



ELSEVIER

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta


Isotropical linear spaces and valuated Delta-matroids

Felipe Rincón

University of California, Berkeley, Department of Mathematics, 1045 Evans Hall, Berkeley, CA, United States

ARTICLE INFO

Article history:

Received 3 June 2010

Available online 27 August 2011

Keywords:

Tropical linear space

Isotropic subspace

Delta matroid

Coxeter matroid

Valuated matroid

Spinor variety

Wick relations

Matroid polytope

Tropical basis

ABSTRACT

The spinor variety is cut out by the quadratic Wick relations among the principal Pfaffians of an $n \times n$ skew-symmetric matrix. Its points correspond to n -dimensional isotropic subspaces of a $2n$ -dimensional vector space. In this paper we tropicalize this picture, and we develop a combinatorial theory of tropical Wick vectors and tropical linear spaces that are tropically isotropic. We characterize tropical Wick vectors in terms of subdivisions of Δ -matroid polytopes, and we examine to what extent the Wick relations form a tropical basis. Our theory generalizes several results for tropical linear spaces and valuated matroids to the class of Coxeter matroids of type D .

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Let n be a positive integer, and let V be a $2n$ -dimensional vector space over an algebraically closed field K of characteristic 0. Fix a basis $e_1, e_2, \dots, e_n, e_{1^*}, e_{2^*}, \dots, e_{n^*}$ for V , and consider the symmetric bilinear form on V defined as

$$Q(x, y) = \sum_{i=1}^n x_i y_{i^*} + \sum_{i=1}^n x_{i^*} y_i,$$

for any two $x, y \in V$ with coordinates $x = (x_1, \dots, x_n, x_{1^*}, \dots, x_{n^*})$ and $y = (y_1, \dots, y_n, y_{1^*}, \dots, y_{n^*})$. An n -dimensional subspace $U \subseteq V$ is called (totally) *isotropic* if for all $u, v \in U$ we have $Q(u, v) = 0$, or equivalently, for all $u \in U$ we have $Q(u, u) = 0$. Denote by $2^{[n]}$ the collection of subsets of the set $[n] := \{1, 2, \dots, n\}$. The space of pure spinors $\text{Spin}^\pm(n)$ is an algebraic set in projective space $\mathbb{P}^{2^{[n]}-1}$ that parametrizes totally isotropic subspaces of V . Its defining ideal is generated by very special quadratic equations, known as Wick relations. We describe these relations in detail in Section 2. Since

E-mail address: felipe@math.berkeley.edu.

any linear subspace $W \subseteq K^n$ defines an isotropic subspace $U := W \times W^\perp \subseteq K^{2n}$, all Grassmannians $G(k, n)$ can be embedded naturally into the space of pure spinors, and in fact, Wick relations can be seen as a natural generalization of Plücker relations.

In [21], Speyer studied tropical Plücker relations, tropical Plücker vectors (or valuated matroids [9]), and their relation to tropical linear spaces. In his study he showed that these objects have a beautiful combinatorial structure, which is closely related to matroid polytope decompositions. In this paper we study the tropical variety and prevariety defined by all Wick relations, the combinatorics satisfied by the vectors in these spaces (valuated even Δ -matroids [8]), and their connection with tropical linear spaces that are tropically isotropic (which we call isotropical linear spaces). Much of our work can be seen as a generalization to type D of some of the results obtained by Speyer, or as a generalization of the theory of even Δ -matroids to the “valuated” setup.

In Section 4 we give a brief introduction to the notions of tropical geometry that we discuss in this paper, and we also examine for what values of n the Wick relations form a tropical basis (see Section 4.1 for the definition of tropical basis). We provide an answer for all $n \neq 6$:

Theorem 4.5. *If $n \leq 5$ then the Wick relations are a tropical basis; if $n \geq 7$ then they are not.*

We conjecture that, in fact, for all $n \leq 6$ the Wick relations are a tropical basis.

We will say that a vector $p \in \mathbb{T}^{2[n]}$ with coordinates in the tropical semiring $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is a tropical Wick vector if it satisfies the tropical Wick relations. A central object for our study of tropical Wick vectors is that of an even Δ -matroid [3]. Even Δ -matroids are a natural generalization of classical matroids, and much of the basic theory of matroids can be extended to them. In particular, their associated polytopes are precisely those 0/1 polytopes whose edges have the form $\pm e_i \pm e_j$, with $i \neq j$. In this sense, even Δ -matroids can be seen as Coxeter matroids of type D , while classical matroids correspond to Coxeter matroids of type A . We present the necessary background on even Δ -matroids in Section 3. Tropical Wick vectors will be valuated (even) Δ -matroids: real functions on the set of bases of an even Δ -matroid satisfying certain “valuated exchange property” which is amenable to the greedy algorithm (see [8]). We prove in Section 5 that in fact tropical Wick vectors can be characterized in terms of even Δ -matroid polytope subdivisions:

Theorem 5.3. *The vector $p \in \mathbb{T}^{2[n]}$ is a tropical Wick vector if and only if the regular subdivision induced by p is a subdivision of an even Δ -matroid polytope into even Δ -matroid polytopes.*

We give a complete list of all even Δ -matroids up to isomorphism on a ground set of at most 5 elements, together with their corresponding spaces of valuations, on the website <http://math.berkeley.edu/~felipe/delta/>.

In Section 6 we extend some of the theory of even Δ -matroids to the valuated setup. We will say that a vector $x = (x_1, x_2, \dots, x_n, x_{1^*}, x_{2^*}, \dots, x_{n^*}) \in \mathbb{T}^{2n}$ with coordinates in the tropical semiring $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is admissible if for all i we have that at most one of x_i and x_{i^*} is not equal to ∞ . Based on this notion of admissibility we define duality, circuits, and cycles for a tropical Wick vector p , generalizing the corresponding definitions for even Δ -matroids. Of special importance is the cocycle space of a tropical Wick vector, which can be seen as an analog in type D to the tropical linear space associated to a tropical Plücker vector. We study some of its properties, and in particular, we give a parametric description of it in terms of cocircuits:

Theorem 6.9. *The cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$ of a tropical Wick vector $p \in \mathbb{T}^{2[n]}$ is equal to the set of admissible vectors in the tropical convex hull of the cocircuits of p .*

We then specialize our results to tropical Plücker vectors, unifying in this way several results for tropical linear spaces given by Murota and Tamura [15], Speyer [21], and Ardila and Klivans [1].

In Section 7 we define isotropical linear spaces and study their relation to tropical Wick vectors. We give an effective characterization in Theorem 7.3 for determining when a tropical linear space is isotropical, in terms of its associated Plücker vector. We also show that the correspondence between

isotropic linear spaces and points in the pure spinor space is lost after tropicalizing; nonetheless, we prove that this correspondence still holds when we restrict our attention only to admissible vectors:

Theorem 7.5. *Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series. Let $U \subseteq K^{2n}$ be an isotropic subspace, and let w be its corresponding point in the space of pure spinors $\text{Spin}^\pm(n)$. Suppose $p \in \mathbb{T}^{2[n]}$ is the tropical Wick vector obtained as the valuation of w . Then the set of admissible vectors in the tropicalization of U is the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$ of p .*

2. Isotropic linear spaces and spinor varieties

Let n be a positive integer, and let V be a $2n$ -dimensional vector space over an algebraically closed field K of characteristic 0, with a fixed basis $e_1, e_2, \dots, e_n, e_{1^*}, e_{2^*}, \dots, e_{n^*}$. Denote by $2^{[n]}$ the collection of subsets of the set $[n] := \{1, 2, \dots, n\}$. In order to simplify the notation, if $S \in 2^{[n]}$ and $a \in [n]$ we will write Sa , $S - a$, and $S \Delta a$ instead of $S \cup \{a\}$, $S \setminus \{a\}$, and $S \Delta \{a\}$, respectively. Given an n -dimensional isotropic subspace $U \subseteq V$, one can associate to it a vector $w \in \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates as follows. Write U as the rowspace of some $n \times 2n$ matrix M with entries in K . If the first n columns of M are linearly independent, we can row reduce the matrix M and assume that it has the form $M = [I|A]$, where I is the identity matrix of size n and A is an $n \times n$ matrix. The fact that U is isotropic is equivalent to the property that the matrix A is skew-symmetric. The vector $w \in \mathbb{P}^{2^{[n]}-1}$ is then defined as

$$w_{[n] \setminus S} := \begin{cases} \text{Pf}(A_S) & \text{if } |S| \text{ is even,} \\ 0 & \text{if } |S| \text{ is odd;} \end{cases}$$

where $S \in 2^{[n]}$ and $\text{Pf}(A_S)$ denotes the Pfaffian of the principal submatrix A_S of A whose rows and columns are indexed by the elements of S . If the first n columns of M are linearly dependent then we proceed in a similar way but working over a different affine chart of $\mathbb{P}^{2^{[n]}-1}$. In this case, we can first reorder the elements of our basis (and thus the columns of M) using a permutation of $2n := \{1, 2, \dots, n, 1^*, 2^*, \dots, n^*\}$ consisting of transpositions of the form (j, j^*) for all j in some index set $J \subseteq [n]$, so that we get a new matrix that can be row-reduced to a matrix of the form $M' = [I|A]$ (with A skew-symmetric). We then compute the Wick coordinates as

$$w_{[n] \setminus S} := \begin{cases} (-1)^{\text{sg}(S, J)} \cdot \text{Pf}(A_{S \Delta J}) & \text{if } |S \Delta J| \text{ is even,} \\ 0 & \text{if } |S \Delta J| \text{ is odd;} \end{cases}$$

where $(-1)^{\text{sg}(S, J)}$ is some sign depending on S and J that will not be important for us. The vector $w \in \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates depends only on the subspace U , and the subspace U can be recovered from its vector w of Wick coordinates as

$$U = \bigcap_{T \subseteq [n]} \left\{ x \in V : \sum_{i \in T} (-1)^{\text{sg}(i, T)} w_{T-i} \cdot x_i + \sum_{j \notin T} (-1)^{\text{sg}(j, T)} w_{Tj} \cdot x_{j^*} \right\}, \quad (1)$$

where again the signs $(-1)^{\text{sg}(i, T)}$ and $(-1)^{\text{sg}(j, T)}$ will not matter for us.

