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# Well-Posedness of the Cauchy Problem for a Shallow Water Equation on the Circle

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In this paper we consider the periodic Cauchy problem for a fifth order modification of the Camassa–Holm equation. We prove local well-posedness in appropriate Bourgain spaces for initial data in a Sobolev space $H^s(T)$, $s > 1/2$. We also prove global well-posedness for data in $H^1(T)$ and of arbitrary size. The proofs are based on a priori estimates using Fourier analysis techniques, microlocalization in phase space, an interpolation argument and a fixed point theorem. 
*Key Words:* Cauchy problem; well-posedness, Sobolev spaces; Fourier transform.

## 1. INTRODUCTION

In this paper we consider the periodic initial value problem for the fifth order Camassa–Holm equation

$$\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\, \partial_x u - 2\, \partial_x u\, \partial_x^2 u - u\, \partial_x^3 u - \partial_x^5 u = 0 \tag{1.1}$$

$$u(x, 0) = \varphi(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T}. \tag{1.2}$$

This equation is a modification by a fifth order term $-(k/3)\, \partial_x^5 v$ of the Camassa–Holm equation (CH) derived in [CH]

$$v_t - v_{xxt} + kv_x + 3vv_x - 2v_x v_{xx} - vv_{xxx} = 0. \tag{1.3}$$

Without the fifth order term, Eq. (1.1) can be readily obtained from (1.3) by suitable substitutions (for example, pick for simplicity $k = 3$ and substituting $v = u - 1$). Equation (1.3) is well-known for its interesting properties. It is completely integrable and admits (peaked) soliton solutions (see [CH]). It can be derived as the geodesic equation of the right-invariant metric on the Bott–Virasoro group (the one-dimensional central extension of the group of diffeomorphisms of the circle), which at the identity is given by the $H^1$ inner product (see [M]). It can also be studied using inverse scattering approaches (see Alber, Camassa, Holm, and Marsden [ACHM], Beals,

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Sattinger, and Szmigielski [BSS], Constantin and McKean [CMcK]). These properties make the CH equation similar to the much-studied KdV equation. However, while the Cauchy problem (in both periodic and non-periodic case) for the KdV is known to be globally well-posed (see for example Sjøberg [S], Kato [K], Kenig, Ponce, and Vega [KPV1], Bourgain [B2, B3]), it has been observed that certain solutions to CH blow up in finite time (see [CH], Constantin and Escher [CE] or McKean [McK]). For example, if $\kappa = 0$, then the Cauchy problem (1.3) is not globally well-posed for mean-zero initial data in $H^3$. It is therefore of interest to study higher order modifications of the CH equation.

Another motivation for this work is to extend the methods developed in [B1–B3] to study the periodic Cauchy problem for KdV type equations

$$\partial_t u + \partial_x^{2j+1} u + f(u, \partial_x u, ..., \partial_x^l u) = 0 \tag{1.4}$$

to equations containing mixed derivative terms such as (1.1) or (1.3). The presence of these terms requires modifications of the original approach. Observe, for example, that one can rewrite (1.1) in the form (1.4) however the function $f$ will now depend nonlocally on $u$ and its derivatives. In fact the methods in this paper may be applied to other equations with more general nonlocal and nonlinear terms as well as higher dimensional analogues of the CH equation considered for example in Holm, Marsden, and Ratiu [HMR] or Holm, Kouranbaeva, Marsden, Ratiu, and Shkoller [HKMRS].

The initial value problem (1.1)–(1.2) was considered in a slightly more general form in our previous work [HM]. There we proved local and global well-posedness in appropriate Bourgain function spaces (see [B1, B2]), under the restriction of small initial data. In this work we remove this restriction by replacing the localizing cut-off function $\psi(t)$ used in [HM] with $\psi_\delta(t) = \psi(t/\delta)$ which is supported in the interval $[-\delta, \delta]$ and equal to 1 near zero. This introduces both positive and negative powers of $\delta$ into the constants appearing in the a priori estimates. More precisely, on the one hand we gain a factor of $\delta^{1/12}$ (see estimates in (2.13) and (2.14) and Lemma 3.2) while on the other hand we loose $\delta^{-\varepsilon}$ (see Lemma 2.2). Choosing however $\varepsilon$ sufficiently small allows us to control the size of the initial data in order to apply a fixed point argument (see Lemma 2.4 and (2.18)). The proofs of these estimates are based on appropriate partitions of the phase space needed to control the nonlocal nonlinearity in (2.15) as well as an interpolation argument. We believe that these techniques are of independent interest and may be useful for other equations.

Furthermore, the techniques developed here suggest the following approach to the study of the original CH Eq. (1.3). Introducing a small parameter $\varepsilon$ in front of the fifth order term $\partial_x^5 u$ one can study the dependence of the a priori estimates on $\varepsilon$ in order to obtain the solution $u$ of the initial value problem for (1.3) as a limit in an appropriate space of solutions $u_\varepsilon$ of the $\varepsilon$-problem.

The main results of this paper are the following.

THEOREM 1.1. *For any $s > 1/2$ and any initial data $\varphi \in H^s(\mathbb{T})$ the initial value problem* (1.1)–(1.2) *has a unique local solution in the space $X^s$ of all $L^2$ functions $u$: $\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ with finite norm*

$$|||u|||_s^2 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |n|^{2s} (1 + |\lambda - n^3 + \hat{\varphi}(0) n|) \, |\hat{u}(n, \lambda)|^2 \, d\lambda. \tag{1.5}$$

THEOREM 1.2. *For any initial data $\varphi \in H^1(\mathbb{T})$ the initial value problem* (1.1)–(1.2) *is globally well-posed in the space $X^1$.*

In the next section we prove the two theorems stated above while in Section 3 we prove the main technical propositions and lemmas.

