



ELSEVIER

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta



Beyond Göllnitz' Theorem I: A bijective approach



Isaac Konan

*IRIF, Université de Paris, Bâtiment Sophie Germain, Case courrier 7014, 8 Place
Aurélien Nemours, 75205 Paris Cedex 13, France*

ARTICLE INFO

Article history:

Received 22 October 2019

Received in revised form 30

December 2020

Accepted 25 January 2021

Available online 1 February 2021

Keywords:

Rogers-Ramanujan type identities

Weighted words

Göllnitz' identity

Bijections

ABSTRACT

In 2003, Alladi, Andrews and Berkovich proved an identity for partitions where parts occur in eleven colors: four primary colors, six secondary colors, and one quaternary color. Their work answered a longstanding question of how to go beyond a classical theorem of Göllnitz, which uses three primary and three secondary colors. Their main tool was a deep and difficult four parameter q -series identity. In this paper we take a different approach. Instead of adding an eleventh quaternary color, we introduce forbidden patterns and give a bijective proof of a ten-colored partition identity lying beyond Göllnitz' theorem. Using a second bijection, we show that our identity is equivalent to the identity of Alladi, Andrews, and Berkovich. From a combinatorial viewpoint, the use of forbidden patterns is more natural and leads to a simpler formulation. In fact, in Part II of this series we will show how our method can be used to go beyond Göllnitz' theorem to any number of primary colors.

© 2021 Elsevier Inc. All rights reserved.

E-mail address: konan@irif.fr.

<https://doi.org/10.1016/j.jcta.2021.105426>

0097-3165/© 2021 Elsevier Inc. All rights reserved.

1. Introduction and statements of results

1.1. History

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n . For example, the partitions of 7 are

$$(7), (6, 1), (5, 2), (5, 1, 1), (4, 3), (4, 2, 1), (4, 1, 1, 1), (3, 3, 1), (3, 2, 2), (3, 2, 1, 1), \\ (3, 1, 1, 1, 1), (2, 2, 2, 1), (2, 2, 1, 1, 1), (2, 1, 1, 1, 1, 1) \text{ and } (1, 1, 1, 1, 1, 1, 1).$$

The study of partition identities has a long history, dating back to Euler's proof that there are as many partitions of n into distinct parts as partitions of n into odd parts. The corresponding identity is

$$(-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}}, \quad (1.1)$$

where

$$(x; q)_m = \prod_{k=0}^{m-1} (1 - xq^k),$$

for any $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and x, q such that $|q| < 1$.

One of the most important identities in the theory of partitions is Schur's theorem [9].

Theorem 1.1 (Schur 1926). *For any positive integer n , the number of partitions of n into distinct parts congruent to $\pm 1 \pmod{3}$ is equal to the number of partitions of n where parts differ by at least three and multiples of three differ by at least six.*

There have been a number of proofs of Schur's result over the years, including a q -difference equation proof of Andrews [4] and a simple bijective proof of Bressoud [6]. Another important identity is Göllnitz' theorem [7].

Theorem 1.2 (Göllnitz 1967). *For any positive integer n , the number of partitions of n into distinct parts congruent to $2, 4, 5 \pmod{6}$ is equal to the number of partitions of n into parts different from 1 and 3, and where parts differ by at least six with equality only if parts are congruent to $2, 4, 5 \pmod{6}$.*

Like Schur's theorem, Göllnitz's identity can be proved using q -difference equations [5] and an elegant Bressoud-style bijection [8,10]. Seminal work of Alladi, Andrews, and Gordon in the 90's showed how the theorems of Schur and Göllnitz emerge from more general results on colored partitions [2].

In the case of Schur's theorem, we consider parts in three colors $\{a, b, ab\}$ and order them as follows:

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < \dots \quad (1.2)$$

We then consider the partitions with colored parts different from 1_{ab} and satisfying the minimal difference conditions in the table

$$\begin{array}{c|cc|c} \lambda_i \backslash \lambda_{i+1} & a & b & ab \\ \hline a & 1 & 2 & 1 \\ b & 1 & 1 & 1 \\ ab & 2 & 2 & 2 \end{array}. \quad (1.3)$$

Here, the part λ_i with color in the row and the part λ_{i+1} with color in the column differ by at least the corresponding entry in the table. An example of such a partition is $(7_{ab}, 5_b, 4_a, 3_{ab}, 1_b)$. The Alladi-Gordon refinement of Schur's partition theorem [3] is stated as follows:

Theorem 1.3. *Let u, v, n be non-negative integers. Denote by $A(u, v, n)$ the number of partitions of n into u distinct parts with color a and v distinct parts with color b , and denote by $B(u, v, n)$ the number of partitions of n satisfying the conditions in (1.3), with u parts with color a or ab , and v parts with color b or ab . We then have $A(u, v, n) = B(u, v, n)$ and the identity*

$$\sum_{u, v, n \geq 0} B(u, v, n) a^u b^v q^n = \sum_{u, v, n \geq 0} A(u, v, n) a^u b^v q^n = (-aq; q)_\infty (-bq; q)_\infty. \quad (1.4)$$

Note that a transformation implies Schur's theorem:

$$\begin{cases} \text{dilation :} & q \mapsto q^3 \\ \text{translations :} & a, b \mapsto q^{-2}, q^{-1} \end{cases}. \quad (1.5)$$

In fact, the minimal difference conditions given in (1.3) give after these transformations the minimal differences in Schur's theorem.

In the case of Göllnitz' theorem, we consider parts that occur in six colors $\{a, b, c, ab, ab, bc\}$ with the order

$$1_{ab} < 1_{ac} < 1_a < 1_{bc} < 1_b < 1_c < 2_{ab} < 2_{ac} < 2_a < 2_{bc} < 2_b < 2_c < 3_{ab} < \dots, \quad (1.6)$$

and the partitions with colored parts different from $1_{ab}, 1_{ac}, 1_{bc}$ and satisfying the minimal difference conditions in

$\lambda_i \setminus \lambda_{i+1}$	a	b	c	ab	ac	bc
a	1	2	2	1	1	2
b	1	1	2	1	1	1
c	1	1	1	1	1	1
ab	2	2	2	2	2	2
ac	2	2	2	1	2	2
bc	1	2	2	1	1	2

(1.7)

The Alladi-Andrews-Gordon refinement of Göllnitz's partition theorem can be stated as follows:

Theorem 1.4. *Let u, v, w, n be non-negative integers. Denote by $A(u, v, w, n)$ the number of partitions of n into u distinct parts with color a , v distinct parts with color b and w distinct parts with color c , and denote by $B(u, v, w, n)$ the number of partitions of n satisfying the conditions in (1.7), with u parts with color a, ab or ac , v parts with color b, ab or bc and w parts with color c, ac or bc . We then have $A(u, v, w, n) = B(u, v, w, n)$ and the identity*

$$\sum_{u,v,w,n \geq 0} B(u, v, w, n) a^u b^v c^w q^n = \sum_{u,v,w,n \geq 0} A(u, v, w, n) a^u b^v c^w q^n = (-aq; q)_\infty (-bq; q)_\infty (-cq; q)_\infty. \quad (1.8)$$

Note that a transformation implies Göllnitz' theorem:

$$\begin{cases} \text{dilation :} & q \mapsto q^6 \\ \text{translations :} & a, b, c \mapsto q^{-4}, q^{-2}, q^{-1} \end{cases}. \quad (1.9)$$

Observe that while Schur's theorem is not a direct corollary of Göllnitz' theorem, Theorem 1.3 is implied by Theorem 1.4 by setting $c = 0$. Therefore Göllnitz' theorem may be viewed as a level higher than Schur's theorem, since it requires three primary colors instead of two.

Following the work of Alladi, Andrews, and Gordon, it was an open problem to find a partition identity beyond Göllnitz' theorem, in the sense that it would arise from four primary colors. This was famously solved by Alladi, Andrews, and Berkovich [1]. To describe their result, we consider parts that occur in eleven colors $\{a, b, c, d, ab, ac, ad, bc, bd, cd, abcd\}$ and ordered as follows:

$$1_{abcd} < 1_{ab} < 1_{ac} < 1_{ad} < 1_a < 1_{bc} < 1_{bd} < 1_b < 1_{cd} < 1_c < 1_d < 2_{abcd} < \cdots \quad (1.10)$$

Let us consider the partitions with the size of the secondary parts greater than one and satisfying the minimal difference conditions in

$\lambda_i \setminus \lambda_{i+1}$	ab	ac	ad	a	bc	bd	b	cd	c	d
ab	2	2	2	2	2	2	2	2	2	2
ac	1	2	2	2	2	2	2	2	2	2
ad	1	1	2	2	2	2	2	2	2	2
a	1	1	1	1	2	2	2	2	2	2
bc	1	1	1	1	2	2	2	2	2	2
bd	1	1	1	1	1	2	2	2	2	2
b	1	1	1	1	1	1	1	2	2	2
cd	1	1	1	1	1	1	1	2	2	2
c	1	1	1	1	1	1	1	1	1	2
d	1	1	1	1	1	1	1	1	1	1

(1.11)

and such that parts with color $abcd$ differ by at least 4, and the smallest part with color $abcd$ is at least equal to $4 + 2\tau - \chi(1_a \text{ is a part})$, where τ is the number of primary and secondary parts in the partition. The theorem is then stated as follows.