The space of pure spinors is the set $\text{Spin}^\pm(n) \subseteq \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates of all n -dimensional isotropic subspaces of V , and thus it is a parameter space for these subspaces. It is an algebraic set, and it decomposes into two isomorphic irreducible varieties as $\text{Spin}^\pm(n) = \text{Spin}^+(n) \sqcup \text{Spin}^-(n)$, where $\text{Spin}^+(n)$ consists of all Wick coordinates w whose support $\text{supp}(w) := \{S \in 2^{[n]} : w_S \neq 0\}$ is made of even-sized subsets, and $\text{Spin}^-(n)$ consists of all Wick coordinates whose support is made of odd-sized subsets. The irreducible variety $\text{Spin}^+(n)$ is called the *spinor variety*; it is the projective closure of the image of the map sending an $n \times n$ skew-symmetric matrix to its vector of principal Pfaffians. Its defining ideal consists of all polynomial relations among the principal Pfaffians of a skew-symmetric matrix, and it is generated by the following quadratic relations:

$$\sum_{i=1}^s (-1)^i w_{\tau_1 \sigma_1 \sigma_2 \dots \sigma_r} \cdot w_{\tau_1 \tau_2 \dots \hat{\tau}_i \dots \tau_s} + \sum_{j=1}^r (-1)^j w_{\sigma_1 \sigma_2 \dots \hat{\sigma}_j \dots \sigma_r} \cdot w_{\sigma_j \tau_1 \tau_2 \dots \tau_s}, \quad (2)$$

where $\sigma, \tau \in 2^{[n]}$ have odd cardinalities r, s , respectively, and the variables w_σ are understood to be alternating with respect to a reordering of the indices, e.g. $w_{2134} = -w_{1234}$ and $w_{1135} = 0$. The ideal defining the space of pure spinors is generated by all quadratic relations having the form (2), but now with $\sigma, \tau \in 2^{[n]}$ having any cardinality. These relations are known as *Wick relations*. The shortest nontrivial Wick relations are obtained when $|\sigma \Delta \tau| = 4$, in which case they have the form

$$w_{Sabcd} \cdot w_S - w_{Sab} \cdot w_{Scd} + w_{Sac} \cdot w_{Sbd} - w_{Sad} \cdot w_{Sbc}$$

and

$$w_{Sabc} \cdot w_{Sd} - w_{Sabd} \cdot w_{Sc} + w_{Sacd} \cdot w_{Sb} - w_{Sbcd} \cdot w_{Sa},$$

where $S \subseteq [n]$ and $a, b, c, d \in [n] \setminus S$ are distinct. These relations will be of special importance for us; they will be called the *4-term Wick relations*.

If W is any linear subspace of K^n then $U := W \times W^\perp$ is an n -dimensional isotropic subspace of K^{2n} whose Wick coordinates are the Plücker coordinates of W , so Wick vectors and Wick relations are a generalization of Plücker vectors and Plücker relations, respectively. For more information about spinor varieties and isotropic linear spaces we refer the reader to [13,20,22].

3. Even Delta-matroids

In this section we review some of the basic theory of even Δ -matroids. They generalize matroids in a very natural way, and have also the useful feature of being characterized by many different sets of axioms. For a much more extensive exposition of matroids and even Δ -matroids, the reader can consult [19,3,2,4,5].

3.1. Bases and representability

Our first description of even Δ -matroids is the following.

Definition 3.1. An even Δ -matroid (or even Delta-matroid) is a pair $M = (E, \mathcal{B})$, where E is a finite set and \mathcal{B} is a nonempty collection of subsets of E satisfying the following *symmetric exchange axiom*:

- For all $A, B \in \mathcal{B}$ and for all $a \in A \Delta B$, there exists $b \in A \Delta B$ such that $b \neq a$ and $A \Delta \{a, b\} \in \mathcal{B}$.

Here Δ denotes symmetric difference: $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. The set E is called the *ground set* of M , and \mathcal{B} is called the collection of *bases* of M . We also say that M is an even Δ -matroid over the set E .

An even Δ -matroid is called a Lagrangian orthogonal matroid in [2]. It follows from the definition that all bases of an even Δ -matroid have the same parity, which can be either even or odd. It can also be proved that even Δ -matroids satisfy the following much stronger exchange axiom (see [2]).

Proposition 3.2. Let M be an even Δ -matroid. Then M satisfies the following strong exchange axiom:

- For all $A, B \in \mathcal{B}$ and for any $a \in A \Delta B$, there exists $b \in A \Delta B$ such that $b \neq a$ and both $A \Delta \{a, b\}$ and $B \Delta \{a, b\}$ are in \mathcal{B} .

Even Δ -matroids are a natural generalization of classical matroids; in fact, matroids are precisely those even Δ -matroids whose bases have all the same cardinality.

If $M = (E, \mathcal{B})$ is an even Δ -matroid then the collection $\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$ is also the collection of bases of an even Δ -matroid M^* over E , called the *dual* even Δ -matroid to M .

Our interest in even Δ -matroids comes from the following fact.

Proposition 3.3. Let V be a $2n$ -dimensional vector space over the field K . If $U \subseteq V$ is an n -dimensional isotropic subspace with Wick coordinates w , then the subsets in the support $\text{supp}(w) := \{S \in 2^{[n]} : w_S \neq 0\}$ of w form the collection of bases of an even Δ -matroid over $[n]$. An even Δ -matroid arising in this way is said to be representable (over the field K).

3.2. Matroid polytopes

A very useful way of working with (even Δ -) matroids is via their associated polytopes. These polytopes and their subdivisions will play a very important role for us.

Given any collection \mathcal{B} of subsets of $[n]$ one can associate to it the polytope $\Gamma_{\mathcal{B}} := \text{convex}\{e_S : S \in \mathcal{B}\}$, where $e_S := \sum_{i \in S} e_i$ is the indicator vector of the subset S .

Theorem 3.4. (See [10].) If $\mathcal{B} \subseteq 2^{[n]}$ is nonempty then \mathcal{B} is the collection of bases of a matroid if and only if all the edges of the polytope $\Gamma_{\mathcal{B}}$ have the form $e_i - e_j$, where $i, j \in [n]$ are distinct.

Theorem 3.4 is just a special case of a very general theorem characterizing the associated polytopes of a much larger class of matroids, called Coxeter matroids (see [2]). In the case of even Δ -matroids it takes the following form.

Theorem 3.5. If $\mathcal{B} \subseteq 2^{[n]}$ is nonempty then \mathcal{B} is the collection of bases of an even Δ -matroid if and only if all the edges of the polytope $\Gamma_{\mathcal{B}}$ have the form $\pm e_i \pm e_j$, where $i, j \in [n]$ are distinct.

These results allow us think of matroids and even Δ -matroids in terms of root systems: classical matroids should be thought of as matroids of type A, and even Δ -matroids as matroids of type D.

3.3. Circuits and symmetric matroids

We will now define circuits for even Δ -matroids, generalizing the notion of circuits for classical matroids. In order to simplify the definitions we will first introduce a different way of encoding an even Δ -matroid. A much more detailed description of all these notions can be found in [2].

Consider the sets $[n] := \{1, 2, \dots, n\}$ and $[n]^* := \{1^*, 2^*, \dots, n^*\}$. Define the map $*$: $[n] \rightarrow [n]^*$ by $i \mapsto i^*$ and the map $*$: $[n]^* \rightarrow [n]$ by $i^* \mapsto i$. We can think of $*$ as an involution of the set $2\mathbf{n} := [n] \cup [n]^*$, where for any $j \in 2\mathbf{n}$ we have $j^{**} = j$. If $J \subseteq 2\mathbf{n}$, we define $J^* := \{j^* : j \in J\}$. We say that the set J is *admissible* if $J \cap J^* = \emptyset$, and that it is a *transversal* if it is an admissible set of size n . For any $S \subseteq [n]$, we define its *extension* $\tilde{S} \subseteq 2\mathbf{n}$ to be the transversal given by $\tilde{S} := S \cup ([n] \setminus S)^*$, and for any transversal J we will define its *restriction* to be the set $J \cap [n]$. Extending and restricting are clearly bijections (inverse to each other) between the set $2^{[n]}$ and the set of transversals $\mathcal{V}(n)$ of $2\mathbf{n}$.

Definition 3.6. Given an even Δ -matroid $M = ([n], \mathcal{B})$, the *symmetric matroid* associated to M is the collection $\tilde{\mathcal{B}}$ of transversals defined as $\tilde{\mathcal{B}} := \{\tilde{B} : B \in \mathcal{B}\}$. A subset $S \subseteq 2\mathbf{n}$ is called *independent* in M if it is contained in some transversal $\tilde{B} \in \tilde{\mathcal{B}}$, and it is called *dependent* in M if it is not independent. A subset $C \subseteq 2\mathbf{n}$ is called a *circuit* of M if C is a minimal dependent subset which is admissible. A *cocircuit* of M is a circuit of the dual even Δ -matroid M^* . The set of circuits of M will be denoted by $\mathcal{C}(M)$. An admissible union of circuits of M is called a *cycle* of M . A *cocycle* of M is a cycle of the dual even Δ -matroid M^* .

We will make use of the following proposition in subsequent chapters.

Proposition 3.7. Let $M = ([n], \mathcal{B})$ be an even Δ -matroid. Suppose $\tilde{B} \in \tilde{\mathcal{B}}$ and $j \in 2\mathbf{n} \setminus \tilde{B}$. Then $\tilde{B} \cup j$ contains a unique circuit $C(\tilde{B}, j)$, called the *fundamental circuit* of j over \tilde{B} . It is given by $C(\tilde{B}, j) = \{i \in \tilde{B} : \tilde{B} \Delta \{j, j^*, i, i^*\} \in \tilde{\mathcal{B}}\} \cup j$.