## 2. PROOFS OF THEOREMS

First we shall reduce our initial value problem to the case of mean-zero data. For this observe that $u$ solves (1.1)–(1.2) if and only if $u - \hat{\varphi}(0)$ solves the following problem

$$\partial_t u + \partial_x^3 u + \hat{\varphi}(0) \, \partial_x u + 2\hat{\varphi}(0)(1 - \partial_x^2)^{-1} \, \partial_x u + w = 0 \tag{2.1}$$

$$u(x, 0) = \varphi(x) - \hat{\varphi}(0) = \varphi_o, \tag{2.2}$$

where $w$ is given by

$$w = \tfrac{1}{2} \, \partial_x(u^2) + (1 - \partial_x^2)^{-1} \, [\partial_x(u^2) + \tfrac{1}{2} \, \partial_x((\partial_x u)^2)], \tag{2.3}$$

and

$$\hat{\varphi}(0) = \int_{\mathbb{T}} \varphi(x) \, dx = 0. \tag{2.4}$$

Our initial value problem is now equivalent to the following integral equation

$$u(x, t) = W(t) \, \varphi_o(x) - \int_0^t W(t - \tau) \, w(x, \tau) \, d\tau, \tag{2.5}$$

where $W(t)=\exp\{-t[\partial_x^3+\hat{\varphi}(0)\,\partial_x+2\hat{\varphi}(0)(1-\partial_x^2)^{-1}\,\partial_x]\}$. Using Fourier transform in $t$ and Fourier series in $x$, and setting

$$a(n)=n^3-\hat{\varphi}(0)\,n-2\hat{\varphi}(0)\,\frac{n}{1+n^2} \tag{2.6}$$

we express (2.5) in the following form

$$\begin{aligned} u(x,t)=&\sum_{n\in\mathbb{Z}} e^{i[nx+a(n)\,t]}\hat{\varphi}_o(n)\\ &+i\sum_{n\in\mathbb{Z}} e^{i[nx+a(n)\,t]}\int_{-\infty}^{\infty}\frac{e^{i[\lambda-a(n)]\,t}-1}{\lambda-a(n)}\,\hat{w}(n,\lambda)\,d\lambda, \end{aligned}$$

where

$$\hat{w}(n,\lambda)=\frac{i}{8\pi^2}\left(n+\frac{2n}{n^2+1}\right)\hat{u}*\hat{u}(n,\lambda)-\frac{i}{8\pi^2}\frac{n}{n^2+1}\widehat{\partial_x u}*\widehat{\partial_x u}(n,\lambda). \tag{2.7}$$

Pick a cut-off function $\psi(t)\in C_0^\infty(-1,1)$ with $0\leqslant\psi\leqslant 1$ and such that $\psi(t)\equiv 1$ for $|t|<1/2$. Then for $\delta>0$ let

$$\psi_\delta(t)=\psi\left(\frac{t}{\delta}\right). \tag{2.8}$$

Decompose the expression for $u$ into

$$\psi_\delta(t)\,u(x,t)$$

$$=\psi_\delta(t)\sum_{n\in\dot{\mathbb{Z}}}\hat{\varphi}(n)\,e^{i(nx+a(n)t)} \tag{2.9}$$

$$+i\sum_{k=1}^{\infty}\frac{i^k}{k!}\,t^k\psi_\delta(t)\left\{\sum_{n\in\dot{\mathbb{Z}}}\left(\int_{-\infty}^{\infty}\psi(\lambda-a(n))(\lambda-a(n))^{k-1}\,\hat{w}(n,\lambda)\,d\lambda\right)\right. \tag{2.10}$$

$$\left.\times\, e^{i(nx+a(n)\,t)}\right\}$$

$$+i\psi_\delta(t)\sum_{n\in\dot{\mathbb{Z}}}e^{inx}\int_{-\infty}^{\infty}\frac{(1-\psi)(\lambda-a(n))}{\lambda-a(n)}\,e^{i\lambda t}\hat{w}(n,\lambda)\,d\lambda \tag{2.11}$$

$$-i\psi_\delta(t)\sum_{n\in\dot{\mathbb{Z}}}e^{i(nx+a(n)\,t)}\int_{-\infty}^{\infty}\frac{(1-\psi)(\lambda-a(n))}{\lambda-a(n)}\,\hat{w}(n,\lambda)\,d\lambda, \tag{2.12}$$

where $\dot{\mathbb{Z}}=\mathbb{Z}-\{0\}$.

Let $X^s_o$ be the subspace of $X^s$ consisting of all functions $u(x, t)$ supported on $\mathbb{T} \times [-\delta, \delta]$ and such that

$$\int_{\mathbb{T}} u(x, t)\, dx = 0.$$

Let $T$ be the map defined by the equation in (2.9)–(2.12). Our aim will be to show that $u \to Tu$ is a contraction with respect to the norm (1.5). To achieve this we will need the following estimate.

THEOREM 2.1. *If $s > 1/2$, then for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that*

$$|||Tu|||_s \leqslant C_\varepsilon(\delta^{(1/12)-\varepsilon}(1 + 2\,|\hat{\varphi}(0)|)^{3/2}\, |||u|||_s^2 + \|\varphi_o\|_{H^s})$$

*for all $u \in X^s_o$.*

For the proof of Theorem 2.1 we shall need the following lemma.

LEMMA 2.2. *For $\delta > 0$ let $\psi_\delta(t)$ be as in (2.8). Then for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that*

$$|||\psi_\delta u|||_s \leqslant C_\varepsilon \delta^{-\varepsilon}\, |||u|||_s,$$

*for all $u \in X^s_o$.*

Also we shall need the following proposition.