Theorem 1.5. *Let u, v, w, t, n be non-negative integers. Denote by $A(u, v, w, t, n)$ the number of partitions of n into u distinct parts with color a , v distinct parts with color b , w distinct parts with color c and t distinct parts with color d , and denote by $B(u, v, w, t, n)$ the number of partitions of n satisfying the conditions in (1.11), with u parts with color a, ab, ac, ad or $abcd$, v parts with color b, ab, bc, bd or $abcd$, w parts with color c, ac, bc, cd or $abcd$ and t parts with color d, ad, bd, cd or $abcd$. We then have $A(u, v, w, t, n) = B(u, v, w, t, n)$ and the identity*

$$\sum_{u,v,w,t,n \geq 0} B(u, v, w, t, n) a^u b^v c^w d^t q^n = (-aq; q)_\infty (-bq; q)_\infty (-cq; q)_\infty (-dq; q)_\infty. \quad (1.12)$$

Note that the result of Alladi-Andrews-Berkovich uses four primary colors, the full set of secondary colors, along with one quaternary color $abcd$. When $d = 0$, we recover Theorem 1.4. Their main tool was a difficult q -series identity:

$$\begin{aligned} & \sum_{i,j,k,l - \text{constraints}} \frac{q^{T_\tau + T_{AB} + T_{AC} + T_{AD} + T_{BC} + T_{BD} + T_{CD} - BC - BD - CD + 4T_{Q-1} + 3Q + 2Q\tau}}{(q)_A (q)_B (q)_C (q)_D (q)_{AB} (q)_{AC} (q)_{AD} (q)_{BC} (q)_{BD} (q)_{CD} (q)_Q} \\ & \cdot \{(1 - q^A) + q^{A+BC+BD+Q}(1 - q^B) + q^{A+BC+BD+Q+B+CD}\} \\ & = \frac{q^{T_i + T_j + T_k + T_l}}{(q)_i (q)_j (q)_k (q)_l} \end{aligned} \quad (1.13)$$

where $A, B, C, D, AB, AC, AD, BC, BD, CD, Q$ are variables which count the number of parts with respectively color $a, b, c, d, ab, ac, ad, bc, bd, cd, abcd$,

$$\begin{cases} i = A + AB + AC + AD + Q \\ j = B + AB + BC + BD + Q \\ k = C + AC + BC + CD + Q \\ l = D + AD + BD + CD + Q \\ \tau = A + B + C + D + AB + AC + AD + BC + BD + CD \end{cases},$$

$T_n = \frac{n(n+1)}{2}$ is the n^{th} triangular number and $(q)_n = (q; q)_n$. While this identity is difficult to prove, it is relatively straightforward to show that it is equivalent to the statement in Theorem 1.5.

In this paper we give a bijective proof of Theorem 1.5 (and therefore a bijective proof of the identity (1.13)). Our proof is divided into two steps. First we prove Theorem 1.6 below, which arises more naturally from our methods than Theorem 1.5. Instead of adding a quaternary color, we lower certain minimum differences and add some forbidden patterns. Then, we show how Theorem 1.6 is equivalent to Theorem 1.5.

1.2. Statement of results

Suppose that the parts occur in only primary colors a, b, c, d and secondary colors ab, ac, ad, bc, bd, cd , and are ordered as in (1.10) by omitting quaternary parts:

$$1_{ab} < 1_{ac} < 1_{ad} < 1_a < 1_{bc} < 1_{bd} < 1_b < 1_{cd} < 1_c < 1_d < 2_{ab} < \dots \quad (1.14)$$

Let us now consider the partitions with the size of the secondary parts greater than one and satisfying the minimal difference conditions in

$\lambda_i \setminus \lambda_{i+1}$	ab	ac	ad	a	bc	bd	b	cd	c	d
ab	2	2	2	2	2	2	2	2	2	2
ac	1	2	2	2	2	2	2	2	2	2
ad	1	1	2	2	<u>1</u>	2	2	2	2	2
a	1	1	1	1	2	2	2	2	2	2
bc	1	1	1	1	2	2	2	2	2	2
bd	1	1	1	1	1	2	2	2	2	2
b	1	1	1	1	1	1	1	2	2	2
cd	<u>0</u>	1	1	1	1	1	1	2	2	2
c	1	1	1	1	1	1	1	1	1	2
d	1	1	1	1	1	1	1	1	1	1

(1.15)

and which avoid the forbidden patterns

$$((k+2)_{cd}, (k+2)_{ab}, k_c), ((k+2)_{cd}, (k+2)_{ab}, k_d), ((k+2)_{ad}, (k+1)_{bc}, k_a), \quad (1.16)$$

except the pattern $(3_{ad}, 2_{bc}, 1_a)$ which is allowed. An example of such a partition is

$$(11_{ad}, 10_{bc}, 8_a, 7_{cd}, 7_{ab}, 4_c, 3_{ad}, 2_{bc}, 1_a) \cdot$$

We can now state the main theorem of this paper.

Theorem 1.6. *Let u, v, w, t, n be non-negative integers. Denote by $A(u, v, w, t, n)$ the number of partitions of n into u distinct parts with color a , v distinct parts with color b , w distinct parts with color c and t distinct parts with color d , and denote by $B(u, v, w, t, n)$ the number of partitions of n satisfying the conditions above, with u parts with color a, ab, ac or ad , v parts with color b, ab, bc or bd , w parts with color c, ac, bc or cd and t parts with color d, ad, bd or cd . We then have $A(u, v, w, t, n) = B(u, v, w, t, n)$, and the corresponding q -series identity is given by*

$$\sum_{u,v,w,t,n \in \mathbb{N}} B(u, v, w, t, n) a^u b^v c^w d^t q^n = (-aq; q)_\infty (-bq; q)_\infty (-cq; q)_\infty (-dq; q)_\infty \cdot \quad (1.17)$$

By specializing the variables in Theorem 1.6, one can deduce many partition identities. For example, by considering the following transformation in (1.17)

$$\begin{cases} \text{dilation :} & q \mapsto q^{12} \\ \text{translations :} & a, b, c, d \mapsto q^{-8}, q^{-4}, q^{-2}, q^{-1} \end{cases} \quad (1.18)$$

we obtain a corollary of Theorem 1.6.

Corollary 1.1. *For any positive integer n , the number of partitions of n into distinct parts congruent to $-2^3, -2^2, -2^1, -2^0 \pmod{12}$ is equal to the number of partitions of n into parts not congruent to $1, 5 \pmod{12}$ and different from $2, 3, 6, 7, 9$, such that the difference between two consecutive parts is greater than 12 up to the following exceptions:*

- $\lambda_i - \lambda_{i+1} = 9 \implies \lambda_i \equiv \pm 3 \pmod{12}$ and $\lambda_i - \lambda_{i+2} \geq 24$,
- $\lambda_i - \lambda_{i+1} = 12 \implies \lambda_i \equiv -2^3, -2^2, -2^1, -2^0 \pmod{12}$,

except that the pattern $(27, 18, 4)$ is allowed.

For example, with $n = 49$, the partitions of the first kind are

$$(35, 10, 4), (34, 11, 4), (28, 11, 10), (23, 22, 4), \\ (23, 16, 10), (22, 16, 11) \text{ and } (16, 11, 10, 8, 4)$$

and the partitions of the second kind are

$$(35, 14), (34, 15), (33, 16), (45, 4), (39, 10), (38, 11) \text{ and } (27, 18, 4) \cdot$$

Corollary 1.1 may be compared with Theorem 3 of [1], which is Theorem 1.5 transformed by (1.18) but with the dilation $q \mapsto q^{15}$ instead of $q \mapsto q^{12}$.

The paper is organized as follows. In Section 2, we will present some tools that will be useful for the proof of Theorem 1.6. After that, in Section 3, we will give the bijection for Theorem 1.6. Then, in Section 4, we will prove its well-definedness. Finally, in Section 5, we will present and prove the bijection between the partitions with forbidden patterns considered in Theorem 1.6 and the partitions with quaternary parts given in Theorem 1.5. In Part II of this series, we will show how our method can be used to go beyond Göllnitz' theorem to any number of primary colors.

2. Preliminaries

2.1. The setup

Denote by $\mathcal{C} = \{a, b, c, d\}$ the set of primary colors and $\mathcal{C}_\times = \{ab, ac, ad, bc, bd, cd\}$ the set of secondary colors, and recall the order on $\mathcal{C} \sqcup \mathcal{C}_\times$:

$$ab < ac < ad < a < bc < bd < b < cd < c < d. \quad (2.1)$$

We can then define the strict lexicographic order \succ on colored parts by

$$k_p \succ l_q \iff k - l \geq \chi(p \leq q). \quad (2.2)$$

Explicitly, this gives the order

$$1_{ab} \prec 1_{ac} \prec 1_{ad} \prec 1_a \prec 1_{bc} \prec 1_{bd} \prec 1_b \prec 1_{cd} \prec 1_c \prec 1_d \prec 2_{ab} \prec \dots, \quad (2.3)$$

previously established in (1.14). We denote by \mathcal{P} the set of positive integers with primary color.

