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A method for proving polynomial enumeration formulas

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Abstract

We present an elementary method for proving enumeration formulas which are polynomials in certain parameters if others are fixed and factorize into distinct linear factors over $\mathbb{Z}$. Roughly speaking the idea is to prove such formulas by "explaining" their zeros using an appropriate combinatorial extension of the objects under consideration to negative integer parameters. We apply this method to prove a new refinement of the Bender–Knuth (ex-)Conjecture, which easily implies the Bender–Knuth (ex-)Conjecture itself. This is probably the most elementary way to prove this result currently known. Furthermore we adapt our method to q-polynomials, which allows us to derive generating function results as well. Finally we use this method to give another proof for the enumeration of semistandard tableaux of a fixed shape which differs from our proof of the Bender–Knuth (ex-)Conjecture in that it is a multivariate application of our method.
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1. Introduction

1.1. A simple example

Let $F(r,k)$ denote the number of partitions $(\lambda_1,\ldots,\lambda_r)$, i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$, of length r, with parts in $\{0,1,\ldots,k\}$. It is basic combinatorial knowledge

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that

$$F(r,k) = \binom{k+r}{r} = \frac{(k+1)\cdot(k+2)\cdots(k+r)}{r!}.$$

For fixed r this expression is a polynomial in k with distinct integer zeros. In this paper we present an elementary method for proving polynomial enumeration formulas of that type, together with some non-trivial applications. The underlying idea is to find the appropriate extension of the combinatorial objects under consideration to (typically) negative integer parameters and with this "explain" the zeros of the enumeration polynomial.

To be more concrete let us first demonstrate this 3 step method in terms of our simple example.

(1) In the first step we extend the combinatorial interpretation of $F(r,k)$ to negative integer k's. For $k < 0$ we define

$$F(r,k) = (-1)^r[\#(\lambda_1,\ldots,\lambda_r) \in \mathbb{Z}^r \text{ with } k < \lambda_1 < \lambda_2 < \ldots < \lambda_r < 0].$$

This definition seems to appear from nowhere, however, the following step should convince us that it was a good choice.

(2) In this step we show that for fixed r the function $k \to F(r,k)$ can be expressed by a polynomial in k of degree at most r. This is equivalent to $\Delta^{r+1}F(r,k) = 0$, where the differences are taken with respect to the parameter k. In order to show this we use induction with respect to r. The initial step follows from $F(1,k) = k+1$. Assume that $r > 1$ and $k \geqslant 0$. Then

$$\begin{aligned}
\Delta F(r,k) &= F(r,k+1) - F(r,k)\\
&= [\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_r \geqslant 0]\\
&\quad -[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_r \geqslant 0]\\
&= [\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_r \geqslant 0 \text{ and } \lambda_1 = k+1]\\
&= F(r-1,k+1).
\end{aligned}$$

If $k < 0$ we have

$$\begin{aligned}
\Delta F(r,k) &= F(r,k+1) - F(r,k)\\
&= (-1)^r[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 < \lambda_1 < \ldots < \lambda_r < 0]\\
&\quad -(-1)^r[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k < \lambda_1 < \ldots < \lambda_r < 0]\\
&= (-1)^{r-1}[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k < \lambda_1 < \ldots < \lambda_r < 0 \text{ and } \lambda_1 = k+1]\\
&= F(r-1,k+1).
\end{aligned}$$

The induction hypothesis implies $\Delta^r F(r-1,k+1) = 0$ and thus $\Delta^{r+1}F(r,k) = 0$.

(3) In the final step we explore the integer zeros of $F(r,k)$ in k. Consider the definition of $F(r,k)$ for negative k's and observe that $F(r,k) = 0$ for $k = -1,-2,\ldots,-r$. By Step 2 $F(r,k)$ is a polynomial in k and therefore it has the factor $(k+1)_r$, where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = \prod_{i=0}^{n-1}(a+i)$. The degree estimation of Step 2 implies that this factor determines $F(r,k)$ up to a factor independent of k. Observe that $F(r,0) = 1$, and thus this factor is equal to $1/r!$ and the formula is proved.

1.2. The method

We summarize the general strategy in the example above and with this establish our method for proving polynomial enumeration formulas. It applies to the enumeration of combinatorial objects which depend on an integer parameter k and where we suspect the existence of an enumeration formula which is polynomial in k and factorizes into distinct linear factors over $\mathbb{Z}$. The method is divided into the following three steps.

(1) *Extension of the combinatorial interpretation*: Typically the admissible domain of k is a set S of non-negative integers. In the first step of our method we have to find (most likely new) combinatorial objects indexed by an *arbitrary* integer k which are in bijection with the original objects for $k \in S$.
(2) *The extending objects are enumerated by a polynomial*: The extension of the combinatorial interpretation in the previous step has to be chosen so that we are able to prove that the new objects are enumerated by a polynomial in k. In many cases this is done with the help of a recursion. Moreover the degree of this polynomial has to be computed.
(3) *Exploring "natural" linear factors*: Finally one has to find the k's for which there exist none of these objects, i.e. one has to compute the (integer) zeros of the polynomial. [1] Typically these zeros will not lie in S, which made the extension in Step 1 necessary. Moreover one has to find an additional (and thus non-zero) evaluation of the polynomial which is easy to compute. The zeros and the single non-zero evaluation determine the polynomial uniquely and we are finally able to compute it.

The last step shows the limits of this method. Even if one succeeds in the first two steps, it may be that the polynomial has non-integer zeros or multiple zeros and the method as described does not work. On the other hand the enumeration problems which result in polynomials that factorize totally over $\mathbb{Z}$ are exactly the one we are especially interested in and where we are longing for an understanding of the simplicity of the result.

1.3. A refinement of the Bender–Knuth (ex-)Conjecture

Next we explain a plane partition enumeration result we have obtained by using this method. The main purpose of the rest of the paper is the proof of this result. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. A *strict plane partition* of shape λ is an array $\pi_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \lambda_i}$ of non-negative integers such that the rows are weakly decreasing and the columns are strictly decreasing. The *norm* $n(\pi)$ of a strict plane partition is defined as the sum of its parts and π is said to be a strict plane partition of the non-negative integer $n(\pi)$. For instance

7 5 5 4 3 2
6 4 3 2
5 2
3 1

[1] In the first step it may have been necessary to introduce a signed enumeration outside of the admissible domain in order to have the same enumeration polynomial for all k's. In this case we have to find the k's for which objects cancel in pairs with respect to the sign.

is a strict plane partition of shape (6, 4, 2, 2) with norm 52. In [3, p. 50] Bender and Knuth conjectured that the generating function of strict plane partitions with at most c columns, parts in $\{1, 2, \ldots, n\}$ and with respect to this norm is equal to

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1}$ and $[a;q]_n = \prod_{i=0}^{n-1}[a+i;q]$. This conjecture was proved by Andrews [1], Gordon [8], Macdonald [13, Ex. 19, p. 53] and Proctor [15, Proposition 7.2]. For related papers, which mostly include generalizations of the Bender–Knuth (ex-)Conjecture see [5,6,10,11,16,19].