PROPOSITION 2.3. *For all $f, g \in X^s_o$ we have*

$$\left(\sum_{n \in \dot{Z}} |n|^{2s} \int_{\mathbb{R}} \frac{|\hat{w}_{fg}(n, \lambda)|^2}{1 + |\lambda - n^3 - \hat{\varphi}_0 n|}\, d\lambda\right)^{1/2} \lesssim \delta^{1/12}\, |||f|||_s \cdot |||g|||_s, \tag{2.13}$$

$$\left(\sum_{n \in \dot{Z}} |n|^{2s} \left(\int_{\mathbb{R}} \frac{|\hat{w}_{fg}(n, \lambda)|}{1 + |\lambda - n^3 - \hat{\varphi}_0 n|}\, d\lambda\right)^2\right)^{1/2} \lesssim \delta^{1/12}\, |||f|||_s \cdot |||g|||_s, \tag{2.14}$$

*where $\hat{w}_{fg}$ is defined by*

$$\hat{w}_{fg}(n, \lambda) \simeq \left(n + \frac{2n}{1+n^2}\right) \hat{f} * \hat{g}(n, \lambda)$$
$$+ \frac{n}{1+n^2}\, \widehat{\partial_x f} * \widehat{\partial_x g}(n, \lambda). \tag{2.15}$$

*Remark.* In the statement of the above proposition as well as in the rest of the paper we use the notation "$f \lesssim g$" (resp. "$f \simeq g$") to denote "$f \leqslant cg$" (resp. "$f = cg$") where $c$ is a universal constant.

*Proof of Theorem* 2.1. We estimate $Tu$ using the decomposition (2.9)–(2.12).

*Estimate for* 2.9. A straightforward estimation gives

$$|||(2.9)|||_s \lesssim \|\varphi_o\|_{H^s}.$$

*Estimate for* 2.10. We have

$$\begin{aligned} &|||(2.10)|||_s \\ &\quad\lesssim \sum_{k=1}^{\infty} \frac{1}{k!} \|\widehat{t^k\psi_\delta}(\tau)(1+|\tau|)^{1/2}\|_{L^1} \\ &\qquad\times \sup_{k\geqslant 1} \left|\left|\left| \sum_{n\in\dot{Z}} \left( \int_{-\infty}^{\infty} \psi(\lambda-a(n))(\lambda-a(n))^{k-1}\, \hat{w}(n,\lambda)\, d\lambda \right) e^{i(nx+a(n)\,t)} \right|\right|\right|_s \\ &\quad\lesssim \delta^{1/2}(1+2\,|\hat{\varphi}(0)|)^{1/2} \left\{ \sum_{n\in\dot{Z}} |n|^{2s} \left( \int_{|\lambda-a(n)|\leqslant 1} |\hat{w}(n,\lambda)|\, d\lambda \right)^2 \right\}^{1/2}. \end{aligned}$$

The last inequality follows from the definition of the norm $|||\cdot|||_s$ and from the estimate

$$\sup_{k\geqslant 1} \|\widehat{t^k\psi_\delta}(\tau)(1+|\tau|)^{1/2}\|_{L^1} \lesssim \delta^{1/2}.$$

Using $(1+|\lambda-a(n)|)/(1+|\lambda-n^3+\hat{\varphi}(0)\,n|) \leqslant (1+2\,|\hat{\varphi}(0)|)$ we have

$$\int_{|\lambda-a(n)|\leqslant 1} |\hat{w}(n,\lambda)|\, d\lambda \lesssim (1+2\,|\hat{\varphi}(0)|) \int_{\mathbb{R}} \frac{|\hat{w}(n,\lambda)|}{1+|\lambda-n^3+\hat{\varphi}(0)\,n|}\, d\lambda.$$

This together with inequality (2.14) gives

$$|||(2.10)|||_s \lesssim \delta^{1/2}(1+2\,|\hat{\varphi}(0)|)^{3/2}\, |||u|||_s^2.$$

*Estimate for* 2.11. Using Lemma 2.2 we obtain

$$\begin{aligned} |||(2.11)|||_s \lesssim C_\varepsilon \delta^{-\varepsilon} \Bigg\{ &\sum_{n\in\dot{Z}} |n|^{2s} \\ &\times \int_{\mathbb{R}} (1+|\lambda-n^3+\hat{\varphi}(0)\,n|) \left| \frac{(1-\psi)(\lambda-a(n))}{\lambda-a(n)}\, \hat{w}(n,\lambda) \right|^2 d\lambda \Bigg\}^{1/2} \end{aligned}$$

$$\lesssim C_\varepsilon \delta^{-\varepsilon} \left\{ \sum_{n \in \dot{Z}} |n|^{2s} \right.$$
$$\left. \times \int_{|\lambda - a(n)| \geqslant 1/2} \frac{(1 + |\lambda - n^3 + \hat{\varphi}(0)\, n|)^2}{(1 + |\lambda - a(n)|)^2} \frac{|\hat{w}(n, \lambda)|^2 \, d\lambda}{1 + |\lambda - n^3 + \hat{\varphi}(0)\, n|} \right\}^{1/2}$$
$$\lesssim C_\varepsilon \delta^{(1/12) - \varepsilon} (1 + 2\, |\hat{\varphi}(0)|)\, \|\!|u|\!\|_s^2,$$

where the last inequality follows from estimate (2.13).