4. Tropical Wick relations

We now turn to the study of the tropical prevariety and tropical variety defined by the Wick relations. We first start with a very brief introduction to some of the basic concepts in tropical geometry.

4.1. Basic tropical notions

The field of *Puiseux series* on the variable t over the complex numbers is the algebraically closed field $\mathbb{C}\{\{t\}\} := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{\frac{1}{n}}))$ whose elements are formal power series of the form $f = \sum_{k=k_0}^{+\infty} c_k \cdot t^{\frac{k}{N}}$, where N is a positive integer, k_0 is any integer, and the coefficients c_k are complex numbers. The field $\mathbb{C}\{\{t\}\}$ comes equipped with a valuation $\text{val} : \mathbb{C}\{\{t\}\} \rightarrow \mathbb{Q} \cup \{\infty\}$ that makes it a valuated field, where $\text{val}(f)$ is the least exponent r such that the coefficient of t^r in f is nonzero (so $\text{val}(0) = \infty$). If $Y \subseteq \mathbb{C}\{\{t\}\}^n$, we define its *valuation* to be the set

$$\text{val}(Y) := \{(\text{val}(y_1), \text{val}(y_2), \dots, \text{val}(y_n)) \in (\mathbb{Q} \cup \infty)^n : (y_1, y_2, \dots, y_n) \in Y\}.$$

Now, denote by $\mathbb{T} := (\mathbb{R} \cup \infty, \oplus, \odot)$ the *tropical semiring* of real numbers with ∞ together with the binary operations *tropical addition* \oplus and *tropical multiplication* \odot , defined as $x \oplus y = \min(x, y)$ and $x \odot y = x + y$. A tropical polynomial p in n variables is the tropical sum (or minimum) of tropical monomials

$$p = \bigoplus_{a_1, a_2, \dots, a_n} v_{a_1, a_2, \dots, a_n} \odot x_1^{a_1} \odot x_2^{a_2} \odot \dots \odot x_n^{a_n}, \quad (3)$$

where the coefficients v_{a_1, a_2, \dots, a_n} are elements of \mathbb{T} , and only finitely many of them are not equal to ∞ (here exponentiation should be understood as repeated application of tropical multiplication). Given a multivariate classical polynomial

$$P = \sum_{a_1, a_2, \dots, a_n} f_{a_1, a_2, \dots, a_n} \cdot x_1^{a_1} \cdot x_2^{a_2} \cdot \dots \cdot x_n^{a_n} \in \mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n],$$

we define its *tropicalization* to be the tropical polynomial obtained by substituting the operations in P by their tropical counterpart and the coefficients by their corresponding valuations, i.e.,

$$\text{trop}(P) := \bigoplus_{a_1, a_2, \dots, a_n} \text{val}(f_{a_1, a_2, \dots, a_n}) \odot x_1^{a_1} \odot x_2^{a_2} \odot \dots \odot x_n^{a_n}.$$

Given any subset $I \subseteq \mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n]$, we define its tropicalization to be the set of tropical polynomials $\text{trop}(I) := \{\text{trop}(P) : P \in I\}$.

The notion of “tropical zero set” is defined as follows. Given a tropical polynomial p as the minimum of tropical monomials, the *tropical hypersurface* $\mathcal{T}(p) \subseteq \mathbb{T}^n$ is the set of points $(x_1, x_2, \dots, x_n) \in \mathbb{T}^n$ such that this minimum is attained by at least two different monomials of p (or it is equal to ∞). If T is a set of tropical polynomials in n variables, the *tropical prevariety* described by them is $\mathcal{T}(T) := \bigcap_{p \in T} \mathcal{T}(p)$. If $I \subseteq \mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n]$ is an ideal then the tropical prevariety $\mathcal{T}(\text{trop}(I))$ is called a *tropical variety*.

If the ideal I is generated by some set of polynomials $S \subseteq I$, it is *not* necessarily true that the tropical variety defined by I is equal to the tropical prevariety defined by S , not even if we impose the condition that S be a universal Gröbner basis for I . When it does happen that $\mathcal{T}(\text{trop}(I)) = \mathcal{T}(\text{trop}(S))$ we say that S is a *tropical basis* for I . The notion of tropical basis is very subtle, and it is general very hard (both theoretically and computationally) to determine if a given set of generators forms such a basis. For an excellent example illustrating these difficulties, the reader is invited to see [6].

The Fundamental Theorem of Tropical Geometry establishes the connection between the “algebraic tropicalization” of an ideal and the “geometric tropicalization” of its corresponding variety. A proof of it can be found in [14].

Theorem 4.1 (Fundamental Theorem of Tropical Algebraic Geometry). *Let I be an ideal of $\mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n]$ and $X := V(I) \subseteq \mathbb{C}\{\{t\}\}^n$ its associated algebraic set. Then $\mathcal{T}(\text{trop}(I)) \cap (\mathbb{Q} \cup \infty)^n = \text{val}(X)$. Moreover, if I is a prime ideal then $\mathcal{T}(\text{trop}(I)) \cap \mathbb{R}^n$ is a pure connected polyhedral complex of the same dimension as the irreducible variety X .*

In this way, tropical geometry allows us to get information about the variety X just by studying the combinatorially defined polyhedral complex $\mathcal{T}(\text{trop}(I))$. This approach has been very fruitful in many cases, and has led to many beautiful results. The reader is invited to consult [14] for much more on tropical geometry.

4.2. Tropical Wick relations

We now focus our attention on the Wick relations and the ideal they generate.

Definition 4.2. A vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is called a *tropical Wick vector* if it satisfies the tropical Wick relations, that is, for all $S, T \in 2^{[n]}$ the minimum

$$\min_{i \in S \Delta T} (p_{S \Delta i} + p_{T \Delta i}) \quad (4)$$

is achieved at least twice (or it is equal to ∞). The Δ -Dressian $\Delta\text{Dr}(n) \subseteq \mathbb{T}^{2^{[n]}}$ is the space of all tropical Wick vectors in $\mathbb{T}^{2^{[n]}}$, i.e., the tropical prevariety defined by the Wick relations.

Tropical Wick vectors have also been studied in the literature under the name of *valuated (even) Δ -matroids* (see [8,16]), and in a more general setup under the name of *M -convex functions on jump systems* (see [18]).

The *support* of a vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is the collection $\text{supp}(p) := \{S \subseteq [n] : p_S \neq \infty\}$. We will later see (Theorem 5.1) that the support of any tropical Wick vector consists of subsets whose cardinalities have all the same parity, so the Δ -Dressian decomposes as the disjoint union of two tropical prevarieties: the *even Δ -Dressian* $\Delta\text{Dr}^+(n) \subseteq \mathbb{T}^{2^{[n]}}$ (consisting of all tropical Wick vectors whose support has only subsets of even cardinality) and the *odd Δ -Dressian* $\Delta\text{Dr}^-(n) \subseteq \mathbb{T}^{2^{[n]}}$ (defined analogously).

One of the main advantages of allowing our vectors to have ∞ entries is that tropical Wick vectors can be seen as a generalization of tropical Plücker vectors (or valuated matroids), as explained below.

Definition 4.3. A tropical Wick vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is called a *tropical Plücker vector* (or a *valuated matroid*) if all the subsets in $\text{supp}(p)$ have the same cardinality r_p , called the *rank* of p . The name is justified by noting that in this case, the tropical Wick relations become just the tropical Plücker relations: For all $S, T \in 2^{[n]}$ such that $|S| = r_p - 1$ and $|T| = r_p + 1$, the minimum $\min_{i \in T \setminus S} (p_{S \cup i} + p_{T - i})$ is achieved at least twice (or it is equal to ∞). The space of tropical Plücker vectors of rank k is called the *Dressian* $\text{Dr}(k, n)$; it is the tropical prevariety defined by the Plücker relations of rank k .

Tropical Plücker vectors play a central role in the combinatorial study of tropical linear spaces done by Speyer (see [21]). In his paper he only deals with tropical Plücker vectors whose support is the collection of *all* subsets of $[n]$ of some fixed size k ; we will later see that our definition is the “correct” generalization to more general supports.

Definition 4.4. The *tropical pure spinor space* $\text{TSpin}^\pm(n) \subseteq \mathbb{T}^{2^{[n]}}$ is the tropicalization of the space of pure spinors, i.e., it is the tropical variety defined by the ideal generated by all Wick relations. A tropical Wick vector in the tropical pure spinor space is said to be *realizable*. The decomposition of the Δ -Dressian into its even and odd parts induces a decomposition of the tropical pure spinor space as the disjoint union of two “isomorphic” tropical varieties $\text{TSpin}^+(n)$ and $\text{TSpin}^-(n)$, namely, the tropicalization of the spinor varieties $\text{Spin}^+(n)$ and $\text{Spin}^-(n)$ described in Section 2. The tropicalization $\text{TSpin}^+(n) \subseteq \mathbb{T}^{2^{[n]}}$ of the even part $\text{Spin}^+(n)$ will be called the *tropical spinor variety*.

By definition, the tropical pure spinor space $\text{TSpin}^\pm(n)$ is contained in the Δ -Dressian $\Delta\text{Dr}(n)$. A first step in studying representability of tropical Wick vectors (i.e. valuated Δ -matroids) is to determine when these two spaces are equal, or equivalently, when the Wick relations form a tropical basis. Our main result in this section answers this question for almost all values of n .