We can easily see that for any $pq \in \mathcal{C}_\times$, with $p < q$, and any $k \geq 1$, we have that

$$(2k)_{pq} = k_q + k_p \quad (2.4)$$

$$(2k+1)_{pq} = (k+1)_p + k_q. \quad (2.5)$$

In fact, any part greater than 1 with a secondary color pq can be uniquely written as the sum of two consecutive parts in \mathcal{P} with colors p and q . We then denote by \mathcal{S} the set of secondary parts greater than 1, and define the functions α and β on \mathcal{S} by

$$\alpha : \begin{cases} 2k_{pq} & \mapsto k_q \\ (2k+1)_{pq} & \mapsto (k+1)_p \end{cases} \quad \text{and} \quad \beta : \begin{cases} 2k_{pq} & \mapsto k_p \\ (2k+1)_{pq} & \mapsto k_q \end{cases}, \quad (2.6)$$

respectively named upper and lower halves. One can check that for any $k_{pq} \in \mathcal{S}$,

$$\alpha((k+1)_{pq}) = \beta(k_{pq}) + 1 \quad \text{and} \quad \beta((k+1)_{pq}) = \alpha(k_{pq}). \quad (2.7)$$

In the previous sum, adding an integer to a part does not change its color. We can then deduce by induction that for any $m \geq 0$,

$$\alpha((k+m)_{pq}) \preceq \alpha(k_{pq}) + m \quad \text{and} \quad \beta((k+m)_{pq}) \preceq \beta(k_{pq}) + m. \quad (2.8)$$

Recall the table (1.11)

$\lambda_i \setminus \lambda_{i+1}$	ab	ac	ad	a	bc	bd	b	cd	c	d
ab	2	2	2	2	2	2	2	2	2	2
ac	1	2	2	2	2	2	2	2	2	2
ad	1	1	2	2	2	2	2	2	2	2
a	1	1	1	1	2	2	2	2	2	2
bc	1	1	1	1	2	2	2	2	2	2
bd	1	1	1	1	1	2	2	2	2	2
b	1	1	1	1	1	1	1	2	2	2
cd	1	1	1	1	1	1	1	2	2	2
c	1	1	1	1	1	1	1	1	1	2
d	1	1	1	1	1	1	1	1	1	1

It can be viewed as an order \triangleright on $\mathcal{P} \sqcup \mathcal{S}$ defined by

$$k_p \triangleright l_q \iff k - l \geq 1 + \begin{cases} \chi(p < q) & \text{if } p \text{ or } q \in \mathcal{C} \\ \chi(p \leq q) & \text{if } p \text{ and } q \in \mathcal{C}_\infty \end{cases}. \quad (2.9)$$

By considering the lexicographic order \succ , (2.9) becomes

$$k_p \triangleright l_q \iff \begin{cases} k_p \succeq (l+1)_q & \text{if } p \text{ or } q \in \mathcal{C} \\ k_p \succ (l+1)_q & \text{if } p \text{ and } q \in \mathcal{C}_\infty \end{cases}. \quad (2.10)$$

We can observe that for any primary colors p, q

$$k_p \succ l_q \quad \text{and} \quad k_p \not\succ l_q \iff k - l = \chi(p < q) \quad \text{and} \quad p \neq q, \quad (2.11)$$

and we easily check that in this case, $(k_p, l_q) = (\alpha(k_p + l_q), \beta(k_p + l_q))$, for $k_p + l_q$ viewed as an element of \mathcal{S} (see (2.4), (2.5)).

We recall that the tables (1.15)

$$\Delta =$$

$\lambda_i \setminus \lambda_{i+1}$	ab	ac	ad	a	bc	bd	b	cd	c	d
ab	2	2	2	2	2	2	2	2	2	2
ac	1	2	2	2	2	2	2	2	2	2
ad	1	1	2	2	1	2	2	2	2	2
a	1	1	1	1	2	2	2	2	2	2
bc	1	1	1	1	2	2	2	2	2	2
bd	1	1	1	1	1	2	2	2	2	2
b	1	1	1	1	1	1	1	2	2	2
cd	0	1	1	1	1	1	1	2	2	2
c	1	1	1	1	1	1	1	1	1	2
d	1	1	1	1	1	1	1	1	1	1

and (1.11) differ only when we have a pair (p, q) of secondary colors such that $(p, q) \in \{(cd, ab), (ad, bc)\}$. In these cases, the difference in (1.15) is one less.

We will now define a relation \gg on $\mathcal{P} \sqcup \mathcal{S}$ in such a way that,

$$k_p \gg l_q \iff k - l \geq \Delta(p, q). \quad (2.12)$$

Using (2.10), this relation can be summarized by the following equivalence:

$$k_p \gg l_q \iff \begin{cases} k_p \succeq (l+1)_q & \text{if } p \text{ or } q \in \mathcal{C} \\ k_p \succ (l+1)_q & \text{if } p \text{ and } q \in \mathcal{C}_\times \text{ and } (p, q) \notin \{(cd, ab), (ad, bc)\} \\ k_p \succ l_q & \text{if } (p, q) \in \{(cd, ab), (ad, bc)\} \end{cases} \quad (2.13)$$

We denote by \mathcal{O} the set of partitions with parts in \mathcal{P} and well-ordered by \succ . We then have that $\lambda \in \mathcal{O}$ if and only if there exist $\lambda_1 \succ \dots \succ \lambda_t \in \mathcal{P}$ such that $\lambda = (\lambda_1, \dots, \lambda_t)$. We set $c(\lambda_i)$ to be the color of λ_i in \mathcal{C} , and $C(\lambda) = c(\lambda_1) \cdots c(\lambda_t)$ as a commutative product of colors in \mathcal{C} . We denote by \mathcal{E} the set of partitions with parts in $\mathcal{P} \sqcup \mathcal{S}$ and well-ordered by \gg . We then have that $\nu \in \mathcal{E}$ if and only if there exist $\nu_1 \gg \dots \gg \nu_t \in \mathcal{P} \sqcup \mathcal{S}$ such that $\nu = (\nu_1, \dots, \nu_t)$. We set colors $c(\nu_i) \in \mathcal{C} \sqcup \mathcal{C}_\times$ depending on whether ν_i is in \mathcal{P} or \mathcal{S} , and we also define $C(\nu) = c(\nu_1) \cdots c(\nu_t)$ seen as a commutative product of colors in \mathcal{C} . In fact, a secondary color is just a product of two primary colors. For both kinds of partitions, their size is the sum of their part sizes.

We also denote by \mathcal{E}_1 the subset of partitions of \mathcal{E} without the forbidden patterns,

$$((k+2)_{cd}, (k+2)_{ab}, k_c), ((k+2)_{cd}, (k+2)_{ab}, k_d), ((k+2)_{ad}, (k+1)_{bc}, k_a), \quad (2.14)$$

except the pattern $(3_{ad}, 2_{bc}, 1_a)$ which is allowed. We finally define \mathcal{E}_2 as the subset of partitions of \mathcal{E} with parts well-ordered by \triangleright in (2.10), and we observe that \mathcal{E}_2 is indeed a subset of \mathcal{E}_1 .

2.2. Technical lemmas

We will state and prove some important lemmas for the proof of Theorem 1.6.

Lemma 2.1 (Ordering primary and secondary parts). *For any $(l_p, k_q) \in \mathcal{P} \times \mathcal{S}$, we have the following equivalences:*

$$l_p \not\gg k_q \iff (k+1)_q \gg (l-1)_p, \quad (2.15)$$

$$l_p \gg \alpha(k_q) \iff \beta((k+1)_q) \not\succ (l-1)_p. \quad (2.16)$$

Lemma 2.2 (Ordering secondary parts). *Let us consider the table Δ in (1.15). Then, for any secondary colors $p, q \in \mathcal{C}_\times$,*

$$\Delta(p, q) = \min\{k - l : \beta(k_p) \succ \alpha(l_q)\}. \quad (2.17)$$

Moreover, if the secondary parts k_p, l_q are such that $\beta(k_p) \succ \beta(l_q)$, then

$$(k+1)_p \gg l_q. \quad (2.18)$$

Furthermore, if $k_p \gg l_q$, we then have either $\beta(k_p) \succ \alpha(l_q)$ or

$$\alpha(l_q) + 1 \gg \alpha((k-1)_p) \succ \beta((k-1)_p) \succ \beta(l_q). \quad (2.19)$$

Lemma 2.3 (Reversibility $\mathcal{O} \leftarrow \mathcal{E}_1$). *Let us consider a partition $\nu = (\nu_1, \dots, \nu_t) \in \mathcal{E}$. Then, for any $i \in [1, t-2]$ such that $(\nu_{i+1}, \nu_{i+2}) \in \mathcal{S} \times \mathcal{P}$ and $(c(\nu_i), c(\nu_{i+1})) \notin \{(ad, bc), (cd, ab)\}$, we have*

$$\nu_i \succ \nu_{i+2} + 2. \quad (2.20)$$

Furthermore, the following are equivalent:

- (1) $\nu \in \mathcal{E}_1$,
- (2) For any $i \in [1, t-2]$ such that (ν_i, ν_{i+1}) is a pattern in $\{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$ different from $(3_{ad}, 2_{bc})$, we have that

$$\nu_i \succeq \nu_{i+2} + 2. \quad (2.21)$$

Proof of Lemma 2.1. To prove (2.15), we observe that, for any $(l_p, k_q) \in \mathcal{P} \times \mathcal{S}$, by (2.13),

$$l_p \not\gg k_q \iff l_p \not\prec (k+1)_q,$$

and

$$\begin{aligned}(k+1)_q \gg (l-1)_p &\iff (k+1)_q \succ l_p \\ &\iff (k+1)_q \not\preceq l_p.\end{aligned}$$

To prove (2.16), we first remark that, by (2.7), $\alpha(k_q) = \beta((k+1)_q)$. We then obtain by (2.13) that

$$l_p \gg \alpha(k_q) \iff (l-1)_p \succeq \alpha(k_q)$$

and

$$\begin{aligned}\beta((k+1)_q) \not\succ (l-1)_p &\iff \alpha(k_q) \not\succ (l-1)_p \\ &\iff \alpha(k_q) \preceq (l-1)_p. \quad \square\end{aligned}$$

Proof of Lemma 2.2. Let us consider $\min\{k-l : \beta(k_p) \succ \alpha(l_q)\}$. We just check for the 36 pairs (p, q) in \mathcal{C}_\times^2 . As an example, we take the pairs $(cd, ab), (ad, bc)$.