Using a "q-extension" of our method we have obtained the following new refinement of this result. As an additional parameter k we introduce the number of parts equal to n in the strict plane partition.

Theorem 1. *The generating function of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$, *at most c columns and k parts equal to n is*

$$\sum q^{n(\pi)} = \frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

If we sum this generating function over all k's, $0 \leqslant k \leqslant c$, we easily obtain the Bender–Knuth (ex-)Conjecture. Probably this detour via Theorem 1 is the easiest and most elementary way to prove the Bender–Knuth (ex-)Conjecture currently known. In [12, Section 3] the authors come to the conclusion that all other proofs of the Bender–Knuth (ex-)Conjecture "share more or less explicitly an identity, which relates Schur functions and odd orthogonal characters of the symmetric group of rectangular shape". In our elementary proof this is not the case.

We first prove the special case $q = 1$ of Theorem 1, i.e. we compute the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n, see Theorem 2. (Observe that for $q = 1$ the formula in Theorem 1 is a polynomial in k, which factorizes into distinct linear factors over $\mathbb{Z}$.) This result is new as well. Later we will see that the method can be extended to q-polynomials in order to prove the general result.

1.4. Outlook and outline of the paper

We plan to apply this method to other enumeration problems in the future. The most ambitious project in this direction is probably our current effort to give another proof of the Refined Alternating Sign Matrix Theorem. There is some hope for a proof along the lines of the proof of Theorem 1, for details see Section 7. Moreover we plan to extend our method to polynomial enumeration formulas that do not factor into distinct linear factors over $\mathbb{Z}$. Hopefully the lack of integer zeros can be compensated by other properties of the polynomial.

The rest of the paper is organized as follows. In Section 2 we introduce our combinatorial extension with respect to k of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c

columns and k parts equal to n as proposed in *Step 1* of our method above. In Section 3 we show that these objects are enumerated by a polynomial in k which is of degree at most $2n-2$ (*Step 2*) and in Section 4 we show that the polynomial has the predicted zeros (*Step 3*). This concludes the proof of Theorem 1 for $q=1$. In Section 5 we apply the method to give another proof of the formula for the number of semistandard tableaux of a fixed shape. This application of our method is of interest since in this case we have to work with more than just one polynomial parameter. Finally we extend our method to what we call "q-polynomials" and prove Theorem 1 in its full strength in Section 6. In Section 7 a connection of our result to the Refined Alternating Sign Matrix Theorem is presented.

Throughout the whole article we use the extended definition of the summation symbol, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a)+f(a+1)+\cdots+f(b) & \text{if } a \leqslant b, \\ 0 & \text{if } b=a-1, \\ -f(b+1)-f(b+2)-\cdots-f(a-1) & \text{if } b+1 \leqslant a-1. \end{cases} \tag{1.1}$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I containing $\mathbb{Q}$ there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y. We usually write $\sum_{x=0}^{y} p(x)$ for $q(y)$. We use the analogous extended definition of the product symbol. In particular this extends the definition of the Pochhammer symbol, for instance $(a)_{-1} = 1/(a-1)$.

2. From strict plane partitions to generalized $(n-1, n, c)$ Gelfand–Tsetlin patterns

Let n be a positive integer. A Gelfand–Tsetlin pattern with n rows is a triangular array of integers, say

$$\begin{array}{ccccccccccccc} & & & & & & a_{n,n} & & & & & & \\ & & & & & a_{n-1,n-1} & & a_{n-1,n} & & & & & \\ & & & & \cdots & & \cdots & & \cdots & & & & \\ & & & a_{3,3} & & \cdots & & \cdots & & a_{3,n} & & & \\ & & a_{2,2} & & a_{2,3} & & \cdots & & \cdots & & a_{2,n} & & \\ & a_{1,1} & & a_{1,2} & & a_{1,3} & & \cdots & & \cdots & & a_{1,n} & \end{array} ,$$

such that $a_{i,j} \leqslant a_{i-1,j}$ for $1 < i \leqslant j \leqslant n$ and $a_{i,j} \leqslant a_{i+1,j+1}$ for $1 \leqslant i \leqslant j < n$, see [18, p. 313] or [7, (3)] for the original reference. An example of a Gelfand–Tsetlin pattern with 7 rows is given below.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 1 & & 3 & & & & \\ & & & 0 & & 1 & & 2 & & 4 & & & \\ & & 0 & & 1 & & 1 & & 3 & & 5 & & \\ & 0 & & 0 & & 1 & & 2 & & 4 & & 6 & \\ 0 & & 0 & & 0 & & 2 & & 2 & & 4 & & 6 \end{array}$$

The following correspondence between Gelfand–Tsetlin patterns and strict plane partitions is crucial for our paper.

Lemma 1. *There is a bijection between Gelfand–Tsetlin patterns* $(a_{i,j})$ *with n rows, parts in* $\{0, 1, \ldots, c\}$ *and fixed* $a_{n,n} = k$*, and strict plane partitions with parts in* $\{1, 2, \ldots, n\}$*, at most c columns and k parts equal to n. In this bijection* $(a_{1,n}, a_{1,n-1}, \ldots, a_{1,1})$ *is the shape of the strict plane partition.*

Proof. Given such a Gelfand–Tsetlin pattern, the corresponding strict plane partition is such that the shape filled by entries greater than i corresponds to the partition given by the $(n-i)$th row of the Gelfand–Tsetlin pattern, counting from the top. As an example, consider the strict plane partition in the introduction. If we choose $n = 7$ and $c = 6$ then this strict plane partition corresponds to the Gelfand–Tsetlin pattern above. □

Therefore it suffices to enumerate Gelfand–Tsetlin patterns $(a_{i,j})$ with n rows, parts in $\{0, 1, \ldots, c\}$ and fixed $a_{n,n} = k$. Why should this be easier than enumerating the corresponding strict plane partitions? Recall that k is the polynomial parameter in our refinement of the Bender–Knuth (ex-)Conjecture which we want to make use of when applying our method. In order to accomplish Step 1 of the method we have to find a "natural" extended definition of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n, where k is an *arbitrary* integer which does not necessarily lie in $\{0, 1, \ldots, c\}$. (Parts equal to n may only appear in the first row of a (column-)strict plane partition with parts in $\{1, 2, \ldots, n\}$ and thus, when considering strict plane partitions, we must have $k \in \{0, 1, \ldots, c\}$.) "Natural" stands for the fact that the extension has to be chosen such that the extending objects are enumerated by a polynomial in k. In order to find this extension it seems easier to work with Gelfand–Tsetlin patterns rather than with strict plane partitions. Next we define generalized Gelfand–Tsetlin patterns which turn out to be the right extension.