Theorem 4.5. *If $n \leq 5$ then the tropical pure spinor space $\text{TSpin}^\pm(n)$ is equal to the Δ -Dressian $\Delta\text{Dr}(n)$, i.e., the Wick relations form a tropical basis for the ideal they generate. If $n \geq 7$ then $\text{TSpin}^\pm(n)$ is strictly smaller than $\Delta\text{Dr}(n)$; in fact, there is a vector in the even Δ -Dressian $\Delta\text{Dr}^+(n)$ whose support consists of all even-sized subsets of $[n]$ which is not in the tropical spinor variety $\text{TSpin}^+(n)$.*

Corollary 4.6. *Let M be an even Δ -matroid on a ground set of at most 5 elements. Then M is a representable even Δ -matroid over any algebraically closed field of characteristic 0.*

We will postpone the proof of Theorem 4.5 and Corollary 4.6 until Section 5, after we have studied some of the combinatorial properties of tropical Wick vectors. To show that the tropical pure spinor space and the Δ -Dressian agree when $n \leq 5$ we will make use of Anders Jensen's software Gfan [12]. It is still unclear what happens when $n = 6$. In this case, the spinor variety is described by 76 nontrivial Wick relations (60 of which are 4-term Wick relations) on 32 variables, and a Gfan computation requires a long time to finish. We state the following conjecture.

Conjecture 4.7. *The tropical pure spinor space $\text{TSpin}^\pm(6)$ is equal to the Δ -Dressian $\Delta\text{Dr}(6)$.*

5. Tropical Wick vectors and Delta-matroid subdivisions

In this section we provide a description of tropical Wick vectors in terms of polytopal subdivisions. We start with a useful local characterization, which was basically proved by Murota in [18].

Theorem 5.1. *Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ has nonempty support. Then p is a tropical Wick vector if and only if the following two conditions are satisfied:*

- (a) *The support $\text{supp}(p)$ of p is the collection of bases of an even Δ -matroid over $[n]$.*
- (b) *The vector p satisfies the 4-term tropical Wick relations: For all $S \in 2^{[n]}$ and all $a, b, c, d \in [n] \setminus S$ distinct, the minima*

$$\begin{aligned} & \min(p_{Sabcd} + p_S, p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc}), \\ & \min(p_{Sabc} + p_{Sd}, p_{Sabd} + p_{Sc}, p_{Sacd} + p_{Sb}, p_{Sbcd} + p_{Sa}) \end{aligned} \quad (5)$$

are achieved at least twice (or are equal to ∞).

Proof. If p is a tropical Wick vector then, by definition, p satisfies the 4-term tropical Wick relations. To show that $\text{supp}(p)$ is an even Δ -matroid, suppose $A, B \in \text{supp}(p)$ and $a \in A \Delta B$. Take $S = A \Delta a$ and $T = B \Delta a$. The minimum in Eq. (4) is then a finite number and thus it is achieved at least twice, so there exists $b \in S \Delta T = A \Delta B$ such that $b \neq a$ and $p_{S\Delta b} + p_{T\Delta b} = p_{A\Delta\{a,b\}} + p_{B\Delta\{a,b\}} < \infty$. This implies that $A \Delta \{a, b\}$ and $B \Delta \{a, b\}$ are both in $\text{supp}(p)$, showing that $\text{supp}(p)$ satisfies the strong exchange axiom for even Δ -matroids.

The reverse implication is basically a reformulation of the following characterization given by Murota (done in greater generality for M -convex functions on jump systems; for details see [18]): If $\text{supp}(p)$ is the collection of bases of an even Δ -matroid over $[n]$ then p is a tropical Wick vector if and only if for all $A, B \in \text{supp}(p)$ such that $|A \Delta B| = 4$, there exist $a, b \in A \Delta B$ distinct such that $p_A + p_B \geq p_{A\Delta\{a,b\}} + p_{B\Delta\{a,b\}}$. \square

As a corollary, we get the following local description of tropical Plücker vectors.

Corollary 5.2. Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ has nonempty support. Then p is a tropical Plücker vector if and only if the following two conditions are satisfied:

- (a) The support $\text{supp}(p)$ of p is the collection of bases of matroid over $[n]$ (of rank r_p).
- (b) The vector p satisfies the 3-term tropical Plücker relations: For all $S \in 2^{[n]}$ such that $|S| = r_p - 2$ and all $a, b, c, d \in [n] \setminus S$ distinct, the minimum

$$\min(p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc})$$

is achieved at least twice (or it is equal to ∞).

Proof. The 3-term tropical Plücker relations are just the 4-term tropical Wick relations in the case where all the subsets in $\text{supp}(p)$ have the same cardinality. \square

Corollary 5.2 shows that our notion of tropical Plücker vector is indeed a generalization of the one given by Speyer in [21] to the case where $\text{supp}(p)$ is not necessarily the collection of bases of a uniform matroid.

It is worth mentioning that the assumptions on the support of p are essential in the local descriptions given above. As an example of this, consider the vector $p \in \mathbb{T}^{2^{[6]}}$ defined as

$$p_I := \begin{cases} 0 & \text{if } I = 123 \text{ or } I = 456, \\ \infty & \text{otherwise.} \end{cases}$$

The vector p satisfies the 3-term tropical Plücker relations, but its support is not the collection of bases of a matroid and thus p is not a tropical Plücker vector.

Given a vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$, denote by $\Gamma_p \subseteq \mathbb{R}^n$ its associated polytope $\Gamma_p := \text{convex}\{e_S : S \in \text{supp}(p)\}$. The vector p induces naturally a regular subdivision \mathcal{D}_p of Γ_p in the following way. Consider the vector p as a height function on the vertices of Γ_p , so “lift” vertex e_S of Γ_p to height p_S to obtain the *lifted polytope* $\Gamma'_p = \text{convex}\{(e_S, p_S) : S \in \text{supp}(p)\} \subseteq \mathbb{R}^{n+1}$. The *lower faces* of Γ'_p are the faces of Γ'_p minimizing a linear form $(v, 1) \in \mathbb{R}^{n+1}$; their projection back to \mathbb{R}^n form the polytopal subdivision \mathcal{D}_p of Γ_p , called the *regular subdivision induced by p* .

We now come to the main result of this section. It describes tropical Wick vectors as the height vectors that induce “nice” polytopal subdivisions. After finishing this paper, it was pointed out to the author that an equivalent formulation of this result had already been proved by Murota in [17], under the language of maximizers of an even Δ -matroid.

Theorem 5.3. Let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$. Then p is a tropical Wick vector if and only if the regular subdivision \mathcal{D}_p induced by p is an even Δ -matroid subdivision, i.e., it is a subdivision of an even Δ -matroid polytope into even Δ -matroid polytopes.

Proof. Assume p is a tropical Wick vector. By condition (a) in Theorem 5.1, we know that Γ_p is an even Δ -matroid polytope. Let $Q \subseteq \mathbb{R}^n$ be one of the polytopes in \mathcal{D}_p . By definition, Q is the projection back to \mathbb{R}^n of the face of the lifted polytope $\Gamma'_p \subseteq \mathbb{R}^{n+1}$ minimizing some linear form $(v, 1) \in \mathbb{R}^{n+1}$, and thus

$$\text{vertices}(Q) = \left\{ e_R \in \{0, 1\}^n : p_R + \sum_{j \in R} v_j \text{ is minimal} \right\}.$$

To show that Q is an even Δ -matroid polytope, suppose e_A and e_B are vertices of Q , and assume $a \in A \Delta B$. Let $S = A \Delta a$ and $T = B \Delta a$. Since p is a tropical Wick vector, the minimum $\min_{i \in S \Delta T} (p_{S \Delta i} + p_{T \Delta i})$ is achieved at least twice or it is equal to ∞ . Adding $\sum_{j \in S} v_j + \sum_{j \in T} v_j$, we get that the minimum

$$\min_{i \in S \Delta T} \left(\left(p_{S \Delta i} + \sum_{j \in S \Delta i} v_j \right) + \left(p_{T \Delta i} + \sum_{j \in T \Delta i} v_j \right) \right) \quad (6)$$

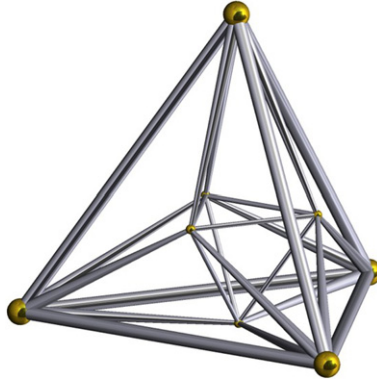


Fig. 1. Schlegel diagram of the 4-demicube.

is achieved at least twice or it is equal to ∞ . Since the minimum over all $R \in 2^{[n]}$ of $p_R + \sum_{j \in R} v_j$ is achieved when $R = A$ and $R = B$, it follows that the minimum (6) is achieved when $i = a$ and it is finite. Therefore, there exists $b \in S \triangle T = A \triangle B$ such that $b \neq a$ and

$$\left(p_{S\Delta b} + \sum_{j \in S\Delta b} v_j\right) + \left(p_{T\Delta b} + \sum_{j \in T\Delta b} v_j\right) = \left(p_A + \sum_{j \in A} v_j\right) + \left(p_B + \sum_{j \in B} v_j\right),$$

so $e_{S\Delta b}$ and $e_{T\Delta b}$ are also vertices of Q . This shows that the subsets corresponding to vertices of Q satisfy the strong exchange axiom (see Proposition 3.2), and thus Q is an even Δ -matroid polytope.

Now, suppose \mathcal{D}_p is an even Δ -matroid subdivision. We have that $\text{supp}(p)$ is the collection of bases of an even Δ -matroid, so by Theorem 5.1 it is enough to prove that p satisfies the 4-term tropical Wick relations. If this is not the case then for some $S \in 2^{[n]}$ and $a, b, c, d \in [n] \setminus S$ distinct, one of the two minima in (5) is achieved only once (and it is not equal to infinity). It is easy to check that the corresponding sets

$$\begin{aligned} &\{e_{Sabcd}, e_S, e_{Sab}, e_{Scd}, e_{Sac}, e_{Sbd}, e_{Sad}, e_{Sbc}\} \cap \{e_S : S \in \text{supp}(p)\}, \\ &\{e_{Sabc}, e_{Sd}, e_{Sabd}, e_{Sc}, e_{Sacd}, e_{Sb}, e_{Sbcd}, e_{Sa}\} \cap \{e_S : S \in \text{supp}(p)\} \end{aligned} \quad (7)$$

are the set of vertices of faces of Γ_p . This implies that \mathcal{D}_p contains an edge joining the two vertices that correspond to the term where this minimum is achieved, which is not an edge of the form $\pm e_i \pm e_j$, so by Theorem 3.5 the subdivision \mathcal{D}_p is not an even Δ -matroid subdivision. \square

Note that Theorem 5.1 can now be seen as a local criterion for even Δ -matroid subdivisions: the regular subdivision induced by p is an even Δ -matroid subdivision if and only if the subdivisions it induces on the polytopes whose vertices are described by the sets of the form (7) are even Δ -matroid subdivisions. These polytopes are all isometric (when p has maximal support), and they are known as the 4-demicube. This is a regular 4-dimensional polytope with 8 vertices and 16 facets; a picture of its Schlegel diagram, created using Robert Webb's Great Stella software [23], is shown in Fig. 1. The 4-demicube plays the same role for even Δ -matroid subdivisions as the hypersimplex $\Delta(2, 4)$ (an octahedron) for classical matroid subdivisions.