- For $k = 2k' + 1$, we have $(\alpha(k_{cd}), \beta(k_{cd})) = ((k' + 1)_c, k'_d)$. Then, to minimize $k - l$, $\alpha(l_{ab})$ and $\beta(l_{ab})$ have to be the greatest primary parts with color a, b less than k'_d . So we obtain k'_b and k'_a . We then have

$$k - l = 2k' + 1 - 2k' = 1.$$

For $k = 2k'$, we have $(\alpha(k_{cd}), \beta(k_{cd})) = (k'_d, k'_c)$. Then to minimize $k - l$, $\alpha(l_{ab})$ and $\beta(l_{ab})$ have to be the greatest primary parts with color a, b less than k'_c . So we obtain k'_b and k'_a . We then have

$$k - l = 2k' - 2k' = 0.$$

Therefore, $\Delta(cd, ab) = 0$.

- We check with the same reasoning by taking for (ad, bc) consecutive parts

$$(k+1)_a \succ k_d \succ k_c \succ k_b$$

and

$$(k+1)_d \succ (k+1)_a \succ k_c \succ k_b,$$

and we obtain $\Delta(ad, bc) = 1$.

To prove (2.18), we have by (2.7) that $\alpha((l-1)_q) = \beta(l_q)$. Since $\beta(k_p) \succ \beta(l_q) = \alpha((l-1)_q)$, this then implies by (2.17) that $k_p \gg (l-1)_q$, and this is equivalent to $(k+1)_p \gg l_q$.

Let us now suppose that $k - l \geq \Delta(p, q)$. We just saw that this minimum value was

reached at k or $k - 1$. Then if we do not have $\beta(k_p) \succ \alpha(l_q)$, we necessarily have $\beta((k - 1)_p) \succ \alpha((l - 1)_q) = \beta(l_q)$ by (2.7). Moreover, by (2.13), we have

$$\beta(k_p) \not\succ \alpha(l_q) \iff \alpha(l_q) + 1 \gg \alpha((k - 1)_p),$$

so that we obtain (2.19). \square

Proof of Lemma 2.3. For any $\nu = (\nu_1, \dots, \nu_t) \in \mathcal{E}$ and any $i \in [1, t - 2]$, we have $\nu_i \gg \nu_{i+1} \gg \nu_{i+2}$. By (2.13), the fact that $(\nu_{i+1}, \nu_{i+2}) \in \mathcal{S} \times \mathcal{P}$ implies that

$$\nu_{i+1} \succ \nu_{i+2} + 1$$

and $(c(\nu_i), c(\nu_{i+1})) \notin \{(ad, bc), (cd, ab)\}$ implies that

$$\nu_i \succ \nu_{i+1} + 1,$$

and we thus have (2.20).

To prove the second part, we have to show that not having the forbidden patterns in (2.14) is equivalent to the second condition.

- If we suppose that $(\nu_i, \nu_{i+1}) = (k_{cd}, k_{ab})$, we then have by (2.13) that

$$\begin{aligned} \nu_{i+1} \gg \nu_{i+2} &\iff \nu_{i+1} \succ \nu_{i+2} + 1 \\ &\iff k_{ab} \succ \nu_{i+2} + 1 \\ &\iff (k - 1)_d \succeq \nu_{i+2} + 1 \quad (\text{by (2.3)}). \end{aligned}$$

By (2.3), we then have that the fact that the patterns $k_{cd}, k_{ab}, (k - 2)_d$ and $k_{cd}, k_{ab}, (k - 2)_c$ are forbidden for $k \geq 3$ is equivalent to $(k - 1)_{cd} \succeq \nu_{i+2} + 1$, which means that $k_{cd} \succeq \nu_{i+2} + 2$.

- If we suppose that $(\nu_i, \nu_{i+1}) = ((k + 1)_{ad}, k_{bc})$ with $k \geq 3$, we then have by (2.13) that

$$\begin{aligned} \nu_{i+1} \gg \nu_{i+2} &\iff \nu_{i+1} \succ \nu_{i+2} + 1 \\ &\iff k_{cb} \succ \nu_{i+2} + 1 \\ &\iff k_a \succeq \nu_{i+2} + 1 \quad (\text{by (2.3)}). \end{aligned}$$

We then have by (2.3) that the fact that the pattern $(k + 1)_{ad}, k_{bc}, (k - 1)_a$ is forbidden for $k \geq 3$ is equivalent to $k_{ad} \succeq \nu_{i+2} + 1$, which means that $(k + 1)_{ad} \succeq \nu_{i+2} + 2$. \square

3. Bressoud's algorithm

Here we adapt the algorithm given by Bressoud in his bijective proof of Schur's partition theorem [6]. The bijection is easy to describe and execute, but its justification is more subtle and is given in the next section.

3.1. From \mathcal{O} to \mathcal{E}_1

Let us consider the following machine Φ :

Step 1: For a sequence $\lambda = \lambda_1, \dots, \lambda_t$, take the smallest $i < t$ such that $\lambda_i, \lambda_{i+1} \in \mathcal{P}$ and $\lambda_i \succ \lambda_{i+1}$ but $\lambda_i \not\geq \lambda_{i+1}$, if it exists, and replace

$$\begin{aligned} \lambda_i &\leftarrow \lambda_i + \lambda_{i+1} \text{ as a part in } \mathcal{S} \\ \lambda_j &\leftarrow \lambda_{j+1} \quad \text{for all } i < j < t \end{aligned} \quad (3.1)$$

and move to **Step 2**. We call such a pair of parts a *troublesome* pair. We observe that λ loses two parts in \mathcal{P} and gains one part in \mathcal{S} . The new sequence is $\lambda = \lambda_1, \dots, \lambda_{t-1}$. Otherwise, exit from the machine.

Step 2: For $\lambda = \lambda_1, \dots, \lambda_t$, take the smallest $i < t$ such that $(\lambda_i, \lambda_{i+1}) \in \mathcal{P} \times \mathcal{S}$ and $\lambda_i \not\geq \lambda_{i+1}$ if it exists, and replace

$$(\lambda_i, \lambda_{i+1}) \mapsto (\lambda_{i+1} + 1, \lambda_i - 1) \in \mathcal{S} \times \mathcal{P} \quad (3.2)$$

and redo **Step 2**. We say that the parts λ_i, λ_{i+1} are *crossed*. Otherwise, move to **Step 1**.

Let $\Phi(\lambda)$ be the resulting sequence after putting any $\lambda = (\lambda_1, \dots, \lambda_t) \in \mathcal{O}$ in Φ . This transformation preserves the size and the commutative product of primary colors of partitions. Let us apply this machine on the partition $(11_c, 8_d, 6_a, 4_d, 4_c, 4_b, 3_a, 2_b, 2_a, 1_d, 1_c, 1_b, 1_a)$.

$$\begin{array}{cccccccccccc} \begin{array}{l} 11_c \\ 8_d \\ 6_a \\ 4_d \\ 4_c \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 8_d \\ \mathbf{6_a} \\ \mathbf{8_{cd}} \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ \mathbf{8_d} \\ \mathbf{9_{cd}} \\ 5_a \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 7_d \\ \mathbf{5_a} \\ \mathbf{4_b} \\ \mathbf{3_a} \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ \mathbf{7_d} \\ \mathbf{9_{ab}} \\ 3_a \\ \mathbf{2_b} \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ \mathbf{3_a} \\ \mathbf{2_b} \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ \mathbf{2_a} \\ \mathbf{1_d} \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ \mathbf{1_c} \\ \mathbf{1_b} \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} . \end{array}$$

3.2. From \mathcal{E}_1 to \mathcal{O}

Let us consider the following machine Ψ :

Step 1: For a sequence $\nu = \nu_1, \dots, \nu_t$, take the greatest $i \leq t$ such that $\nu_i \in \mathcal{S}$ if it exists. If $\nu_{i+1} \in \mathcal{P}$ and $\beta(\nu_i) \not\succ \nu_{i+1}$, then replace

$$(\nu_i, \nu_{i+1}) \mapsto (\nu_{i+1} + 1, \nu_i - 1) \in \mathcal{P} \times \mathcal{S} \quad (3.3)$$

and redo **Step 1**. We say that the parts ν_i, ν_{i+1} are *crossed*. Otherwise, move to **Step 2**. If there are no more parts in \mathcal{S} , exit from the machine.

