at least 3 edges.*

By Lemma 3.3 there exist leaves v_1 and v'_1 of Γ and Γ' such that $\ell(v_1) = \ell(v'_1)$ and $f(e_1) = e'_1$ where e_1 and e'_1 are the unique edges attached to v_1 and v'_1 . By the induction hypothesis, upon contracting e_1 and $f(e_1) = e'_1$ we get that there exists σ such that $\sigma(\Gamma/\{e_1\}) \simeq \Gamma'/\{e'_1\}$ and $f(e) = \sigma(e)$ for every $e \in E(\Gamma/\{e_1\}) = E(\Gamma) \setminus \{e_1\}$. Let v_2 and v'_2 be the other vertices connected to e_1 and e'_1 . Since Γ is a tree, there exists a unique path from v_2 to $\sigma^{-1}(v'_2)$. If $\sigma(v_2) = v'_2$, then the result follows. Otherwise, we have 2 cases.

In the first case the path has at least 2 edges. Let e_2 and e_3 be the edges of the path attached respectively to v_2 and $\sigma^{-1}(v'_2)$, and let $e'_2 := f(e_2)$ and $e'_3 := f(e_3)$. Contracting all edges of Γ , except for e_1, e_2, e_3 and all edges of Γ' , except for e'_1, e'_2, e'_3 , we get two stable n -legged chains K and K' with 4 vertices such that $f(C(K)) = C(K')$. The edges

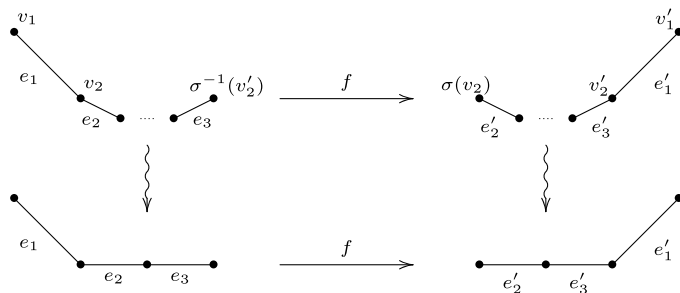


Fig. 4. The first case.

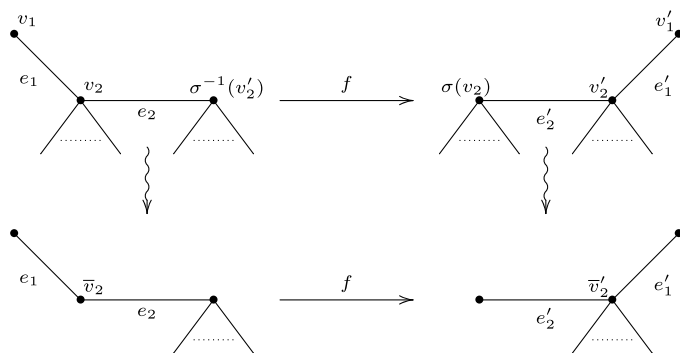


Fig. 5. The second case: first contraction.

e_2 of K and e'_3 of K' are attached to no leaf. However [Step 2](#) applied to K and K' implies that $f(e_2) = e'_3$, contradiction (see [Fig. 4](#)).

In the second case the path has only one edge e_2 . Let $e'_2 := f(e_2)$. We claim that, up to switching f with f^{-1} and Γ with Γ' , we can assume that v'_2 has no other edge attached to it other than e'_1 and e'_2 . Note that this also implies that $\sigma^{-1}(v'_2)$ is a leaf. To prove the claim, contract the set S of all edges that belong to the connected component of $\Gamma \setminus \{e_1, e_2\}$ that contains v_2 . Let \bar{v}_2 and \bar{v}'_2 be the vertices of Γ/S and $\Gamma'/f(S)$, respectively, to which v_2 and v'_2 contract. If at least one edge was contracted, i.e., S is nonempty, then, by the induction hypothesis, there exists a permutation σ' of I_n , such that $\sigma'(\Gamma/S) \simeq \Gamma'/f(S)$ and $f(e) = \sigma'(e)$ for every $e \in E(\Gamma/S) = E(\Gamma) \setminus S$. This implies that $\sigma'(e_1) = f(e_1) = e'_1$, hence $\sigma'(\bar{v}_2) = \bar{v}'_2$, and since \bar{v}_2 has no edges attached to it other than e_1 and e_2 , the only edges attached to \bar{v}'_2 are e'_1 and e'_2 . However, no edge attached to v'_2 was contracted, hence we have proved that $E(v'_2) = \{e'_1, e'_2\}$. If $S = \emptyset$, then we get that the only edges attached to v_2 are e_1 and e_2 , which proves our claim after switching Γ with Γ' and f with f^{-1} (see [Fig. 5](#)).