If we restrict Theorem 5.3 to the case where all subsets in $\text{supp}(p)$ have the same cardinality, we get the following corollary. It generalizes the results of Speyer in [21] for subdivisions of a hypersimplex.

Corollary 5.4. *Let $p \in \mathbb{T}^{2^{[n]}}$. Then p is a tropical Plücker vector if and only if the regular subdivision \mathcal{D}_p induced by p is a matroid subdivision, i.e., it is a subdivision of a matroid polytope into matroid polytopes.*

We are now in position to prove Theorem 4.5 and its corollary.

Proof of Theorem 4.5. For $n \leq 5$, we used Anders Jensen's software Gfan [12] to compute both the tropical spinor variety $\text{TSpin}^+(n)$ and the even Δ -Dressian $\Delta\text{Dr}^+(n)$, and we then checked that they were equal. At the moment, Gfan does not support computations with vectors having coordinates equal to ∞ , so we split our computation into several parts. We first computed all possible even Δ -matroids on a ground set of at most 5 elements, getting a list of 35 even Δ -matroids up to isomorphism. We then used Gfan to compute for each of these even Δ -matroids M , the set of vectors in the tropical spinor variety and in the even Δ -Dressian whose support is the collection of bases of M . We finally checked that for all M these two sets were the same. A complete list of the 35 even Δ -matroids up to isomorphism and their corresponding spaces can be found on the website <http://math.berkeley.edu/~felipe/delta/>.

The most important of these spaces is obtained when M is the even Δ -matroid whose bases are all even-sized subsets of the set [5]. It is the finite part of the even Δ -Dressian $\Delta\text{Dr}^+(5)$ (and the tropical spinor variety $\text{TSpin}^+(5)$), and it is described by 10 nontrivial Wick relations on 16 variables. Using Gfan we computed this space to be a pure simplicial 11-dimensional polyhedral fan with a 6-dimensional lineality space. After modding out by this lineality space we get a 5-dimensional polyhedral fan whose f -vector is $(1, 36, 280, 960, 1540, 912)$. By Theorem 5.3, all vectors in this fan induce an even Δ -matroid subdivision of the polytope Γ_M associated to M , which is known as the 5-demicube. As an example of this, the 36 rays in the fan correspond to the coarsest non-trivial even Δ -matroid subdivisions of Γ_M , which come in two different isomorphism classes: 16 isomorphic hyperplane splits of Γ_M into 2 polytopes, and 20 isomorphic subdivisions of Γ_M into 6 polytopes. The 912 maximal cones in the fan correspond to the finest even Δ -matroid subdivisions of Γ_M , which come in four different isomorphism classes: 192 isomorphic subdivisions into 11 pieces, and 720 subdivisions into 12 pieces, divided into 3 distinct isomorphism classes of sizes 120, 120, and 480, respectively. A complete description of all these subdivisions can also be found on the website <http://math.berkeley.edu/~felipe/delta/>; they were computed with the aid of the software polymake [11].

In order to deal with the case $n \geq 7$ we will make use of the notions of minors and rank for even Δ -matroids and symmetric matroids, which are discussed for example in [4,5]. We will prove that for any even Δ -matroid M with rank function r_M , the vector $p = (p_T) \in \mathbb{R}^{2^{[n]}}$ defined as

$$p_T := \begin{cases} -r_M(\bar{T}) & \text{if } |T| \text{ is even,} \\ \infty & \text{otherwise;} \end{cases}$$

is a tropical Wick vector (where $\bar{T} := T \cup ([n] \setminus T)^*$). By Theorem 5.1, it is enough to prove that for any $S \in 2^{[n]}$ and any $a, b, c, d \in [n] \setminus S$ distinct, p satisfies the 4-term tropical Wick relations given in (5). Since the rank function of M satisfies $r_M(S \cup I) = r_{M/S}(I) + r_M(S)$, we can assume that $S = \emptyset$. In a similar way, by restricting our matroid to the ground set $\{a, b, c, d\}$ we see that it is enough to prove our claim for even Δ -matroids over a ground set of at most 4 elements. There are 11 even Δ -matroids up to isomorphism in this case (see <http://math.berkeley.edu/~felipe/delta/>), and it is not hard to check that for all of them the assertion holds.

Now, take M to be an even Δ -matroid which is not representable over \mathbb{C} (for example, let M be any matroid having the Fano matroid as a direct summand). In this case, the linear form $(0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ attains its minimum on the lifted polytope Γ'_p at the vertices corresponding to the bases of M , so the corresponding even Δ -matroid subdivision \mathcal{D}_p has as one of its faces the even Δ -matroid polytope of M . Since M is not representable over \mathbb{C} , the tropical Wick vector p is not in the tropical pure spinor space, by Lemma 5.5 below. \square

Lemma 5.5. *If $p \in \mathbb{T}^{2^{[n]}}$ is a representable tropical Wick vector then all the faces in the regular subdivision \mathcal{D}_p induced by p are polytopes associated to even Δ -matroids which are representable over \mathbb{C} .*

Proof. Suppose p is a representable tropical Wick vector. Without loss of generality, we can assume that all the entries of p are in $\mathbb{Q} \cup \infty$, so by the Fundamental Theorem of Tropical Geometry, p can

be obtained as the valuation of the vector of Wick coordinates corresponding to some n -dimensional isotropic subspace $U \subseteq \mathbb{C}\{\{t\}\}^{2n}$. Applying a suitable change of coordinates, we might assume as well that U is the row space of some $n \times 2n$ matrix of the form $[I|A]$, where I is the identity matrix of size n and A is an $n \times n$ skew-symmetric matrix. Let $v \in \mathbb{R}^n$, and suppose the face of the lifted polytope Γ'_p minimizing the linear form $(v, 1) \in \mathbb{R}^{n+1}$ projects back to \mathbb{R}^n to the polytope of an even Δ -matroid M . The bases of M are then the subsets $S \in 2^{[n]}$ at which $p'_S := p_S + \sum_{i \in S} v_i$ is minimal. Multiplying the rows and columns of the matrix A by appropriate powers of t (namely, multiplying row i and column i by t^{-v_i}), we see that the vector $(p'_S) \in \mathbb{T}^{2^{[n]}}$ is also a representable tropical Wick vector, so we might assume that $v = \vec{0} \in \mathbb{R}^n$. We can also add a scalar to all entries of p and assume that $\min_{S \in 2^{[n]}} p_S = 0$. Now, if $w = w(t)$ is a Wick vector in $\mathbb{C}\{\{t\}\}^{2^{[n]}}$ whose valuation is p then the vector $w(0)$ obtained by substituting in w the variable t by 0 is a Wick vector with entries in \mathbb{C} whose support is precisely the collection of bases of M , thus M is representable over \mathbb{C} . \square

Proof of Corollary 4.6. The proof of Theorem 4.5 shows that the existence of an even Δ -matroid over the ground set $[n]$ which is not representable over \mathbb{C} implies that the tropical pure spinor $\text{TSpin}^\pm(n)$ space is strictly smaller than the Δ -Dressian $\Delta\text{Dr}(n)$, so all even Δ -matroids on a ground set of at most 5 elements are representable over \mathbb{C} . Moreover, since the representability of an even Δ -matroid M over a field K is a first order property of the field K , any even Δ -matroid which is representable over \mathbb{C} is also representable over any algebraically closed field of characteristic 0. \square

6. The cocycle space

In this section we define circuits, cocircuits and duality for tropical Wick vectors (i.e. valuated Δ -matroids), and study the space of vectors which are “tropically orthogonal” to all circuits. The admissible part of this space will be called the cocycle space, for which we give a parametric representation.

Definition 6.1. Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is a tropical Wick vector. The vector $p^* = (p_S^*) \in \mathbb{T}^{2^{[n]}}$ defined as $p_S^* := p_{[n] \setminus S}$ is also a tropical Wick vector, called the *dual tropical Wick vector* to p . Note that the even Δ -matroid associated to p^* is the dual even Δ -matroid to the one associated to p .

