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Infinite Multiplicity of Positive Entire Solutions for a Semilinear Elliptic Equation[1]

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We establish that the elliptic equation $\Delta u + K(x)u^p + \mu f(x) = 0$ in $\mathbf{R}^n$ has infinitely many positive entire solutions for small $\mu \geqslant 0$ under suitable conditions on K, p, and f.

Key Words: semilinear elliptic equations; positive solutions; asymptotic behavior; infinite multiplicity.

1. INTRODUCTION

In this paper, we study the elliptic equation

$$\Delta u + K(x)u^p + \mu f(x) = 0, \tag{1.1}$$

where $n \geqslant 3$, $\Delta = \sum_{i=1}^{n} (\partial^2/\partial x_i^2)$ is the Laplace operator, $p > 1$, $\mu \geqslant 0$ is a parameter, and f as well as K is a given locally Hölder continuous function in $\mathbf{R}^n \backslash \{0\}$. By an entire solution of Eq. (1.1), we mean a positive weak solution of (1.1) in $\mathbf{R}^n$ satisfying (1.1) pointwise in $\mathbf{R}^n \backslash \{0\}$.

Inhomogeneous elliptic equations have been studied to afford an understanding of the effects of the inhomogeneous term in the existence and properties of solutions, compared with those of homogeneous equations; see [1, 3, 7].

The purpose of this paper is to study the asymptotic behavior of positive entire solutions and to establish infinite multiplicity for (1.1) which has diverse physical and geometrical backgrounds. In particular, Eq. (1.1) in the case $K = 1$ and $p = 2$ arises naturally in establishing occupation time limit theorems for super-Brownian motions which requires analyzing cumulant generating functions satisfying some integral equations equivalent to the parabolic counterparts of (1.1). We refer the interested readers to [3, 7] and

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the references therein. On the other hand, the corresponding homogeneous equation

$$\Delta u + K(x)u^p = 0 \tag{1.2}$$

stands for the prescribing scalar curvature problem in Riemannian geometry when p is the critical Sobolev exponent $\frac{n+2}{n-2}$ or for the Lane–Emden equation in astrophysics when $K(x) = |x|^l$. There have been many works devoted to studying the existence of positive solutions of (1.2) in $\mathbf{R}^n$ after the first contribution by Ni [10] in 1982; see [2, 5, 9–11]. One of the remarkable features of the equation is that (1.2) can possess infinitely many solutions as long as the exponent p and the dimension n are large enough. Recent studies in [1, 5] paid special attention to this phenomenon.

To illuminate the motivations of this paper in detail, we need the following notations. Set

$$p_c = p_c(n,l) = \begin{cases} \frac{(n-2)^2 - 2(l+2)(n+l) + 2(l+2)\sqrt{(n+l)^2 - (n-2)^2}}{(n-2)(n-10-4l)} & \text{if } n > 10 + 4l, \\ \infty & \text{if } n \leqslant 10 + 4l, \end{cases} \tag{1.3}$$

for some $l > -2$. Let $m = \frac{2+l}{p-1}$ and

$$\lambda_1 = \lambda_1(n,p,l) = \frac{(n-2-2m) - \sqrt{(n-2-2m)^2 - 4(l+2)(n-2-m)}}{2}, \tag{1.4}$$

$$\lambda_2 = \lambda_2(n,p,l) = \frac{(n-2-2m) + \sqrt{(n-2-2m)^2 - 4(l+2)(n-2-m)}}{2}. \tag{1.5}$$

