Journal of Combinatorial Theory, Series A 150 (2017) 108–136

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Gowers' Ramsey theorem with multiple operations and dynamics of the homeomorphism group of the Lelek fan

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A R T I C L E I N F O

Article history:
Received 27 August 2015
Available online xxxx

Keywords:
Gowers' Ramsey Theorem
Lelek fan
Fraïssé limits
Extreme amenability

A B S T R A C T

We generalize the finite version of Gowers' Ramsey theorem to multiple tetris-like operations and apply it to show that a group of homeomorphisms that preserve a "typical" linear order of branches of the Lelek fan, a compact connected metric space with many symmetries, is extremely amenable.

1. Introduction

In [7], Gowers proved a generalisation of Hindman's finite sums theorem in order to show the oscillation stability of the unit sphere in the Banach space c_0. Recently, Tyros in [22] and Ojeda-Aristizabal in [18] independently gave constructive combinatorial proofs of the finite version of Gowers' theorem.

This article aims to give a new Ramsey theorem, which generalizes the finite version of Gowers' Ramsey theorem to multiple operations (Theorem 2.8), and most importantly

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http://dx.doi.org/10.1016/j.jcta.2017.02.002
0097-3165/

to establish a surprising connection between our Ramsey theorem and the dynamics of the homeomorphism group of the Lelek fan – a compact connected metric space with many symmetries.

Our work was motivated by a striking correspondence between structural Ramsey theory, Fraïssé theory, and topological dynamics of automorphism groups, which was established by Kechris, Pestov and Todorčević in [10], and further extended by Nguyen van Thé in [17]. In these articles, they characterized a strong fixed point property, called extreme amenability, of automorphism groups in terms of the Ramsey property. Here a topological group is *extremely amenable* if every continuous action on a compact Hausdorff space admits a fixed point. For instance, using the Ramsey property for linearly ordered finite metric spaces by Nešetřil [14], the authors of [10] showed that the isometry group of the separable Urysohn metric space is extremely amenable. This result was originally proved by Pestov in [19] using concentration of measure techniques. Further, applying the Ramsey property for finite linearly ordered graphs (Nešetřil–Rödl [15] and [16]), finite linearly ordered hypergraphs (Nešetřil–Rödl [15] and [16]; Abramson–Harrington [1]), and finite naturally ordered vector spaces over a finite field (Graham–Leeb–Rothschild [8]), Kechris, Pestov and Todorčević showed that the groups of automorphisms of the random ordered graph, the random ordered hypergraph, and the ordered $\aleph_0$-dimensional vector space over a finite field, respectively, are extremely amenable.

In this article, we dualize the Kechris–Pestov–Todorčević correspondence from [10] to the projective Fraïssé setting (Section 4) and give its first application, namely to the dynamics of a certain natural group of homeomorphisms of the Lelek fan (Section 7). The projective Fraïssé theory was originally developed by Irwin and Solecki in [9] in order to capture a well-known compact and connected metric space – the pseudo-arc.

2. Discussion of results

2.1. Dynamics of the homeomorphism group of the Lelek fan

A *continuum* is a compact connected metric space. Denoting by C the Cantor set and by $[0,1]$ the unit interval, one defines the *Cantor fan* to be the quotient of $C \times [0,1]$ by the equivalence relation $\sim$ given by $(a,b) \sim (c,d)$ if and only if either $(a,b) = (c,d)$ or $b = d = 0$. For a continuum X, a point $x \in X$ is an *endpoint* in X if for every homeomorphic embedding $h : [0,1] \to X$ with x in the image of h either $x = h(0)$ or $x = h(1)$. The *Lelek fan* L, constructed by Lelek in [11], can be characterized as the unique non-degenerate subcontinuum of the Cantor fan whose endpoints are dense (see [5] and [6]). Denote by v the *top* $(0,0)/{\sim}$ of the Lelek fan. The "endpoint" and the "top" belong to the standard terminology in continuum theory. We point out that when we think of the Cantor fan, the top point is often really at the bottom.

We will use the description of the Lelek fan via the class of finite fans as in [3], where by a *fan* we mean an undirected connected simple graph with all loops, with no cycles of the length greater than one, and with a distinguished point r, called the *root*, such that all elements other than r have degree at most 2. We will study the class $\mathcal{F}$ of finite fans and the class $\mathcal{F}_<$ of finite fans expanded by a linear order on the set of branches. Families $\mathcal{F}$ and $\mathcal{F}_<$ form projective Fraïssé classes and therefore have projective Fraïssé limits, as defined by Irwin and Solecki [9] dualizing the classical (injective) Fraïssé theory from model theory. A natural quotient of a projective Fraïssé limit of $\mathcal{F}$ is the Lelek fan, as proved in [3] (see also Section 3.1), and a natural quotient of a projective Fraïssé limit of $\mathcal{F}_<$ will turn out to be a "branch-ordered" Lelek fan. We provide necessary basics about projective Fraïssé classes and projective Fraïssé limits in Section 3.1.

For a class $\mathcal{G}$ of finite structures and $A, B \in \mathcal{G}$, we denote by $\binom{B}{A}$ the set of all epimorphisms (that is, surjective maps preserving the structure; see Section 3.1 for the definition of a structure and of an epimorphism) from B onto A. We say that $\mathcal{G}$ is a *Ramsey class* or that it has the *Ramsey property* if for every $A, B \in \mathcal{G}$ and every natural number $r \geq 2$ there exists $C \in \mathcal{G}$ such that for every colouring c of $\binom{C}{A}$ with r colours there exists $g \in \binom{C}{B}$ such that $\binom{B}{A} \circ g = \{f \circ g : f \in \binom{B}{A}\}$ is *c-monochromatic*, that is, c restricted to $\binom{B}{A} \circ g$ is constant.

Typically a projective Fraïssé class is not a Ramsey class, however, it can become one when expanded by more relations such as a linear order. This is the case also for the class $\mathcal{F}$, while $\mathcal{F}$ is not a Ramsey class, the natural expansion $\mathcal{F}_<$, as we show using Theorem 2.8 and Corollary 5.5, is a Ramsey class.

Theorem 2.1. *The class $\mathcal{F}_<$ is a Ramsey class.*

Let $\mathcal{G}$ be a projective Fraïssé family with the projective Fraïssé limit $\mathbb{G}$. Let $G = \mathrm{Aut}(\mathbb{G})$ be the automorphism group of $\mathbb{G}$. We say that $\mathcal{G}$ is *rigid* if for every $A \in \mathcal{G}$, $\mathrm{Aut}(A)$ is trivial. In Section 4, we discuss and dualize the Kechris–Pestov–Todorčević correspondence to the projective setting showing the following.

Theorem 2.2. *The following are equivalent:*

(1) the group G is extremely amenable;
(2) the family $\mathcal{G}$ is a Ramsey class and it consists of rigid elements.

For a topological group G, a *G-flow* (or a *flow* if there is no ambiguity) is a continuous action of G on a compact Hausdorff space X, i.e. a continuous map $\pi : G \times X \to X$ such that $\pi(e, x) = x$ for every $x \in X$ and e the identity in G, and $\pi(gh, x) = \pi(g, \pi(h, x))$ for every $x \in X$ and $g, h \in G$. When the action is understood, we write gx instead of $\pi(g, x)$. We call G *extremely amenable, if every G-flow has a fixed point.* A G-flow is called *minimal* if it has no non-trivial closed G-invariant subsets. A continuous map $\psi : X \to Y$ between two G-flows is a *homomorphism* if $\psi(gx) = g(\psi(x))$ for every $g \in G$ and $x \in X$. The *universal minimal flow* of G is the unique minimal G-flow that has all other minimal G-flows as its homomorphic images. The universal minimal flow exists for every topological group and it is unique up to isomorphism. It is easy to see that G is extremely amenable if and only if the universal minimal flow of G is a singleton.

Let $\mathbb{L}$ and $\mathbb{L}_<$ denote, respectively, the projective Fraïssé limits of $\mathcal{F}$ and $\mathcal{F}_<$, and let $\mathrm{Aut}(\mathbb{L}_<)$ be the automorphism group of $\mathbb{L}_<$. Then $\mathrm{Aut}(\mathbb{L}_<)$ is a closed subgroup of $\mathrm{Aut}(\mathbb{L})$, the automorphism group of $\mathbb{L}$; see Section 3 for more details. Since $\mathcal{F}_<$ is a rigid Ramsey class, Theorem 2.2 provides the following.

Theorem 2.3. *The group* $\mathrm{Aut}(\mathbb{L}_<)$ *is extremely amenable.*

Let $H(L)$ denote the homeomorphism group of the Lelek fan L with the compact-open topology. The group $\mathrm{Aut}(\mathbb{L})$ continuously embeds as a dense subgroup into $H(L)$; see Section 3. Let H be the closure of $\mathrm{Aut}(\mathbb{L}_<)$ via this embedding. Then Theorem 2.3 will imply Theorem 2.4.

Theorem 2.4. *The group H is extremely amenable.*

In Proposition 7.4, we identify H with the group $H(L_<)$ of homeomorphisms that preserve the order coming from the one on $\mathbb{L}_<$.

2.2. A generalisation of Gowers' Ramsey theorem to multiple operations

To prove theorems stated in Section 2.1, we will need to generalize the finite version of Gowers' Ramsey theorem (Theorem 2.6). For this we will prove Theorem 2.8 and Corollary 5.5. In this section, we will state Theorem 2.8. First we will introduce the necessary notation.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers (we will follow the convention that 0 is not a natural number) and, for the remainder of this section, fix $k \in \mathbb{N}$. For a function $p : \mathbb{N} \to \{0, 1, \ldots, k\}$, we define the *support* $\mathrm{supp}(p)$ of p to be the set $\{l \in \mathbb{N} : p(l) \neq 0\}$. Let

$$\mathrm{FIN}_k = \{p : \mathbb{N} \to \{0, 1, \ldots, k\} : |\mathrm{supp}(p)| < \infty \text{ and } (\exists l \in \mathbb{N})\, (p(l) = k)\},$$

and, for each $n \in \mathbb{N}$, let

$$\mathrm{FIN}_k(n) = \{p : \mathbb{N} \to \{0, 1, \ldots, k\} : \mathrm{supp}(p) \subset \{1, 2, \ldots, n\}\}.$$

We equip FIN_k and each $\mathrm{FIN}_k(n)$ with a partial semigroup operation $+$ defined for p and q whenever $\max(\mathrm{supp}(p)) < \min(\mathrm{supp}(q))$ by $(p + q)(x) = p(x) + q(x)$.

Gowers' Theorem (Theorem 2.5, below) involves a *tetris* operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ defined by

$$T(p)(l) = \max\{0, p(l) - 1\}.$$

We define, for every $0 < i \leq k$, an operation $T_i^{(k)} : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ that behaves like the identity up to the value $i-1$ and like tetris above it as follows.

$$T_i^{(k)}(p)(l) = \begin{cases} p(l) & \text{if } p(l) < i \\ p(l) - 1 & \text{if } p(l) \geq i. \end{cases}$$

We also define $T_0^{(k)} = \mathrm{id} \restriction_{\mathrm{FIN}_k}$. It may seem more natural to denote the identity by $T_{k+1}^{(k)}$ or $T_\infty^{(k)}$, only for notational convenience later on we will be using $T_0^{(k)}$. Note that in our notation, $T_1^{(k)}$ is the usual Gowers tetris operation. When the context is clear we will usually drop superscripts and write T_i rather than $T_i^{(k)}$.

A sequence $B = (b_s)_{s\in\mathbb{N}}$ is called a *block sequence* if for every $i \in \mathbb{N}$

$$\max(\mathrm{supp}(b_i)) < \min(\mathrm{supp}(b_{i+1})).$$

Analogously, we define a finite block sequence $B = (b_s)_{s=1}^m$ and we call m the *length* of the sequence. We let $\mathrm{FIN}_k^{[d]}$ denote the set of all block sequences in FIN_k of length d and similarly we define $\mathrm{FIN}_k^{[d]}(n)$.

Let B be a block sequence in FIN_k (finite or infinite). Let P_k denote the product $\prod_{j=1}^k \{0, 1, \ldots, j\}$. For any I such that $(0, \ldots, 0) \in I \subset P_k$ and for $\vec{i} = (i(1), \ldots, i(k)) \in I$, denote

$$T_{\vec{i}} = T_{i(1)} \circ \ldots \circ T_{i(k)}.$$

Let $\langle B \rangle_I$ denote the partial subsemigroup of FIN_k consisting of elements of the form

$$\sum_{s=1}^{l} T_{\vec{i}_s}(b_s),$$

where l is a natural number, $\vec{i}_s \in I$, $b_s \in B$, for $s = 1, \ldots, l$, and there is some s such that all the entries of $\vec{i}_s$ are 0.