Step 2: For $\nu = \nu_1, \dots, \nu_t$, take the greatest $i \leq t$ such that $\nu_i \in \mathcal{S}$. By **Step 1**, it satisfies $\beta(\nu_i) \succ \nu_{i+1}$. Then replace

$$\begin{aligned} \nu_{j+1} &\leftarrow \nu_j & \text{for all } t \geq j > i \\ (\nu_i) &\Rightarrow (\alpha(\nu_i), \beta(\nu_i)) \text{ as a pair of parts in } \mathcal{P}, \end{aligned} \quad (3.4)$$

and move to **Step 1**. We say that the part ν_i *splits*. We observe that ν gains two parts in \mathcal{P} and loses one part in \mathcal{S} . The new sequence is $\nu = \nu_1, \dots, \nu_{t+1}$.

Let $\Psi(\nu)$ be the resulting sequence after putting any $\nu = (\nu_1, \dots, \nu_t) \in \mathcal{E}_1$ in Ψ . This transformation preserves the size and the product of primary colors of partitions. For example, applying this to $(11_c, 10_{cd}, 10_{ab}, 6_d, 5_{ab}, 3_{ad}, 2_{bc}, 1_a)$ gives

$$\begin{array}{cccccccccccc} \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 1_c + 1_b \\ 1_a \end{array} & \Rightarrow & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 1_c \\ 1_b \\ 1_a \end{array} & \Rightarrow & \begin{array}{l} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 3_a + 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \Rightarrow & \begin{array}{l} 11_c \\ 10_{cd} \\ 5_b + 5_a \\ 6_d \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 10_{cd} \\ 7_d \\ 5_a + 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 8_d \\ 5_c + 4_d \\ 5_a \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \mapsto & \begin{array}{l} 11_c \\ 8_d \\ 6_a \\ 4_d + 4_c \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} & \Rightarrow & \begin{array}{l} 11_c \\ 8_d \\ 6_a \\ 4_d \\ 4_c \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \end{array}$$

4. Proof of the well-definedness of Bressoud's maps

In this section, we will show the following proposition.

Proposition 4.1. *The transformation Φ describes a mapping from \mathcal{O} to \mathcal{E}_1 such that $\Psi \circ \Phi = Id_{\mathcal{O}}$, and Ψ describes a mapping from \mathcal{E}_1 to \mathcal{O} such that $\Phi \circ \Psi = Id_{\mathcal{E}_1}$.*

4.1. Well-definedness of Φ

In this subsection, we will show the following proposition.

Proposition 4.2. *Let us consider any $\lambda = (\lambda_1, \dots, \lambda_t) \in \mathcal{O}$, and set $\gamma^0 = 0$, $\mu^0 = \lambda$. Then, in the process Φ on λ , at the u^{th} passage from **Step 2** to **Step 1**, there exists a pair of partitions $\gamma^u, \mu^u \in \mathcal{E}_1 \times \mathcal{O}$ such that the sequence obtained is γ^u, μ^u . Moreover, if we denote by $l(\gamma^u)$ and $g(\mu^u)$ respectively the smallest part of γ^u and the greatest part of μ^u , we then have that*

- (1) $l(\gamma^u)$ is the u^{th} element in \mathcal{S} of γ^u ,
- (2) $l(\gamma^u) \gg g(\mu^u)$ so that the partition $(\gamma^u, g(\mu^u))$ is in \mathcal{E}_1 ,
- (3) for any u , γ^u is the beginning of the partition γ^{u+1} and the number of parts of μ^{u+1} is at least two less than the number of parts of μ^u .

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition in \mathcal{O} . Let us set c_1, \dots, c_t to be the primary colors of the parts $\lambda_1, \dots, \lambda_t$. Now consider the first troublesome pair $\lambda_i, \lambda_{i+1} \in \mathcal{P}$ obtained at **Step 1** in Φ , and the first resulting secondary part $\lambda_i + \lambda_{i+1}$. Note that this is reversible by **Step 2** of Ψ .

- If there is a part λ_{i+2} after λ_{i+1} , we have that

$$\begin{aligned} \lambda_i + \lambda_{i+1} - \lambda_{i+2} &= \chi(c_i < c_{i+1}) + 2\lambda_{i+1} - \lambda_{i+2} \quad \text{by (2.11)} \\ &\geq \chi(c_i < c_{i+1}) + 2\chi(c_{i+1} \leq c_{i+2}) + \lambda_{i+2} \quad \text{by (2.2)} \\ &\geq 1 + \chi(c_i \leq c_{i+2}) + \chi(c_{i+1} \leq c_{i+2}). \end{aligned}$$

Since by (2.1), we have that $c_i > c_{i+2}$ and $c_{i+1} > c_{i+2}$ implies $c_i c_{i+1} > c_{i+2}$, we then have that $\lambda_i + \lambda_{i+1} - \lambda_{i+2} \geq 1 + \chi(c_i c_{i+1} \leq c_{i+2})$, and we conclude that $\lambda_i + \lambda_{i+1} \gg \lambda_{i+2}$.

- The primary parts to the left of λ_i are well-ordered by \gg . We then have

$$\lambda_1 \gg \dots \gg \lambda_{i-1} \gg \lambda_i.$$

We obtain by (2.13) and (2.8) that for any $j < i$,

$$\lambda_j \gg \lambda_i + i - j - 1 \succeq \alpha(\lambda_i + \lambda_{i+1} + i - j - 1)$$

so that by (2.13), $\lambda_j \gg \alpha(\lambda_i + \lambda_{i+1} + i - j - 1)$. If after $i - j$ iterations of **Step 2**, $\lambda_i + \lambda_{i+1}$ crosses λ_j , we will have at the same time that

$$(\lambda_i + \lambda_{i+1} + i - j) \gg (\lambda_j - 1) \gg \dots \gg \lambda_{i-1} - 1 \quad \text{(by (2.15))} \quad (4.1)$$

$$\beta(\lambda_i + \lambda_{i+1} + i - j) \not\gg (\lambda_j - 1) \gg \dots \gg \lambda_{i-1} - 1 \quad \text{(by (2.16))}, \quad (4.2)$$

and so these iterations are reversible by **Step 1** in Ψ (recursively on $j \leq j' < i$).

- We also have by (2.13) that

$$\begin{aligned}\lambda_{i-1} \gg \lambda_i \succ \lambda_{i+1} \succ \lambda_{i+2} &\implies \lambda_{i-1} - 1 \succeq \lambda_i \succ \lambda_{i+1} \succ \lambda_{i+2} \\ &\implies \lambda_{i-1} - 1 \succ \lambda_{i+2}.\end{aligned}$$

- If we can no longer apply **Step 2** after $i - j$ iterations, we then obtain

$$\lambda_1 \gg \cdots \gg \lambda_{j-1} \gg (\lambda_i + \lambda_{i+1} + i - j) \gg (\lambda_j - 1) \gg \cdots \gg \lambda_{i-1} - 1 \succ \lambda_{i+2} \succ \cdots \succ \lambda_t$$

and we set

$$\gamma^1 = \lambda_1 \gg \cdots \gg (\lambda_i + \lambda_{i+1} + i - j) \quad (4.3)$$

$$\mu^1 = (\lambda_j - 1) \gg \cdots \gg \lambda_{i-1} - 1 \succ \lambda_{i+2} \succ \cdots \succ \lambda_t, \quad (4.4)$$

and the conditions in the proposition are respected. In fact, even if $j = i$, we saw that $\lambda_i + \lambda_{i+1} \gg \lambda_{i+2}$.

Now, by applying **Step 1** for the second time, we see by (4.4) that the next troublesome pair is either $\lambda_{i-1} - 1, \lambda_{i+2}$, or $\lambda_{i+2+x}, \lambda_{i+3+x}$ for some $x \geq 0$.

- If $\lambda_{i-1} - 1 \not\gg \lambda_{i+2}$, this means that $\lambda_{i-1} - 1, \lambda_{i+2}$ are consecutive for \succ , and **Step 1** occurs there. By (2.18), we have that $(\lambda_i + \lambda_{i+1} + 1) \gg (\lambda_{i-1} + \lambda_{i+2} - 1)$. Then, even if $(\lambda_{i-1} + \lambda_{i+2} - 1)$ crosses the primary parts $(\lambda_j - 1) \gg \cdots \gg \lambda_{i-2} - 1$ after $i - j - 1$ iterations of **Step 2**, by (2.13), we will still have that

$$(\lambda_i + \lambda_{i+1} + i - j) \gg (\lambda_{i-1} + \lambda_{i+2} + i - j - 2).$$

- If $\lambda_{i-1} - 1 \gg \lambda_{i+2}$, then the next troublesome pair appears at $\lambda_{i+2+x}, \lambda_{i+3+x}$ for some $x \geq 0$, and it forms the secondary part $\lambda_{i+2+x} + \lambda_{i+3+x}$. We also have

$$\lambda_i \succ \lambda_{i+1} \succ \lambda_{i+2} \gg \cdots \gg \lambda_{i+2+x} \succ \lambda_{i+3+x}. \quad (4.5)$$