Let r, n, c be integers, r non-negative and n positive. In this paper a *generalized* (r, n, c) *Gelfand–Tsetlin pattern* (for short: (r, n, c)-pattern) is an array $(a_{i,j})_{1 \leqslant i \leqslant r+1, i-1 \leqslant j \leqslant n+1}$ of integers with

(1) $a_{i,i-1} = 0$ and $a_{i,n+1} = c$,
(2) if $a_{i,j} \leqslant a_{i,j+1}$ then $a_{i,j} \leqslant a_{i-1,j} \leqslant a_{i,j+1}$,
(3) if $a_{i,j} > a_{i,j+1}$ then $a_{i,j} > a_{i-1,j} > a_{i,j+1}$.

A $(3, 6, c)$-pattern for example is of the form

$$\begin{array}{ccccccccccc}
 & & & 0 & & a_{4,4} & & a_{4,5} & & a_{4,6} & & c \\
 & & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & a_{3,6} & & c \\
 & 0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & a_{2,6} & & c \\
0 & & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} & & a_{1,6} & & c,
\end{array}$$

such that every entry not in the top row is between its northwest neighbour w and its northeast neighbour e, if $w \leqslant e$ then weakly between, otherwise strictly between.

Thus

$$\begin{array}{ccccccccccc} & & 0 & & 3 & & -5 & & 10 & & 4 \\ & 0 & & 2 & & -2 & & 3 & & 8 & \\ 0 & & 2 & & -1 & & 2 & & 4 & & 7 \\ & 0 & & 0 & & 1 & & 2 & & 5 & \end{array} \quad 6 \quad 4$$

is an example of an $(3, 6, 4)$-pattern. Note that a generalized $(n-1, n, c)$ Gelfand–Tsetlin pattern $(a_{i,j})$ with $0 \leqslant a_{n,n} \leqslant c$ is a Gelfand–Tsetlin pattern with n rows and parts in $\{0, 1, \ldots, c\}$ as defined at the beginning of this section. This is because $0 \leqslant a_{n,n} \leqslant c$ implies that the third possibility in the definition of a generalized Gelfand–Tsetlin pattern never occurs.

Next we introduce the sign of an (r, n, c)-pattern $a = (a_{i,j})$, since we actually have to work with a signed enumeration if $a_{n,n} \notin \{0, 1, \ldots, c\}$. A pair $(a_{i,j}, a_{i,j+1})$ with $a_{i,j} > a_{i,j+1}$ and $i \neq 1$ is called an *inversion* of the (r, n, c)-pattern and $(-1)^{\#\text{ of inversions}}$ is said to be the *sign* of the pattern, denoted by $\operatorname{sgn}(a)$. The $(3, 6, 4)$-pattern in the example above has 6 inversions altogether and thus its sign is 1. We define the following expression

$$F(r, n, c; k_1, k_2, \ldots, k_{n-r}) = \sum_a \operatorname{sgn}(a),$$

where the sum is over all (r, n, c)-patterns $a = (a_{i,j})$ with top row defined by $a_{r+1,r+i} = k_i$ for $i = 1, \ldots, n-r$. Now it is important to observe that for $0 \leqslant k \leqslant c$ the number of $(n-1, n, c)$-patterns with $a_{n,n} = k$ is given by $F(n-1, n, c; k)$. This is because an $(n-1, n, c)$-pattern with $0 \leqslant a_{n,n} \leqslant c$ has no inversions. Thus $F(n-1, n, c; k)$ is the quantity we want to compute. It has the advantage that it is defined for all integers k, whereas our original enumeration problem was only defined for $0 \leqslant k \leqslant c$.

3. $F(n-1, n, c; k)$ is a polynomial in k of degree at most $2n-2$

In this section we establish Step 2 of the method above for our refinement of the Bender–Knuth (ex-)Conjecture. The following recursion for $F(r, n, c; k_1, k_2, \ldots, k_{n-r})$ is fundamental.

$$\begin{aligned} &F(r, n, c; k_1, k_2, \ldots, k_{n-r}) \\ &\quad = \sum_{l_1=0}^{k_1} \sum_{l_2=k_1}^{k_2} \sum_{l_3=k_2}^{k_3} \cdots \sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}} \sum_{l_{n-r+1}=k_{n-r}}^{c} F(r-1, n, c; l_1, l_2, \ldots, l_{n-r+1}). \end{aligned} \tag{3.1}$$

It is obvious for $(k_1, k_2, \ldots, k_{n-r})$ with $0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_{n-r} \leqslant c$. After recalling the extended definition of the summation symbol (1.1) one observes that the generalized (r, n, c) Gelfand–Tsetlin patterns and $F(r, n, c; k_1, \ldots, k_{n-r})$ were simply defined in such a way that this recursion holds for arbitrary integer tuples $(k_1, \ldots, k_{n-r})$. This recursion together with the initial condition

$$F(0, n, c; k_1, k_2, \ldots, k_n) = 1$$

implies the following lemma.

Lemma 2. *Let r, n be integers, r non-negative and n positive. Then $F(r, n, c; k_1, \ldots, k_{n-r})$ can be expressed by a polynomial in the k_i's and in c.*

In the following $F(r, n, c; k_1, \ldots, k_{n-r})$ is identified with this polynomial. In particular $F(n-1, n, c; k)$ is a polynomial in k and with this we have established the first half of Step 2 in our method. Next we aim to show that $F(r, n, c; k_1, \ldots, k_{n-r})$ is of degree at most $2r$ in every k_i. This will imply that $F(n-1, n, c; k)$ is of degree at most $2n-2$ in k and completing Step 2. However, this degree estimation is complicated and takes Lemmas 3–6.

The degree of $F(r, n, c; k_1, \ldots, k_{n-r})$ in k_i is the degree of

$$\sum_{l_i=k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} F(r-1, n, c; l_1, \ldots, l_{n-r+1}), \tag{3.2}$$

in k_i, where $k_0 = 0$ and $k_{n-r+1} = c$. Let us assume by induction with respect to r that the degree of $F(r-1, n, c; l_1, \ldots, l_{n-r+1})$ is at most $2r-2$ in each of l_i and l_{i+1}. By (3.2) we can easily conclude that the degree of $F(r, n, c; k_1, \ldots, k_{n-r})$ in k_i is at most $4r-2$; however, we want to establish that the degree is at most $2r$. The following lemma shows how to obtain a sharper degree estimation in summations of our type.