Let us come back to the proof of the second case. Contracting all edges of Γ except e_1 and e_2 and all edges of Γ' except e'_1 and e'_2 , we end up in the conditions of [Step 1](#). Denote by \bar{v}'_1 , \bar{v}'_2 and $\overline{\sigma^{-1}(v'_2)}$ the vertices to which v'_1 , v'_2 and $\sigma^{-1}(v'_2)$ contract. Since v'_1 , v'_2 and

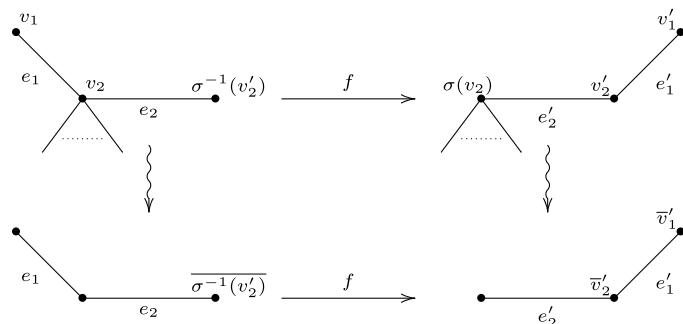


Fig. 6. The second case: second contraction.

$\sigma^{-1}(v'_2)$ have no edges attached to them that are contracted, we get that $L(\overline{v'_1}) = L(v'_1)$, $L(\overline{v'_2}) = L(v'_2)$ and $L(\overline{\sigma^{-1}(v'_2)}) = L(\sigma^{-1}(v'_2))$. Moreover, since $f(e_2) = e'_2$, we have that

$$\ell(\sigma^{-1}(v'_2)) = n - \ell(v'_2) - \ell(v'_1) \quad (2)$$

(see Fig. 6).

Applying Proposition 3.1 to Γ and Γ' , we must have that

$$\begin{aligned} \sum_{v \in V(\Gamma)} \left(2^{\ell(v) + \text{val}(v) - 1} - (\ell(v) + \text{val}(v) + 1) \right) &= \\ &= \sum_{v \in V(\Gamma')} \left(2^{\ell(v) + \text{val}(v) - 1} - (\ell(v) + \text{val}(v) + 1) \right). \end{aligned}$$

However, if $v_0 \in V(\Gamma) \setminus \{v_1, v_2, \sigma^{-1}(v'_2)\}$, then $\sigma(v_0) \in V(\Gamma') \setminus \{v'_1, v'_2, \sigma(v_2)\}$. Since $\sigma(\Gamma/\{e_1\}) \simeq \Gamma'/\{e'_1\}$ and since $e_1 \notin E(v_0)$ and $e'_1 \notin E(\sigma(v_0))$, we have that $\ell(v_0) = \ell(\sigma(v_0))$ and $\text{val}(v_0) = \text{val}(\sigma(v_0))$. We also know that

$$\ell(v_1) = \ell(v'_1), \quad \ell(v_2) + \ell(\sigma^{-1}(v'_2)) = \ell(\sigma(v_2)) + \ell(v'_2)$$

and

$$\text{val}(v_1) = 1 = \text{val}(v'_1), \quad \text{val}(\sigma(v_2)) = \text{val}(v_2) - 1, \quad \text{val}(\sigma^{-1}(v'_2)) = \text{val}(v'_2) - 1 = 1.$$

So we obtain

$$2^{\ell(v_2) + \text{val}(v_2) - 1} + 2^{\ell(\sigma^{-1}(v'_2))} = 2^{\ell(\sigma(v_2)) + \text{val}(\sigma(v_2)) - 1} + 2^{\ell(v'_2) + 1}. \quad (3)$$