For a set X and $r \in \mathbb{N}$, we will often call a function $c : X \to \{1, 2, \ldots, r\}$ a *colouring*. We say that $A \subset X$ is *c-monochromatic*, or just *monochromatic*, if $c \restriction A$ is constant.

Let us state Gowers' Ramsey theorem in this language.

Theorem 2.5 *(Gowers [7]). Let $c : \mathrm{FIN}_k \to \{1, 2, \ldots, r\}$ be a colouring. Then there exists an infinite block sequence B in FIN_k such that $\langle B \rangle_{\prod_{i=1}^k \{0,1\}}$ is c-monochromatic.*

The finite version of Gowers' theorem (Theorem 2.6) can be deduced by a simple compactness argument.

Theorem 2.6. *Let k, m, r be natural numbers. Then there exists n such that for every colouring $c : \mathrm{FIN}_k(n) \to \{1, 2, \ldots, r\}$ there is a block sequence B of length m in $\mathrm{FIN}_k(n)$ such that $\langle B \rangle_{\prod_{i=1}^k \{0,1\}}$ is c-monochromatic.*

As a consequence of our main Ramsey result, Theorem 2.8, we will obtain the following generalisation of the finite Gowers theorem to all T_i's.

Corollary 2.7. *Let k, m, r be natural numbers. Then there exists a natural number n such that for every colouring $c : \mathrm{FIN}_k(n) \to \{1, 2, \ldots, r\}$ there is a block sequence B of length m in $\mathrm{FIN}_k(n)$ such that $\langle B\rangle_{P_k}$ is c-monochromatic.*

In order to state Theorem 2.8 in full generality, we need a few more pieces of notation. From the proof of Theorem 2.1 it will be clear why this is the theorem we need. For $l > k$, let $P^l_{k+1} = \prod_{j=k+1}^{l}\{1, 2, \ldots, j\}$, and let P^k_{k+1} contain only the constant sequence $(0, \ldots, 0)$. Note that if $p \in \mathrm{FIN}_l$ and $\vec{i} \in P^l_{k+1}$, then $T_{\vec{i}}(p) \in \mathrm{FIN}_k$.

Let $l \geq k$ and let $B = (b_s)_{s=1}^m$ be a block sequence in FIN_l. Let $\left\langle \bigcup_{\vec{i}\in P^l_{k+1}} T_{\vec{i}}(B)\right\rangle_{P_k}$ denote the partial subsemigroup of FIN_k consisting of elements of the form

$$\sum_{s=1}^{m} T_{\vec{t}_s} \circ T_{\vec{i}_s}(b_s),$$

where $\vec{i}_1, \ldots, \vec{i}_m \in P^l_{k+1}$, $\vec{t}_1, \ldots, \vec{t}_m \in P_k$, and there is an s such that all entries of $\vec{t}_s$ are 0. Let $\left\langle \bigcup_{\vec{i}\in P^l_{k+1}} T_{\vec{i}}(B)\right\rangle^{[d]}_{P_k}$ be the set of all block sequences in $\left\langle \bigcup_{\vec{i}\in P^l_{k+1}} T_{\vec{i}}(B)\right\rangle_{P_k}$ of length d.

Theorem 2.8. *Let $k \geq 1$. Then for every d, every $m \geq d$, every $l \geq k$, and every r, there exists a natural number n such that for every colouring $c : \mathrm{FIN}^{[d]}_k(n) \to \{1, 2, \ldots, r\}$, there is a block sequence B in $\mathrm{FIN}_l(n)$ of length m such that the partial semigroup $\left\langle \bigcup_{\vec{i}\in P^l_{k+1}} T_{\vec{i}}(B)\right\rangle^{[d]}_{P_k}$ is c-monochromatic. Denote the smallest such n by $G_d(k, l, m, r)$.*

Notice that setting $k = l$ and $d = 1$ in Theorem 2.8, and observing that $\left\langle \bigcup_{\vec{i}\in P^k_{k+1}} T_{\vec{i}}(B)\right\rangle_{P_k} = \langle B\rangle_{P_k}$, we obtain Corollary 2.7.

We prove Theorem 2.8 in Section 6 and use it to derive Theorem 2.4 in Section 7.

To motivate here the statement of Theorem 2.8, let us see how to an epimorphism between structures in $\mathcal{F}_<$ we can associate an element in $\mathrm{FIN}^{[d]}_k$. To each $f \in \binom{C}{A}$, we associate $f^* = (p^f_i)_{i=1}^d \in \mathrm{FIN}^{[d]}_k(n)$ such that

$$\mathrm{supp}(p^f_i) = \{j : a^1_i \in f(c_j)\}$$

and for $j \in \mathrm{supp}(p^f_i)$

$$p^f_i(j) = z \iff f(c^N_j) = a^z_i,$$

where $a_1, \ldots, a_d$ and $c_1, \ldots, c_n$ are the increasing enumerations of branches in A and C, respectively, $(a^z_i)_{z=0}^k$ is the increasing enumeration of a_i and $(c^y_j)_{y=0}^N$ is the increasing enumeration of c_j. As $f \to f^*$ is not injective, to prove Theorem 2.1, we will need not only Theorem 2.8, but also another Ramsey theoretic statement – Corollary 5.5.

In the proof of Theorem 2.8, we generalize methods introduced by Tyros [22], who recently gave a direct constructive proof of the finite version of Gowers' Ramsey theorem, providing upper bounds on n. Independently of Tyros, a proof of the finite version of Gowers' theorem was presented by Ojeda-Aristizabal [18]. On the other hand, the only known proof of the infinite Gowers Ramsey theorem [7] uses the Galvin–Glazer method of idempotents in a compact right-topological semigroup of ultrafilters. During the time this paper was under revision, Lupini [12] proved the infinite version of Corollary 2.7.

3. Preliminaries

3.1. A construction of the Lelek fan

For completeness, we include the construction of the Lelek fan from [3], and we refer the reader to that article for any details we omit here.

Given a first-order language $\mathcal{L}$ that consists of relation symbols r_i with arity m_i, $i \in I$, and function symbols f_j, with arity n_j, $j \in J$, a *topological $\mathcal{L}$-structure* is a compact zero-dimensional second-countable space A equipped with closed relations $r_i^A \subset A^{m_i}$ and continuous functions $f_j^A : A^{n_j} \to A$, $i \in I, j \in J$. A continuous surjection $\phi : B \to A$ between two topological $\mathcal{L}$-structures is an *epimorphism* if it preserves the structure, that is, for a function symbol f in $\mathcal{L}$ of arity n and $x_1, \dots, x_n \in B$ we require:

$$f^A(\phi(x_1), \dots, \phi(x_n)) = \phi(f^B(x_1, \dots, x_n));$$

and for a relation symbol r in $\mathcal{L}$ of arity m and $x_1, \dots, x_m \in A$ we require:

$$\begin{aligned} &r^A(x_1, \dots, x_m) \\ &\quad \iff \exists y_1, \dots, y_m \in B \left(\phi(y_1) = x_1, \dots, \phi(y_m) = x_m, \text{ and } r^B(y_1, \dots, y_m) \right). \end{aligned}$$

The if and only if condition in preservation of relations by epimorphism allows us to obtain connected spaces as natural quotients of inverse limits of (finite) topological structures.

By an *isomorphism* we mean a bijective epimorphism.

Let $\mathcal{G}$ be a countable family of finite topological $\mathcal{L}$-structures. We say that $\mathcal{G}$ is a *projective Fraïssé family* if the following two conditions hold:

(JPP) (the joint projection property) for any $A, B \in \mathcal{G}$ there are $C \in \mathcal{G}$ and epimorphisms from C onto A and from C onto B;

(AP) (the amalgamation property) for $A, B_1, B_2 \in \mathcal{G}$ and any epimorphisms $\phi_1 : B_1 \to A$ and $\phi_2 : B_2 \to A$, there exists $C \in \mathcal{G}$ with epimorphisms $\phi_3 : C \to B_1$ and $\phi_4 : C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

A topological $\mathcal{L}$-structure $\mathbb{G}$ is a *projective Fraïssé limit* of a projective Fraïssé family $\mathcal{G}$ if the following three conditions hold:

(L1) (the projective universality) for any $A \in \mathcal{G}$ there is an epimorphism from $\mathbb{G}$ onto A;

(L2) for any finite discrete topological space X and any continuous function $f : \mathbb{G} \to X$ there are $A \in \mathcal{G}$, an epimorphism $\phi : \mathbb{G} \to A$, and a function $f_0 : A \to X$ such that $f = f_0 \circ \phi$;

(L3) (the projective ultrahomogeneity) for any $A \in \mathcal{G}$ and any epimorphisms $\phi_1 : \mathbb{G} \to A$ and $\phi_2 : \mathbb{G} \to A$ there exists an isomorphism $\psi : \mathbb{G} \to \mathbb{G}$ such that $\phi_2 = \phi_1 \circ \psi$.

Remark 3.1. It follows from (L2) above that if $\mathbb{G}$ is the projective Fraïssé limit of $\mathcal{G}$, then every finite open cover can be *refined by an epimorphism, i.e. for every open cover* $\mathcal{U}$ of $\mathbb{G}$ there is an epimorphism $\phi : \mathbb{G} \to A$, for some $A \in \mathcal{G}$, such that for every $a \in A$, $\phi^{-1}(a)$ is contained in an open set in $\mathcal{U}$.

Theorem 3.2 *(Irwin–Solecki [9]). Let $\mathcal{G}$ be a projective Fraïssé family of finite topological $\mathcal{L}$-structures. Then:*

(1) there exists a projective Fraïssé limit of $\mathcal{G}$;
(2) any two projective Fraïssé limits of $\mathcal{G}$ are isomorphic.

Let $\mathcal{G}$ be a projective Fraïssé family of topological $\mathcal{L}$-structures and let $\mathbb{G}$ be a topological $\mathcal{L}$-structure. We say that $\mathbb{G}$ has the *extension property* (with respect to $\mathcal{G}$) if for every $A, B \in \mathcal{G}$ and epimorphisms $\phi_1 : B \to A$ and $\phi_2 : \mathbb{G} \to A$, there is an epimorphism $\psi : \mathbb{G} \to B$ such that $\phi_2 = \phi_1 \circ \psi$.

Similarly as for the (injective) Fraïssé theory, one can show the following.

Proposition 3.3. *Let $\mathcal{G}$ be a projective Fraïssé family. If a topological $\mathcal{L}$-structure $\mathbb{G}$ satisfies properties (L1) and (L2), and it has the extension property with respect to $\mathcal{G}$, then $\mathbb{G}$ is the projective Fraïssé limit of $\mathcal{G}$.*

Below we describe the projective Fraïssé family $\mathcal{F}$ that we used to construct the Lelek fan in [3].

Recall that by a *fan* we mean an undirected connected simple graph with all loops, with no cycles of the length greater than one, with a distinguished point r, called the *root*, such that all elements other than r have degree at most 2. On a fan T, there is a natural partial tree order $\preceq_T$: for $t, s \in T$ we let $s \preceq_T t$ if and only if s belongs to the path connecting t and the root. We say that t is a *successor* of s if $s \preceq_T t$ and $s \neq t$. It is an *immediate successor* if additionally there is no $p \in T$, $p \neq s, t$ with $s \preceq_T p \preceq_T t$.

A *chain* in a fan T is a subset of T on which the order $\preceq_T$ is linear. A *branch* of a fan T is a maximal chain in $(T, \preceq_T)$. If b is a branch in T, we will sometimes write $b = (b^0, \ldots, b^n)$, where b^0 is the root of T, and b^i is an immediate successor of b^{i-1}, for every $i = 1, 2, \ldots, n$. In that case, n will be called the *height* of the branch b.

Let $\mathcal{L} = \{R\}$ be the language with R a binary relation symbol. For $s, t \in T$ we let $R^T(s,t)$ if and only if $s = t$ or t is an immediate successor of s. Let $\mathcal{F}$ be the family of all finite fans with all branches of the same height, viewed as topological $\mathcal{L}$-structures, equipped with the discrete topology. Every fan in $\mathcal{F}$ is specified by the height of its branches and its width, that is, the number of its branches.

Remark 3.4. For two fans (S, R^S) and (T, R^T) in $\mathcal{F}$, a function $\phi : (S, R^S) \to (T, R^T)$ is an epimorphism if and only if it is a surjective homomorphism, i.e. for every $s_1, s_2 \in S$, $R^S(s_1, s_2)$ implies $R^T(\phi(s_1), \phi(s_2))$.

We list a few relevant results obtained in [3].