In order to state this lemma we have to define an operator D_i which turns out to be crucial for the analysis of the recursion in (3.1). Let $G(k_1, k_2, \ldots, k_m)$ be a function in m variables and $1 \leqslant i \leqslant m-1$. We set

$$\begin{aligned} D_i G(k_1, \ldots, k_m) := {} & G(k_1, \ldots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \ldots, k_m) \\ & + G(k_1, \ldots, k_{i-1}, k_{i+1}+1, k_i - 1, k_{i+2}, \ldots, k_m). \end{aligned}$$

Lemma 3. *Let $F(x_1, x_2)$ be a polynomial in x_1 and x_2 which is of degree at most R in each of x_1 and x_2. Moreover assume that $D_1 F(x_1, x_2)$ is of degree at most R as a polynomial in x_1 and x_2, i.e. a linear combination of monomials $x_1^m x_2^n$ with $m + n \leqslant R$. Then $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F(x_1, x_2)$ is of degree at most $R+2$ in y. If $D_1 F(x_1, x_2) = 0$ then $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F(x_1, x_2)$ is of degree at most $R+1$ in y.*

Proof. Set $F_1(x_1, x_2) = D_1 F(x_1, x_2)/2$ and $F_2(x_1, x_2) = (F(x_1, x_2) - F(x_2+1, x_1-1))/2$. Clearly $F(x_1, x_2) = F_1(x_1, x_2) + F_2(x_1, x_2)$. Observe that $F_2(x_2+1, x_1-1) = -F_2(x_1, x_2)$. Thus $F_2(x_1, x_2)$ is a linear combination of terms of the form $(x_1)_m (x_2+1)_n - (x_1)_n (x_2+1)_m$ with $m, n \leqslant R$. Now observe that

$$\begin{aligned} \sum_{x_1=a}^{y} \sum_{x_2=y}^{b} (x_1)_m (x_2+1)_n - (x_1)_n (x_2+1)_m = {} & \frac{1}{m+1} \frac{1}{n+1} \\ & \times ((a-1)_{n+1} (b+1)_{m+1} - (a-1)_{m+1} (b+1)_{n+1} - (a-1)_{n+1} (y)_{m+1} \\ & + (b+1)_{n+1} (y)_{m+1} + (a-1)_{m+1} (y)_{n+1} - (b+1)_{m+1} (y)_{n+1}) \end{aligned}$$

and thus $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b} F_2(x_1,x_2)$ is a polynomial of degree at most $R+1$ in y. By the assumption in the lemma $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b} F_1(x_1,x_2)$ is of degree at most $R+2$ in y and the assertion follows. □

Thus it suffices to show that $D_i F(r,n,c;.)(k_1,\dots,k_{n-r})$ is of degree at most $2r$ as a polynomial in k_i and k_{i+1}. In Lemma 5 we show a much stronger assertion, namely we prove a formula which expresses $D_i F(r,n,c;k_1,\dots,k_{n-r})$ as a product of $F(r,n-2,c+2;k_1,\dots,k_{i-1},k_{i+2}+2,\dots,k_{n-r}+2)$ and an (explicit) polynomial in k_i and k_{i+1} which is obviously of degree $2r$. For the proof of Lemma 5 we need another lemma.

Lemma 4. *Let $G(l_1,l_2,l_3)$ be a function on $\mathbb{Z}^3$. Then*

$$\begin{aligned} &D_1\left(\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3} G(l_1,l_2,l_3)\right)(k_0,k_1,k_2,k_3)\\ &\quad=-\frac{1}{2}\left(\sum_{l_1=k_1+1}^{k_2+1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} D_1G(l_1,l_2,l_3)\right.\\ &\qquad\left.+\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_2-1} D_2G(l_1,l_2,l_3)\right). \end{aligned}$$

If $H(l_1,l_2)$ is a function on $\mathbb{Z}^2$ then

$$D_1\left(\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3} H(l_1,l_2)\right)(k_1,k_2,k_3)=-\frac{1}{2}\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_1-1}^{k_2-1} D_1H(l_1,l_2).$$

Proof. The left-hand side in the first statement of the lemma is equal to

$$\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3} G(l_1,l_2,l_3)+\sum_{l_1=k_0}^{k_2+1}\sum_{l_2=k_2+1}^{k_1-1}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3).$$

In this formula, we reverse the middle sum of the second triple sum and split up the first sum in this triple sum to obtain

$$\begin{aligned} &\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3} G(l_1,l_2,l_3)-\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3)\\ &\quad-\sum_{l_1=k_1+1}^{k_2+1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3). \end{aligned}$$

Next we cancel some common terms in the first and the second sum and obtain

$$-\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_2-1} G(l_1,l_2,l_3) - \sum_{l_1=k_1+1}^{k_2+1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3),$$

which is equal to the right-hand side of the first statement. The proof of the second formula is easy. □

Let $r \geqslant 1$. We need the following identity.

$$\sum_{x'=x}^{y}\sum_{y'=x-1}^{y-1} (y'-x'-r+3)_{2r-3}(y'-x'+1) = \frac{1}{r(2r-1)}(y-x-r+2)_{2r-1}(y-x+1). \tag{3.3}$$

It follows from

$$(y'-x'-r+3)_{2r-3}(y'-x'+1) = \frac{1}{2}((y'-x'-r+2)_{2r-2} - (y'-x'-r+3)_{2r-2}),$$

the summation formula

$$\sum_{z=a}^{b}(z+w)_n = \frac{1}{n+1}((b+w)_{n+1} - (a-1+w)_{n+1})$$

and the fact that $(-r+1)_{2r} = 0$ and $(-r)_{2r} = 0$.

Lemma 5. *Let r, n, i be integers, r non-negative, n positive, $2 \leqslant n-r$ and $1 \leqslant i \leqslant n-r-1$. Then*

$$\begin{aligned} &D_i F(r,n,c;.)(k_1,\dots,k_{n-r}) \\ &\quad = (-1)^r \frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1) \\ &\qquad \times F(r,n-2,c+2;k_1,\dots,k_{i-1},k_{i+2}+2,\dots,k_{n-r}+2). \end{aligned}$$

Proof. We show the assertion by induction with respect to r. For $r = 0$ there is nothing to prove. We assume $r > 0$. By (3.1) and Lemma 4 the left-hand side in the statement is equal to

$$-\frac{1}{2}\left(\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_i+1}^{k_{i+1}+1}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}\right. \\ \left. D_i F(r-1,n,c;.)(l_1,\dots,l_{n-r+1})\right.$$

$$+\sum_{l_1=0}^{k_1}\cdots\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+1}-1}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}$$
$$D_{i+1}F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})\Bigg).$$

In this formula, we replace $D_iF(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})$ and $D_{i+1}F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})$ by the expressions implied by the induction hypothesis. Furthermore we apply (3.3) to obtain

$$\begin{aligned}
&(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)\\
&\times\Bigg(\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}\\
&\quad F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_{i+2}+2,l_{i+3}+2,\ldots,l_{n-r+1}+2)\\
&+\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}\\
&\quad F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_i,l_{i+3}+2,\ldots,l_{n-r+1}+2)\Bigg).
\end{aligned}$$

We shift the range of summation of $l_{i+2},l_{i+3},\ldots,l_{n-r+1}$ by two and compensate this shift with the appropriate change in the summand. In the first multiple sum, we rename l_{i+2} to l_i and merge the two multiple sums. We obtain

$$\begin{aligned}
&(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)\\
&\times\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_{i+2}+2}\sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2}\cdots\sum_{l_{n-r+1}=k_{n-r}+2}^{c+2}\\
&\quad F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_i,l_{i+3},\ldots,l_{n-r+1})\\
&=(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)\\
&\quad\times F(r,n-2,c+2;k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2)
\end{aligned}$$

and the assertion follows. □

We are finally able to prove the degree lemma.