On the other hand $\text{val}(\sigma(v_2)) = \text{val}(v_2) - 1$ and $\ell(\sigma(v_2)) = \ell(v_2) + \ell(v_1)$ (because $\sigma(\Gamma/\{e_1\}) \simeq \Gamma'/\{e'_1\}$), from which we deduce that

$$\ell(\sigma(v_2)) + \text{val}(\sigma(v_2)) - 1 = \ell(v_2) + \text{val}(v_2) - 2 + \ell(v_1) > \ell(v_2) + \text{val}(v_2) - 1.$$

Hence it follows from Equation (3) that

$$\ell(v_2) + \text{val}(v_2) - 1 = \ell(v'_2) + 1. \quad (4)$$

Now, the number n of legs in Γ must be at least

$$\ell(v_1) + \ell(v_2) + \ell(\sigma^{-1}(v'_2)) + 2(\text{val}(v_2) - 2)$$

and using Equations (4) and (2), we deduce that

$$\begin{aligned} n &\geq \ell(v_1) + \ell(v'_2) + \ell(\sigma^{-1}(v'_2)) + \text{val}(v_2) - 2 \\ &= n + \text{val}(v_2) - 2. \end{aligned}$$

Therefore $\text{val}(v_2) = 2$, which implies that Γ has only 3 vertices, in which case the result follows from Step 1. \square

Remark 3.5. In this remark, for a n -legged tree Γ , we denote by $\tilde{\Gamma}$ its underlying tree. When $f(C(\Gamma)) = C(\Gamma')$, Proposition 3.4 and Remark 2.1 shows that f induces a unique isomorphism $g_\Gamma: \tilde{\Gamma} \rightarrow \tilde{\Gamma}'$ unless Γ has exactly 2 vertices with the same number of legs incident to them, in which case the two isomorphisms between $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ can be induced by some permutation σ that satisfies Proposition 3.4. The unique isomorphisms induced by f are compatible with specializations, namely, for every $S \subset E(\Gamma)$ the isomorphism $g_{\Gamma/S}: \widetilde{\Gamma/S} \rightarrow \widetilde{\Gamma'/f(S)}$ is induced by g_Γ via specialization. Moreover, in the case where Γ has exactly 2 vertices with the same number of legs incident to them, only one of the two possible isomorphisms is compatible with specializations, unless $n = 4$. To prove such a claim just choose a n -legged tree with 3 vertices that specializes to Γ .

We will abuse notation and denote $f(v) = g_\Gamma(v)$ for every $v \in V(\Gamma)$.

Let A be a subset of I_n , with $2 \leq |A| \leq n - 2$. Define Γ_A the n -legged tree with exactly 2 vertices v_A and \bar{v}_A such that $L(v_A) = A$.

Lemma 3.6. *If $B \subset A$, with $A, B \subset I_n$ with $2 \leq |A|, |B| \leq n - 2$, then $L(f(v_B)) \subset L(f(v_A))$.*

Proof. If $B = A$, there is nothing to do, so we can assume that $B \neq A$. Let Γ be the n -legged tree with exactly 3 vertices w_1, w_2 and w_3 , with w_1 and w_3 being the leaves, such that $L(w_1) = B$ and $L(w_2) = A \setminus B$. By Proposition 3.4 we can write $L(f(w_1)) = B'$ and $L(f(w_2)) = C'$, with $|B'| = |B|$ and $|C'| = |A| - |B|$. Contracting the edge between w_2 and w_3 in Γ we get Γ_B , hence, by Remark 3.5, we get $L(f(v_B)) = B'$. Contracting the edge between w_1 and w_2 in Γ we get Γ_A , hence, again by Remark 3.5, $L(f(v_A)) = B' \cup C'$. Therefore we have $L(f(v_B)) \subset L(f(v_A))$. \square

Corollary 3.7. *Let $n \geq 5$ and Γ_i be a stable n -legged trees with exactly 2 vertices v_i and $\overline{v_i}$ for $i = 1, 2, 3$, such that $\ell(v_1) = \ell(v_2) = 2$, $|L(v_1) \cap L(v_2)| = 1$ and $L(v_3) = L(v_1) \cup L(v_2)$. Then $|L(f(v_1)) \cap L(f(v_2))| = 1$ and $L(f(v_3)) = L(f(v_1)) \cup L(f(v_2))$.*