Observe that $\lambda_1, \lambda_2 \in \mathbf{R}^+$ if and only if $n > 10 + 4l$ and $p \geqslant p_c$. The two numbers, λ_1 and λ_2, play important roles in describing the asymptotic behavior at ∞ of positive radial solutions to the Lane–Emden equation with $p \geqslant p_c(n,l)$,

$$\Delta u + c|x|^l u^p = 0 \tag{1.6}$$

in $\mathbf{R}^n$ for $l > -2$ and $c > 0$. It is known that when $p > \frac{n+2+2l}{n-2}$ and $l > -2$, (1.6) has a positive radial solution $\bar{u}_\alpha$ with $\bar{u}_\alpha(0) = \alpha$ for each $\alpha > 0$ and

$$\lim_{r\to\infty} r^m \bar{u}_\alpha(r) = L, \tag{1.7}$$

where

$$L = L(n, p, l, c) = \left[\frac{l+2}{p-1}\left(n - 2 - \frac{l+2}{p-1}\right)\frac{1}{c}\right]^{\frac{1}{p-1}} \tag{1.8}$$

(see [4, 12]). Furthermore, $p \geqslant p_c(n, l)$ if and only if any two positive radial solutions of (1.6) cannot intersect each other [12]. By analogy with (1.6), it is natural to expect that (1.2) with $p \geqslant p_c$ has infinitely many positive solutions under suitable conditions on K. In [5], Gui studied conditions on K to guarantee infinite multiplicity for (1.2) and established the following

THEOREM A. *Suppose that* $K \geqslant 0$ *satisfies*

(K1) $K(x) = O(|x|^{\sigma})$ *at* $x = 0$ *for some* $\sigma > -2$, *and*

(K2) $K(x) = c|x|^{l} + O(|x|^{-d})$ *near* $|x| = \infty$

for some $c > 0$, $l > -2$, *and*

$$d > \frac{n-2}{2} - l + \sqrt{(n-2-2m)^2 - 4(l+2)(n-2-m)}. \tag{1.9}$$

Then, Eq. (1.2) *with* $p \geqslant p_c(n, l)$ *possesses infinitely many positive entire solutions satisfying*

$$\lim_{|x| \to \infty} |x|^m u(x) = L(n, p, l, c).$$

The first objective of this paper is to improve Theorem A. We have found by a barrier method (see [1, 5]) that Theorem A is true for a wider class of K. For example, we may replace (1.9) in Theorem A with

$$d > n - \lambda_2(n, p, l) - mp.$$

In fact, a more general assumption on K of integral form shall be given without any pointwise condition at ∞ like (K2). Furthermore, we weaken the integrability condition again up to the form covering the case

$$d > n - \lambda_2(n, p, l) - m(p+1)$$

by imposing an upper bound on K near ∞. The monotonicity of $\bar{u}_\alpha$ with respect to α is essential for the constructions of infinitely many pairs of super- and subsolutions. It, therefore, seems interesting to examine multiplicity for (1.2) satisfying $L|x|^{-m}$ at ∞ when $p < p_c$.

On the other hand, another natural question is whether (1.1) still could have infinitely many entire solutions. Bae and Ni [1] recently confirmed the question positively for (1.1) with $K \equiv 1$, combining the modified version of the barrier method initiated by Gui [5] and the asymptotic behavior near

∞ of positive solutions of suitable homogeneous equations. The existence results in [1] for the equation

$$\Delta u + u^p + \mu f(x) = 0 \qquad \text{in } \mathbf{R}^n, \tag{1.10}$$

where $\mu > 0$ is a parameter, can be summarized as follows.

THEOREM B. (i) *If* $p > p_c(n, 0)$, $f \not\equiv 0$ *and near* ∞,

$$\max(\pm f(x), 0) \leqslant |x|^{-q_\pm},$$

where $q_+ > n - \lambda_2(n, p, 0)$ *and* $q_- > n - \lambda_2(n, p, 0) - \frac{2}{p-1}$, *then there exists* μ_* > 0 *such that for every* $\mu \in (0, \mu_*)$ *Eq.* (1.10) *possesses infinitely many positive entire solutions with the asymptotic behavior* $L(n, p, 0, 1)|x|^{-2/(p-1)}$ *at* ∞.