*Estimate for* 2.12. Similarly, using Lemma 2.2 we obtain

$$\|\!|(2.12)|\!\|_s \lesssim C_\varepsilon \delta^{-\varepsilon} (1 + 2\, |\hat{\varphi}(0)|)^{1/2}$$
$$\times \left\{ \sum_{n \in \dot{Z}} |n|^{2s} \left| \int_{\mathbb{R}} \frac{(1 - \psi)(\lambda - a(n))}{\lambda - a(n)} \hat{w}(n, \lambda) \, d\lambda \right|^2 \right\}^{1/2}$$
$$\lesssim C_\varepsilon \delta^{-\varepsilon} (1 + 2\, |\hat{\varphi}(0)|)^{3/2} \left\{ \sum_{n \in \dot{Z}} |n|^{2s} \right.$$
$$\left. \times \left[ \int_{\mathbb{R}} \frac{|\hat{w}(n, \lambda)|}{1 + |\lambda - n^3 + \hat{\varphi}(0)\, n|} \, d\lambda \right]^2 \right\}^{1/2}$$
$$\lesssim C_\varepsilon \delta^{(1/12) - \varepsilon} (1 + 2\, |\hat{\varphi}(0)|)^{3/2}\, \|\!|u|\!\|_s^2,$$

where the last inequality follows from estimate (2.14). This completes the proof of Theorem 2.1.

Next lemma states that $T$ defines a contraction on a closed ball in $X_o^s$.

LEMMA 2.4. *For $s > 1/2$ there is a constant $c > 0$ such that*

$$\|\!|Tu|\!\|_s \leqslant c(\delta^{1/24} (1 + 2\, \|\varphi\|_{H^s})^{3/2}\, \|\!|u|\!\|_s^2 + \|\varphi\|_{H^s}) \tag{2.16}$$

*and*

$$\|\!|Tu - Tv|\!\|_s \leqslant c\delta^{1/24} (1 + 2\, \|\varphi\|_{H^s})^{3/2}\, (\|\!|u|\!\|_s + \|\!|v|\!\|_s)(\|\!|u - v|\!\|_s), \tag{2.17}$$

*for all $u, v \in X_o^s$. Moreover $T$ is a contraction on the closed ball $B(0, r) \subset X_o^s$, where*

$$r = 2c\, \|\varphi\|_{H^s}, \qquad \text{and} \qquad 0 < \delta \leqslant [(1 + 2\, \|\varphi\|_{H^s})^{3/2}\, 4c^2]^{-24}. \tag{2.18}$$

*Proof.* Inequality (2.16) follows from Theorem 2.1 by choosing $\varepsilon = 1/24$, and using the fact that $|\hat{\varphi}(0)| \leqslant \|\varphi\|_{H^s}$. For the proof of inequality (2.17) we have

$$\begin{aligned}
|||Tu - Tv|||_s &\leqslant \left|\left|\left| \psi_\delta(t) \int_0^t W(t-\tau)[w_u(\tau) - w_v(\tau)]\, d\tau \right|\right|\right|_s \\
&\leqslant \left|\left|\left| \psi_\delta(t) \int_0^t W(t-\tau)(w_{fg}(\tau))\, d\tau \right|\right|\right|_s \\
&\leqslant c\delta^{1/24}(1 + 2\, \|\varphi\|_{H^s})^{3/2}\, |||u+v|||_s \cdot |||u-v|||_s .
\end{aligned}$$

Here we have used the fact that

$$\widehat{w_u - w_v} = \hat{w}_{fg}, \qquad \text{where} \quad f = u+v \quad \text{and} \quad g = u - v,$$

and then proceeded as in the proof of Theorem 2.1 with $\varphi = 0$. Now it can easily be checked that (2.16) implies that $T$ maps the closed ball $B(0, r)$ into itself and by (2.17) satisfies

$$|||Tu - Tv|||_s \leqslant \tfrac{1}{2}\, |||u - v|||_s .$$

This completes the proof of Lemma 2.4 and therefore Theorem 1.1.

Theorem 1.2 follows now in the standard way. Observe that mean zero data are preserved since

$$\begin{aligned}
\partial_t \int_{\mathbb{T}} u(t)\, dx = \int_{\mathbb{T}} \partial_t u = \int_{\mathbb{T}} \partial_t u - \partial_t\, \partial_x^2 u \\
= \int_{\mathbb{T}} -3u\, \partial_x u + 2\, \partial_x u\, \partial_x^2 u + u\, \partial_x^3 u = 0.
\end{aligned}$$

Therefore for any $t$

$$\int_{\mathbb{T}} u(t)\, dx = \hat{\varphi}(0).$$

Observe also that the $H^1$ norm of the initial data is preserved. Since the existence time of the local solution depends only on the $H^s$ norm of the initial data (see (2.18)), we get a global solution for data of arbitrary size. Theorem 1.2 follows.

## 3. PROOFS OF LEMMA 2.2 AND PROPOSITION 2.3

*Proof of Lemma* 2.2. It suffices to show

$$\int_{\mathbb{R}} (1 + |\lambda - \sigma(n)|)\, |\widehat{\psi_\delta u}(n, \lambda)|^2\, d\lambda \lesssim \delta^{-2\varepsilon} \int_{\mathbb{R}} (1 + |\lambda - \sigma(n)|)\, |\hat{u}(n, \lambda)|^2\, d\lambda,$$

where $\sigma(n) \doteq n^3 - \hat{\varphi}(0)\, n$. If we let $\tau = \lambda - \sigma(n)$ then this inequality can be written in the form

$$\int_{\mathbb{R}} (1+|\tau|)\, |[e^{-i\sigma(n)\, t} \psi_\delta(t)\, \hat{u}(n, t)]^\wedge(\tau)|^2\, d\tau$$

$$\lesssim \delta^{-2\varepsilon} \int_{\mathbb{R}} (1+|\tau|)\, |[e^{-i\sigma(n)\, t} \hat{u}(n, t)]^\wedge(\tau)|^2\, d\tau.$$