Definition 6.2. Let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. We will keep referring the notions introduced in Section 3.3. It is convenient for us to work with the natural extension $\bar{p} \in \mathbb{T}^{\mathcal{V}(n)}$ of p defined as $\bar{p}_{\bar{S}} := p_S$. For any $T \in 2^{[n]}$ we define the vector $c_T \in \mathbb{T}^{2^n}$ (also denoted $c_{\bar{T}}$) as

$$(c_T)_i = (c_{\bar{T}})_i := \begin{cases} \bar{p}_{\bar{T} \Delta \{i, i^*\}} & \text{if } i \in \bar{T}, \\ \infty & \text{otherwise.} \end{cases}$$

If $\text{supp}(c_T) \neq \emptyset$ then $\text{supp}(c_T)$ is one of the fundamental circuits of the even Δ -matroid M_p whose collection of bases is $\text{supp}(p)$ (see Proposition 3.7). We will say that the vector $c \in \mathbb{T}^{2^n}$ is a *circuit* of the tropical Wick vector p if $\text{supp}(c) \neq \emptyset$ and there is some $T \in 2^{[n]}$ and some $\lambda \in \mathbb{R}$ such that $c = \lambda \odot c_T$ (or in classical notation, $c = c_T + \lambda \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the vector in \mathbb{T}^{2^n} whose coordinates are all equal to 1). Since every circuit of M_p is a fundamental circuit, we have $\mathcal{C}(M_p) = \{\text{supp}(c) : c \text{ is a circuit of } p\}$, so this notion of circuits indeed generalizes the notion of circuits for even Δ -matroids to the “valuated” setup. The collection of circuits of p will be denoted by $\mathcal{C}(p) \subseteq \mathbb{T}^{2^n}$. A *cocircuit* of the tropical Wick vector p is just a circuit of the dual vector p^* , i.e., a vector of the form $\lambda \odot c_T^*$, where $c_T^* \in \mathbb{T}^{2^n}$ (also denoted $c_{\bar{T}}^*$) is the vector

$$(c_T^*)_i = (c_{\bar{T}}^*)_i := \begin{cases} \bar{p}_{\bar{T} \Delta \{i, i^*\}} & \text{if } i \notin \bar{T}, \\ \infty & \text{otherwise.} \end{cases}$$

Definition 6.3. Two vectors $x, y \in \mathbb{T}^N$ are said to be *tropically orthogonal*, denoted by $x \top y$, if the minimum $\min(x_1 + y_1, x_2 + y_2, \dots, x_N + y_N)$ is achieved at least twice (or it is equal to ∞). If $X \subseteq \mathbb{T}^N$ then its *tropically orthogonal set* is $X^\top := \{y \in \mathbb{T}^N : y \top x \text{ for all } x \in X\}$.

With these definitions, tropical Wick relations take a very simple form:

Proposition 6.4. *Let $p \in \mathbb{T}^{2[n]}$ be a tropical Wick vector. Then any circuit of p is tropically orthogonal to any cocircuit of p .*

We now turn to the study of the space of admissible vectors which are tropically orthogonal to all circuits. As we will see later, this space can be thought of as a tropical linear space of type D.

Definition 6.5. A vector $x \in \mathbb{T}^{2n}$ is said to be *admissible* if $\text{supp}(x)$ is an admissible subset of $2n$ (see Section 3.3). Let $p \in \mathbb{T}^{2[n]}$ be a tropical Wick vector. If $x \in \mathcal{C}(p)^\top$ is admissible then x will be called a *cocycle* of p . The set of all cocycles of p will be called the *cocycle space* of p , and will be denoted by $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$.

Proposition 6.6. *Suppose $p \in \mathbb{T}^{2[n]}$ is a tropical Wick vector, and let M be the even Δ -matroid whose collection of bases is $\text{supp}(p)$. Then*

- If $x \in \mathcal{C}(p)^\top$ has nonempty support then $\text{supp}(x)$ is a dependent subset in M^* .
- The cocycles of p having minimal nonempty support (with respect to inclusion) are precisely the cocircuits of p .
- For any two cocircuits c_1^* and c_2^* of p with the same support there is a $\lambda \in \mathbb{R}$ such that $c_1^* = \lambda \odot c_2^*$.

Proof. Assume that $x \in \mathcal{C}(p)^\top$ has nonempty independent support in M^* , so there exists a basis $B \in \mathcal{B}(M)$ such that $\text{supp}(x) \cap \bar{B} = \emptyset$. Take $j \in \text{supp}(x)$, and consider the admissible subset $J := \bar{B} \Delta \{j, j^*\}$. The circuit c_J of p satisfies $j \in \text{supp}(c_J) \subseteq \bar{B} \cup j$, so $\text{supp}(x) \cap \text{supp}(c_J) = \{j\}$ and thus x cannot be tropically orthogonal to c_J .

Now, Proposition 6.4 tells us that all cocircuits of p are cocycles of p . Suppose x is a cocycle with minimal nonempty support, and fix $j \in \text{supp}(x)$. Since $\text{supp}(x)$ is an admissible dependent subset in M^* and $\mathcal{Q}(p)$ contains all cocircuits of p , we have that $\text{supp}(x)$ must be a cocircuit of M . Therefore, there is a basis $B \in \mathcal{B}(M)$ such that $(\text{supp}(x) - j) \cap \bar{B} = \emptyset$. For any $k \in \text{supp}(x) - j$, consider the admissible subset $J_k := \bar{B} \Delta \{k, k^*\}$. We have that $k \in \text{supp}(x) \cap \text{supp}(c_{J_k})$, $\text{supp}(x) \subseteq (2n \setminus \bar{B}) \cup j$ and $\text{supp}(c_{J_k}) \subseteq \bar{B} \cup k$, so we must have $\text{supp}(x) \cap \text{supp}(c_{J_k}) = \{j, k\}$, since $x \top c_{J_k}$. We thus have $x_j + (c_{J_k})_j = x_k + (c_{J_k})_k$, so

$$x_k - x_j = (c_{J_k})_j - (c_{J_k})_k = p_{\bar{B} \Delta \{k, k^*\} \Delta \{j, j^*\}} - p_{\bar{B}}. \quad (8)$$

Since Eq. (8) is true for any $k \in \text{supp}(x) - j$ (and also for $k = j$), it follows that

$$x = c_{\bar{B} \Delta \{j, j^*\}}^* + (x_j - p_{\bar{B}}) \cdot \mathbf{1}, \quad (9)$$

so x is a cocircuit of p as required. Finally, the above discussion shows that if c_1^* and c_2^* are cocircuits of p with the same support then both of them can be written in the form given in Eq. (9) (using the same B and j), so there is a $\lambda \in \mathbb{R}$ such that $c_1^* = c_2^* + \lambda \cdot \mathbf{1}$. \square

We will now give a parametric description for the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$ of a tropical Wick vector $p \in \mathbb{T}^{2[n]}$. For this purpose we first introduce the concept of tropical convexity. More information about this topic can be found in [7].

Definition 6.7. A set $X \subseteq \mathbb{T}^N$ is called *tropically convex* if it is closed under tropical linear combinations, i.e., for any $x_1, \dots, x_r \in X$ and any $\lambda_1, \dots, \lambda_r \in \mathbb{T}$ we have that $\lambda_1 \odot x_1 \oplus \dots \oplus \lambda_r \odot x_r \in X$. For any $a_1, \dots, a_r \in \mathbb{T}^N$, their *tropical convex hull* is defined to be

$$\text{tconvex}(a_1, \dots, a_r) := \{\lambda_1 \odot a_1 \oplus \dots \oplus \lambda_r \odot a_r : \lambda_1, \dots, \lambda_r \in \mathbb{T}\};$$

it is the smallest tropically convex set containing the vectors a_1, \dots, a_r . A set of the form $\text{tconvex}(a_1, \dots, a_r)$ is usually called a *tropical polytope*.

Lemma 6.8. Let $p = (p_S) \in \mathbb{T}^{2[n]}$ be a tropical Wick vector. If $x \in \mathbb{T}^{2n}$ is in the cocycle space $\mathcal{Q}(p)$ of p then x is in the tropical convex hull of the cocircuits of p .

Proof. Let M denote the even Δ -matroid whose collection of bases is $\text{supp}(p)$. Let $x \in \mathcal{Q}(p)$, and suppose $j \in \text{supp}(x)$.

Assume first that $\{j\}$ is an independent set in M^* , and take a basis $B \in \mathcal{B}(M^*)$ such that $j \in \bar{B}$, the number of elements in $\bar{B} \cap \text{supp}(x)$ is as large as possible, and

$$p'_B := p_B^* + \sum_{l \in \text{supp}(x) \cap \bar{B}} x_l \quad (10)$$

is as small as possible (using that order of precedence). Now, consider the admissible subset $J := (2n \setminus \bar{B}) \Delta \{j, j^*\}$, and denote $c_j := c_J$. Since x is a cocycle of p , we have that $x \top c_j$, so there is a $k \in 2n - j$ such that the minimum $\min_{l \in 2n} (x_l + (c_j)_l)$ is attained when $l = k$. It follows that

$$x_k + (c_j)_k \leq x_j + (c_j)_j < \infty, \quad (11)$$

so in particular $k \in \text{supp}(x) \cap \text{supp}(c_j)$. Note that, since $\text{supp}(c_j) \subseteq J$, we have that $k \in 2n \setminus \bar{B}$. Let $J' := (2n \setminus \bar{B}) \Delta \{k, k^*\}$, and consider the cocircuit $c_{j'}^* := c_{J'}^*$ of p . The support of $c_{j'}^*$ is the fundamental circuit in M^* of k over \bar{B} , so our choice of B and the fact that x is admissible imply that $\text{supp}(c_{j'}^*) \subseteq \text{supp}(x)$ (see Proposition 3.7). Moreover, since $k \in \text{supp}(c_j) \cap \text{supp}(c_{j'}^*)$, $\text{supp}(c_{j'}^*) \subseteq \bar{B} \cup k$, and $\text{supp}(c_j) \subseteq (2n \setminus \bar{B}) \cup j$, by Proposition 6.4 we must have $\text{supp}(c_j) \cap \text{supp}(c_{j'}^*) = \{j, k\}$ and

$$(c_j)_j + (c_{j'}^*)_j = (c_j)_k + (c_{j'}^*)_k. \quad (12)$$

Now, note that for any $l \in \text{supp}(c_{j'}^*) - j$, our choice of B minimizing (10) implies that $p'_B \leq p'_{B \Delta \{k, k^*\} \Delta \{l, l^*\}}$. Since x is admissible, this means that

$$(c_{j'}^*)_k - (c_{j'}^*)_l = p_B^* - p_{B \Delta \{k, k^*\} \Delta \{l, l^*\}}^* \leq x_k - x_l. \quad (13)$$

Moreover, (11) and (12) tell us that

$$(c_{j'}^*)_j - (c_{j'}^*)_k = (c_j)_k - (c_j)_j \leq x_j - x_k, \quad (14)$$

and adding (13) and (14) we get

$$(c_{j'}^*)_j - (c_{j'}^*)_l \leq x_j - x_l. \quad (15)$$

Now, consider the cocircuit $d_j^* := c_j^* - ((c_j)_j - x_j) \cdot \mathbf{1}$ of p . We have $(d_j^*)_j = x_j$, and if $l \in \text{supp}(d_j^*) - j = \text{supp}(c_j^*) - j$ then (15) implies that $(d_j^*)_l \geq x_l$.