Lemma 6. *Let r,n,i be integers, r non-negative, n positive and $1\leqslant i\leqslant n-r$. Then $F(r,n,c;k_1,\ldots,k_{n-r})$ is a polynomial in k_i of degree at most $2r$.*

Proof. We prove the assertion by induction with respect to r. For $r=0$ it is trivial. Assume $r>0$ and $1\leqslant i\leqslant n-r$. The degree of $F(r,n,c;k_1,\ldots,k_{n-r})$ in k_i is the degree of (3.2)

in k_i. By Lemma 5 the degree of $D_i F(r-1, n, c; l_1, \ldots, l_{n-r+1})$ as a polynomial in l_i and l_{i+1} is $2r-2$. Moreover the degree of $F(r-1, n, c; l_1, \ldots, l_{n-r+1})$ in l_i as well as in l_{i+1} is at most $2r-2$ by the induction hypothesis. The assertion follows from Lemma 3. □

4. Exploring the zeros of $F(n-1, n, c; k)$

We finally establish Step 3 of our method for the refinement of the Bender–Knuth (ex-)Conjecture.

Lemma 7. *Let r, n, i be integers, r non-negative, n positive and $1 \leqslant i \leqslant n-r$. Then there exists no (r, n, c)-pattern with first row $(0, k_1, \ldots, k_{n-r}, c)$, if $k_1 = -1, -2, \ldots, -r$ or $k_{n-r} = c+1, c+2, \ldots, c+r$.*

Proof. Suppose $(a_{i,j})$ is an (r, n, c)-pattern with $a_{r+1,r+1} \in \{-1, -2, \ldots, -r\}$. In particular we have $0 > a_{r+1,r+1}$ and thus the definition of (r, n, c)-patterns implies that $0 > a_{r,r} > a_{r+1,r+1}$. In a similar way we obtain $0 > a_{1,1} > a_{2,2} > \ldots > a_{r,r} > a_{r+1,r+1}$. This is, however, a contradiction, since there exist no r distinct integers strictly between 0 and $a_{r+1,r+1}$ if $a_{r+1,r+1} \in \{-1, -2, \ldots, -r\}$. The case that $a_{r+1,n} \in \{c+1, c+2, \ldots, c+r\}$ is similar. □

We obtain the following corollary.

Corollary 1. *$F(n-1, n, c; k)/((1+k)_{n-1}(1+c-k)_{n-1})$ is independent of k.*

Proof. By Lemma 7, $(1+k)_{n-1}(1+c-k)_{n-1}$ is a factor of $F(n-1, n, c; k)$. By Lemma 6, $F(n-1, n, c; k)$ is of degree at most $2n-2$ in k and the assertion follows. □

Theorem 2. *The number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n is given by*

$$F(n-1, n, c; k) = \frac{(1+k)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$

Proof. We prove the assertion by induction with respect to n. Observe that the formula is true for $n = 1$ since $F(0, 1, c; k) = 1$. Assume $n > 1$. By Corollary 1

$$F(n-1, n, c; k) = (1+k)_{n-1}(1+c-k)_{n-1} \frac{F(n-1, n, c; c)}{(1+c)_{n-1}(1)_{n-1}}.$$

Observe that if we have $a_{n,n} = c$ in an $(n-1, n, c)$-pattern $(a_{i,j})_{1 \leqslant i \leqslant n, i-1 \leqslant j \leqslant n+1}$ then $a_{i,n} = c$ for all i. This implies the recursion

$$F(n-1, n, c; c) = \sum_{k=0}^{c} F(n-2, n-1, c; k). \tag{4.1}$$

We need one other ingredient, namely the following hypergeometric identity

$$\begin{aligned}
&\sum_{k=0}^{c} (1+k)_{m-1}(1+c-k)_{m-1} \\
&\quad = (1)_{m-1}^2 \sum_{k=0}^{c} \binom{m+k-1}{m-1} \binom{c-k+m-1}{m-1} \\
&\quad = (1)_{m-1}^2 \binom{c+2m-1}{2m-1} = \frac{(1)_{m-1}^2 (c+1)_{2m-1}}{(1)_{2m-1}},
\end{aligned} \tag{4.2}$$

where the second equality is equivalent to the Chu–Vandermonde identity; see [9, p. 169, (5.26)]. With the help of the recursion (4.1), the induction hypothesis for $F(n-2, n-1, c; k)$ and the hypergeometric identity we are able to compute $F(n-1, n, c; c)$ and with this $F(n-1, n, c; k)$. □

Remark 1. By the symmetry of Schur functions, the number of strict plane partitions of a fixed shape with x_i parts equal to i is equal to the number of strict plane partitions with $x_{\pi(i)}$ parts equal to i for every permutation π. Thus Theorem 2 gives the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to i for arbitrary $i \in \{1, 2, \ldots, n\}$. However, note that this does not generalize to the generating function of these objects.

Corollary 2 (*Andrews [1], Gordon [8], Macdonald [13], Proctor [15]*). *The number of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$ *and at most c columns is*

$$\prod_{i=1}^{n} \frac{(c+i)_i}{(i)_i}.$$

Proof. By Theorem 2 the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$ and at most c columns equals

$$\sum_{k=0}^{c} \frac{(k+1)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$

The assertion now follows from (4.2). □

5. Semistandard tableaux of a fixed shape

In this section we apply our method to the enumeration of semistandard tableaux of a fixed shape. This result is certainly well-known. Nonetheless we think it might be interesting for

the reader to see another application of our method which moreover uses more than just one "polynomial parameter" as opposed to the single parameter k in the example above. (At this point the reader may wonder what we mean by a multivariate application of our method, since we have only described the case of a single polynomial parameter in the introduction. However, it is straightforward to generalize this method to a multivariate version, as should become clear in this section.)

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition and r a positive integer. A semistandard tableau of shape λ with entries between 1 and r is a filling of the Ferrers diagram of shape λ with entries in $\{1, 2, \ldots, r\}$ such that rows are weakly increasing and columns are strictly increasing. (Semistandard tableaux and strict plane partitions are equivalent objects. Indeed, if we replace every entry e in a semistandard tableau with entries between 1 and r with $r - e$ we clearly obtain a strict plane partition. However, we choose to use the notion of semistandard tableaux in this section for historical reasons.) It is well-known [18, p. 375, in (7.105) $q \to 1$] that the number of semistandard tableaux of shape λ with entries between 1 and r is

$$\prod_{1 \leqslant i < j \leqslant k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{i=1}^{k} \frac{(\lambda_i + k + 1 - i)_{r-k}}{(k + 1 - i)_{r-k}}$$

if $k \leqslant r$, otherwise this number is obviously zero by columnstrictness. If $k = r$ the formula simplifies to

$$\prod_{1 \leqslant i < j \leqslant r} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{5.1}$$

It suffices to prove this formula, for the number of semistandard tableaux of shape $(\lambda_1, \ldots, \lambda_k)$ is equal to the number of semistandard tableaux of shape $(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ ($r - k$ zeros).