Proof. By Lemma 3.6, we have $L(f(v_1)) \subset L(f(v_3))$ and $L(f(v_2)) \subset L(f(v_3))$. Since $L(f(v_1)) \neq L(f(v_2))$ (otherwise $f(C(\Gamma_1)) = f(C(\Gamma_2))$) and $\ell(f(v_i)) = \ell(v_i)$ for $i = 1, 2, 3$ (by Corollary 3.2), the result follows. \square

Proposition 3.8. *There exists a permutation σ of I_n such that, for every stable n -legged tree Γ with exactly 2 vertices, we have $L(f(v)) = \sigma(L(v))$ for all $v \in V(\Gamma)$.*

Proof. First, let us prove that the statement holds true if $\ell(v) = 2$. By Corollary 3.7, there exist distinct elements i_1, i_2 and i_3 of I_n such that

$$L(f(v_{\{1,2\}})) = \{i_1, i_2\}, \quad L(f(v_{\{1,3\}})) = \{i_1, i_3\}, \quad L(f(v_{\{1,2,3\}})) = \{i_1, i_2, i_3\}.$$

Then, by Lemma 3.6, we have that $L(f(v_{\{2,3\}})) \subset \{i_1, i_2, i_3\}$. In this way, since

$$L(f(v_{\{2,3\}})) \neq L(f(v_{\{1,2\}})) \quad \text{and} \quad L(f(v_{\{2,3\}})) \neq L(f(v_{\{1,3\}})),$$

we get $L(f(v_{\{2,3\}})) = \{i_2, i_3\}$. By Corollary 3.7 we have

$$L(f(v_{\{1,4\}})) \cap L(f(v_{\{1,2\}})) \neq \emptyset \quad \text{and} \quad L(f(v_{\{1,4\}})) \cap L(f(v_{\{1,3\}})) \neq \emptyset.$$

Thus, we get that either $i_1 \in L(f(v_{\{1,4\}}))$ or $\{i_2, i_3\} = L(f(v_{\{1,4\}}))$. The latter case can not happen because $L(f(v_{\{1,4\}})) \neq L(f(v_{\{2,3\}}))$. Hence, there exists $i_4 \in I_n \setminus \{i_1, i_2, i_3\}$ such that $L(f(v_{\{1,4\}})) = \{i_1, i_4\}$. Analogously, $i_2 \in L(f(v_{\{2,4\}}))$, and since $L(f(v_{\{1,4\}})) \cap L(f(v_{\{2,4\}})) \neq \emptyset$, we get that $L(f(v_{\{2,4\}})) = \{i_2, i_4\}$. Iterating the argument, we find indices $i_j \in I_n$ such that $L(f(v_{\{j,k\}})) = \{i_j, i_k\}$ and we can define $\sigma(j) := i_j$. This ends the proof in the case $\ell(v) = 2$.

If $\ell(v) \geq 3$, let $B \subset L(v)$ such that $|B| = 2$. Then, by previous case, there exists a permutation σ of I_n , which does not depend on B , such that $\sigma(B) = L(f(v_B))$. By Lemma 3.6, we have that $\sigma(B) \subset L(f(v))$. Since this holds for all B with $|B| = 2$, then $L(f(v)) = \sigma(L(v))$. \square

Theorem 3.9. *For $n \geq 5$ the automorphism groups of $M_{0,n}^{\text{trop}}$ and $\overline{M}_{0,n}^{\text{trop}}$ are isomorphic to S_n . The automorphism groups of $M_{0,4}^{\text{trop}}$ and $\overline{M}_{0,4}^{\text{trop}}$ are isomorphic to S_3 .*

Proof. Note that by Proposition 2.3 and Example 2.4 we reduce to prove the theorem for $M_{0,n}^{\text{trop}}$ and $n \geq 5$. So, assume throughout the proof that $n \geq 5$. First of all, we show that there is an injective group homomorphism from S_n to the automorphism group of $M_{0,n}^{\text{trop}}$. Clearly, given a permutation $\sigma \in S_n$, there is an automorphism of $M_{0,n}^{\text{trop}}$ defined by the map sending Γ to $\sigma(\Gamma)$ and preserving the lengths of the edges. If σ

induces the identity in $M_{0,n}^{\text{trop}}$, then $\sigma(\Gamma_A) = \Gamma_A$ for every $A \subset I_n$ with $2 \leq |A| \leq n-2$ (recall the definition of Γ_A before [Lemma 3.6](#)). This implies that $\sigma(A) = A$ for every $A \subset I_n$ with $|A| \neq n/2$. In particular, since $n \geq 5$, this holds for every $A \subset I_n$ such that $|A| = 2$. This clearly implies that σ is the identity.