(ii) *If* $p = p_c(n, 0)$, *the conclusion in* (i) *holds with the additional assumption that either* f *has a compact support in* $\mathbf{R}^n$ *or* f *does not change sign in* $\mathbf{R}^n$.

Our next objective is to extend the result of Theorem B to the more general Eq. (1.1) and to remove the extra condition in the *critical* case $p = p_c$. The main difference between (1.10) and (1.1) lies in the fact that the part $\Delta u + Ku^p$ of (1.1) does not possess any scaling property in general. Hence, the barrier method used in [1] cannot apply to the problem (1.1) directly. We formulate a new approach to managing infinite multiplicity for (1.1), which is to verify certain *continuity* of a limiting function demonstrating the asymptotic behavior at ∞ of positive solutions of Eq. (1.2) (see Proposition 4.2 below). This observation makes it possible for the infinitely many pairs of positive solutions of (1.1) constructed by super- and subsolution arguments to have specific behaviors at ∞ in order to discern one another, which is, in fact, the key idea in [1] to get infinite multiplicity for the inhomogeneous problem (1.10). When $\frac{n+2}{n-2} < p < p_c$, the multiplicity question for (1.10) is fundamental, but left unanswered.

The main result of this paper is the following

THEOREM 1.1. *Let* $p \geqslant p_c(n, l)$ *with* $l > -2$. *Assume that* $K \geqslant 0$ *holds* (K1), (K2) *for some constants* $c > 0$ and $d > n - \lambda_2(n, p, l) - m(p+1)$ *while* f *holds*:

(f1) $f(x) = O(|x|^\tau)$ *at* $x = 0$ *for some* $\tau > -2$,

(f2) $-(1 + |x|^{mp})f(x) \leqslant \min_{|z|=|x|} K(z)$, *and*

(f3) *near* $|x| = \infty$, $f(x) = O(|x|^{-q})$ *for some constant* $q > n - \lambda_2(n, p, l) - m$.

Then, there exists $\mu_* > 0$ *such that for every* $\mu \in [0, \mu_*)$, *Eq.* (1.1) *possesses infinitely many positive entire solutions with the asymptotic behavior* $L(n, p, l, c)|x|^{-m}$ *at* ∞.

This paper is organized as follows. Some preliminaries are reviewed in Section 2. In Section 3, we study the homogeneous equation (1.2) and apply multiplicity results to Riemannian geometry. Finally, the asymptotic behavior of positive solutions of (1.2) is investigated and then Theorem 1.1 is established in Section 4.

2. PRELIMINARIES

In this section, we consider positive radial solutions of (1.2) with a radial function K. The radial version of Eq. (1.2) is of the form

$$u'' + \frac{n-1}{r}u' + K(r)u^p = 0; \quad u(0) = \alpha > 0. \tag{2.1}$$

Under the assumption

($\mathscr{A}1$) $K(r) \geqslant 0$, $K(r) \in \mathbf{C}((0,\infty))$, and $\int_0 rK(r)dr < \infty$,

Eq. (2.1) has a unique solution $u \in \mathbf{C}^2((0,\varepsilon)) \cap \mathbf{C}([0,\varepsilon))$ for some $\varepsilon > 0$ (see [11]). For each $\alpha > 0$, the local solution u_α of (2.1) is decreasing and extended locally wherever it exists and remains positive.

We first recall the asymptotic behavior of positive radial solutions $\bar{u}_\alpha$ of Eq. (1.6) (see [8, 6; Theorem 2.5, Lemma 4.13, and (4.15)] for details).