The expression in (5.1) is a polynomial in the λ_i's which is up to a constant determined by its zeros $\lambda_i = \lambda_j - j + i$, $1 \leqslant i < j \leqslant r$. Clearly the number of semistandard tableaux of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with entries between 1 and r can be interpreted to be zero if $\lambda_i = \lambda_j - j + i$ for some i, j with $1 \leqslant i < j \leqslant r$, since λ is not a partition in this case. However, we have to find a combinatorial extension to arbitrary $\lambda \in \mathbb{Z}^r$ such that the number of objects is zero if and only if $\lambda_i = \lambda_j - j + i$ for some i, j with $1 \leqslant i < j \leqslant r$, and thus this number is non-zero for many $(\lambda_1, \ldots, \lambda_r)$ that are not partitions. Again the proof is divided into three steps.

5.1. Extension

We extend the combinatorial interpretation of the number of semistandard tableaux of shape λ to arbitrary $\lambda \in \mathbb{Z}^r$. The situation is somehow "reversed" to that in our Bender–Knuth (ex-)Conjecture refinement. In Lemma 1 we have showed that semistandard tableaux of shape λ with enries in $\{1, 2, \ldots, r\}$ are in bijection with Gelfand–Tsetlin patterns with r rows and prescribed bottom row $(\lambda_r, \lambda_{r-1}, \ldots, \lambda_1)$.

A *generalized reversed Gelfand–Tsetlin pattern of size r* (for short: reversed *r*-pattern) is a triangular array $(a_{i,j})_{1\leqslant i\leqslant r, i\leqslant j\leqslant r}$ of integers with

(1) if $a_{i,j-1}\leqslant a_{i,j}$ then $a_{i,j-1}\leqslant a_{i+1,j}\leqslant a_{i,j}$,
(2) if $a_{i,j-1} > a_{i,j}$ then $a_{i,j-1} > a_{i+1,j} > a_{i,j}$.

For instance,

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 4 & & 4 & & \\ & 5 & & 3 & & 4 & \\ 6 & & 1 & & 5 & & 3 \end{array}$$

is a reversed 4-pattern. Note that a reversed *r*-pattern $(a_{i,j})_{1\leqslant i\leqslant r, i\leqslant j\leqslant r}$ with $a_{1,1}\leqslant a_{1,2}\leqslant \cdots \leqslant a_{1,r}$ is a Gelfand–Tsetlin pattern with *r* rows in the original sense. A pair $(a_{i,j-1}, a_{i,j})$ with $a_{i,j-1} > a_{i,j}$ is called an *inversion* and $(-1)^{\#\text{of inversions}}$ is said to be the sign of the pattern, denoted by $\operatorname{sgn}(a)$. We define

$$A_r(k_1,\ldots,k_r) = \sum_a \operatorname{sgn}(a),$$

where the sum is over all reversed *r*-patterns $a = (a_{i,j})$ with prescribed bottom row $a_{1,i} = k_i$. If $k_1\leqslant k_2\leqslant\cdots\leqslant k_r$ then $A_r(k_1,\ldots,k_r)$ is the number of semistandard tableaux of shape $(k_r,\ldots,k_1)$ with entries in $\{1,2,\ldots,r\}$. This is because a reversed *r*-pattern with $a_{1,1}\leqslant a_{1,2}\leqslant\cdots\leqslant a_{1,r}$ has no inversion.

5.2. Polynomial enumeration formula

Observe that the following recursion holds for $A_r(k_1,\ldots,k_r)$.

$$A_r(k_1,\ldots,k_r) = \sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\cdots\sum_{l_{r-1}=k_{r-1}}^{k_r} A_{r-1}(l_1,l_2,\ldots,l_{r-1}). \tag{5.2}$$

It is similar to (3.1). Since $A_1(k_1) = 1$ we can conclude that $A_k(k_1,\ldots,k_r)$ is a polynomial in $(k_1,\ldots,k_r)$.

Next we show that $A_r(k_1,\ldots,k_r)$ is of degree at most $r-1$ in every k_i. This is done by induction with respect to *r*. Moreover, we show that $D_i A_r(k_1,\ldots,k_r) = 0$ for all *i* and $r\geqslant 2$. The assertion on the degree is obviously true if $r = 1, 2$ since $A_2(k_1,k_2) = k_2 - k_1 + 1$. Moreover $D_1A_2(k_1,k_2) = 0$. Assume that $r > 2$. By induction $A_{r-1}(k_1,\ldots,k_{r-1})$ is of degree at most $r-2$ in every k_i and $D_iA_{r-1}(k_1,\ldots,k_{r-1}) = 0$ for all *i*. Thus, by (5.2) and Lemma 3, $A_r(k_1,\ldots,k_r)$ is of degree at most $r-1$ in k_i. Moreover, by Lemma 4, $D_iA_r(k_1,\ldots,k_r) = 0$ for all *i*.

5.3. Linear factors

In this step we find the zeros of $A_r(k_1,\ldots,k_r)$ and use them to deduce the formula. By definition there exists no reversed *r*-pattern with $a_{1,j} = a_{1,j+1}+1$ and thus $A_r(k_1,\ldots,k_{i-1},$

$k_{i+1}+1, k_{i+1}, k_{i+2}, \dots, k_r) = 0$ for all i. In fact $A_r(k_1, \dots, k_r) = 0$ if $k_i = k_j + j - i$ and $i < j$:

$$\begin{aligned} &A_r(k_1, \dots, k_{i-1}, k_j + j - i, k_{i+1}, \dots, k_j, \dots, k_r) \\ &\quad = -A_r(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_j + j - i - 1, k_{i+2} + 1, \dots, k_j, \dots, k_r) \\ &\quad = \cdots = (-1)^{j-i-1} A_r(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_{i+2} + 1, \dots, k_j + 1, k_j, \dots, k_r) \\ &\quad = 0, \end{aligned}$$

where the lth equality follows from $D_{i+l-1}A_r(k_1, \dots, k_r)$=0. Therefore $A_r(k_1, \dots, k_r)$ has the factor $\prod_{1 \leqslant i < j \leqslant r}(k_j - k_i + j - i)$. This factor is a polynomial of degree $r-1$ in every k_i and thus

$$A_r(k_1, \dots, k_r) = C \prod_{1 \leqslant i < j \leqslant r} (k_j - k_i + j - i),$$

where C does not depend on $(k_1, \dots, k_r)$. Since $A_r(0, \dots, 0) = 1$ we conclude that $C = \prod_{1 \leqslant i < j \leqslant r} \frac{1}{j-i}$ and (5.1) is proved.

Remark 2. The extension of our method introduced in the following section can be used to derive the q-version of (5.1), see [18, p. 375, (7.105)].

6. Extension of the method to q-polynomials

A natural question to ask is whether it is possible to obtain a generating function version of Theorem 2. Of course only this would refine the Bender–Knuth (ex-)Conjecture. Clearly our generating function (see Theorem 1) is not a polynomial in k, however, we introduce the notion of a *q-polynomial* below and find that the generating function is such a q-polynomial. Thus we adapt our method to q-polynomials in this section.