Setting

$$h(t) = \hat{u}(n, t)$$

the last inequality becomes

$$\|e^{-i\sigma(n)\, t} \psi_\delta(t)\, h(t)\|^2_{H^{1/2}} \lesssim \delta^{-2\varepsilon}\, \|e^{-i\sigma(n)\, t} h(t)\|^2_{H^{1/2}}. \tag{3.1}$$

The strategy for proving (3.1) is to apply interpolation. It will be obtained in the form

$$\|e^{-i\sigma(n)\, t} \psi_\delta(t)\, h(t)\|^2_{A_\rho} \leqslant C_1^{(1-\rho)} C_2^\rho \delta^{(1-2b)\, \rho}\, \|e^{-i\sigma(n)\, t} h(t)\|_{B_\rho} \tag{3.2}$$

for a special value of $\rho$ with $0 < \rho < 1$. The parameter $b > \frac{1}{2}$ will be chosen later. We pick $A_\rho = B_\rho = L^2_{\rho_b}(\mathbb{R}) \doteq H^{\rho b}(\mathbb{R})$ and observe that for $0 \leqslant \rho \leqslant 1$ we have

$$\frac{1}{2} = \frac{1-\rho}{2} + \frac{\rho}{2}.$$

From Stein's interpolation theorem (see [SW, p. 212, Section 5.7]) with $\alpha_o = 0$, $\alpha_1 = b$ and

$$\alpha = (1-\rho)\, \alpha_0 + \rho\alpha_1 = \rho b$$

inequality (3.2) will be valid for $\rho = 1/2b$ as soon as we prove it for $\rho = 0$ and for $\rho = 1$. The operator under consideration here is multiplication by $\psi_\delta$, that is

$$e^{-i\sigma(n)\, t} h(t) \mapsto \psi_\delta(t)\, e^{-i\sigma(n) t} h(t).$$

*The case* $\rho = 0$. We have $A_0 = B_0 = L^2(\mathbb{R})$ and (3.2) takes the form

$$\|e^{-i\sigma(n) t} \psi_\delta(t)\, h(t)\|^2_{L^2(\mathbb{R})} \leqslant C_1\, \|e^{-i\sigma(n) t} h(t)\|^2_{L^2(\mathbb{R})},$$

which is obvously true with $C_1 = 1$.

*The case* $\rho = 1$. We have $A_1 = B_1 = H^b(\mathbb{R})$ and (3.2) takes the form

$$\|e^{-i\sigma(n)t}\psi_\delta(t)\,h(t)\|^2_{L^2_b(\mathbb{R})} \leqslant C_2 \delta^{1-2b}\,\|e^{-i\sigma(n)t}h(t)\|^2_{L^2_b(\mathbb{R})}. \tag{3.3}$$

This inequality has been proved by Kenig, Ponce, and Vega in [KPV3]. Therefore (3.1) follows from (3.2) by choosing $b = \frac{1}{2} + \varepsilon/2$ and $\rho = 1/2b$. Then $\rho b = 1/2$ and

$$\frac{1-2b}{2b} = -\varepsilon\left(\frac{1}{1+\varepsilon}\right) \simeq -\varepsilon.$$

This completes the proof of Lemma 2.2.

*Proof of Proposition* 2.3. It suffices to prove Proposition 2.3 separately for $\hat{w}_1(n, \lambda)$ and $\hat{w}_2(n, \lambda)$, where

$$\hat{w}_1(n, \lambda) \doteq \left(n + \frac{2n}{1+n^2}\right) \hat{f} * \hat{g}(n, \lambda), \tag{3.4}$$

$$\hat{w}_2(n, \lambda) \doteq \frac{n}{1+n^2}\,\widehat{\partial_x f} * \widehat{\partial_x g}(n, \lambda). \tag{3.5}$$

For $\hat{\varphi}(0) = 0$ and without the factor $\delta^{1/12}$ this proposition was proved in [HM]. For the KdV (which corresponds to $\hat{w}(n, \lambda) = n\hat{f} * \hat{g}(n, \lambda)$) it was proved in [B2]. Since the term $\hat{w}_1(n, \lambda)$ is similar to the KdV term, we shall consider only the term $\hat{w}_2(n, \lambda)$, which requires a different partition of the $(n, \lambda)$-space for proving the corresponding estimates. By (3.5) this term satisfies the inequality

$$\hat{w}_2(n, \lambda)| \lesssim \frac{|n|}{n^2+1} \sum_{n_1} \int_{\mathbb{R}} |(n-n_1)\,\hat{f}(n-n_1, \lambda-\lambda_1)|\;|n_1\,\hat{g}(n_1, \lambda_1)|\,d\lambda_1. \tag{3.6}$$

*Proof of* (2.13). Using (3.6) we have

$$\begin{aligned}
&\frac{|n|^s\,|\hat{w}_2(n, \lambda)|}{(1+|\lambda - n^3 + \hat{\varphi}(0)\,n|)^{1/2}} \\
&\lesssim \sum_{n_1 \in \dot{Z}} \int_{\mathbb{R}} \frac{|n|^{s+1}\,|n-n_1|^{1-s}\,|n_1|^{1-s}}{(1+n^2)(1+|\lambda - n^3 + \hat{\varphi}(0)n|)^{1/2}} \\
&\quad \cdot \frac{c_f(n-n_1, \lambda-\lambda_1)\,c_g(n_1, \lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3+\hat{\varphi}(0)(n-n_1)|)^{1/2}\,(1+|\lambda_1 - n_1^3 + \hat{\varphi}(0)\,n_1|)^{1/2}}\,d\lambda_1,
\end{aligned} \tag{3.7}$$

where for a function $h$, $c_h$ is defined by

$$c_h(n, \lambda) = |n|^s \,(1 + |\lambda - n^3 + \hat{\varphi}(0)\, n|)^{1/2}\, |\hat{h}(n, \lambda)|. \tag{3.8}$$

Observe that using notation (3.8) the norm $|||h|||_s$ is written as

$$|||h|||_s = \left( \sum_{n \in \dot{Z}_1} \int_{\mathbb{R}} c_h(n, \lambda)^2 \, d\lambda \right)^{1/2} \tag{3.9}$$

To estimate the denominators in (3.7) we shall partition the $(n, \lambda; n_1, \lambda_1)$-space using the following elementary lemma.