Proposition 2.1. *Let $l > -2$ and $c > 0$. For $p \geqslant p_c(n,l)$, we have that for arbitrarily given $\varepsilon > 0$,*

$$\bar{u}_\alpha(r) = \frac{L}{r^m} + \frac{a_\alpha}{r^{m+\lambda_1}} + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right) \qquad if \;\; p > p_c, \tag{2.2}$$

$$\bar{u}_\alpha(r) = \frac{L}{r^m} + \frac{a_\alpha \log r}{r^{m+\lambda_1}} + \cdots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right) \qquad if \;\; p = p_c, \tag{2.3}$$

near ∞, where L is given by (1.8), *λ_1 is given by* (1.4), *and*

$$a_\alpha = \alpha^{-\lambda_1/m}a_1 < 0. \tag{2.4}$$

Although Theorem 2.5 in [6] deals only with the case $l = 0$, the arguments in the proof can proceed similarly to conclude Proposition 2.1. Another direct consequence of Theorem 2.5 in [6] is the following

PROPOSITION 2.2. *Let v_1, v_2 be two positive radial solutions of the equation*

$$\Delta u + cr^l u^p = 0$$

near ∞, *where* $c > 0$ *and* $l > -2$. *Suppose that*

$$\lim_{r\to\infty} r^m v_1(r) = L = \lim_{r\to\infty} r^m v_2(r)$$

and

$$\lim_{r\to\infty} r^{\lambda_1}(r^m v_1(r) - L) = \lim_{r\to\infty} r^{\lambda_1}(r^m v_2(r) - L) \qquad \text{if } \ p > p_c,$$

$$\lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(r^m v_1(r) - L) = \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(r^m v_2(r) - L) \qquad \text{if } \ p = p_c.$$

Then, $v_1(r) - v_2(r) = O(r^{-m-\lambda_2})$ *near* ∞, *where* λ_2 *is given by* (1.5).

The existence of a positive radial supersolution of (1.6) having the following asymptotic behavior is verified similarly as in [6] (see [6; Theorems 2.5 and 4.1 and Lemmas 4.11 and 4.13]).

PROPOSITION 2.3. *Let* $p \geqslant p_c(n, l)$ *with* $l > -2$ *and* $c > 0$. *Then*, *for each* $\alpha > 0$, *there exists a positive radial supersolution* $\bar{u}_\alpha^+(r)$ *of* (1.6) *such that* $\bar{u}_\alpha^+(r) > \bar{u}_\alpha(r)$ *for* $r \in [0, \infty)$ *and* $\bar{u}_\alpha^+(r) - \bar{u}_\alpha(r) = O(r^{-m-\lambda_2})$ *as* $r \to \infty$.

We now remark that Proposition 2.3 produces an improved form of a result in [5]. In [5], Gui used an estimation for $\bar{u}_\alpha^+(r) - \bar{u}_\alpha(r)$, but by replacing Theorem 2.3 in [5] with Proposition 2.3 we modify Theorem 3.4 of [5] and write the following, to be improved again by Theorem 3.2 in the next section.

THEOREM 2.4. *Let* $p \geqslant p_c(n, l)$ *with* $l > -2$. *Assume that in the radial case*, *K holds* ($\mathscr{A}1$) *and*, *for some* $c > 0$,

$$\int_1^\infty |K(r) - cr^l| r^{n-1-mp-\lambda_2}\, dr < \infty,$$

while in the nonradial case, *K holds* ($K1$) *and*, *for some* $c > 0$,

$$\int_1^\infty |K_i(r) - cr^l| r^{n-1-mp-\lambda_2}\, dr < \infty, \qquad i = 1, 2,$$

where $K_1(r) := \inf_{|x|=r} K(x)$, $K_2(r) := \sup_{|x|=r} K(x)$, *and* $\lambda_2 = \lambda_2(n, p, l)$. *Then, there exist infinitely many positive entire solutions of* (1.2) (*which are radial if* K *is radial*) *and no two of them can intersect. Furthermore, every solution* $u(x)$ *obtained above satisfies*

$$\lim_{|x|\to\infty} |x|^m u(x) = L(n, p, l, c).$$

For the radial case, the solution u_α of (2.1) exists globally for every $\alpha > 0$ small under the assumptions of Theorem 2.4 and is monotone with respect to α. In general, the existence of three separated positive radial solutions of (2.1) leads to a one-parameter family of positive radial solutions indexed by initial data.