Let I be an integral domain containing $\mathbb{Q}$. A *q-polynomial* over I in the variables $X_1, X_2, \dots, X_n$ is an ordinary polynomial over $I(q)$, the field of rational functions in q over I, in $q^{X_1}, q^{X_2}, \dots, q^{X_n}$. The ring of these q-polynomials is denoted by $I_q[X_1, \dots, X_n]$. For expressions of the form

$$q^{a_0}(q^{X_1})^{a_1}(q^{X_2})^{a_2} \dots (q^{X_n})^{a_n}$$

in a q-polynomial, where the a_i are non-negative integers for $1 \leqslant i \leqslant n$, we also write

$$q^{a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n}.$$

We define $[X; q] = (1 - q^X)/(1 - q)$ and $[X; q]_n = \prod_{i=0}^{n-1} [X + i; q]$. Observe that

$$[X_1; q]_{m_1} [X_2; q]_{m_2} \dots [X_n; q]_{m_n},$$

$(m_1, m_2, \dots, m_n) \in \mathbb{Z}, m_i \geqslant 0$, is a basis of $I_q[X_1, \dots, X_n]$ over $I(q)$. This basis is the most convenient for our purpose.

If we review the proof of Theorem 2 we see that the following two basic properties of polynomials were crucial.

- If $p(X)$ is a polynomial over an integral domain containing $\mathbb{Q}$, then there exists a (unique) polynomial $r(X)$ with $\deg\, r = \deg\, p + 1$ and

$$\sum_{x=1}^{y} p(x) = r(y)$$

for every integer y.
- If $p(X)$ is a polynomial over an integral domain containing $\mathbb{Q}$ and $a_1, a_2, \ldots, a_r$ are distinct zeros of $p(X)$, then there exists a polynomial $r(X)$ with

$$p(X) = (X - a_1)(X - a_2)\ldots(X - a_r)r(X).$$

The following analogs holds for q-polynomials.

- If $p(X)$ is a q-polynomial, then there exists a (unique) q-polynomial $r(X)$ with $\deg r = \deg p + 1$ and

$$\sum_{x=1}^{y} p(x)\, q^x = r(y)$$

for all integers y. (The degree of a q-polynomial in X is defined as the degree of the corresponding ordinary polynomial in q^X.) In order to see that note

$$[X; q]_{n+1} - [X - 1; q]_{n+1} = q^{X-1}[n + 1; q][X; q]_n,$$

which implies

$$\sum_{x=1}^{y} [x; q]_n q^x = \frac{q}{[n + 1; q]}[y; q]_{n+1} \tag{6.1}$$

for all integers y.
- If $p(X)$ is a q-polynomial and $a_1, a_2, \ldots, a_r$ are distinct integer zeros of $p(X)$, then there exists a q-polynomial $r(X)$ with

$$\begin{aligned} p(X) &= ([X; q] - [a_1; q])([X; q] - [a_2; q]) \cdots ([X; q] - [a_r; q])r(X) \\ &= q^{a_1+a_2+\ldots+a_r}[X - a_1; q][X - a_2; q] \cdots [X - a_r; q]r(X). \end{aligned}$$

The proof is analogous to the proof for ordinary polynomials, namely the fundamental identity is

$$\begin{aligned} [X; q]^n - [a; q]^n &= ([X; q] - [a; q]) \sum_{i=0}^{n-1} [X; q]^i [a; q]^{n-1-i} \\ &= q^a [X - a; q] \sum_{i=0}^{n-1} [X; q]^i [a; q]^{n-1-i}. \end{aligned}$$

Remark 3. In the following we often use expressions like $[y-x+i;q]_j$ which are no q-polynomials in x and y. However, in our formulas these expressions always come together with a factor $q^{x\cdot j}$ and thus $[y-x+i;q]_j q^{x\cdot j}$ is used as a shorthand for the degree j polynomial $\prod_{l=0}^{j-1}(q^x-q^{y+i+l})/(1-q)$. With this interpretation our formulas are indeed q-polynomials in x and y.

Using these q-analogs it is quite straightforward to modify the proof of Theorem 2 in order to prove Theorem 1. In the following we sketch it by stating the q-versions of the definitions and lemmas that are necessary to prove Theorem 2.

The norm of an (r,n,c)-pattern is defined as the sum of its parts, where we omit the first and the last part in each row. Our first observation is that the bijection in Lemma 1 is norm-preserving. We introduce a q-analog of $F(r,n,c;k_1,\dots,k_{n-r})$. Let

$$F_q(r,n,c;k_1,\dots,k_{n-r})=\left(\sum_a \operatorname{sgn}(a)q^{\operatorname{norm}(a)}\right)/(q^{k_1+k_2+\dots+k_{n-r}}),$$

where the sum is over all (r,n,c)-patterns $a=(a_{i,j})$ with $a_{r+1,r+i}=k_i$ for $i=1,2,\dots,n-r$. Observe that $F_q(n-1,n,c;k)\,q^k$ is the generating function of strict plane partitions with parts in $\{1,2,\dots,n\}$, at most c columns and k parts equal to n. We have $F_q(0,n,c;k_1,\dots,k_n)=1$ and

$$\begin{aligned}&F_q(r,n,c;k_1,k_2,\dots,k_{n-r})\\&\quad=\sum_{l_1=0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3}\cdots\sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}}\sum_{l_{n-r+1}=k_{n-r}}^{c}\\&\qquad F_q(r-1,n,c;l_1,l_2,\dots,l_{n-r+1})\,q^{l_1+l_2+\dots+l_{n-r+1}}.\end{aligned}\tag{6.2}$$

This shows that $F_q(r,n,c;k_1,k_2,\dots,k_{n-r})$ is a q-polynomial in $(k_1,\dots,k_{n-r})$. Next we have to show that $F_q(r,n,c;k_1,\dots,k_{n-r})$ is of degree at most $2r$ in k_i. For that purpose we need the following q-analog of Lemma 3.