By (2.13), we can easily check that

$$\lambda_i \succ \lambda_{i+1} \succ \lambda_{i+2} \succeq \lambda_{i+2+x} + x \succ \lambda_{i+3+x} + x$$

so that, by (2.17),

$$(\lambda_i + \lambda_{i+1}) \gg (\lambda_{i+2+x} + \lambda_{i+3+x} + 2x).$$

This means by (2.13) that,

$$(\lambda_i + \lambda_{i+1}) \gg (\lambda_{i+2+x} + \lambda_{i+3+x} + x) \quad (4.6)$$

and, as soon as $x \geq 1$, by (2.9)

$$(\lambda_i + \lambda_{i+1}) \triangleright (\lambda_{i+2+x} + \lambda_{i+3+x} + x). \quad (4.7)$$

We then obtain that, even if the secondary part $\lambda_{i+2+x} + \lambda_{i+3+x}$ crosses, after $x+i-j$ iterations of **Step 2**, the primary parts

$$\lambda_j - 1 \gg \cdots \gg (\lambda_{i-1} - 1) \gg \lambda_{i+2} \gg \cdots \gg \lambda_{i+1+x},$$

we will still have

$$(\lambda_i + \lambda_{i+1} + i - j) \gg (\lambda_{i+2+x} + \lambda_{i+3+x} + x + i - j).$$

However, as soon as $x \geq 1$, we directly have

$$(\lambda_i + \lambda_{i+1} + i - j) \triangleright (\lambda_{i+2+x} + \lambda_{i+3+x} + x + i - j).$$

In that case, the pair $(\lambda_i + \lambda_{i+1} + i - j, \lambda_{i+2+x} + \lambda_{i+3+x} + x + i - j)$ cannot have the form (k_{cd}, k_{ab}) or $((k+1)_{ad}, k_{bc})$. In order to have these patterns, we must necessarily have that the second troublesome pair is either $(\lambda_{i-1} - 1, \lambda_{i+2})$ or $(\lambda_{i+2}, \lambda_{i+3})$. In both cases, we can see that either both parts crossed the primary part l_s to the right of the pattern, or they do not move backward, so that the lower half of the second secondary part is greater than the primary part l_s to the right of the pattern in terms of \succ . In the first case, we have that

$$\begin{aligned} (l+2)_s &\not\gg \lambda_i + \lambda_{i+1} + i - j - 1 \\ \iff (l+2)_s &\not\prec \lambda_i + \lambda_{i+1} + i - j \quad \text{by (2.13)} \end{aligned}$$

and then $\lambda_i + \lambda_{i+1} + i - j \succ (l+2)_s$, so that the forbidden patterns in (2.14) do not occur. In the second case, we check the different subcases:

$$\begin{aligned} (2k'_{cd}, 2k'_{ab}, l_s) &\implies k'_a \succ l_s \\ &\implies k' - l \geq \chi(a \leq s) = 1 \\ &\implies 2k' - l \geq l + 2 \geq 3 \\ &\implies 2k' - (l+2) \geq 1 \geq \chi(cd \leq s), \\ ((2k' + 1)_{cd}, (2k' + 1)_{ab}, l_s) & \\ \implies k'_b &\succ l_s \\ \implies k' - l &\geq \chi(b \leq s) \\ \implies 2k' + 1 - l &\geq l + 1 + 2\chi(b \leq s) \\ \implies 2k' + 1 - l &\geq 2 + 2\chi(b \leq s) \end{aligned}$$

$$\begin{aligned}
&\implies (2k' + 1) - (l + 2) \geq \chi(b \leq s) \geq \chi(cd \leq s) \quad \text{since } b < cd. \\
((2k' + 2)_{ad}, (2k' + 1)_{bc}, l_s) &\implies k'_c \succ l_s \\
&\implies k' - l \geq \chi(c \leq s) \\
&\implies 2k' - l \geq l + 2\chi(c \leq s) \\
&\implies 2k' - l \geq 1 + 2\chi(c \leq s) \\
&\implies (2k' + 2) - (l + 2) \geq 1 + \chi(c \leq s) \geq \chi(ad \leq s). \\
((2k' + 1)_{ad}, 2k'_{bc}, l_s) &\implies k'_b \succ l_s \\
&\implies k' - l \geq \chi(b \leq s) \\
&\implies 2k' + 1 - l \geq l + 1 + 2\chi(b \leq s) \\
&\implies 2k' + 1 - (l + 2) \geq l - 1 + \chi(b \leq s) \geq 1 - \chi(l = 1)\chi(s = a)
\end{aligned}$$

We can see that only $(3_{ad}, 2_{bc}, 1_a)$ does not satisfy the fact that $(l + 2)_s$ is less than the first part $\lambda_i + \lambda_{i+1} + i - j$. Recall that the second part needs to be greater than $(l + 1)_s$. By Lemma 2.3, we then have the forbidden patterns

$$((k + 2)_{cd}, (k + 2)_{ab}, k_c), ((k + 2)_{cd}, (k + 2)_{ab}, k_d), ((k + 2)_{ad}, (k + 1)_{bc}, k_a),$$

with only $(3_{ad}, 2_{bc}, 1_a)$ allowed. The conditions in the proposition are satisfied after the second move from **Step 2** to **Step 1**.

By induction, Proposition 4.2 follows. Moreover, by (4.2), every single step is reversible by Ψ , since by its application the sequence γ^{u+1}, μ^{u+1} becomes exactly after the iterations of **Step 1** and splitting in **Step 2** the sequence γ^u, μ^u (with the last part of γ^{u+1}). \square

The fact that $\Phi(\mathcal{O}) \subset \mathcal{E}_1$ follows from Proposition 4.2 since μ^u strictly decreases in terms of number of parts and the process stops as soon as either μ^u has at most one part, or all its primary parts are well-ordered by \gg . And the reversibility implies that $\Psi \circ \Phi|_{\mathcal{O}} = Id_{\mathcal{O}}$.

4.2. Well-definedness of Ψ

In this subsection, we will show the following proposition.

Proposition 4.3. *Let us consider any $\nu = \nu_1, \dots, \nu_t \in \mathcal{E}_1$, and set $\gamma^0 = \nu$, $\mu^0 = 0$. Then, in the process Ψ on ν , at the u^{th} passage from **Step 2** to **Step 1**, there exists a pair of partitions $\gamma^u, \mu^u \in \mathcal{E}_1 \times \mathcal{O}$ such that the sequence obtained is γ^u, μ^u . Moreover, if we denote by $l(\gamma^u)$ and $g(\mu^u)$ respectively the smallest part of γ^u and the greatest part of μ^u , we then have that*

$$(1) \quad l(\gamma^u) \in \mathcal{P},$$

- (2) $l(\gamma^u)$ and $g(\mu^u)$ are consecutive for \succ ,
 (3) for any u , μ^u is the tail of the partition μ^{u+1} and the number of secondary parts of γ^u decreases by one at each step.

Proof. If the pattern $(3_{ad}, 2_{bc}, 1_a)$ is in ν , these parts are then the last ones. By applying Ψ , we obtain after the second passage at the tail of the partition the sequence $2_a, 1_d, 1_c, 1_b, 1_a$. Now suppose that this pattern does not occur in ν . Let us consider the last secondary part ν_i of ν .

- Suppose that **Step 1** does not occur and we directly have **Step 2**. If there is a part ν_{i-1} to its left, and $(\nu_{i-1}, \nu_i) \notin \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, we then have $\nu_{i-1} \triangleright \nu_i$ and

$$\begin{aligned} \nu_{i-1} - \alpha(\nu_i) &= \nu_{i-1} - \nu_i + \beta(\nu_i) \\ &\geq 2 \quad (\text{by (2.9) and the fact that } \beta(\nu_i) \geq 1), \end{aligned}$$

so that $\nu_{i-1} \gg \alpha(\nu_i)$. In the case that $(\nu_{i-1}, \nu_i) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, a quick check according to the parity of k shows that we also have $\nu_{i-1} \gg \alpha(\nu_i)$. If we have the pattern $(\nu_{i-2}, \nu_{i-1}) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, then $\nu_i \preceq \nu_{i-2} - 2$, and

$$\begin{aligned} \nu_{i-2} - \alpha(\nu_i) &= \nu_{i-2} - \nu_i + \beta(\nu_i) \\ &\geq 3 \quad \text{by (2.9) and the fact that } \beta(\nu_i) \geq 1 \end{aligned}$$

so that $\nu_{i-2} \succ (\alpha(\nu_i) + 2)$. Note that by Lemma 2.3, this implies that $\nu_{i-2}, \nu_{i-1}, \alpha(\nu_i)$ cannot be a forbidden pattern.

- If ν_i crosses after iteration of **Step 1** the primary parts $\nu_{i+1} \gg \dots \gg \nu_j$, we then have

$$\nu_{i-1} \gg \nu_{i+1} + 1 \gg \dots \gg \nu_j + 1 \gg \alpha(\nu_i - j + i) \quad (\text{by (2.16)}) \quad (4.8)$$

$$\nu_{i-1} \gg \nu_{i+1} + 1 \gg \dots \gg \nu_j + 1 \not\gg (\nu_i - j + i). \quad (4.9)$$

In fact, by (2.20), if $(\nu_{i-1}, \nu_i) \notin \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, we necessarily have that $\nu_{i-1} \gg \nu_i \gg \nu_{i+1}$ so that $\nu_{i-1} \gg \nu_{i+1} + 1$. If $(\nu_{i-1}, \nu_i) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, since $\nu_{i-1} \succ (\nu_{i+1} + 2) \in \mathcal{P}$, we necessarily have by (2.13) that $\nu_{i-1} \gg (\nu_{i+1} + 1)$. If we have the pattern $(\nu_{i-2}, \nu_{i-1}) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$, then $\nu_i \preceq \nu_{i-2} - 2$, and

$$\nu_{i-2} \succeq \nu_i + 2 \succ \nu_{i+1} + 3.$$

So $\nu_{i-2} \succ \nu_{i+1} + 3$, and the pattern $\nu_{i-2}, \nu_{i-1}, \nu_{i+1} + 1$ is not forbidden.