In the case $\{j\}$ is a cocircuit of M , take d_j^* to be the cocircuit of p given by

$$(d_j^*)_l := \begin{cases} x_j & \text{if } l = j, \\ \infty & \text{otherwise.} \end{cases}$$

By the above discussion, we have that $x = \min_{j \in \text{supp}(x)} d_j^*$, so x is in the tropical convex hull of the cocircuits of p as desired. \square

We now state the main theorem of this section, which describes cocycle spaces as the admissible part of tropical polytopes.

Theorem 6.9. Let $p \in \mathbb{T}^{2[n]}$ be a tropical Wick vector. Then the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$ of p is the set of admissible vectors in the tropical convex hull of the cocircuits of p .

Proof. One implication is given by Lemma 6.8. For the reverse implication, it is not hard to see that if $y \in \mathbb{T}^{2n}$ then the set $\{y\}^\top$ is tropically convex, and since any intersection of tropically convex sets is tropically convex, any set of the form Y^\top with $Y \subseteq \mathbb{T}^{2n}$ is tropically convex. Therefore, since the space $\mathcal{C}(p)^\top$ contains all the cocircuits of p , it contains their tropical convex hull, so the result follows. \square

Theorem 6.9 implies that if p is a tropical Wick vector and M is its associated even Δ -matroid then the set of supports of all cocycles of p is precisely the set of cocycles of M (see Definition 3.6), showing that our definition of cocycles for tropical Wick vectors extends the usual definition of cocycles for even Δ -matroids to the valuated setup.

Corollary 6.10. Let $p \in \mathbb{T}^{2[n]}$ be a tropical Wick vector. Then $\mathcal{Q}(p^*) \subseteq \mathbb{T}^{2n}$ is the set of admissible vectors in $\mathcal{Q}(p)^\top$.

Proof. Since $\mathcal{Q}(p)$ contains all cocircuits of p , taking orthogonal sets we get that all admissible vectors in $\mathcal{Q}(p)^\top$ are also in $\mathcal{Q}(p^*)$. On the other hand, by definition, we have that $\mathcal{Q}(p)^\top$ contains all the circuits of p , and since $\mathcal{Q}(p)^\top$ is tropically convex, $\mathcal{Q}(p)^\top$ contains their tropical convex hull. Applying Theorem 6.9 to p^* we get that $\mathcal{Q}(p^*)$ is contained in the set of admissible vectors of $\mathcal{Q}(p)^\top$. \square

6.1. Tropical linear spaces

We will now specialize some of the results presented above to tropical Plücker vectors (i.e. valuated matroids). In this way we unify several results for tropical linear spaces given by Murota and Tamura in [15], Speyer in [21], and Ardila and Klivans in [1]. Unless otherwise stated, all matroidal terminology in this section will refer to the classical matroidal notions and not to the Δ -matroidal notions discussed above.

Definition 6.11. Let $p = (p_S) \in \mathbb{T}^{2[n]}$ be a tropical Plücker vector of rank r_p . For $T \in 2^{[n]}$ of size $r_p + 1$, we define the vector $d_T \in \mathbb{T}^n$ as

$$(d_T)_i := \begin{cases} p_{T-i} & \text{if } i \in T, \\ \infty & \text{otherwise.} \end{cases}$$

If $\text{supp}(d_T) \neq \emptyset$ then $\text{supp}(d_T)$ is one of the fundamental circuits of the matroid M_p whose collection of bases is $\text{supp}(p)$. We will say that the vector $d \in \mathbb{T}^n$ is a *Plücker circuit* of p if $\text{supp}(d) \neq \emptyset$ and there is some $T \in 2^{[n]}$ of size $r_p + 1$ and some $\lambda \in \mathbb{R}$ such that $d = \lambda \odot d_T$ (or in classical notation, $d = d_T + \lambda \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the vector in \mathbb{T}^n whose coordinates are all equal to 1). Since every circuit of M_p is a fundamental circuit, we have $\mathcal{C}(M_p) = \{\text{supp}(d) : d \text{ is a Plücker circuit of } p\}$, so this notion of Plücker circuits generalizes the notion of circuits for matroids to the “valuated” setup. The collection of Plücker circuits of p will be denoted by $\mathcal{PC}(p)$. A *Plücker cocircuit* of p is a Plücker circuit of the dual vector p^* .

The reason we are using the name “Plücker circuits” is just so that they are not confused with the circuits of p in the Δ -matroidal sense; a more appropriate name (but not very practical for the purposes of this paper) would be “circuits of type A” (while the Δ -matroidal circuits are “circuits of type D”).

The following definition was introduced by Speyer in [21].

Definition 6.12. Let $p \in \mathbb{T}^{2[n]}$ be a tropical Plücker vector. The space $L_p := \mathcal{PC}(p)^\top \subseteq \mathbb{T}^n$ is called the *tropical linear space* associated to p .

The tropical linear space L_p should be thought of as the space of cocycles of p “of type A” (while $\mathcal{Q}(p)$ is the space of cocycles of p “of type D”).

Tropical linear spaces have a very special geometric importance that we now describe. We will only mention some of the basic facts, the reader can consult [21] for much more information and proofs. Consider the n -dimensional vector space $V := \mathbb{C}\{\{t\}\}^n$ over the field $K := \mathbb{C}\{\{t\}\}$, and suppose W is a k -dimensional linear subspace of V with Plücker coordinates $P \in K^{\binom{n}{k}}$. Let $p \in \mathbb{T}^{\binom{n}{k}} \subseteq \mathbb{T}^{2^{[n]}}$ be the valuation of the vector P . Since P satisfies the Plücker relations, the vector p is a tropical Plücker vector. Under this setup, Speyer proved that the tropicalization of the linear space W is precisely the tropical linear space L_p . Also, if W^\perp is the corresponding orthogonal linear subspace then the tropicalization of W^\perp is the tropical linear space L_{p^*} . It is also shown in [21] that if p is any tropical Plücker vector (not necessarily realizable by a subspace W of V) of rank r_p then the polyhedral complex $L_p \cap \mathbb{R}^n$ is a pure polyhedral complex of dimension r_p .

The following proposition will allow us to apply the “type D” results that we got in previous sections to tropical linear spaces.

Proposition 6.13. *Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Plücker vector, and let $L_p \subseteq \mathbb{T}^n$ be its associated tropical linear space. Then, under the natural identification $\mathbb{T}^{2^n} \cong \mathbb{T}^n \times \mathbb{T}^n$, we have $\mathcal{C}(p)^\top = L_p \times L_{p^*}$.*

Proof. It is not hard to check that the circuits of p are precisely the vectors of the form $(d, \infty) \in \mathbb{T}^{2^n}$ with $d \in \mathbb{T}^n$ a Plücker circuit of p (where ∞ denotes the vector in \mathbb{T}^n with all coordinates equal to ∞), and of the form $(\infty, d^*) \in \mathbb{T}^{2^n}$ with $d^* \in \mathbb{T}^n$ a Plücker cocircuit of p ; so the result follows directly from the definitions. \square

The following theorem describes tropical linear spaces as tropical polytopes. It was first proved by Murota and Tamura in [15]. In the case of realizable tropical linear spaces it also appears in work of Yu and Yuster [24].

Corollary 6.14. *Suppose $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Plücker vector. Then the tropical linear space $L_p \subseteq \mathbb{T}^n$ is the tropical convex hull of the Plücker cocircuits of p .*

Proof. The cocircuits of p are the vectors of the form $(d^*, \infty) \in \mathbb{T}^{2^n}$ with $d^* \in \mathbb{T}^n$ a Plücker cocircuit of p , and of the form $(\infty, d) \in \mathbb{T}^{2^n}$ with $d \in \mathbb{T}^n$ a Plücker circuit of p ; so the result follows from Proposition 6.13 and Theorem 6.9. \square

It is instructive to see what Corollary 6.14 is saying when applied to tropical Plücker vectors with only zero and infinity entries (what is usually called the “constant coefficient case” in tropical geometry). In this case, since the complements of unions of cocircuits of the associated matroid M are exactly the flats of M , we get precisely the description of the tropical linear space in terms of the flats of M that was given by Ardila and Klivans in [1].

Another useful application to the study of tropical linear spaces is the following. It was also proved by Murota and Tamura in [15].

Corollary 6.15. *If $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Plücker vector then $L_{p^*} = L_p^\top$. In particular, for any tropical linear space L , we have $(L^\top)^\top = L$.*

Proof. By Proposition 6.13 we have that $L_{p^*} = \mathcal{C}(p^*)^\top \cap (\mathbb{T}^n \times \{\infty\}) = \mathcal{Q}(p^*) \cap (\mathbb{T}^n \times \{\infty\})$, so the result follows from Corollary 6.10. \square

One can also apply these ideas to prove the following result of Speyer in [21].

Corollary 6.16. *There is a bijective correspondence between tropical linear spaces and tropical Plücker vectors (up to tropical scalar multiplication).*

Proof. Proposition 6.13 and Corollary 6.15 show that one can recover $\mathcal{C}(p)^\top$ from the tropical linear space L_p . Proposition 6.6 shows that one can recover the cocircuits of p (and thus p , up to a scalar multiple of $\mathbf{1}$) from $\mathcal{C}(p)^\top$. \square

7. Isotropical linear spaces

Definition 7.1. Let $L \subseteq \mathbb{T}^{2n}$ be an n -dimensional tropical linear space. We say that L is (totally) *isotropic* if for any two $x, y \in L$ we have that the minimum

$$\min(x_1 + y_1^*, \dots, x_n + y_n^*, x_1^* + y_1, \dots, x_n^* + y_n)$$

is achieved at least twice (or it is equal to ∞). In this case, we also say that L is an *isotropical linear space*. Note that if $K = \mathbb{C}\{\{t\}\}$ and $V = K^{2n}$, the tropicalization of any n -dimensional isotropic subspace U of V (see Section 2) is an isotropical linear space $L \subseteq \mathbb{T}^{2n}$. In this case we say that L is *isotropically realizable* by U .