Lemma 8. *Let $F(x_1,x_2)$ be a q-polynomial in x_1 and x_2 which is of degree at most R in each of x_1 and x_2. Moreover assume that $D_1F(x_1,x_2)$ is of degree at most R as a q-polynomial in x_1 and x_2, i.e. a linear combination of monomials $(q^{x_1})^m(q^{x_2})^n$ with $m+n\leqslant R$. Then $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b}F(x_1,x_2)q^{x_1+x_2}$ is of degree at most $R+2$ in y.*

Next we state the q-analog of (3.3). Let $r\geqslant 1$ be an integer. Then

$$\begin{aligned}&\sum_{x'=x}^{y}\sum_{y'=x-1}^{y-1}[y'-x'-r+3;q]_{2r-3}[y'-x'+1;q]\,q^{(2r-2)x'}\,(1+q^{r-1})\,q^{x'+y'}\\&\quad=2\frac{[y-x-r+2;q]_{2r-1}[y-x+1;q]q^{2rx}}{[2r-1;q][2r;q]}(1+q^r)q^{r-2}.\end{aligned}\tag{6.3}$$

The proof is analogous to the proof of (3.3). First one has to use the following transformation.

$$\begin{aligned}&[y'-x'-r+3;q]_{2r-3}[y'-x'+1;q]\,(1+q^{r-1})\\&\quad=[y'-x'-r+2;q]_{2r-2}q^{r-1}+[y'-x'-r+3;q]_{2r-2}.\end{aligned}$$

Then the fundamental identities are

$$\sum_{z=a}^{b}[z+w;q]_n q^z=\frac{q^{-w+1}}{[n+1;q]}\left([b+w;q]_{n+1}-[a-1+w;q]_{n+1}\right),$$

which is an easy consequence of (6.1) and

$$[z;q]_n=(-1)^n q^{n(z+(n-1)/2)}[-z-n+1;q]_n. \qquad \square$$

Lemma 4 and (6.3) imply the q-analog of Lemma 5.

Lemma 9. *Let r, n, i be integers, r non-negative, n positive and $1\leqslant i\leqslant n-r-1$. Then*

$$\begin{aligned}&D_i F_q(r,n,c;.)(k_1,\dots,k_{n-r})\\&\quad=(-1)^r\frac{(1+q^r)}{[1;q]_{2r}}[k_{i+1}-k_i-r+2;q]_{2r-1}[k_{i+1}-k_i+1;q]q^{2rk_i}\\&\qquad\times q^{r(1+4i-4n+5r)/2}F_q(r,n-2,c+2;k_1,\dots,k_{i-1},k_{i+2}+2,\dots,k_{n-r}+2).\end{aligned}$$

Lemma 9 shows that $D_i F_q(r,n,c;.)(k_1,\dots,k_{n-r})$ is of degree $2r$ in k_i and in k_{i+1}. In the next lemma we see that this is also true for $F_q(r,n,c;k_1,\dots,k_{n-r})$ itself.

Lemma 10. *Let r, n, i be integers, r non-negative, n positive and $1\leqslant i\leqslant n-r$. Then $F_q(r,n,c;k_1,\dots,k_{n-r})$ is a q-polynomial in k_i of degree at most $2r$.*

Proof. Use (6.2), Lemma 9 and Lemma 8 in the same way as their analogs in Lemma 6. $\square$

This lemma, Lemma 7 and the second property of q-polynomials imply the following q-analog of Corollary 1.

Corollary 3. *$F_q(n-1,n,c;k)/([1+k;q]_{n-1}[1+c-k;q]_{n-1}q^{(n-1)k})$ is independent of k.*

We are now able to prove our main theorem.

Proof of Theorem 1. We prove the assertion by induction with respect to n. Observe that the formula is true for $n=1$ since $F_q(0,1,c;k)=1$. Applying Corollary 3 in the same way as Corollary 1 was applied in the proof of Theorem 2, it suffices to check the formula for $F_q(n-1,n,c;c)$. For that purpose we need the recursion

$$F_q(n-1,n,c;c)=q^{c\,n-c}\sum_{k=0}^{c}F_q(n-2,n-1,c;k)q^k$$

and the following identity

$$\sum_{k=0}^{c} [1+k;q]_{m-1}\,[1+c-k;q]_{m-1}\,q^{mk} = \frac{[1;q]_{m-1}^2\,[c+1;q]_{2m-1}}{[1;q]_{2m-1}}, \tag{6.4}$$

which can be deduced from the q-Chu–Vandermonde identity, see [2, (3.3.10)]. □

Finally we are able to prove the Bender–Knuth (ex-)Conjecture.

Corollary 4. *The generating function of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$ *and at most c columns is*

$$\prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i}.$$

Proof. By Theorem 1 the generating function is equal to

$$\sum_{k=0}^{c} \frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

The assertion follows from (6.4). □

7. A final observation

A monotone triangle of size n, see [4, p. 58], is an $(n-1, n, n+1)$-pattern with strictly increasing rows. Monotone triangles of size n with the central part of the first row equal to k are easily seen to be in bijection with alternating sign matrices of size n, where the unique 1 in the first row is in the kth column. Let $A(n,k)$ denote the number of these objects. It was conjectured by Mills et al. [14] (well-known as the Refined Alternating Sign Matrix Theorem) and proved by Zeilberger [21] that

$$A(n,k) = \frac{(k)_{n-1}(1+n-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(1)_{3i-2}}{(1)_{n+i-1}}.$$

Surprisingly it turns out that the number of $(n-1, n, n-1)$-patterns $(a_{i,j})$ with $a_{n,n} = k-1$ divided by $A(n,k)$ is independent of k. In fact it is equal to

$$\prod_{1 \leqslant i \leqslant j \leqslant n-1} \frac{i+j+n-2}{i+2j-2},$$

the number of $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions, see [20]. As in the case of the enumeration of $(n-1, n, c)$-patterns, it suffices to show that

$$A(n,k)/((k)_{n-1}(1+n-k)_{n-1})$$

is independent of k in order to prove the formula for $A(n,k)$, see [4, Section 5.2] for an explanation. Therefore we hope to find another proof of the Refined Alternating Sign Matrix

Theorem which is along the lines of the proof of Theorem 2. The situation is similar to that of strict plane partitions which is under consideration in this paper. First, one would have to find an extension of the combinatorial interpretation of alternating sign matrices of order n such that the unique 1 in the first row is in the kth column to *arbitrary* integers k. That is to say that one would have to find combinatorial objects indexed by a positive integer n and an arbitrary integer k which are in bijection with alternating sign matrices of order n, where the unique 1 in the first row is in the kth column for $k \in \{1, 2, \ldots, n\}$. In the view of the fact that generalized $(n-1, n, c)$ Gelfand–Tsetlin patterns were the right extension of the strict plane partitions, one would rather work with monotone triangles than with alternating sign matrices. Next it would have to be shown that for fixed n these objects are enumerated by a polynomial $P_n(k)$ in k of degree $2n-2$. Typically, this could be done by a recursion similar to (3.1). Finally it would have to be shown that there exist none of these extending combinatorial objects if $k = 0, -1, \ldots, -n+2$ or $k = n+1, n+2, \ldots, 2n-1$.

We also wish to remark another way one could bijectively prove the Refined Alternating Sign Matrix Theorem. We have already seen that it would suffice to show that the number of $(n-1, n, n-1)$-patterns $(a_{i,j})$ with $a_{n,n} = k-1$ divided by the number of alternating sign matrices of order n, where the unique 1 in the first row is in the kth column is independent of k. Thus, a bijection between $(n-1, n, n-1)$-patterns with $0 \leqslant a_{n,n} = k-1 \leqslant n-1$ on one side and pairs consisting of a monotone triangle of size n with the central part in the first row equal to k and $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions would simultaneously prove the formula for $A(n, k)$ and for the number of $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions.

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