Proposition 3.5. *The family $\mathcal{F}$ is a projective Fraïssé family.*

By Theorem 3.2, there exists a unique Fraïssé limit of $\mathcal{F}$, which we denote by $\mathbb{L} = (\mathbb{L}, R^{\mathbb{L}})$. Let $R^{\mathbb{L}}_S$ be the symmetrization of $R^{\mathbb{L}}$, that is, $R^{\mathbb{L}}_S(s,t)$ if and only if $R^{\mathbb{L}}(s,t)$ or $R^{\mathbb{L}}(t,s)$, for $s, t \in \mathbb{L}$.

Theorem 3.6. *The relation $R^{\mathbb{L}}_S$ is an equivalence relation which has only one and two element equivalence classes.*

Theorem 3.7. *The quotient space $\mathbb{L}/R^{\mathbb{L}}_S$ is homeomorphic to the Lelek fan L.*

We denote by $\mathrm{Aut}(\mathbb{L})$ the group of all automorphisms of $\mathbb{L}$, that is, the group of all homeomorphisms of $\mathbb{L}$ that preserve the relation $R^{\mathbb{L}}$. This is a topological group when equipped with the compact-open topology inherited from $H(\mathbb{L})$, the group of all homeomorphisms of the Cantor set underlying the structure $\mathbb{L}$. Since $R^{\mathbb{L}}$ is closed in $\mathbb{L} \times \mathbb{L}$, the group $\mathrm{Aut}(\mathbb{L})$ is closed in $H(\mathbb{L})$.

Note that every $h \in \mathrm{Aut}(\mathbb{L})$ induces a homeomorphism $h^* \in H(L)$ satisfying $h^* \circ \pi(x) = \pi \circ h(x)$ for $x \in \mathbb{L}$. We will frequently identify $\mathrm{Aut}(\mathbb{L})$ with the corresponding subgroup $\{h^* : h \in \mathrm{Aut}(\mathbb{L})\}$ of $H(L)$. Observe that the compact-open topology on $\mathrm{Aut}(\mathbb{L})$ is finer than the topology on $\mathrm{Aut}(\mathbb{L})$ that is inherited from the compact-open topology on $H(L)$.

3.2. Ultrafilters

In this section, we introduce the notion of an ultrafilter, which we will use in Section 4.

Definition 3.8. Let X be a set and let $\mathcal{E}$ be a family of subsets of X. We say that $\mathcal{E}$ is a *filter* on X if

(1) whenever $A \in \mathcal{E}$ and $B \supset A$, then also $B \in \mathcal{E}$ and
(2) for every $A, B \in \mathcal{E}$ also $A \cap B \in \mathcal{E}$.

The family $\mathcal{E}$ is an *ultrafilter* if in addition

(3) for every $A \subset X$ either $A \in \mathcal{E}$ or $X \setminus A \in \mathcal{E}$ (but not both).

An ultrafilter is *free* if it does not contain a singleton.

Remark 3.9. Note that for every ultrafilter $\mathcal{E}$ on X, $A \in \mathcal{E}$ and a partition of A into $A_1, \ldots, A_n$, there is exactly one $i = 1, \ldots, n$ such that $A_i \in \mathcal{E}$.

Remark 3.10. Any family satisfying the condition (2) in Definition 3.8 can be extended to a filter by simply adding all supersets, and every filter can be extended to an ultrafilter by Zorn's lemma.

4. Dualization of the Kechris–Pestov–Todorčević correspondence

In this section, we prove Theorem 2.2 that dualizes the Kechris–Pestov–Todorčević correspondence between extreme amenability of automorphism groups of countable ultrahomogeneous linearly ordered structures and the structural Ramsey theory (Theorem 4.5 in [10]), which was further extended by Nguyen van Thé (Theorem 1 in [17]) to structures that need not be linearly ordered.

Let $\mathcal{G}$ be a projective Fraïssé family with the projective Fraïssé limit $\mathbb{G}$. Let $G = \mathrm{Aut}(\mathbb{G})$ be the automorphism group of $\mathbb{G}$ equipped with the compact-open topology.

We first prove an analogue of Proposition 3 in [17].

Proposition 4.1. *Suppose that $\mathcal{G}$ is rigid. Then the following are equivalent.*

(1) *The class $\mathcal{G}$ is a Ramsey class.*
(2) *For every $A, B \in \mathcal{G}$ and every colouring $c : \binom{\mathbb{G}}{A} \to \{1, 2, \ldots, r\}$ there exists $\psi \in \binom{\mathbb{G}}{B}$ such that $\binom{B}{A} \circ \psi$ is monochromatic.*

Proof. Since $\mathbb{G}$ is projectively universal, (1) easily implies (2).

For the reverse implication, suppose that (2) holds, but there are $A, B \in \mathcal{G}$ for which the Ramsey property fails, i.e. for every $C \in \mathcal{G}$ there exists a colouring $\chi_C : \binom{C}{A} \to \{1, 2, \ldots, r\}$ such that for no $\gamma \in \binom{C}{B}$ the set $\binom{B}{A} \circ \gamma$ is monochromatic.

We first show that there is a free ultrafilter $\mathcal{U}$ on $\bigcup_{D \in \mathcal{G}} \binom{\mathbb{G}}{D}$ such that for every $D \in \mathcal{G}$ and every $\phi \in \binom{\mathbb{G}}{D}$ we have

$$K_\phi = \bigcup_{C \in \mathcal{G}} \{\psi : \psi \in \binom{\mathbb{G}}{C}\ \exists \psi' \in \binom{C}{D} \text{ such that } \ \phi = \psi' \circ \psi\} \in \mathcal{U}.$$

Suppose that we have $\phi_0 : \mathbb{G} \to D_0$ and $\phi_1 : \mathbb{G} \to D_1$. By (L2), we can find an $E \in \mathcal{G}$ and an epimorphism $\psi : \mathbb{G} \to E$ such that $\{\psi^{-1}(e) : e \in E\}$ refines both

$\{\phi_i^{-1}(d) : d \in D_i\}$ for $i = 1, 2$. Then clearly $K_\psi \subset K_{\phi_0} \cap K_{\phi_1}$, so by Remark 3.10 such a $\mathcal{U}$ exists.

Now, for $\phi \in \binom{\mathbb{G}}{A}$, we will write K_ϕ as a disjoint union $K_\phi^1 \cup K_\phi^2 \cup \ldots \cup K_\phi^r$ where

$$K_\phi^\varepsilon = \bigcup_{C \in \mathcal{G}} \{\psi \in \binom{\mathbb{G}}{C} : \exists \psi' \in \binom{C}{A} \text{ such that } (\psi' \circ \psi = \phi) \ \& \ (\chi_C(\psi') = \varepsilon)\},$$

for $\varepsilon = 1, 2, \ldots, r$. Note that since ψ is surjective, then ψ' if it exists, it is unique. We define a colouring $c : \binom{\mathbb{G}}{A} \to \{1, 2, \ldots, r\}$ by $c(\phi) = \varepsilon$ if and only if $K_\phi^\varepsilon \in \mathcal{U}$. Note that c is well defined by Remark 3.9. We claim that for no $\delta \in \binom{\mathbb{G}}{B}$, the collection $\binom{B}{A} \circ \delta$ is c-monochromatic. Suppose on the contrary that there is a δ such that $\binom{B}{A} \circ \delta$ is c-monochromatic in a colour ε_0. Then the set

$$\bigcap_{\alpha \in \binom{B}{A}} K_{\alpha \circ \delta}^{\varepsilon_0} \cap K_\delta$$

belongs to $\mathcal{U}$; in particular it is nonempty containing an element $\xi \in \binom{\mathbb{G}}{C}$ for some $C \in \mathcal{G}$. Since $\xi \in K_\delta$, that there is $\xi' \in \binom{C}{B}$ such that $\delta = \xi' \circ \xi$. Therefore for every $\alpha \in \binom{B}{A}$, we have $\alpha \circ \delta = (\alpha \circ \xi') \circ \xi$. Since $\xi \in K_{\alpha \circ \delta}^{\varepsilon_0}$, we obtain that $\chi_C(\alpha \circ \xi') = \varepsilon_0$ for every $\alpha \in \binom{B}{A}$. This implies that the set $\binom{B}{A} \circ \xi'$ is χ_C-monochromatic in the colour ε_0 – a contradiction. $\square$

To prove Theorem 2.2, we follow the approach to extreme amenability via syndetic sets from the dissertation [2] of the first author.

For $A \in \mathcal{G}$ and an epimorphism $\phi : \mathbb{G} \to A$, let

$$G_\phi = \{g \in G : \forall a \in A \ g(\phi^{-1}(a)) = \phi^{-1}(a)\}$$

be the pointwise stabilizer of ϕ. Equivalently, $G_\phi = \{h \in G : \phi \circ h = \phi\}$. It is easy to see that G_ϕ is an open, and therefore clopen subgroup of G. The collection

$$\{G_\phi : \phi \in \binom{\mathbb{G}}{A}, A \in \mathcal{G}\}$$

forms a basis at the identity of G.

Note that for every $g \in G$, $G_\phi g = \{h : \forall a \in A \ h^{-1}(\phi^{-1}(a)) = g^{-1}(\phi^{-1}(a))\}$, that is, $h \in G_\phi g$ if and only if $\phi \circ h = \phi \circ g$. Projective ultrahomogeneity of $\mathbb{G}$ provides a natural bijective identification of G/G_ϕ with $\binom{\mathbb{G}}{A}$ via $G_\phi g \mapsto \phi \circ g$. Similarly, $gG_\phi = \{h \in G : \forall a \in A \ h(\phi^{-1}(a)) = g(\phi^{-1}(a))\}$, that is, $h \in gG_\phi$ if and only if $\phi \circ h^{-1} = \phi \circ g^{-1}$.

We also introduce the setwise stabilizer $G_{(\phi)}$ of ϕ, that is, the clopen subgroup

$$G_{(\phi)} = \{h : h(\{\phi^{-1}(a) : a \in A\}) = \{\phi^{-1}(a) : a \in A\}\}.$$

By the projective ultrahomogeneity of $\mathbb{G}$, $h \in G_{(\phi)}$ if and only if for some automorphism ψ of A we have $\phi \circ h = \phi \circ \psi$.

Definition 4.2. A subset A of a group G is called *syndetic* if there exist finitely many $g_1, \dots, g_n \in G$ such that $\bigcup_{i=1}^n g_i A = G$.

The following lemma characterizes extreme amenability in terms of syndetic sets.

Lemma 4.3 *(Bartošová [2], Lemma 11). A topological group G is extremely amenable if and only if for every pair A, B of syndetic subsets of G and every open neighbourhood V of the identity in G we have $VA \cap VB \neq \emptyset$.*

In the lemma above, it is sufficient to only consider open sets V taken from a neighbourhood basis of the identity in G. Since for an epimorphism $\phi : \mathbb{G} \to A$ we have $G_\phi(G_\phi A) = G_\phi A$, we immediately obtain the following equivalence.

Lemma 4.4. *Let G be a topological group that admits a neighbourhood basis at the identity consisting of open subgroups. Then the following are equivalent.*

(1) *G is extremely amenable.*
(2) *For every clopen subgroup H of G and every $K \subset G$, at most one of HK and $G \backslash HK$ is syndetic.*

Proof of Theorem 2.2. (1) $\Rightarrow$ (2) We first prove that $\mathcal{G}$ is rigid. Let $A \in \mathcal{G}$ and pick an epimorphism $\phi : \mathbb{G} \to A$ by projective universality. Then $G_{(\phi)}/G_\phi$ is a finite discrete space of cardinality $\mathrm{Aut}(A)$ with a natural transitive continuous action of $G_{(\phi)}$ given by $g(G_\phi h) = G_\phi h g^{-1}$. Being an open subgroup of G, $G_{(\phi)}$ is extremely amenable by Lemma 13 in [4], and therefore $|\mathrm{Aut}(A)| = |G_{(\phi)}/G_\phi| = 1$.

Secondly, we show that $\mathcal{G}$ is a Ramsey class. Let $A \in \mathcal{G}$ and let $c : \binom{\mathbb{G}}{A} \to \{1, 2, \dots, r\}$ be a colouring. We view c as a point in the compact space $X = \{1, 2, \dots, r\}^{\binom{\mathbb{G}}{A}}$ of all colourings of $\binom{\mathbb{G}}{A}$ by r colours. We consider X with the natural action of G given by $g \cdot d(\phi) = d(\phi \circ g^{-1})$. Let Y be the closure of the orbit of c. Since G is extremely amenable, the induced action of G on Y has a fixed point e. By the projective ultrahomogeneity of $\mathbb{G}$, G acts transitively on $\binom{\mathbb{G}}{A}$, and consequently e must be constant, say with the range $\{i\} \subset \{1, 2 \dots, r\}$. Let $B \in \mathcal{G}$ and pick a $\gamma \in \binom{\mathbb{G}}{B}$, which exists by the projective universality of $\mathbb{G}$. Since $e \in \overline{Gc}$, there is $g \in G$ such that $c \restriction \binom{B}{A} \circ \gamma \circ g = e \restriction \binom{B}{A} \circ \gamma$, and therefore c on $\binom{B}{A} \circ (\gamma \circ g)$ is constant. Since $\mathcal{G}$ is rigid, Proposition 4.1 concludes the proof.