LEMMA 3.1. *For all $n \neq 0, n_1 \neq 0$ and $n - n_1 \neq 0$ the quantity*

$$\begin{aligned} d_3 \doteq (\lambda - n^3 + \hat{\varphi}(0)\, n) - [(\lambda_1 - n_1^3 + \hat{\varphi}(0)\, n_1) \\ + (\lambda - \lambda_1 - (n - n_1)^3 + \hat{\varphi}(0)(n - n_1))] \end{aligned} \tag{3.10}$$

*satisfies*

$$|d_3| \geqslant \tfrac{3}{2}\, |n|^2, \qquad \textit{and} \qquad |d_3| \geqslant \tfrac{3}{2}\, |n_1|^2.$$

If $s \geqslant 1$ then $\hat{w}_2$ is treated in the same way as $\hat{w}_1$. We shall therefore consider the case

$$1/2 < s < 1.$$

From Lemma 3.1 it suffices to consider the following three cases separately.

I. $|\lambda - n^3 + \hat{\varphi}(0)\, n| \geqslant \frac{3}{8} n_1^2$
II. $|\lambda_1 - n_1^3 + \hat{\varphi}(0)\, n_1| \geqslant \frac{3}{8} n_1^2$
III. $|\lambda - \lambda_1 - (n - n_1)^3 + \hat{\varphi}(0)(n - n_1)| \geqslant \frac{3}{8} n_1^2$.

*Case* I. In (3.7) we replace $(|\lambda - n^3 + \hat{\varphi}(0)n|)^{1/2}$ with $\frac{3}{8} n_1^2$, and use the inequality

$$\frac{|n|^{s+1}\, |n - n_1|^{1-s}\, |n_1|^{1-s}}{(1 + n^2)\, |n_1|} \leqslant \frac{|n - n_1|^{1-s}}{|n|^{1-s}\, |n_1|^s} \leqslant \frac{(|n| + |n_1|)^{1-s}}{|n|^{1-s}\, |n_1|^{1-s}} \leqslant 2^{1-s}, \tag{3.11}$$

to obtain

$$\begin{aligned} \frac{|n|^s\, |\hat{w}_2(n, \lambda)|}{(1 + |\lambda - n^3 + \hat{\varphi}(0)n|)^{1/2}} &\lesssim \sum_{n \in \dot{Z}_1} \int_{\mathbb{R}} \frac{c_f(n - n_1, \lambda - \lambda_1)}{(1 + |\lambda - \lambda_1 - (n - n_1)^3 + \hat{\varphi}(0)(n - n_1)|)^{1/2}} \\ &\quad \times \frac{c_g(n_1, \lambda_1)}{(1 + |\lambda_1 - n_1^3 + \hat{\varphi}(0)\, n_1|)^{1/2}} \, d\lambda_1 \\ &= \hat{F}_f * \hat{F}_g(n, \lambda), \end{aligned}$$

where

$$F_h(x,t)=\sum_m \int_{\mathbb{R}} \frac{c_h(m,\mu)}{(1+|\mu-m^3+\hat{\varphi}(0)m|)^{1/2}}\, e^{i(mx+\mu t)}\, d\mu. \tag{3.12}$$

Taking $L^2$-norms, using Parseval's equality, and Hölder's inequality we obtain

$$\begin{aligned}(\text{LHS of } 2.13)_I &\lesssim \left(\sum_{n\in \dot{Z}} \int_{\mathbb{R}} |\widehat{F_f F_g}(n,\lambda)|^2\, d\lambda\right)^{1/2}\\ &\simeq \|F_f F_g\|_{L^2}\\ &\lesssim \|F_f\|_{L^4}\, \|F_g\|_{L^4}, \qquad (\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2})\\ &\lesssim \delta^{1/12}\, |||f|||_s\, \delta^{1/12}\, |||g|||_s.\end{aligned}$$

In the last inequality we used the following lemma with $v=3$ and $a=\hat{\varphi}(0)$.

LEMMA 3.2. *Let $v\geqslant 2$, and denote*

$$|||h|||_s^2 \doteq \sum_n |n|^{2s} \int_{\mathbb{R}} (1+|\lambda-n^v+an|)\, |\hat{h}(n,\lambda)|^2\, d\lambda$$

*and*

$$F_h(x,t)=\sum_m \int_{\mathbb{R}} \frac{c_h(m,\mu)}{(1+|\mu-m^v+am|)^{1/2}}\, e^{i(mx+\mu t)}\, d\mu,$$

*where $c_h(m,\mu)=|m|^s\,(1+|\mu-m^v+am|)^{1/2}\,|\hat{h}(m,\mu)|$. Then*

$$\|F_h\|_{L^4} \lesssim \delta^{(v-1)/8v}\, |||h|||_s.$$

The proof of Lemma 3.2 will be given at the end of this section.