Finally, since $\nu_i \gg \nu_{i+1} \gg \dots \gg \nu_j$ and $\nu_{i+1}, \dots, \nu_j \in \mathcal{P}$, we then have by (2.13)

that $\nu_i \gg \nu_j + j - i - 1$, and this is equivalent by (2.15) to $\nu_j + 1 \not\gg (\nu_i - j + i)$. This implies that all these iterations of **Step 1** are reversible by **Step 2** of Φ .

- In the case $j = t$, we have by (4.9), (2.15) and (2.13) that

$$\nu_i - t + i \succ \nu_t.$$

If we suppose that $\nu_i - t + i$ has size 1, then ν_t has also size 1 and a color smaller than the color of ν_i . But by (2.4) and (2.1), we necessarily have that $\beta(\nu_i - t + i + 1)$ has size 1 and a color greater than the color of ν_i . We then obtain by (2.3) that

$$\beta(\nu_i - t + i + 1) \succ \nu_i - t + i \succ \nu_t,$$

so that we do not cross $\nu_i - t + i + 1$ and ν_t . This is absurd by assumption. In any case, after crossing, we still have that the secondary part size is greater than 1, so that after splitting, its upper and lower halves stay in \mathcal{P} .

- If we stop the iteration of **Step 1** just before ν_{j+1} , this means by (4.8) that

$$\nu_{i-1} \gg \nu_{i+1} + 1 \gg \cdots \gg \nu_j + 1 \gg \alpha(\nu_i - j + i) \succ \beta(\nu_i - j + i) \succ \nu_{j+1} \gg \cdots \gg \nu_t. \quad (4.10)$$

We then set

$$\begin{aligned} \gamma^1 &= \nu_1 \gg \cdots \gg \nu_{i-1} \gg \nu_{i+1} + 1 \gg \cdots \gg \nu_j + 1 \gg \alpha(\nu_i - j + i), \\ \mu^1 &= \beta(\nu_i - j + i) \succ \nu_{j+1} \succ \cdots \succ \nu_t, \end{aligned}$$

and we saw with all the different cases that the conditions of Proposition 4.3 are respected.

Let us now consider the secondary part ν_{i-x} before ν_i , for some $x \geq 1$. Then, by iteration of **Step 1**, it can never cross $\beta(\nu_i - j + i)$. In fact, suppose that it crosses all primary parts $\nu_{i-x+1} \gg \cdots \gg \nu_{i-1} \gg \nu_{i+1} + 1 \gg \cdots \gg \nu_j + 1$. We then obtain $\nu_{i-x} - x + 1 + i - j$, and since

$$\nu_{i-x} \gg \nu_{i-x+1} \gg \cdots \gg \nu_{i-1} \gg \nu_i,$$

and $\nu_{i-x+1}, \dots, \nu_{i-1} \in \mathcal{P}$, we have by (2.13) that $\nu_{i-x} - x + 1 \gg \nu_i$, which is equivalent to $\nu_{i-x} - x + 1 + i - j \gg \nu_i - j + i$. We obtain by (2.19) that

$$\begin{aligned} \text{either } & \beta(\nu_{i-x} - x + 1 + i - j) \succ \alpha(\nu_i - j + i) \\ \text{or } & \alpha(\nu_i - j + i) + 1 \gg \alpha(\nu_{i-x} - x + i - j) \succ \beta(\nu_{i-x} - x + i - j) \succ \beta(\nu_i - j + i). \end{aligned}$$

In any case, the splitting in **Step 2** occurs before $\beta(\nu_i - j + i)$. We set then

$$\begin{aligned}\gamma^2 &= \nu_1 \gg \cdots \gg \alpha(\nu_{i-x} - y), \\ \mu^2 &= \beta(\nu_{i-x} - y) \succ \cdots \succ \beta(\nu_i - j + i) \succ \nu_{j+1} \succ \cdots \succ \nu_t,\end{aligned}$$

where y is the number of iterations of **Step 1** before moving to **Step 2**, and by reasoning as before for the different cases, we can easily see that the conditions of Proposition 4.3 are respected. We obtain the result recursively. We also observe that the sequence γ^u, μ^u is exactly what we obtain by applying successively iteration of **Step 1** and **Step 2** of the transformation Φ on γ^{u+1}, μ^{u+1} . \square

By the lemma, since the number of secondary parts decreases by one at each passage from **Step 2** to **Step 1**, we will stop after exactly the number of secondary parts in ν . And the result is of the form $\gamma^U, \mu^U \in \mathcal{O}$ with γ^U well-ordered by \gg , and the last part of the first partition and the first of the second partition are consecutive in terms of \succ . We then conclude that $\Psi(\mathcal{E}_1) \subset \mathcal{O}$. Since all the steps are reversible by Φ , we also have $\Phi \circ \Psi|_{\mathcal{E}_1} = Id_{\mathcal{E}_1}$.

5. Bijective proof of Theorem 1.5

In this section, we will describe a bijection for Theorem 1.5. For brevity, we refer to the partitions in Theorem 1.5 as quaternary partitions.

5.1. From \mathcal{E}_1 to quaternary partitions

We consider the patterns $((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})$ and we sum them as follows:

$$\begin{aligned}(k+1)_{ad} + k_{bc} &= (2k+1)_{abcd} \\ k_{cd} + k_{ab} &= 2k_{abcd}.\end{aligned}\tag{5.1}$$

Let us now take a partition ν in \mathcal{E}_1 . We then identify all the patterns $(M^i, m^i) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$ and suppose that

$$\nu = \nu_1, \dots, \nu_x, M^1, m^1, \nu_{x+1}, \dots, \nu_y, M^2, m^2, \nu_{y+1}, \dots, M^t, m^t, \dots, \nu_s.$$

As long as we have a pattern ν_j, M^i, m^i , we cross the parts by replacing them using

$$\nu_j, M^i, m^i \mapsto M^i + 1, m^i + 1, \nu_j - 2.\tag{5.2}$$

At the end of the process, we obtain a final sequence

$$N^1, n^1, N^2, n^2, \dots, N^t, n^t, \nu'_1, \dots, \nu'_s.$$

Finally, the associated pair of partitions is set to be $(K^1, \dots, K^t), \nu' = (\nu'_1, \dots, \nu'_t)$, where $K^i = N^i + n^i$ according to (5.1).

To sum up the previous transformation, we only remark that, for each quaternary part K^i obtained by summing of the original pattern M^i, m^i , we add twice the number of the remaining primary and secondary parts in ν to the left of the pattern that gave K^i , while we subtract from these parts two times the number of quaternary parts obtained by patterns that occur to their right.

With the example $11_c, 10_{cd}, 10_{ab}, 6_d, 5_{ab}, 3_{ad}, 2_{bc}, 1_a$,

$$\begin{array}{ccccccc}
 \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_c \\ 10_{cd}, 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 4_{ad}, 3_{bc} \\ 3_{ab} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 5_{ad}, 4_{bc} \\ 4_d \\ 3_{ab} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 6_{ad}, 5_{bc} \\ 7_c \\ 4_d \\ 3_{ab} \\ 1_a \end{array} .
 \end{array}$$

we obtain $[(22_{abcd}, 11_{abcd}), (7_c, 4_d, 3_{ab}, 1_a)]$. We now proceed to show that the image of this mapping is indeed a quaternary partition. The inverse mapping will be presented in the next subsection.

- (1) **Quaternary parts are well-ordered.** Let us consider two consecutive patterns $(M^j, m^j) = (k_p, l_q)$ and $(M^{j+1}, m^{j+1}) = (k'_{p'}, l'_{q'})$. Since ν is well-ordered by \gg , we have by (2.13) and (2.10) that

$$l_q \triangleright l_{p_1}^1 \triangleright \cdots \triangleright l_{p_i}^i \triangleright k'_{p'} . \quad (5.3)$$

By (2.10), we then have that $l_q \succ k'_{p'} + i + 1$ so that $l - k' \geq i + 1 + \chi(q \leq p')$. Since by (2.13), $k - l = \chi(p \leq q)$ and $k' - l' = \chi(p' \leq q')$, we then have that

$$\begin{aligned}
 k + l - (k' + l') &= \chi(p \leq q) + \chi(p' \leq q') + 2(l - k') \\
 &\geq \chi(p \leq q) + \chi(p' \leq q') + 2\chi(q \leq p') + 2i + 2
 \end{aligned}$$

and we obtain that

$$\begin{aligned}
 \chi(cd \leq ab) + \chi(cd \leq ab) + 2\chi(ab \leq cd) &= 2 \\
 \chi(cd \leq ab) + \chi(ad \leq bc) + 2\chi(ab \leq ad) &= 3 \\
 \chi(ad \leq bc) + \chi(cd \leq ab) + 2\chi(bc \leq cd) &= 3 \\
 \chi(ad \leq bc) + \chi(ad \leq bc) + 2\chi(bc \leq ad) &= 2,
 \end{aligned}$$

so that $k + l - (k' + l') \geq 4 + 2i$. We will then have, after adding twice the remaining primary and secondary elements to their left, that the difference between two consecutive quaternary parts will be at least 4.