(2) $\Rightarrow$ (1) Striving for a contradiction, suppose that G is not extremely amenable. In the light of Lemma 4.4, it means that there are $A \in \mathcal{G}$, an epimorphism $\phi : \mathbb{G} \to A$ and $K_0, K_1 \subset G$ such that both $G_\phi K_0$ and $G \backslash G_\phi K_0 = G_\phi K_1$ are syndetic. Let $g_1, \dots, g_n \in G$ witness syndeticity of both, i.e.

$$\bigcup_{i=1}^{n} g_i G_\phi K_0 = G = \bigcup_{i=1}^{n} g_i G_\phi K_1.$$

Let $\phi_i : \mathbb{G} \to A$ be given by $\phi_i = \phi \circ g_i^{-1}$. Since $\mathcal{G}$ is rigid, we can apply the property (L2) to a disjoint clopen refinement of the cover $\{\phi_i^{-1}(a) : a \in A, i = 1, \ldots, n\}$ of $\mathbb{G}$ and find $B \in \mathcal{G}$, an epimorphism $\gamma : \mathbb{G} \to B$ and surjections $\gamma_i : B \to A$ such that $\phi_i = \gamma_i \circ \gamma$. Since γ and ϕ_i's are epimorphisms, so are γ_i's.

Define a colouring $c : \binom{\mathbb{G}}{A} \to \{0, 1\}$ by $c(\psi) = \varepsilon$ if and only if whenever k satisfies $\phi \circ k = \psi$ we have $k \in G_\phi K_\varepsilon$. Let us remark that c is well-defined as $\phi \circ k = \phi \circ l$ if and only if $G_\phi k = G_\phi l$. By the Ramsey property, there is an epimorphism $\gamma' : \mathbb{G} \to B$ such that $\binom{B}{A} \circ \gamma'$ is monochromatic in a colour $\varepsilon_0 \in \{0, 1\}$, in particular, $c(\gamma_i \circ \gamma') = \varepsilon_0$ for every i. Since $\mathbb{G}$ is projectively ultrahomogeneous, there is $g \in G$ such that $\gamma' = \gamma \circ g$. We have that

$$\gamma_i \circ \gamma' = \gamma_i \circ \gamma \circ g = \phi_i \circ g = \phi \circ g_i^{-1} g,$$

which implies $g_i^{-1} g \in G_\phi K_{\varepsilon_0}$ and consequently $g \in g_i G_\phi K_{\varepsilon_0}$ for every i. It means that $g \notin \bigcup_{i=1}^{n} g_i G_\phi K_{1-\varepsilon_0}$, which is a contradiction. $\square$

5. $\mathcal{F}_<$ is a Ramsey class

The goal of this section is to prove Theorem 2.1. We will use Theorem 2.8, which will be proved in the next section.

Let $\mathcal{G}$ be a projective Fraïssé family. Recall that $\binom{B}{A}$ is the set of all epimorphisms from B onto A, and denote by $\mathcal{G}^{[2]}$ the set of all ordered pairs (A, B) of elements in $\mathcal{G}$ such that $\binom{B}{A} \neq \emptyset$. An ordered pair $(A, B) \in \mathcal{G}^{[2]}$ is a *Ramsey pair* if there exists $C \in \mathcal{G}$ such that for every colouring $c : \binom{C}{A} \to \{1, 2, \ldots, r\}$ there is $g \in \binom{C}{B}$ such that

$$\binom{B}{A} \circ g = \left\{ h \circ g : h \in \binom{B}{A} \right\}$$

is c-monochromatic. Note that the class $\mathcal{G}$ is a Ramsey class if every pair $(A, B) \in \mathcal{G}^{[2]}$ is a Ramsey pair.

As in Section 3.1, let $\mathcal{L}$ be the language consisting of one binary relation symbol R and let $\mathcal{F}$ denote the family of finite fans considered as $\mathcal{L}$-structures. Given $A \in \mathcal{F}$, we always keep in mind the underlying natural partial order $\preceq_A$.

Take the language $\mathcal{L}_< = \{R, S\}$ expanding $\mathcal{L}$ with one binary relation symbol S and let $\mathcal{F}_<$ be the family of all $A_< = (A, R^A, S^A)$ such that $(A, R^A) \in \mathcal{F}$ and for some ordering $a_1 < a_2 < \ldots < a_n$ of branches in A we have $S^A(x, y)$ if and only if there are $i \leq j$ such that $x \in a_i$ and $y \in a_j$. Note that the root r of A belongs to every branch so $S^A(r, x)$ for every $x \in A$ and whenever x, y belong the same branch we have $S^A(x, y)$. Observe that S^A extends the natural partial order on A.

We will frequently use the following lemma. Its proof is straightforward.

Lemma 5.1. *Let $A_<, B_< \in \mathcal{F}_<$ and let $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_n$ be the increasing enumerations of the branches in A and B, respectively. A function $f : B_< \to A_<$ is an epimorphism if and only if $f : B \to A$ is an epimorphism and there exist $1 = k_1 < \ldots < k_{m+1} = n + 1$ such that for every $i = 1, \ldots, n$ and $s = 1, \ldots, m$, if $k_s \leq i < k_{s+1}$ then $f(b_i) \subset a_s$.*

Let us start with the following special case of Theorem 2.1.

Lemma 5.2. *Let $A, B \in \mathcal{F}_<$ both consist of a single branch. If $(A, B) \in \mathcal{F}^{[2]}$, then (A, B) is a Ramsey pair.*

We will see in a moment that Lemma 5.2 is essentially a reformulation of the classical Ramsey theorem. We let $N^{[j]}$ denote the collection of all j-element subsets of $\{1, \ldots, N\}$. We will often write N instead of $N^{[1]}$.

Theorem 5.3 *(Ramsey). Let k, l, r be natural numbers. Then there exists a natural number N such that for every colouring $c : N^{[k]} \to \{1, 2, \ldots, r\}$ there exists a subset X of N of size l such that $X^{[k]}$ is c-monochromatic. Denote by $R(k, l, r)$ the minimal such N.*

Proof of Lemma 5.2. Suppose that A has $k+1$ vertices $r_A = a^0 \prec_A \ldots \prec_A a^k$, B has $l+1$ vertices $r_B = b^0 \prec_B \ldots \prec_B b^l$, and r is given. Let $N = R(k, l, r)$ and let $C \in \mathcal{F}_<$ consist of a single branch with $N+1$ vertices $r_C = c^0 \prec_C \ldots \prec_C c^N$. Every epimorphism $f \in \binom{C}{A}$ can be identified with a k-element subset of C via $\{\min\{f^{-1}(a^i)\} : i = 1, \ldots, k\}$, where the min is taken with respect to $\prec_C$. Therefore any colouring $c : \binom{C}{A} \to \{1, 2, \ldots, r\}$ induces a colouring $d : N^{[k]} \to \{1, 2, \ldots, r\}$. Let $X = \{x^1 \prec \ldots \prec x^l\} \in N^{[l]}$ be such that d restricted to $X^{[k]}$ is constant. Define an epimorphism $\phi : C \to B$ by $\phi(c^i) = b^j$ if $x^j \preceq i \prec x^{j+1}$ and $\phi(c^i) = b^0$ if $i \prec x^1$. Identifying $\binom{B}{A}$ with k-element subsets of B in the same manner as above, we can deduce that $\binom{B}{A} \circ \phi$ corresponds to all l-element subsets of X and therefore is monochromatic. $\square$

For a natural number N let $N^{[\leq j]}$ denote the collection of all at most j-element subsets of $\{1, \ldots, N\}$. Note that $N^{[\leq j]} = \bigcup_{i=0}^{j} N^{[j]}$. Let m, r be natural numbers and $k_1, \ldots, k_m$ be non-negative integers and let

$$c : \prod_{i=1}^{m} N^{[\leq k_i]} \to \{1, 2, \ldots, r\}$$

be a colouring. Given $B_i \subset N$, $i = 1, 2, \ldots, m$, we say that c is *size-determined* on $(B_i)_{i=1}^{m}$ if whenever $A_i, A_i' \subset B_i$ with $0 \leq |A_i| = |A_i'| \leq k_i$ for $i = 1, 2, \ldots, m$, then

$$c(A_1, \ldots, A_m) = c(A_1', \ldots, A_m').$$

For $f \in \prod_{i=1}^m N^{[\leq k_i]}$, define $\mathrm{supp}(f) = \{i : f(i) \neq \emptyset\}$. Given a natural number $d \leq m$, let $\left(\prod_{i=1}^m N^{[\leq k_i]}\right)^{[d]}$ be the set of all sequences $(f_s)_{s=1}^d$ with $f_s \in \prod_{i=1}^m N^{[\leq k_i]}$ and $\max(\mathrm{supp}(f_s)) < \min(\mathrm{supp}(f_{s+1}))$, for $s < d$. Then, more generally, if

$$\chi : \left(\prod_{i=1}^m N^{[\leq k_i]}\right)^{[d]} \to \{1, 2, \dots, r\}$$

is a colouring and $B_i \subset N$ for $i = 1, 2, \dots, m$, we say that χ is *size-determined* on $(B_i)_{i=1}^m$ if whenever $(f_s)_{s=1}^d$ and $(g_s)_{s=1}^d$ are such that $\mathrm{supp}(f_s) = \mathrm{supp}(g_s)$, $|f_s(i)| = |g_s(i)|$ and $f_s(i), g_s(i) \subset B_i$ for every $s \leq d$ and $i \leq m$, then

$$\chi\left((f_s)_{s=1}^d\right) = \chi\left((g_s)_{s=1}^d\right).$$

At the end of this section, we will prove the following theorem, whose corollary essentially reduces Theorem 2.1 to Theorem 2.8.

Theorem 5.4. *Let m, r be natural numbers and let $k_1, \dots, k_m, l_1, \dots, l_m$ be non-negative integers such that $k_i \leq l_i$ for every $i = 1, 2, \dots, m$. Then there exists N such that for every colouring*

$$c : \prod_{i=1}^m N^{[\leq k_i]} \to \{1, 2, \dots, r\}$$

there are $B_1, \dots, B_m \subset N$ with $|B_i| = l_i$ such that c is size-determined on $(B_i)_{i=1}^m$. Denote by $S(m, k_1, \dots, k_m, l_1, \dots, l_m, r)$ the minimal such N.

We are almost ready to prove Theorem 2.1 that the class $\mathcal{F}_<$ is a Ramsey class. We will use Corollary 5.5 to reduce the proof to an application of Theorem 2.8. Corollary 5.5 is a multidimensional version of Theorem 5.4.

Corollary 5.5. *Let d, m, r be natural numbers and let $k_1, \dots, k_m, l_1, \dots, l_m$ be non-negative integers such that $k_i \leq l_i$ for every $i = 1, 2, \dots, m$. Then there exists N such that for every colouring*

$$\chi : \left(\prod_{i=1}^m N^{[\leq k_i]}\right)^{[d]} \to \{1, 2, \dots, r\}$$

there are $B_1, \dots, B_m \subset N$ with $|B_i| = l_i$ such that χ is size-determined on $(B_i)_{i=1}^m$. Denote by $S_d(k_1, \dots, k_m, l_1, \dots, l_m, m, r)$ the minimal such N.

Proof. Let $\Gamma = \{\gamma = (\gamma(1), \dots, \gamma(d+1)) \in \mathbb{N}^{d+1} : \gamma(1) = 1 < \dots < \gamma(d+1) = m\}$. To $\gamma \in \Gamma$ and $(A_1, \dots, A_m) \in \prod_{i=1}^m N^{[\leq k_i]}$, we associate

$$\gamma_{(A_1,\dots,A_m)} = ((A_1,\dots,A_{\gamma(2)-1}),\dots,(A_{\gamma(d)},\dots,A_m)) \in \prod_{i=1}^{m} \left(N^{[\leq k_i]}\right)^{[d]},$$

where for $1 \leq i < j \leq m$ by $(A_i,\dots,A_j)$ we mean the function supported on $[i,j]$ with the respective values $A_i,\dots,A_j$.