*Case* II. Using (3.11) inequality (3.7) gives

$$\frac{|n|^s\,|\hat{w}_2(n,\lambda)|}{(1+|\lambda-n^3-\hat{\varphi}_0 n|)^{1/2}} \lesssim \frac{1}{(1+|\lambda-n^3-\hat{\varphi}_0 n|)^{1/2}}\, \widehat{F_f G_g}(n,\lambda),$$

where $F_f$ is as in (3.12) and

$$G_h(x,t)=\sum_m \int_{\mathbb{R}} c_h(m,\mu)\, e^{i(mx+\mu t)}. \tag{3.13}$$

Taking $L^2$-norms in the last inequality and using the fact $(1+|\lambda-n^3+\varphi(0)|)^{-1} \leqslant (1+|\lambda-n^3+\varphi(0)|)^{-2/3}$ we obtain

$$(\text{LHS of } 2.13)_{\text{II}} \lesssim \left( \sum_n \int_{\mathbb{R}} (1+|\lambda-n^3+\varphi(0)|)^{-2/3} \, |\widehat{F_f G_g}(n, \lambda)|^2 \, d\lambda \right)^{1/2} \tag{3.14}$$

Next we shall need the following inequality

$$\sum_{m,\, n \in Z} (1+|m-n^3+\hat{\varphi}_0 n|)^{-2/3} \, |\hat{h}(m, n)|^2 \leqslant c \, \|h\|^2_{4/3}, \tag{3.15}$$

which is the dual of

$$\|f\|^2_{L^4(\mathbb{T}^2)} \leqslant c \sum_{m,\, n \in Z} (1+|m-n^3+\hat{\varphi}_0 n|)^{2/3} \, |\hat{f}(m, n)|^2. \tag{3.16}$$

These estimates are proved in Bourgain [B1, B2] (see also Fang and Grillakis [FG] for a different approach). In order to apply apriori estimate (3.15) to the right hand-side of (3.14) we need to localize in $t$. This is done by replacing $w(x, t)$ in (2.3) with $w(x, t)\,\chi(t)$, where $\chi(t)$ is an appropriate cut-off function which is equal to 1 near $t=0$. Here and in the rest of the paper we will not carry out the details of this localization, but we shall indicate it by using the notation $L^p(dx, dt(\text{loc}))$. For more details we refer the reader to Bourgain [B2, p. 216]. Using Hölder's inequality and Lemma 3.2 we obtain

$$\begin{aligned}(\text{LHS of } 2.13)_{\text{II}} &\lesssim \|F_f G_g\|_{L^{4/3}(dx,\, dt(\text{loc}))} \\ &\lesssim \|F_f\|_{L^4} \|G_g\|_{L^2} \lesssim \delta^{1/12} \, |||f|||_s \, |||g|||_s.\end{aligned}$$

*Case* III. In this case, inequality (3.7) gives

$$\frac{|n|^s \, |\hat{w}_2(n, \lambda)|}{(1+|\lambda-n^3+\hat{\varphi}_0 n|)^{1/2}} \lesssim \frac{1}{(1+|\lambda-n^3+\hat{\varphi}_0 n|)^{1/2}} \, \widehat{G_f F_g}(n, \lambda),$$

which gives the same estimate as in case (II). This completes the proof of (2.13).

*Proof of* (2.14). The proof of (2.14) for the $\hat{w}_1$ term is like the KdV case (see [B2]) and holds for $s \geqslant 0$. The proof of (2.14) for $\hat{w}_2$ and for $s \geqslant 1$ can again be reduced to the KdV case. Therefore below we shall only present the proof of (2.14) for the $\hat{w}_2$ term when

$$1/2 < s < 1.$$

For this we again use the partition (I)–(III). We have

$$\frac{|n|^s\,|\hat{w}_2(n,\lambda)|}{(1+|\lambda-n^3+\hat{\varphi}(0)n|)}$$
$$\lesssim \sum_{n_1\in\dot{Z}}\int_{\mathbb{R}}\frac{|n|^{s+1}\,|n-n_1|^{1-s}\,|n_1|^{1-s}}{(1+n^2)(1+|\lambda-n^3+\hat{\varphi}(0)n|)}$$
$$\times\frac{c_f(n-n_1,\lambda-\lambda_1)\,c_g(n_1,\lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3+\hat{\varphi}(0)(n-n_1)|)^{1/2}\,(1+|\lambda_1-n_1^3+\hat{\varphi}(0)\,n_1|)^{1/2}}\,d\lambda_1. \tag{3.17}$$

*Case* I. Let $r$ be a real number such that

$$\tfrac{1}{2}<r\leqslant s. \tag{3.18}$$

Then

$$\frac{|n|^{s+1}\,|n-n_1|^{1-s}\,|n_1|^{1-s}}{(1+n^2)(1+|\lambda-n^3|)^{1-r}}\lesssim 2^{1-s}.$$

This together with (3.17) gives

$$(\text{LHS of } 2.14)_I\lesssim\left(\sum_n\left(\int_{\mathbb{R}}\frac{1}{(1+|\lambda-n^3|)^r}\,\widehat{F_fF_g}(n,\lambda)\,d\lambda\right)^2\right)^{1/2}. \tag{3.19}$$

Using duality in the $l^2$ space, for any sequence $\{a_n\}$ with $l^2$-norm equal to 1, we have

$$(\text{LHS of } 2.14)_I\lesssim\sum_n\int_{\mathbb{R}}\frac{a_n}{(1+|\lambda-n^3|)^r}\,\widehat{F_fF_g}(n,\lambda)\,d\lambda.$$

If we let

$$H(x,t)=\sum_n\int_{\mathbb{R}}\frac{a_n}{(1+|\lambda-n^3|)^r}\,e^{i(nx+\lambda t)}\,d\lambda,$$

then the last inequality gives

$$(\text{LHS of } 2.14)_I\lesssim\|\hat{H}\|_{L^2}\,\|\widehat{F_fF_g}\|_{L^2}\lesssim\|\hat{H}\|_{L^2}\cdot\|F_f\|_{L^4}\cdot\|F_g\|_{L^4}.$$

Using Lemma 3.2 and the fact that $\|\hat{H}\|_{L^2}^2=2/(1-2r)$ we obtain

$$(\text{LHS of } 2.14)_I\lesssim\delta^{1/12}\,|||f|||_s\,|||g|||_s,$$

which is the desired estimate.