Not all isotropical linear spaces that are realizable are isotropically realizable. As an example of this, take $n = 2$ and let $L \subseteq \mathbb{T}^{2n}$ be the tropicalization of the rowspace of the matrix

$$\begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{1}^* & \mathbf{2}^* \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right). \end{array}$$

The tropical linear space L is a realizable isotropical linear space (as can be seen from Theorem 7.3), but it is easy to check that it cannot be isotropically realizable.

We mentioned in Section 2 that if U is an isotropic linear subspace then its vector of Wick coordinates w carries all the information of U . One might expect something similar to hold tropically, that is, that the valuation of the Wick vector w still carries all the information of the tropicalization of U . This is not true, as the next example shows.

Example 7.2. We present two isotropic linear subspaces of $\mathbb{C}\{\{t\}\}^{2n}$ whose corresponding tropicalizations are distinct isotropical linear spaces, but whose Wick coordinates have the same valuation. Take $n = 4$. Let U_1 and U_2 be the 4-dimensional isotropic linear subspaces of $\mathbb{C}\{\{t\}\}^8$ defined as the rowspace of the matrices

$$\begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* & \mathbf{4}^* \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -2 & -1 & 0 \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* & \mathbf{4}^* \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 & -2 & -1 & 0 \end{array} \right), \end{array}$$

respectively. Their corresponding tropical linear spaces L_1 and L_2 are distinct since, for example, the Plücker coordinate indexed by the subset 343^*4^* is nonzero for U_1 but zero for U_2 . However, the Wick coordinates of U_1 and U_2 are all nonzero scalars (the ones indexed by even subsets), and thus their valuations give rise to the same tropical Wick vector.

It is important to have an effective way of deciding if a tropical linear space is isotropical or not. For this purpose, if $v \in \mathbb{T}^{2n}$, we call its *reflection* to be the vector $v^r \in \mathbb{T}^{2n}$ defined as $v_i^r := v_{i^*}$. If $X \subseteq \mathbb{T}^{2n}$ then its reflection is the set $X^r := \{x^r : x \in X\}$.

Theorem 7.3. Let $L \subseteq \mathbb{T}^{2n}$ be a tropical linear space with associated tropical Plücker vector p (whose coordinates are indexed by subsets of $2n$). Then the following are equivalent:

- (1) L is an n -dimensional isotropical linear space.
- (2) $L^\top = L^r$.
- (3) $p_{2n \setminus T} = p_{T^*}$ for all $T \subseteq 2n$.

Proof. By Corollary 6.16 we know that two tropical linear spaces are equal if and only if their corresponding tropical Plücker vectors are equal, so (2) \leftrightarrow (3) follows from Corollary 6.15. To see that (1) \leftrightarrow (2), note that L is an isotropical linear space if and only if L is tropically orthogonal to the reflected tropical linear space L^r , that is, if and only if $L^r \subseteq L^\top$. Since $\dim(L^\top) = 2n - \dim(L) = 2n - \dim(L^r)$, the result follows from Lemma 7.4 below. \square

Lemma 7.4. *If $L_1 \subseteq L_2$ are two tropical linear spaces of the same dimension then $L_1 = L_2$.*

Proof. Let p_1 and p_2 be the corresponding tropical Plücker vectors, and let M_1 and M_2 be their associated matroids. By Corollary 6.14 we have that every Plücker cocircuit of p_1 is in the tropical convex hull of the Plücker cocircuits of p_2 , so in particular, any cocircuit of M_1 is a union of cocircuits of M_2 . This is saying that M_2^* is a quotient of M_1^* (see [19], Proposition 7.3.6), and since M_1^* and M_2^* have the same rank, we have $M_1^* = M_2^*$ [19, Corollary 7.3.4]. But then, in view of Proposition 6.6 and Proposition 6.13, the (Plücker) cocircuits of p_1 and p_2 are the same, so in fact $L_1 = L_2$. \square

Note that Theorem 7.3 describes the set of isotropical linear spaces (or more precisely, their associated tropical Plücker vectors) as an intersection of the Dressian $\text{Dr}(n, 2n)$ with a linear subspace.

If L is an isotropical linear space which is isotropically realizable by U then we have seen that the valuation p of the Wick vector w associated to U does not determine L . Nonetheless, the following theorem shows that p does determine the admissible part of L .

Theorem 7.5. *Let $L \subseteq \mathbb{T}^{2n}$ be an n -dimensional isotropical linear space which is isotropically realizable by the subspace $U \subseteq \mathbb{C}(\{t\})^{2n}$. Let $p \in \mathbb{T}^{2[n]}$ be the tropical Wick vector obtained as the valuation of the Wick vector w associated to U . Then the set of admissible vectors in L is the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{2n}$.*

Proof. Eq. (1) in Section 2 implies that the circuits of p are tropically orthogonal to all the elements of L , so $L \subseteq \mathcal{C}(p)^\top$ and thus the admissible vectors of L are in $\mathcal{Q}(p)$. On the other hand, it can be easily checked that the valuation of the Wick vector associated to the isotropic subspace U^\perp is precisely the dual tropical Wick vector p^* , so repeating the same argument we have that $L^\top \subseteq \mathcal{C}(p^*)^\top$. Taking orthogonal sets we get that $L \supseteq (\mathcal{C}(p^*)^\top)^\top \supseteq \mathcal{C}(p^*)$, and since L is tropically convex, Theorem 6.9 implies that the set of admissible vectors in L contains $\mathcal{Q}(p)$. \square

Acknowledgments

I am grateful to Federico Ardila and Mauricio Velasco for fruitful discussions that got me started in this project. I am also indebted to Bernd Sturmfels for many helpful comments and suggestions, and for supporting me as a Graduate Student Researcher through the U.S. National Science Foundation (DMS-0456960 and DMS-0757207).

References

- [1] Federico Ardila, Caroline J. Klivans, The Bergman complex of a matroid and phylogenetic trees, *J. Combin. Theory Ser. B* 96 (1) (2006) 38–49.
- [2] Alexandre V. Borovik, Israel M. Gelfand, Neil White, *Coxeter Matroids*, Progr. Math., vol. 216, Birkhäuser Boston Inc., Boston, MA, 2003.
- [3] André Bouchet, Greedy algorithm and symmetric matroids, *Math. Program.* 38 (2) (1987) 147–159.
- [4] André Bouchet, Multimatroïds. I. Coverings by independent sets, *SIAM J. Discrete Math.* 10 (4) (1997) 626–646.
- [5] André Bouchet, Multimatroïds. II. Orthogonality, minors and connectivity, *Electron. J. Combin.* 5 (1998), Research Paper 8, 25 pp. (electronic).
- [6] Melody Chan, Anders N. Jensen, Elena Rubei, The 4×4 minors of a $5 \times n$ matrix are a tropical basis, *Linear Algebra Appl.* 435 (7) (2011) 1598–1611.

- [7] Mike Develin, Bernd Sturmfels, Tropical convexity, *Doc. Math.* 9 (2004) 1–27 (electronic).
- [8] Andreas W.M. Dress, Walter Wenzel, A greedy-algorithm characterization of valuated Δ -matroids, *Appl. Math. Lett.* 4 (6) (1991) 55–58.
- [9] Andreas W.M. Dress, Walter Wenzel, Valuated matroids, *Adv. Math.* 93 (2) (1992) 214–250.
- [10] I.M. Gelfand, R.M. Goresky, R.D. MacPherson, V.V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, *Adv. Math.* 63 (3) (1987) 301–316.
- [11] Ewgenij Gawrilow, Michael Joswig, Polymake: a framework for analyzing convex polytopes, in: *Polytopes—Combinatorics and Computation*, Oberwolfach, 1997, in: *DMV Sem.*, vol. 29, Birkhäuser, Basel, 2000, pp. 43–73.
- [12] Anders N. Jensen, Gfan, a software system for Gröbner fans and tropical varieties, available at <http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html>.
- [13] Laurent Manivel, On spinor varieties and their secants, *SIGMA Symmetry Integrability Geom. Methods Appl.* 5 (2009), Paper 078, 22.
- [14] Diane Maclagan, Bernd Sturmfels, Introduction to tropical geometry, 2011, in preparation.
- [15] Kazuo Murota, Akihisa Tamura, On circuit valuation of matroids, *Adv. in Appl. Math.* 26 (3) (2001) 192–225.
- [16] Kazuo Murota, On exchange axioms for valuated matroids and valuated delta-matroids, *Combinatorica* 16 (4) (1996) 591–596.
- [17] Kazuo Murota, Characterizing a valuated delta-matroid as a family of delta-matroids, *J. Oper. Res. Soc. Japan* 40 (4) (1997) 565–578.
- [18] Kazuo Murota, M -convex functions on jump systems: a general framework for minsquare graph factor problem, *SIAM J. Discrete Math.* 20 (1) (2006) 213–226 (electronic).
- [19] James G. Oxley, *Matroid Theory*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.
- [20] Claudio Procesi, *Lie Groups. An Approach Through Invariants and Representations*, Universitext, Springer, New York, 2007.
- [21] David E. Speyer, Tropical linear spaces, *SIAM J. Discrete Math.* 22 (4) (2008) 1527–1558.
- [22] Bernd Sturmfels, Mauricio Velasco, Blow-ups of \mathbb{P}^{n-3} at n points and spinor varieties, *J. Commut. Algebra* 2 (2) (2010) 223–244.
- [23] Robert Webb, Great stella, available at <http://www.software3d.com/Stella.php>.
- [24] Josephine Yu, Debbie S. Yuster, Representing tropical linear spaces by circuits, in: *Proceedings of Formal Power Series and Algebraic Combinatorics*, Tianjin, China, 2007.