Given $\chi : (\prod_{i=1}^{m} N^{[\leq k_i]})^{[d]} \to \{1,2,\dots,r\}$, we define $c : \prod_{i=1}^{m} N^{[\leq k_i]} \to \{1,2,\dots,r\}^{\Gamma}$ by

$$c(A_1,\dots,A_m) = (\chi(\gamma_{(A_1,\dots,A_m)}))_{\gamma\in\Gamma}.$$

Applying Theorem 5.4, we get $B_1,\dots,B_m \subset N$ with $|B_i| = l_i$ such that c is size-determined on $(B_i)_{i=1}^m$. Since whenever $\gamma_{(A_1,\dots,A_m)} = \gamma'_{(A_1,\dots,A_m)}$ we have $c(A_1,\dots,A_m)(\gamma) = c(A_1,\dots,A_m)(\gamma')$, it follows that also χ is size-determined on $(B_i)_{i=1}^m$. $\square$

Proof of Theorem 2.1. Let $S \in \mathcal{F}_<$ be of height k and width d, and let $T \in \mathcal{F}_<$ be of height $l \geq k$ and width $m \geq d$ (so that $\binom{T}{S} \neq \emptyset$). Let $n = G_d(k,l,m,r)$ be as in Theorem 2.8 and let $N = S_d(n,k,\dots,k,l,\dots,l,r)$ be as in Corollary 5.5. Let $U \in \mathcal{F}_<$ consists of n branches of height N. We will show that U works for S,T and r colours.

Let $a_1,\dots,a_d$ and $c_1,\dots,c_n$ be the increasing enumerations of branches in S and U respectively. Let $(a_j^i)_{i=0}^k$ be the increasing enumeration of the branch a_j, $j = 1,\dots,d$, and let $(c_j^i)_{i=0}^N$ be the increasing enumeration of the branch c_j for $j = 1,\dots,n$.

To each $f \in \binom{U}{S}$, we associate $f^* = (p_i^f)_{i=1}^d \in \mathrm{FIN}_k^{[d]}(n)$ such that

$$\mathrm{supp}(p_i^f) = \{j : a_i^1 \in f(c_j)\}$$

and for $j \in \mathrm{supp}(p_i^f)$

$$p_i^f(j) = z \iff f(c_j^N) = a_i^z.$$

We moreover associate to f a block sequence of functions $(F_i^f)_{i=1}^d \in (\prod_{j=1}^n (c_j \setminus \{c_j^0\})^{[\leq k]})^{[d]}$ to fully code f as follows. For $j \in \mathrm{supp}(p_i^f)$, we let

$$F_i^f(j) = \{\min\{c_j^y \in c_j : f(c_j^y) = a_i^x\} : 0 \prec x \preceq p_i^f(j)\},$$

where the min is taken with respect to the partial order on the fan U. Note that $\mathrm{supp}(p_i^f) = \mathrm{supp}(F_i^f)$. By definition $p_i^f(j) = |F_i^f(j)|$ and since f is onto, for each i there is a j such that $p_i^f(j) = k$. Therefore if $f_1^* = f_2^*$, then $|F_i^{f_1}(j)| = |F_i^{f_2}(j)|$ for all i,j.

Similarly, to any $g \in \binom{U}{T}$, we associate $g^* \in \mathrm{FIN}_l^{[m]}(n)$ and $(F_i^g)_{i=1}^m \in (\prod_{j=1}^n (c_j \setminus \{c_j^0\})^{[\leq l]})^{[m]}$.

Let $c : \binom{U}{S} \to \{1,\dots,r\}$ be given. Let c_0 be a colouring of $(\prod_{j=0}^n (c_j \setminus \{c_j^0\})^{[\leq k]})^{[d]}$ induced by c via the injection $f \mapsto (F_i^f)_{i=1}^d$, colouring elements not of the form $(F_i^f)_{i=1}^d$

in an arbitrary way. We first apply Corollary 5.5 to find $C_j \subset c_j \setminus \{c_j^0\}$ of size l for $j = 1, \ldots, n$ such that c_0 is size-determined on $(C_j)_{j=1}^n$. It follows that the colouring $c^* : \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$ given by $c^*(f^*) = c(f)$ for $f \in \binom{U}{S}$ which satisfy $(F_i^f)_{i=1}^d \in (\prod_{j=1}^n C_j^{[\leq k]})^{[d]}$ is well-defined. Second, we apply Theorem 2.8 to obtain a block sequence $D = (d_j)_{j=1}^m$ in $\mathrm{FIN}_l^{[m]}(n)$ such that $\left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(D) \right\rangle_{P_k}^{[d]}$ is c^*-monochromatic.

Let $g \in \binom{U}{T}$ be any epimorphism such that $g^* = D$ and $(F_i^g)_{i=1}^m \in (\prod_{j=1}^n C_j^{[\leq l]})^{[m]}$. Then for every $h \in \binom{T}{S}$, we have $(h \circ g)^* \in \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(D) \right\rangle_{P_k}^{[d]}$ and $(F_i^{h \circ g})_{i=1}^d \in (\prod_{j=1}^n C_j^{[\leq k]})^{[d]}$. Since $\left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(D) \right\rangle_{P_k}^{[d]}$ is c^*-monochromatic, we can conclude that $\binom{T}{S} \circ g$ is c-monochromatic. $\square$

Proof of Theorem 5.4. Let m, r be natural numbers and $k_1, \ldots, k_m, l_1, \ldots, l_m$ be fixed non-negative integers such that $k_i \leq l_i$ for every $i = 1, 2, \ldots, m$. We proceed by a double induction on $m' < m$ and on $k' \leq k_{m'}$, where at each step we apply Theorem 5.3. We prove the following statement: Given $1 \leq m' \leq m$ and $0 \leq k' \leq k_{m'}$, there exists N such that for every colouring

$$c : \prod_{i=1}^{m'-1} N^{[\leq k_i]} \times N^{[\leq k']} \to \{1, \ldots, r\}$$

there are $B_1, \ldots, B_{m'} \subset N$ with $|B_i| = l_i$ such that c is size-determined on $(B_i)_{i=1}^{m'}$.

When $m' = 1$ and $k' = 0$, there is nothing to prove. Let $m' = 1$ and assume that the statement of the theorem holds for $0 \leq k' < k_1$, we will prove it for $k' + 1$. Let $N' = S(1, k', l_1, r)$ and $N = R(k' + 1, N', r)$. Let

$$c : N^{[\leq (k'+1)]} \to \{1, 2, \ldots, r\}$$

be a given colouring. Consider the restricted colouring $d = c \restriction_{N^{[k'+1]}}$ and apply Theorem 5.3 to find $B \subset N$ of size N' such that $B^{[k'+1]}$ is d-monochromatic. By the inductive hypothesis applied to $c \restriction B^{[\leq k']}$, we obtain the desired B_1.

Suppose that the statement of the theorem is true for $m' - 1$ and we shall prove it for m'. When $k' = 0$, simply take $N = S(m' - 1, k_1, \ldots, k_{m'-1}, l_1, \ldots, l_{m'-1}, r)$. Assume that the result is true for $k' < k_{m'}$, and we will prove it for $k' + 1$. Set

$$N' = S(m', k_1, \ldots, k_{m'-1}, k', l_1, \ldots, l_{m'-1}, l_{m'}, r)$$

and

$$N'' = S(m' - 1, k_1, \ldots, k_{m'-1}, N', \ldots, N', r).$$

Denote by α the set of all colourings of $\prod_{i=1}^{m'-1} N''^{[\leq k_i]}$ with colours $1, \ldots, r$. Let

$$N = \max\{N'', R(k'+1, N', |\alpha|)\}.$$

We will show that N works.

Let

$$c : \left(\prod_{i=1}^{m'-1} N^{[\leq k_i]}\right) \times \left(N^{[\leq (k'+1)]}\right) \to \{1, 2, \dots, r\}$$

be an arbitrary colouring. For every $A \subset N$ of size $k'+1$, let c_A be the colouring of $\prod_{i=1}^{m'-1} N''^{[\leq k_i]}$ induced by c and A in the last coordinate, i.e.

$$c_A(A_1, \dots, A_{m'-1}) = c(A_1, \dots, A_{m'-1}, A).$$

Define a colouring

$$d : N_{m'}^{[k'+1]} \to \alpha$$

by $d(A) = c_A$.

By Theorem 5.3, there is a subset $B'_{m'} \subset N$ of size N' such that $B'^{[k'+1]}_{m'}$ is d-monochromatic in a colour $c_0 : \left(\prod_{i=1}^{m'-1} N''^{[\leq k_i]}\right) \to \{1, 2, \dots, r\}$. Applying the induction hypothesis for $m'-1, k_1, \dots, k_{m'-1}, N', \dots, N', r$, we obtain $B'_i \subset N''_i$ of size N' for $i = 1, 2, \dots, m'-1$ such that c_0 is size-determined on $(B'_i)_{i=1}^{m'-1}$.

Finally, we define

$$b = c \restriction_{\left(\prod_{i=1}^{m'-1} B_i'^{[\leq k_i]}\right) \times \left(B'^{[\leq k']}_{m'}\right)} .$$

Using the induction hypothesis for $m'-1, k_1, \dots, k_{m'-1}, k', l_1, \dots, l_{m'}, r$, we obtain $B_i \subset B'_i$ for $i = 1, 2, \dots, m'$ such that $|B_i| = l_i$ and b and therefore c are size-determined on $(B_i)_{i=1}^{m'}$. $\square$

6. Gowers' Ramsey theorem for multiple operations

In this section, we provide a proof by induction of our main Ramsey result, Theorem 2.8. In order to perform the induction, we generalize Tyros' notions of a type and of a pyramid in $\mathrm{FIN}_k(n)$ to sequences in $\mathrm{FIN}_k^{[d]}(n)$.

Let $A = (a_i)_{i=1}^m$ be a block sequence in FIN_1. We can identify each a_i with the characteristic function $\chi(a_i)$ of its support. We define

$$\mathrm{FIN}_k(A) = \left\{ \sum_{i=1}^m j_i \cdot \chi(a_i) : j_i \in \{0, 1, \dots, k\} \ \& \ \exists i \ (j_i = k) \right\}.$$

Let $\mathrm{FIN}_k^{[d]}(A)$ denote the set of all block sequences in $\mathrm{FIN}_k(A)$ of length d.

A function $\phi : \{1, \ldots, m\} \to \{1, \ldots, k\}$ is a *type of length* m *over* k if $\phi(i) \neq \phi(i+1)$ for $i = 1, 2, \ldots, m-1$, and for some i, $\phi(i) = k$. Note that $\phi \in \mathrm{FIN}_k(m)$. Let n be a natural number. For every type ϕ of length $m \leq n$ over k and every block sequence $B = (b_i)_{i=1}^m$ in $\mathrm{FIN}_1(n)$,

$$\mathrm{map}(\phi, B) = \sum_{i=1}^{m} \phi(i) \cdot \chi(b_i)$$

belongs to $\mathrm{FIN}_k(n)$.

On the other hand, for every $p \in \mathrm{FIN}_k(n)$, there exist a unique natural number $m \leq n$, a type ϕ of length m over k, and a block sequence $B = (b_i)_{i=1}^m$ in $\mathrm{FIN}_1(n)$ of length m such that

$$p = \mathrm{map}(\phi, B).$$

We call this ϕ the *type* of p and denote it by $\mathrm{tp}(p)$. We say that $p, q \in \mathrm{FIN}_k$ are of the same type if $\mathrm{tp}(p) = \mathrm{tp}(q)$.

Tyros [22] used the following lemma about types to obtain his constructive proof of the finite version of Gowers' theorem.

Lemma 6.1 *(Tyros [22]). For every triple k, m, r of natural numbers, there exists n such that for every colouring $c : \mathrm{FIN}_k(n) \to \{1, 2, \ldots, r\}$, there is a block sequence A of length m in $\mathrm{FIN}_1(n)$ such that any two elements in $\mathrm{FIN}_k(A)$ of the same type have the same colour.*

We extend the notion of a type to sequences in $\mathrm{FIN}_k^{[d]}$. We say that $\phi = (\phi_1, \ldots, \phi_d)$ is a *type of length* m *over* k if each ϕ_i is a type of length m_i over k such that $\sum_{i=1}^d m_i = m$. For $\bar{p} = (p_1, \ldots, p_d) \in \mathrm{FIN}_k^{[d]}$, we let the *type* of $\bar{p}$ be $(\mathrm{tp}(p_1), \ldots, \mathrm{tp}(p_d))$.

Note that given $d \leq n$ and $A \in \mathrm{FIN}_1^{[n]}$, there is a natural bijection between $\mathrm{FIN}_k^{[d]}(A)$ and the set of pairs (B, ϕ), where $B \in \mathrm{FIN}_1^{[m]}(A)$ and $\phi \in \prod_{i=1}^d \mathrm{FIN}_k(m_i)$, for some $m_1, \ldots, m_d$ such that $\sum_{i=1}^d m_i = m$, is a type of length m over k for some $m \leq n$. As in the case of dimension 1, the $p \in \mathrm{FIN}_k^{[d]}(A)$ that corresponds to (B, ϕ) will be denoted by $\mathrm{map}(\phi, B)$.