*Case* II. In this case we have

$$\frac{|n|^{s+1}\,|n-n_1|^{1-s}\,|n_1|^{1-s}}{(1+n^2)\,|n_1|} \lesssim 2^{1-s}.$$

Choosing a number $\rho$ such that

$$\tfrac{1}{3} \leqslant \rho < \tfrac{1}{2}, \tag{3.20}$$

and using (3.17) we obtain

$$(\text{LHS of } 2.14)_{II} \lesssim \left[ \sum_n \left( \int_{\mathbb{R}} \frac{1}{(1+|\lambda-n^3|)^{1-\rho}} \cdot \frac{\widehat{F_f G_g}(n,\lambda)\, d\lambda}{(1+|\lambda-n^3|)^{\rho}} \right)^2 \right]^{12}.$$

Applying Cauchy–Schwarz inequality and taking into consideration (3.20) we obtain

$$(\text{LHS of } 2.14)_{II} \lesssim \left[ \sum_n \int_{\mathbb{R}} (1+|\lambda-n^3|)^{-2/3}\, (\widehat{F_f G_g}(n,\lambda))^2\, d\lambda \right]^{1/2}.$$

Using (3.16), Hölder's inequality, and Lemma 3.2 we get

$$\begin{aligned}(\text{LHS of } 2.14)_{II} &\lesssim \|F_f G_g\|_{L^{4/3}(dx,\, dt(\mathrm{loc}))} \\ &\lesssim \|F_f\|_{L^4}\, \|G_g\|_{L^2} \lesssim \delta^{1/12}\, |||f|||_s\, |||g|||_s.\end{aligned}$$

which is the desired estimate.

*Case* III. This case is very similar to case (II) the only deference being that $F_f G_g$ is replaced with $G_f F_g$.

This completes the proof of Proposition 2.3.

*Proof of Lemma* 3.2. Using a more general version of (3.16), which is the first inequality below (see [B1, B2]), we obtain

$$\begin{aligned}\|F_h\|_{L^4}^2 &\lesssim \sum_m \int_{\mathbb{R}} (1+|\mu-m^\nu+am|)^{(\nu+1)/2\nu}\, |\hat{F}_h(m,\mu)|^2\, d\mu \\ &= \sum_m \int_{\mathbb{R}} (1+|\mu-m^\nu+am|)^{(\nu+1)/2\nu}\, \frac{c_h^2(m,\mu)}{(1+|\mu-m^\nu+am|)}\, d\mu \\ &= \sum_m \int_{\mathbb{R}} (1+|\mu-m^\nu+am|)^{(\nu+1)/2\nu}\, |m|^{2s}\, |\hat{h}(m,\mu)|^2\, d\mu.\end{aligned}$$

Then, observe that $2=(\nu+1)/\nu+(\nu-1)/\nu$ so that

$$\|F_h\|_{L^4}^2 \lesssim \sum_m \int_{\mathbb{R}} |m|^{s(\nu-1)/\nu}\, |\hat{h}(m,\mu)|^{(\nu-1)/\nu}$$
$$\times (1+|\mu-m^\nu+am|)^{(\nu+1)/2\nu}\, |m|^{s(\nu+1)/\nu}\, |\hat{h}(m,\mu)|^{(\nu+1)/\nu}\, d\mu.$$

Applying Hölder's inequality with $p=2\nu/(\nu-1)$ and $q=2\nu/(\nu+1)$ gives

$$\|F_h\|_{L^4}^2 \lesssim \|\partial_x^s h\|_{L^2}^{(\nu-1)/\nu} \cdot \left(\sum_m \int_{\mathbb{R}} c_h^2(m,\mu)\, d\mu\right)^{(\nu+1)/2\nu}$$
$$\simeq |||h|||_s^{(\nu+1)/\nu}\, \|\partial_x^s h\|_{L^2}^{(\nu-1)/\nu}.$$

Considering that the $t$-support of $h$ is in the interval $(-\delta,\delta)$ and using the Cauchy–Schwarz inequality we obtain

$$\|\partial_x^s h\|_{L^2}^{(\nu-1)/\nu} \lesssim \delta^{(\nu-1)/4\nu}\, \|\partial_x^s h\|_{L^4}^{(\nu-1)/\nu}.$$

Also we have

$$\|\partial_x^s h\|_{L^4}^2 \lesssim \sum_m \int_{\mathbb{R}} (1+|\mu-m^\nu+am|)^{(\nu+1)/2\nu}\, |m|^{2s}\, |\hat{h}(m,\mu)|^2\, d\mu$$
$$\lesssim |||h|||_s^2, \qquad \text{since} \quad \frac{\nu+1}{2\nu}<1.$$

Putting it all together gives

$$\|F_h\|_{L^4}^2 \lesssim \delta^{(\nu-1)/4\nu}\, |||h|||_s^{(\nu+1)/\nu+(\nu-1)/\nu} = \delta^{(\nu-1)/4\nu}\, |||h|||_s^2,$$

which is the desired inequality. This completes the proof of Lemma 3.2.

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