Now we prove that every automorphism f of $M_{0,n}^{\text{trop}}$ is induced by some permutation $\sigma \in S_n$. Let σ be the permutation of I_n as in the statement of [Proposition 3.8](#). All that is left to do is to prove that σ satisfies $L(f(v)) = \sigma(L(v))$ for all Γ stable n -legged tree and $v \in V(\Gamma)$.

We will proceed by induction on the number of edges. Let Γ_1 and Γ_2 be stable n -legged trees, such that $f(C(\Gamma_1)) = C(\Gamma_2)$. Let v be a leaf of Γ_1 and e be the only edge attached to it. Contracting all edges of Γ_1 except e , and all edges of Γ_2 except $f(e)$, we get two stable n -legged trees Γ'_1 and Γ'_2 with exactly 2 vertices. Let v' be the vertex in Γ'_1 to which v contracts. By [Remark 3.5](#), [Proposition 3.8](#) and the fact that $L(v) = L(v')$, we get that

$$L(f(v)) = L(f(v')) = \sigma(L(v')) = \sigma(L(v)). \quad (5)$$

Contracting e and using the induction hypothesis, we get that

$$L(f(w)) = \sigma(L(w)) \quad \text{for } w \in V(\Gamma_1/\{e\}). \quad (6)$$

Let v_1 be the vertex in Γ_1 connected to v and w_1 be the vertex in $\Gamma_1/\{e\}$ to which v and v_1 contract. We have $L(w_1) = L(v) \cup L(v_1)$ and therefore, by [Remark 3.5](#), $L(f(w_1)) = L(f(v)) \cup L(f(v_1))$. Combining Equations (5) and (6), we get $\sigma(L(w_1)) = \sigma(L(v)) \cup \sigma(L(v_1))$, and since $\sigma(L(w_1)) = \sigma(L(v)) \cup \sigma(L(v_1))$, we have that $L(f(v_1)) = \sigma(L(v_1))$. Finally, if $v_2 \in V(\Gamma) \setminus \{v, v_1\}$, then there exists $w \in V(\Gamma/\{e\})$ such that $L(v_2) = L(w)$, and the result follows by Equation (6). \square

4. Final remarks and further questions

We conclude the paper by noting that, by [\[1, Proposition 6.1.8\]](#), we have that a toroidal automorphism of a toroidal scheme X induces an automorphism of its skeleton $\overline{\Sigma}(X)$, and that we get a group homomorphism

$$\alpha_X: \text{Aut}^{\text{tor}}(X) \rightarrow \text{Aut}(\overline{\Sigma}(X)),$$

where $\text{Aut}^{\text{tor}}(X)$ denotes the group of toroidal automorphisms of X . Hence, by [\[1, Theorem 1.2.1\]](#), we have group homomorphisms

$$\text{Aut}(\overline{M}_{0,n}) \hookrightarrow \text{Aut}^{\text{tor}}(\overline{M}_{0,n}) \rightarrow \text{Aut}(\overline{M}_{0,n}^{\text{trop}})$$

which are, a posteriori, isomorphisms for $n \geq 5$ due to the result [\[3\]](#) of Bruno and Mella and [Theorem 3.9](#). Nevertheless it would be interesting to address the following questions

- (1) when the homomorphism α_X is an isomorphism or at least injective?
- (2) when the natural inclusion $\text{Aut}^{\text{tor}}(X) < \text{Aut}(X)$ is an isomorphism?

We note that $\text{Aut}^{\text{tor}}(\overline{M}_{0,4}) = S_3$, while $\text{Aut}(\overline{M}_{0,4}) = \text{PGL}(2)$, hence Question (2) does not always have a positive answer.

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