We will prove a multidimensional version of Lemma 6.1 using a finite version of the Milliken–Taylor theorem [13,20].

Theorem 6.2 *(Milliken–Taylor). Given natural numbers $m \geq d$ and r, there exists n with the following property: For every finite block sequence $A \in \mathrm{FIN}_1$ of length at least n and every colouring of $\mathrm{FIN}_1^{[d]}(A)$ by r colours there exists $B \in \mathrm{FIN}_1^{[m]}(A)$ such that $\mathrm{FIN}_1^{[d]}(B)$ is monochromatic. We denote the smallest such n by $MT_d(m, r)$.*

Lemma 6.3. *Let k and $d \leq m$, and r be natural numbers. Then there exists n such that for every colouring $c : \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$, there is a block sequence A in $\mathrm{FIN}_1(n)$*

of length m such that any two elements in $\mathrm{FIN}_k^{[d]}(A)$ of the same type have the same colour. We denote the smallest such n by $T_d(k,m,r)$.

Proof. Let $\mathcal{T}$ be the set of all types of sequences in $\mathrm{FIN}_k^{[d]}$ of length at most m and let α be the cardinality of the set X of all colourings of $\mathcal{T}$ by r colours. Let $n = MT_m(2m-d,\alpha)$ be as in Theorem 6.2 and let $c : \mathrm{FIN}_k^{[d]}(n) \to \{1,2,\dots,r\}$ be a colouring. Let $q : \mathrm{FIN}_1^{[m]}(n) \to \{1,\dots,r\}^{\mathcal{T}}$ be the colouring given by

$$q(B)(\phi) = c(\mathrm{map}(\phi, (b_i)_{i=1}^{l_\phi})),$$

where l_ϕ denotes length of the type ϕ, $B = (b_i)_{i=1}^m \in \mathrm{FIN}_1^{[m]}(n)$, and $\{1,\dots,r\}^{\mathcal{T}}$ denotes the set of functions from $\mathcal{T}$ to $\{1,\dots,r\}$. By Theorem 6.2, we can find a block sequence A' of length $2m-d$ such that $\mathrm{FIN}_1^{[m]}(A')$ is q-monochromatic and let A be the initial segment of A' of length m. We will show that A is as desired. Indeed, let $\bar{p}_1, \bar{p}_2 \in \mathrm{FIN}_k^{[d]}(A)$ be of the same type ϕ, and let A_1, A_2 be the block sequences in $\mathrm{FIN}_1(A)$ for which $\bar{p}_1 = \mathrm{map}(\phi, A_1)$ and $\bar{p}_2 = \mathrm{map}(\phi, A_2)$. Since ϕ has length between d and m and since A is an initial segment of $A' \in \mathrm{FIN}_k^{[2m-d]}(n)$, we can choose $A_1', A_2' \in \mathrm{FIN}_1^{[m]}(A')$ such that A_1 is an initial segment of A_1' and A_2 is an initial segment of A_2'. It follows that

$$c(\bar{p}_1) = q(A_1')(\phi) = q(A_2')(\phi) = c(\bar{p}_2). \quad \square$$

Another piece needed for the induction in the proof of Theorem 2.8 is a pair of two lemmas capturing how T_1 commutes with T_i's. The proof of the first lemma is an immediate calculation.

Lemma 6.4.

(1) If $1 \le j < l$ and $p \in \mathrm{FIN}_l$, we have $T_j \circ T_1(p) = T_1 \circ T_{j+1}(p)$ and $T_0 \circ T_1 = T_1 \circ T_0$.
(2) For $\vec{t} \in P_k$, $p \in \mathrm{FIN}_l$, $T_{\vec{t}} \circ T_1(p) = T_1 \circ T_{\vec{t}+1}(p)$, where $\vec{t}+1 \in P_{k+1}$ is such that

$$(\vec{t}+1)(1) = 0$$
$$(\vec{t}+1)(x+1) = \vec{t}(x) + 1 \text{ when } 1 \le x \le k.$$

(3) If $\vec{t} = \vec{j}^\frown\vec{i}$ with $\vec{j} \in P_{k-1}$ and $\vec{i} \in P_k^{l-1}$, then $\vec{t}+1 = \vec{j'}^\frown\vec{i'}$ where $\vec{j'} \in P_k$ and $\vec{i'} \in P_{k+1}^l$.
(4) If $\vec{j} \in P_k$ and $\vec{i} \in P_{k+1}^l$, then

$$T_1 \circ T_{\vec{j}} \circ T_{\vec{i}}(p) = T_{\vec{j'}} \circ T_{\vec{i'}} \circ T_1(p)$$

for some $\vec{j'} \in P_{k-1}$, $\vec{i'} \in P_k^{l-1}$.

The second lemma will easily follow from Lemma 6.4 Let us recall the definitions of P_k and P_{k+1}^l from Introduction and observe that for any $1 \le k$

$$\bigcup_{\vec{i}\in P^k_{k+1}} T_{\vec{i}}(B) = B.$$

Lemma 6.5. *Let $B = (b_s)_{s=1}^m$ be a block sequence in $\mathrm{FIN}_l(n)$. Then for $2 \leq k \leq l$, we have*

$$T_1 \left\langle \bigcup_{\vec{i}\in P^l_{k+1}} T_{\vec{i}}(B) \right\rangle_{P_k} = \left\langle \bigcup_{\vec{i}\in P^{l-1}_k} T_{\vec{i}} \circ T_1(B) \right\rangle_{P_{k-1}}.$$

In particular, if $2 \leq k$ and $k = l$

$$T_1 \langle B\rangle_{P_k} = \langle T_1(B)\rangle_{P_{k-1}}.$$

An element $c \in \mathrm{FIN}_l$ is called a *pyramid of height l* if for some block sequence $A = (a_j)_{j=-(l-1)}^{j=l-1}$ in $\mathrm{FIN}_1^{[2l-1]}$, we have

$$c = \sum_{j=-(l-1)}^{l-1} (l - |j|) \cdot \chi(a_j).$$

Observe that if c is a pyramid of height l, $\vec{i} \in P_l$ and j is the number of zero entries in $\vec{i}$, then $T_{\vec{i}}(c)$ is a pyramid of height j. Note that some of the "steps" in the pyramid $T_{\vec{i}}(c)$ may have disappeared and others may have become longer. If $k < l$ and $\vec{i} \in P^l_{k+1}$, then $T_{\vec{i}}(c)$ is a pyramid of height k. In particular, for every $\vec{i}$ in P_k or in P^l_{k+1}, we have that

$$(*) \qquad T_{\vec{i}}(c)(\min \mathrm{supp}(T_{\vec{i}}(c))) = 1 = T_{\vec{i}}(c)(\max \mathrm{supp}(T_{\vec{i}}(c))).$$

Let $C = (c_i)_{i=1}^n$ be a block sequence of n pyramids of height k. For $p \in \langle C\rangle_{P_k}$, let

$$\mathrm{supp}_C(p) = \{i : \mathrm{supp}(p) \cap \mathrm{supp}(c_i) \neq \emptyset\}.$$

For $i \in \mathrm{supp}_C(p)$, we define $\mathrm{ht}_i(p) = \max\{p(x) : x \in \mathrm{supp}(p) \cap \mathrm{supp}(c_i)\}$, while for $i \notin \mathrm{supp}_C(p)$, we let $\mathrm{ht}_i(p) = 0$. Then for $1 \leq i \leq n$, $\mathrm{ht}(p)(i) = \mathrm{ht}_i(p)$ defines a function in $\mathrm{FIN}_k(n)$. For $\bar{p} = (p_1, \ldots, p_d) \in \mathrm{FIN}_k^{[d]}(C)$, we let $\mathrm{ht}(\bar{p}) = (\mathrm{ht}(p_1), \ldots, \mathrm{ht}(p_d)) \in \mathrm{FIN}_k^{[d]}(n)$.

Lemma 6.6. *Let C_1, C_2 be two block sequences of n pyramids of height k, let $p \in \langle C_1\rangle_{P_k}$ and $q \in \langle C_2\rangle_{P_k}$. Then $\mathrm{ht}(T_1(p)) = \mathrm{ht}(T_1(q))$ iff $\mathrm{tp}(p) = \mathrm{tp}(q)$.*

Proof. Observe that $i \in \mathrm{supp}_{C_1}(T_1(p))$ iff $\mathrm{ht}_i(T_1(p)) > 0$ iff $\mathrm{ht}_i(p) = \mathrm{ht}_i(T_1(p)) + 1$, while $i \notin \mathrm{supp}_{C_1}(T_1(p))$ iff $\mathrm{ht}_i(p) = 0$ or $\mathrm{ht}_i(p) = 1$. Analogously for q. The statement therefore follows by equation (*). $\square$

The following lemma shows that if the statement of Theorem 2.8 holds then it remains true after replacing $\mathrm{FIN}_k(n)$ by the partial semigroup generated by a block sequence of n pyramids of height k. It will be essential for the induction in the proof of Theorem 2.8.

Lemma 6.7. *Suppose that*

(**) *for every $1 \leq k$, every d, every $m \geq d$, every $l \geq k$, and every r, there exists n such that for every colouring $c : \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$ there is a block sequence B in $\mathrm{FIN}_l(n)$ of length m such that $\left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(B) \right\rangle_{P_k}^{[d]}$ is c-monochromatic.*

Let $C = (c_i)_{i=1}^n$ be a block sequence of n pyramids of height l. Then for every colouring $e : \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(C) \right\rangle_{P_k}^{[d]} \to \{1, 2, \ldots, r\}$ satisfying that $\mathrm{ht}(\bar{p}) = \mathrm{ht}(\bar{q})$ implies $e(\bar{p}) = e(\bar{q})$, there is a block sequence $D \in \langle C \rangle_{P_l}^{[m]}$ such that $\left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(D) \right\rangle_{P_k}^{[d]}$ is e-monochromatic.

Proof. Let C and e be as in the statement of the theorem. For $1 \leq j \leq l$, we define a one-to-one semigroup homomorphism

$$\iota_j : \mathrm{FIN}_j(n) \to \left\langle \bigcup_{\vec{i} \in P_{j+1}^l} T_{\vec{i}}(C) \right\rangle_{P_j}$$

by

$$q \mapsto \sum_{i \in \mathrm{supp}(q)} T_1^{l-q(i)}(c_i),$$

where $T_1^{l-q(i)}$ denotes the $(l - q(i))$-th iterate of T_1. We naturally extend ι_j to $\iota_j^{[d]}$ on $\mathrm{FIN}_j^{[d]}(n)$ by $\iota_j^{[d]}(q_i)_{i=1}^d = (\iota_j(q_i))_{i=1}^d$.

Let $c : \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$ be the colouring $e \circ \iota_k^{[d]}$. By the hypothesis (**), we can find a block sequence B in $\mathrm{FIN}_l^{[m]}(n)$ such that $\left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(B) \right\rangle_{P_k}^{[d]}$ is c-monochromatic in a colour α. Define $D = \iota_l^{[m]}(B) \in \langle C \rangle_{P_l}^{[m]}$.

It is easy to see that $\mathrm{ht}(\iota_j(q)) = q$ and that $\iota_{j-1} T_r(q) = T_r \iota_j(q)$ for $1 \leq r \leq j$ and $q \in \mathrm{FIN}_j(n)$. It implies that whenever $\bar{p} \in \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(D) \right\rangle_{P_k}^{[d]}$, $\mathrm{ht}(\bar{p}) \in \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(B) \right\rangle_{P_k}^{[d]}$, and therefore $e(\iota_k^{[d]}(\mathrm{ht}(\bar{p}))) = \alpha$, but $\mathrm{ht}(\iota_k^{[d]}(\mathrm{ht}(\bar{p}))) = \mathrm{ht}(\bar{p})$, so also $e(\bar{p}) = \alpha$. $\square$

For $\bar{p} = (p_1, \ldots, p_d) \in \mathrm{FIN}_k^{[d]}$, we define $T_1(\bar{p})$ to be $(T_1(p_1), \ldots, T_1(p_d))$. We are now ready to prove Theorem 2.8. Some of the ideas used in the proof also appeared in the proof of Theorem 1 in [22]. For a block sequence $B = (b_s)_{s=1}^m$, by $T_1(B)$ we will mean the block sequence $(T_1(b_s))_{s=1}^m$.

Proof of Theorem 2.8. We proceed by induction on k. For $k = 1$ and $d, m \geq d, l \geq k, r$ arbitrary, let $n = MT_d(m, r)$. Suppose that $c : \mathrm{FIN}_1^{[d]}(n) \to \{1, 2, \ldots, r\}$ is an arbitrary colouring. By Theorem 6.2, we can find a block sequence $A = (a_s)_{s=1}^m \in \mathrm{FIN}_1^{[m]}(n)$ such that $\mathrm{FIN}_1^{[d]}(A)$ is c-monochromatic. We define $B = (b_s)_{s=1}^m$ by $b_s = l \cdot \chi(a_s)$, so that $T_{\vec{i}}(b_s) = a_s$ for every s and $\vec{i} \in P_2^l$. Then

$$\left\langle \bigcup_{\vec{i} \in P_2^l} T_{\vec{i}}(B) \right\rangle_{P_1}^{[d]} = \langle A \rangle_{P_1}^{[d]} = \mathrm{FIN}_1^{[d]}(A)$$

is c-monochromatic.