- (2) **The partition ν' is in \mathcal{E}_2 .** Let us consider two consecutive elements $\nu_x = k_p, \nu_{x+1} = l_q$. We then have for consecutive patterns M^u, m^u between k_p and l_q that

$$k_p \triangleright M^i \gg m^i \gg \cdots \gg M^j \gg m^j \triangleright l_q. \quad (5.4)$$

For the case $(M^j, m^j, l_q) \neq (3_{ad}, 2_{bc}, 1_a)$, since by Lemma 2.3, $M^u \succeq M^{u+1} + 2$, $M^j \succeq l_q + 2$, and by (2.13), we have that $k_p \succ M^i + 1$, and then

$$k_p \succ 1 + 2(j - i + 1) + l_q \implies k_p \triangleright 2(j - i + 1) + l_q. \quad (5.5)$$

For the case $(M^j, m^j, l_q) = (3_{ad}, 2_{bc}, 1_a)$, we obtain that

$$k_p - 2(j - i + 1) + 1 \succ 3_{ad} \quad (5.6)$$

and this means that $k_p - 2(j - i + 1) + 1 \succeq 3_a$ so that $k_p - 2(j - i + 1) \succeq 2_a \triangleright 1_a$.

In any case, $k_p \triangleright 2(j - i + 1) + l_q$, and this implies that after the subtraction of twice the number of the quaternary parts obtained to their right, these parts will be well-ordered by \triangleright .

- (3) **The minimal quaternary part is well-bounded.** Let us first suppose that the tail of ν consists only of patterns M^u, m^u . We then have that

$$\nu_s \triangleright M^i \gg m^i \gg \cdots \gg M^t \gg m^t$$

and, by Lemma 2.3 and (2.13), $\nu_s - 2(t - i + 1) + 1 \succeq M^t \succeq 2_{cd}$, so that $\nu'_s = \nu_s - 2(t - i + 1) \succeq 1_{cd} \succ 1_a$. This means that $1_a \notin \nu'$. We also obtain that $K^t = M^t + m^t + 2s \geq 2s + 4$.

Now suppose that the tail of ν has the form

$$l_q \triangleright \nu_u \triangleright \cdots \triangleright \nu_s, \quad (5.7)$$

with $M^t, m^t = k_p, l_q$. By (2.10), we obtain that $l_q \succ \nu_s + s - u + 1$.

- If $\nu_s = 1_a$, we then have

$$\begin{aligned} k + l &= \chi(p \leq q) + 2l \\ &\geq \chi(p \leq q) + 2(s - u + 2 + \chi(q \leq a)) \\ &= 2(s - u + 1) + 2 + \chi(p \leq q) + 2\chi(q \leq a), \end{aligned}$$

and with $(p, q) \in \{(ad, bc), (cd, ab)\}$ we have

$$\begin{aligned} \chi(ad \leq bc) + 2\chi(bc \leq a) &= 1 \\ \chi(cd \leq ab) + 2\chi(ab \leq a) &= 2 \end{aligned}$$

so that $k + l \geq 2(s - u + 1) + 3$. Then after the addition of $2(u - 1)$ for the remaining primary and secondary parts of ν to the left of the pattern (M^t, m^t) , we obtain that the smallest quaternary part is at least $2s + 3$. Note that $\nu'_s = \nu_s = 1_a$.

- When $\nu_s = h_r \neq 1_a$, we obtain that

$$\begin{aligned} k + l &\geq \chi(p \leq q) + 2(s - u + 1 + h + \chi(q \leq r)) \\ &= 2(s - u + 1) + 2h + \chi(p \leq q) + 2\chi(q \leq r), \end{aligned}$$

so that if $h \geq 2$, then $k + l \geq 2(s - u + 1) + 4$. If not, $h = 1$, and since there is no secondary part of size 1, we necessarily have that $r \geq b$, so that $\chi(q \leq r) = 1$ whenever $q \in \{ab, bc\}$. We thus obtain $k + l \geq 2(s - u + 1) + 4$. We then conclude that for $\nu_s \neq 1_a$, the smallest quaternary part is at least $2s + 4$.

In any case, we have that the smallest quaternary part is at least $2s + 4 - \chi(1_a \in \nu')$.

5.2. From quaternary partitions to \mathcal{E}_1

Recall by (5.1) that K_{abcd} splits as follows:

$$\begin{aligned} (k + 1)_{ad} + k_{bc} &= (2k + 1)_{abcd} \\ k_{cd} + k_{ab} &= 2k_{abcd} \end{aligned}$$

Let us then consider partitions (K^1, \dots, K^t) and $\nu = (\nu_1, \dots, \nu_s) \in \mathcal{E}_2$, with quaternary part K^u such that $K^t \geq 4 + 2s - \chi(1_a \in \nu)$ and $K^u - K^{u+1} \geq 4$. We also set $K^u = (k^u, l^u)$ the decomposition according to (5.1). We then proceed as follows by beginning with K^t and ν_1 ,

Step 1: If we do not encounter $K^{u+1} = (k^{u+1}, l^{u+1})$ and $\nu_i \neq 1_a$ and $\nu_i + 2 \triangleright k^u - 1$, then replace

$$\begin{aligned} \nu_i &\longmapsto \nu_i + 2 \\ (k^u, l^u) &\longmapsto (k^u - 1, l^u - 1) \end{aligned}$$

and move to $i + 1$ and redo **Step 1**. Otherwise, move to **Step 2**.

Step 2 If we encounter $K^{u+1} = k^{u+1} \gg l^{u+1}$, then split (k^u, l^u) into $k^u \gg l^u$. If not, it means that we have met ν_i such that $\nu_i + 2 \not\triangleright k^u - 1$. Then we split $k^u \gg l^u$. Since we have $\nu_i + 2 \not\triangleright k^u - 1$, which is equivalent by (2.10) to $k^u \succeq \nu_i + 2$, by Lemma 2.3, this is exactly the condition to avoid the forbidden patterns, with $k^u \gg l^u \triangleright \nu_i$.

We can now move to **Step 1** with $u - 1$ and $i = 1$.

With the example $[(22_{abcd}, 11_{abcd}), (7_c, 4_d, 3_{ab}, 1_a)]$, we obtain

$$\begin{array}{ccccccc}
 \begin{array}{c} 11_{cd}, 11_{ab} \\ 6_{ad}, 5_{bc} \\ 7_c \\ 4_d \\ 3_{ab} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 5_{ad}, 4_{bc} \\ 4_d \\ 3_{ab} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 4_{ad}, 3_{bc} \\ 3_{ab} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_c \\ 10_{cd}, 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} & \mapsto & \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} .
 \end{array}$$

It is easy to check that when two quaternary parts meet in **Step 2**, we will always have $l^u \gg k^{u+1}$, since this is exactly the condition for the minimal difference $K^u - K^{u+1} \geq 4$ and they crossed the same number of ν_i . We can also check that even if the minimal part crossed $\nu_1, \dots, \nu_s \neq 1_a$, we will still have at the end $K^t \geq 4$ and for $\nu_s = 1_a$, $K^t \geq 5$. We see with (5.1) that the size of m^t is at least equal to 2, and for the case $\nu_s = 1_a$, m^t is at least equal to $2_{bc} \gg 1_a$. The partition obtained is then in \mathcal{E}_1 .

References

- [1] K. Alladi, G.E. Andrews, A. Berkovich, A new four parameter q -series identity and its partitions implications, *Invent. Math.* 153 (2003) 231–260.
- [2] K. Alladi, G.E. Andrews, B. Gordon, Generalizations and refinements of a partition theorem of Göllnitz, *J. Reine Angew. Math.* 460 (1995) 165–188.
- [3] K. Alladi, B. Gordon, Generalization of Schur's partition theorem, *Manuscr. Math.* 79 (1993) 113–126.
- [4] G.E. Andrews, A new generalization of Schur's second partition theorem, *Acta Arith.* 14 (1968) 429–434.
- [5] G.E. Andrews, On a partition theorem of Göllnitz and related formula, *J. Reine Angew. Math.* 236 (1969) 37–42.
- [6] D. Bressoud, A combinatorial proof of Schur's 1926 partition theorem, *Proc. Am. Math. Soc.* 79 (1980) 338–340.
- [7] H. Göllnitz, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* 225 (1967) 154–190.
- [8] Padmavathamma, M. Rudy Salestina, S.R. Sudarshan, Combinatorial proof of the Göllnitz's theorem on partitions, *Adv. Stud. Contemp. Math.* 8 (1) (2004) 47–54.
- [9] I. Schur, Zur additiven zahlentheorie, *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1926, pp. 488–495.
- [10] J.Y.J. Zhao, A bijective proof of the Alladi-Andrews-Gordon partition theorem, *Electron. J. Comb.* 22 (1) (2015) 1.68.