Now, we assume that the theorem holds for $k-1$ and we shall prove it for k. Let $n' = G_d(k-1, l-1, m, r)$ be given by the induction hypothesis and let $n = T_d(k, n'(2l-1), r)$ be as in Lemma 6.3.

Let $c : \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$ be a given colouring. By Lemma 6.3, we can find a sequence A in $\mathrm{FIN}_1(n)$ of length $n'(2l-1)$ such that any two elements in $\mathrm{FIN}_k^{[d]}(A)$ of the same type have the same colour. Let $C = (c_i)_{i=1}^{n'}$ be the block sequence of n' pyramids in $\mathrm{FIN}_l(A)$, i.e.

$$c_i = \sum_{j=-(l-1)}^{l-1} (l - |j|) \cdot \chi(a_{q_i + j}),$$

where $q_i = (i-1)(2l-1) + l$.

Suppose that $\bar{p}, \bar{q} \in \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(C) \right\rangle_{P_k}^{[d]}$ are such that $\mathrm{ht}(T_1(\bar{p})) = \mathrm{ht}(T_1(\bar{q}))$. Then $\mathrm{tp}(\bar{p}) = \mathrm{tp}(\bar{q})$ by Lemma 6.6 and consequently $c(\bar{p}) = c(\bar{q})$ by the choice of C. Therefore the colouring

$$c' : T_1 \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(C) \right\rangle_{P_k}^{[d]} \to \{1, 2, \ldots, r\},$$

given by $c'(T_1(\bar{p})) = c(\bar{p})$, is well-defined.

By Lemma 6.5,

$$T_1 \left\langle \bigcup_{\vec{i} \in P_{k+1}^l} T_{\vec{i}}(C) \right\rangle_{P_k} = \left\langle \bigcup_{\vec{i} \in P_k^{l-1}} T_{\vec{i}} \circ T_1(C) \right\rangle_{P_{k-1}}.$$

Therefore c' and the sequence of pyramids $T_1(C)$ satisfy the hypothesis of Lemma 6.7.

Applying the induction hypothesis together with Lemma 6.7 we can find a block sequence $B' = (b'_s)_{s=1}^m$ in $\langle T_1(C) \rangle_{P_{l-1}}$ such that $\left\langle \bigcup_{\vec{i} \in P_k^{l-1}} T_{\vec{i}}(B') \right\rangle_{P_{k-1}}^{[d]}$ is c'-monochromatic, say in a colour α. By Lemma 6.5, there is a block sequence B in $\langle C \rangle_{P_l}^{[m]}$ such that

$T_1(B) = B'$. If $\bar{b} \in \left\langle \bigcup_{\vec{i} \in P^l_{k+1}} T_{\vec{i}}(B) \right\rangle^{[d]}_{P_k}$, then $T_1(\bar{b}) \in \left\langle \bigcup_{\vec{i} \in P^{l-1}_k} T_{\vec{i}}(B') \right\rangle^{[d]}_{P_{k-1}}$, so $c(\bar{b}) = c'(T_1(\bar{b})) = \alpha$. We can conclude that B is as required. □

As stated in Section 2.2, setting $k = l$ and $d = 1$ in Theorem 2.8, we obtain Corollary 2.7, a generalisation of the finite version of Gowers' Theorem from the tetris operation T_1 to all the operations T_i. Setting $k = l$ and letting d be arbitrary in Theorem 2.8, we obtain the following generalisation of the finite version of the Milliken–Taylor theorem for FIN_k (see [21], Corollary 5.26).

Corollary 6.8. *Let k, m, r and d be natural numbers. Then there exists a natural number n such that for every colouring $c : \mathrm{FIN}^{[d]}_k(n) \to \{1, 2, \ldots, r\}$ there is a block sequence B of length m in $\mathrm{FIN}_k(n)$ such that $\langle B\rangle^{[d]}_{P_k}$ is c-monochromatic.*

In an earlier version of the article we posed the following question.

Question 6.9. Does Corollary 2.7 admit an infinitary version?

This question was recently solved in positive by Lupini [12].

7. Applications to dynamics of $H(L)$

In this section, we describe a natural closed subgroup H of $H(L)$ and show that it is extremely amenable.

It is not difficult to see that $\mathcal{F}_<$ consists of rigid elements and that it has the JPP. The proposition below thus asserts that $\mathcal{F}_<$ is a projective Fraïssé class.

Proposition 7.1. *The family $\mathcal{F}_<$ has the AP.*

One can deduce this theorem from the JPP and from the Ramsey property, cf. [10], page 20. We will include a direct proof of Proposition 7.1 in Appendix A.

Having shown that $\mathcal{F}_<$ is a projective Fraïssé class, we may now consider its projective Fraïssé limit $\mathbb{L}_<$. Let $G = \mathrm{Aut}(\mathbb{L}_<)$ denote the automorphism group of $\mathbb{L}_<$. Combining the Kechris–Pestov–Todorčević correspondence from Section 4 with $\mathcal{F}_<$ being a rigid Ramsey class, we obtain the that the group $\mathrm{Aut}(\mathbb{L})$ is extremely amenable.

Proof of Theorem 2.3. Follows from Theorems 2.1 and 2.2. □

Observe that the family $\mathcal{F}_<$ is *reasonable* with respect to $\mathcal{F}$, that is, for every $A, B \in \mathcal{F}$, an epimorphism $\phi : B \to A$, and $A_< \in \mathcal{F}_<$ such that $A_< \restriction \mathcal{L} = A$, there is $B_< \in \mathcal{F}_<$ such that $B_< \restriction \mathcal{L} = B$ and $\phi : B_< \to A_<$ is an epimorphism. The proof of the following lemma uses that $\mathcal{F}_<$ is reasonable with respect to $\mathcal{F}$ and implies that $\mathrm{Aut}(\mathbb{L}_<)$ may be identified with a subgroup of $\mathrm{Aut}(\mathbb{L})$, cf. [10], Proposition 5.2. Our proof is slightly more complicated than the one in [10] since we do not have an analogue of the hereditary property for $\mathcal{F}$.

Lemma 7.2. *We have $\mathbb{L}_< \restriction \mathcal{L}$ is a projective Fraïssé limit of $\mathcal{F}$ and therefore isomorphic to $\mathbb{L}$.*

Proof. Set $\mathbb{L}_0 = \mathbb{L}_< \restriction \mathcal{L}$. By Proposition 3.3, it suffices to show that $\mathbb{L}_0$ satisfies the properties (L1) and (L2) in the definition of the projective Fraïssé limit, and it has the extension property with respect to $\mathcal{F}$. Since $\mathbb{L}_<$ has the properties (L1) and (L2) with respect to $\mathcal{F}_<$ and for every $A \in \mathcal{F}$ there is $A_< \in \mathcal{F}_<$ with $A_< \restriction \mathcal{L} = A$, $\mathbb{L}_0$ has the properties (L1) and (L2) with respect to $\mathcal{F}$.

To show the extension property, let $A, B \in \mathcal{F}$ and let $\phi_1 : B \to A$ and $\eta : \mathbb{L}_0 \to A$ be epimorphisms. By property (L2) for $\mathbb{L}_<$, find $C_< \in \mathcal{F}_<$, an epimorphism $\xi : \mathbb{L}_< \to C_<$ and a map $\phi_2 : C \to A$, such that $C = C_< \restriction \mathcal{L}$ and $\phi_2 \circ \xi = \eta$. Note that since $\xi : \mathbb{L}_0 \to C$ and $\eta : \mathbb{L}_0 \to A$ are epimorphisms, so is $\phi_2 : C \to A$. From the AP for $\mathcal{F}$, find $D \in \mathcal{F}$, $\psi_1 : D \to B$ and $\psi_2 : D \to C$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$. Take any $D_< \in \mathcal{F}_<$ with $D_< \restriction \mathcal{L} = D$ such that $\psi_2 : D_< \to C_<$ is an epimorphism. Using the extension property for $\mathcal{F}_<$, find an epimorphism $\rho : \mathbb{L}_< \to D_<$ such that $\psi_2 \circ \rho = \xi$. A simple calculation shows that the epimorphism $(\psi_1 \circ \rho) : \mathbb{L}_0 \to B$ is as needed, i.e. it satisfies $\eta = \phi_1 \circ (\psi_1 \circ \rho)$. $\square$

In the light of Lemma 7.2, from now on we will think of $\mathbb{L}_<$ as $(\mathbb{L}, S^{\mathbb{L}})$ and identify $\mathrm{Aut}(\mathbb{L}_<)$ with a closed subgroup of $\mathrm{Aut}(\mathbb{L})$. To demonstrate that Theorem 2.3 is not trivial, it is appropriate to show that the group $\mathrm{Aut}(\mathbb{L}_<)$ is not a singleton. It will follow from the projective ultrahomogeneity of $\mathbb{L}_<$. Indeed, let $A \in \mathcal{F}_<$ be the fan of height 1 and width 3, with branches $a_1 = (a_1^0, a_1^1)$, $a_2 = (a_2^0, a_2^1)$, and $a_3 = (a_3^0, a_3^1)$, where $r_A = a_1^0 = a_2^0 = a_3^0$ is the root. Let $B \in \mathcal{F}_<$ be the fan of height 1 and width 2 with branches $b_1 = (b_1^0, b_1^1)$ and $b_2 = (b_2^0, b_2^1)$, where $r_B = b_1^0 = b_2^0$ is the root. Let $\phi : \mathbb{L}_< \to A$ be an arbitrary epimorphism. Let $\alpha_1 : A \to B$ be the epimorphism given by $r_A \mapsto r_B$, $a_1^1 \mapsto b_1^1$, $a_2^1 \mapsto b_1^1$, and $a_3^1 \mapsto b_2^1$, and let $\alpha_2 : A \to B$ be the epimorphism given by $r_A \mapsto r_B$, $a_1^1 \mapsto b_1^1$, $a_2^1 \mapsto b_2^1$, and $a_3^1 \mapsto b_2^1$. Then the projective ultrahomogeneity applied to $\alpha_1 \circ \phi, \alpha_2 \circ \phi : \mathbb{L}_< \to B$ provides us with a non-trivial automorphism of $\mathbb{L}_<$.

Let $\pi : \mathbb{L} \to \mathbb{L}/R_S^{\mathbb{L}} \cong L$ be the natural quotient map. Since $\mathbb{L}_< = (\mathbb{L}, S^{\mathbb{L}})$ is a topological $\mathcal{L}_<$-structure, the relation $S^{\mathbb{L}}$ is closed and consequently $\leq_L = \pi(S^{\mathbb{L}})$ is a closed binary relation on L which is reflexive and transitive, that is, it is a preorder. We will call the Lelek fan equipped with $\leq_L$ the *preordered Lelek fan* and denote it by $L_<$.

In Section 3.1, we pointed out that π induces an injective continuous homomorphism which we denote by π^* from $\mathrm{Aut}(\mathbb{L})$ onto a subgroup of $H(L)$.

Definition 7.3. We define the following two subgroups of $H(L)$

$$H = \overline{\pi^*(\mathrm{Aut}(\mathbb{L}_<))}^{H(L)} \tag{1}$$

(2) $\quad H(L_<) = \{h \in H(L) : \text{ for every } x, y \in L \ (x \leq_L y \implies h(x) \leq_L h(y))\}.$

Proposition 7.4. *We have $H = H(L_<)$.*

In the proof of Proposition 7.4, we will use Lemma 7.5, an analog of Lemma 2.14 from [3] but for $L_<$ instead of L.

Lemma 7.5. *Let $d < 1$ be any metric compatible with the topology on L. Let $\varepsilon > 0$ and let v be the top of $L_<$. Then there are $A_< = (A, S^A) \in \mathcal{F}_<$ and an open cover $(U_a)_{a \in A_<}$ of $L_<$ such that*

(C1) *for each $a \in A_<$, diam$(U_a) < \varepsilon$,*
(C2) *for every $I = [v, e]$ where e is an endpoint and v is the top point, $\{U_a \cap I : a \in A_<\}$ is a cover consisting of intervals such that each set $(U_a \cap I) \setminus (U_{a'} \cap I)$ is connected and whenever $U_a \cap I, U_{a'} \cap I$ have a non-empty intersection and there is $y \in U_{a'} \cap I$ with $U_a \cap I \subset [v, y]$ we have $R^{A_<}(a, a')$,*
(C3) *for every $a \in A_<$ there is $x \in L_<$ such that $x \in U_a \setminus (\bigcup\{U_{a'} : a' \in A_<, a' \neq a\})$,*
(C4) *for each $x, y \in L_<$ and $a, b \in A_<$, if $x \leq_L y$, $x \in U_a$ and $y \in U_b$, then $S^{A_<}(a, b)$.*

Proof of Lemma 7.5. Let $\mathcal{U}$ be a finite open $\frac{\varepsilon}{3}$-cover of $L_<$ and let $\mathcal{V} = \{\pi^{-1}(U) : U \in \mathcal{U}\}$. Using (L2) in the definition of the projective Fraïssé limit, find $A_< \in \mathcal{F}_<$ and an epimorphism $\phi : \mathbb{L}_< \to A_<$ that refines $\mathcal{V}$. The set

$$\mathcal{C}_1 = \{V_a = \pi(\phi^{-1}(a)) : a \in A_<\}$$

is a closed $\frac{\varepsilon}{3}$-cover of $L_<$ that satisfies all properties (C1)–(C4). Since L is compact, the distance between any $D, E \in \mathcal{C}_1$, $D \cap E = \emptyset$ is positive, that is, $\inf\{d(x, y) : x \in D, y \in E\} > 0$. So we can find $0 < \delta < \frac{\varepsilon}{3}$ such that for every $D, E \in \mathcal{C}_1$, we have

$$B(D, \delta) \cap B(E, \delta) \neq \emptyset \iff D \cap E \neq \emptyset,$$

where for $X \subset L_<$ we set $B(X, \delta) = \{y \in L_< : \exists_{x \in X} \ d(y, x) < \delta\}$. Then the open cover $\mathcal{C}_2 = \{U_a = B(V_a, \delta) : a \in A_<\}$ satisfies the properties (C1)–(C4) as well. $\square$

Proof of Proposition 7.4. For every $h \in \mathrm{Aut}(\mathbb{L}_<)$, we have that $h^* \in H$. Since $H(L_<)$ is closed, it follows that $H \subset H(L_<)$.

To show the converse, take $h \in H(L_<)$ and $\varepsilon > 0$. Let $d < 1$ be any metric compatible with the topology on L and let $d_{\sup}$ be the corresponding supremum metric on $H(L)$. We will find $\gamma \in \mathrm{Aut}(\mathbb{L}_<)$ such that $d_{\sup}(h, \gamma^*) < \varepsilon$. Let $A_< \in \mathcal{F}_<$ and let $(U_a)_{a \in A_<}$ be an open cover of $L_<$ as in Lemma 7.5. Since h is uniformly continuous, we can assume additionally that for each $a \in A_<$, $\mathrm{diam}(h[U_a]) < \varepsilon$. As h is a homeomorphism, $(h[U_a])_{a \in A_<}$ also satisfies conditions (C2)–(C3) of Lemma 7.5. Finally, $h \in H(L_<)$ ensures that $(h[U_a])_{a \in A_<}$ satisfies (C4).

Consider the open covers $\{V_a^1 := \pi^{-1}(U_a) : a \in A_<\}$ and $\{V_a^2 := \pi^{-1}(h[U_a]) : a \in A_<\}$ of $\mathbb{L}_<$. By the property (L2), we can find $B_< \in \mathcal{F}_<$ and epimorphisms $\phi_i : \mathbb{L}_< \to B_<$ for $i = 1, 2$ that refine the cover $(V_a^i)_{a \in A_<}$. Define $\alpha_i : B_< \to A_<$ by $b \mapsto \max\{a \in A_< : \pi \circ \phi_i^{-1}(b) \subset U_a\}$, where the maximum is taken with respect to the natural partial order on A. Let $\psi_i : \mathbb{L}_< \to A_<$ be the composition $\alpha_i \circ \phi_i$. We will show that ψ_i, $i = 1, 2$ are epimorphisms. Since ϕ_i are continuous, so are ψ_i, and by (C3) they are onto. The property (C2) implies that if $x, y \in \mathbb{L}_<$ satisfy $R^{\mathbb{L}_<}(x, y)$ then $R^{A_<}(\psi_i(x), \psi_i(y))$, $i = 1, 2$. Finally, (C4) provides that if $S^{\mathbb{L}_<}(x, y)$ then $S^{A_<}(\psi_1(x), \psi_1(y))$. Since $(h[U_a])_{a \in A_<}$ also satisfies (C2)–(C4), the same is true for ψ_2, and we can conclude that ψ_1, ψ_2 are epimorphisms.

The projective ultrahomogeneity gives us $\gamma \in \mathrm{Aut}(\mathbb{L}_<)$ such that $\psi_1 = \psi_2 \circ \gamma$. It remains to show that $d_{\sup}(h, \gamma^*) < \varepsilon$. Pick any $x \in \mathbb{L}_<$ and let $a = \psi_1(x)$. Then

$$\gamma^*(\pi(x)) \in \gamma^*(\pi \circ \psi_1^{-1}(a)) = \pi \circ \gamma \circ \psi_1^{-1}(a) = \pi \circ \psi_2^{-1}(a) \subset h[U_a].$$

It means that $\gamma^*(\pi(x))$, $h(\pi(x)) \in h[U_a]$, and since $\mathrm{diam}(h[U_a]) < \varepsilon$, we get the required conclusion. $\square$

Acknowledgments

We would like to thank Miodrag Sokić, Gianluca Basso, and two anonymous referees for a number of suggestions that helped us to improve the presentation of the paper. We thank Martino Lupini for pointing out a mistake in an earlier version of the paper. The first author was supported by FAPESP (2013/14458-9) and FAPESP (2014/12405-8).

Appendix A

We present below a proof of Proposition 7.1.

Proof of Proposition 7.1. Take $A, B, C \in \mathcal{F}_<$ together with epimorphisms $\phi_1 : B \to A$ and $\phi_2 : C \to A$.

For clarity, we start with the simplest case, which will be applied in the induction further on.

Claim. *Assume that A, B and C all consist of one branch only. Then there are $D \in \mathcal{F}_<$ and epimorphisms $\psi_1 : D \to B$ and $\psi_2 : D \to C$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$.*

Proof of Claim. Suppose that A has height l and enumerate it as $a^0, a^1, \ldots, a^l$ with a^0 the root and $R^A(a^i, a^{i+1})$ for $i = 0, 1, \ldots, l-1$. For $0 \le i \le l$ and $\varepsilon = 1, 2$, let $I_i^\varepsilon = \phi_\varepsilon^{-1}(a^i)$ and let $m_i = \max\{|I_i^1|, |I_i^2|\}$. Let $D \in \mathcal{F}_<$ have a single branch of height $M = m_0 + \ldots + m_l$ and write it as $D = \bigcup_{i=0}^l K_i$ with $|K_i| = m_i$ and all elements in K_i preceding elements in K_{i+1} in the natural order. Then ψ_ε mapping K_i onto I_i^ε in an R-preserving manner for $\varepsilon = 1, 2$ finish the argument. $\square$

Second, we deal with the situation when $A = \{a^0, \ldots, a^l\}$ is a single branch and B and C have m and n branches, respectively.

The relation S^B induces an ordering $b_1 < b_2 < \ldots < b_m$ of branches in B, and the relation S^C induces an ordering $c_1 < c_2 < \ldots < c_n$ of branches in C. We will perform induction on $k = 1, \ldots, m+n$ assuming that $\phi_1 \restriction b_m$ and $\phi_2 \restriction c_n$ both map onto A. Later we will see how to eliminate this assumption. In step k, we will construct a branch d_k of D together with homomorphisms $\psi_{1,k} : d_k \to B$ and $\psi_{2,k} : d_k \to C$ such that $\phi_1 \circ \psi_{1,k} = \phi_2 \circ \psi_{2,k}$. We will ensure that the image of $\bigcup\{\psi_{\varepsilon,i} : i = 1, \ldots, k, \varepsilon = 1, 2\}$ contains the first k_B branches of B and the first k_C branches of C for some k_B and k_C such that $k = k_B + k_C$. Each of the steps resembles the proof of Claim.

Step $\boldsymbol{k = 1}$. Since $\phi_1(b_1)$ and $\phi_2(c_1)$ are initial segments of the single branch of A, we may without loss of generality assume that $\phi_1(b_1) \subset \phi_2(c_1)$. Applying Claim with b_1 in place of B and $\phi_2^{-1}(\phi_1(b_1))$ in place of C, we may find a branch d_1 and homomorphisms $\psi_{1,1} : d_1 \to b_1$ and $\psi_{2,1} : d_1 \to c_1$ with $\psi_{1,1}$ onto b_1, $\psi_{2,1}$ onto $\phi_2^{-1}(\phi_1(b_1))$ such that $\phi_1 \circ \psi_{1,1} = \phi_2 \circ \psi_{2,1}$.

Step $\boldsymbol{k + 1}$. Suppose that we have constructed branches d_i with homomorphisms $\psi_{1,i} : d_i \to B$ and $\psi_{2,i} : d_i \to C$ for $i = 1, \ldots, k$ as required. Pick branches b in B and c in C with the smallest indices such that they are not contained in the image of $\bigcup\{\psi_{\varepsilon,i} : i = 1, \ldots, k, \varepsilon = 1, 2\}$. Since $\phi_1(b)$ and $\phi_2(c)$ are initial segments of the only branch of A, we can assume without loss of generality that either ($\phi_1(b) \subset \phi_2(c)$ and $\phi_1(b) \neq \phi_2(c)$) or ($\phi_1(b) = \phi_2(c)$ and $m - k \geq n - k$). With b in place of b_1 and c in place of c_1, we may proceed in the same way as in Step $k = 1$ to obtain a branch d_{k+1} and homomorphisms $\psi_{1,k+1} : d_{k+1} \to B$ and $\psi_{2,k+1} : d_{k+1} \to C$ with $\psi_{1,k+1}$ onto c, $\psi_{2,k+1}$ onto $\phi_2^{-1}(\phi_1(b))$ and $\phi_1 \circ \psi_{1,k+1} = \phi_2 \circ \psi_{2,k+1}$. This finishes the induction step.

Note that $\phi_1 \restriction b_m$, $\phi_2 \restriction c_n$ being onto A allows us to proceed as above for $(n+m)$-many steps, in each step covering a new branch in B or C. We remark that we may ensure that all d_k's have the same height.

Let $D \in \mathcal{F}_<$ be the union of the branches $(d_k)_{k=1}^{m+n}$ with their roots identified and S^D induced by the order $d_1 < \ldots < d_{m+n}$. Then $\psi_1 : D \to B$ and $\psi_2 : D \to C$ given by $\psi_\varepsilon \restriction d_k = \psi_{\varepsilon,k}$ for $\varepsilon = 1, 2$ are as required.

When $\phi_1 \restriction b_1$ and $\phi_2 \restriction c_1$ are onto A, we proceed similarly as above (starting the induction with b_m and c_n and going backwards).

In the case when A has one branch and B and C are arbitrary, let t_B and t_C denote any branch in B and C, respectively, such that $\phi_1 \restriction t_B$ and $\phi_2 \restriction t_C$ are onto A. We split B into two fans, B_1 and B_2, such that all branches in B_1 precede the branches in B_2 in the order corresponding to S^B (which we denote by $B_1 < B_2$) and such that $B_1 \cap B_2 = t_B$. Similarly, we split C into $C_1 < C_2$ with $C_1 \cap C_2 = t_C$.

For B_i, C_i, $\phi_1^i = \phi_1 \restriction B_i$, and $\phi_2^i = \phi_2 \restriction C_i$, we obtain D_i, ψ_1^i, and ψ_2^i, $i = 1, 2$, and observe that $D = D_1 \cup D_2$ with their roots identified and S^D inducing $D_1 < D_2$, $\psi_1 = \psi_1^1 \cup \psi_1^2$, and $\psi_2 = \psi_2^1 \cup \psi_2^2$ are as required.

In a general situation, when A consists of branches $a_1 < \ldots < a_s$, we follow the procedure above for each a_i, $B_i = \phi_1^{-1}(a_i)$, $C_i = \phi_2^{-1}(a_i)$, $\phi_1^i = \phi_1 \restriction B_i$, and $\phi_2^i = \phi_2 \restriction C_i$.

For each i, we obtain D_i, $\psi_1^i : D_i \to B_i$, and $\psi_2^i : D_i \to C_i$, $i = 1, \dots, l$, and conclude that $D = D_1 \cup \ldots \cup D_s$ with roots identified and such that $D_1 < \ldots < D_s$, $\psi_1 = \psi_1^1 \cup \ldots \cup \psi_1^s$, and $\psi_2 = \psi_2^1 \cup \ldots \cup \psi_2^s$ finish the proof. $\square$

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