LEMMA 2.5. *Assume that* $K \not\equiv 0$ *holds* ($\mathscr{A}1$). *Suppose that there exist three solutions* $u_\alpha, u_\beta, u_\gamma$ of (2.1) *such that* $0 < u_\alpha < u_\beta < u_\gamma$ in $[0, \bar{R})$ *for some* $\bar{R} \in (0, \infty]$. *Then, for each* $\alpha < \delta < \beta$, (2.1) *possesses a positive radial solution* u_δ *in* $B_{\bar{R}}$ *satisfying*

$$0 < u_\alpha(r) < u_\delta(r) < u_\beta(r)$$

for $0 \leqslant r < \bar{R}$.

Proof. Suppose that for some $\alpha < \delta < \beta$ and $0 < R < \bar{R}$, $w_2 := u_\delta - u_\alpha > 0$ in $[0, R)$, $w_2(R) = 0$, and $w_1 := u_\beta - u_\delta > 0$ in $[0, R]$. Then, $w_2'(R) \leqslant 0$ and w_2 satisfies $\Delta w_2 + K g_2 w_2 = 0$ with $w_2(0) = \delta - \alpha > 0$, where

$$g_2 := \frac{u_\delta^p - u_\alpha^p}{u_\delta - u_\alpha} < p u_\delta^{p-1}$$

in B_R. We may assume that $K(r) \geqslant 0, \not\equiv 0$ in $[0, R]$. On the other hand, w_1 satisfies $\Delta w_1 + K g_1 w_1 = 0$, where

$$g_1 := \frac{u_\beta^p - u_\delta^p}{u_\beta - u_\delta} > p u_\delta^{p-1}$$

in B_R. It follows from Green's identity that

$$\begin{aligned} \omega_n R^{n-1} w_1(R) w_2'(R) &= \omega_n R^{n-1}(w_1(R) w_2'(R) - w_2(R) w_1'(R)) \\ &= \int_{B_R} (w_1 \Delta w_2 - w_2 \Delta w_1) \geqslant \int_{B_R} (g_1 - g_2) K w_1 w_2 > 0, \end{aligned}$$

where ω_n denotes the surface area of the unit sphere. Then, $w_2'(R) > 0$, a contradiction. Hence, if for some $\alpha < \delta < \beta$, u_δ meets u_α for some $R_\alpha > 0$, then there exists $0 < R_\beta < R_\alpha$ such that $u_\delta > 0$, $w_4 := u_\beta - u_\delta > 0$ in $[0, R_\beta)$ and $w_4(R_\beta) = 0$. Replacing w_2 and w_1 in the above arguments with w_4 and $w_3 := u_\gamma - u_\beta$, respectively, we also have a contradiction. Therefore, for each $\alpha < \delta < \beta$, the local solution u_δ remains between u_α and u_β and exists up to $\bar{R}$. ∎

3. HOMOGENEOUS EQUATION

In this section, we establish the existence of infinitely many positive entire solutions of (1.2). In order to improve Theorem 2.4, we adopt arguments similar to those in [1] and pay attention to the supremum of $r^{-l}K$ near ∞. Later, an interpretation of multiplicity results to Riemannian geometry shall be presented. We first consider the radial case.

PROPOSITION 3.1. *Let $p \geqslant p_c(n, l)$ with $l > -2$. Suppose that K satisfies* $(\mathscr{A}1)$,

$$(\mathscr{A}2) \quad \int_1^\infty (K(r) - cr^l)_- r^{n-1-m(p+1)-\lambda_2} dr < \infty,$$

and either $r^{-l}K(r) \leqslant cp$ near ∞,

$$(\mathscr{A}3) \quad \int_1^\infty (K(r) - cr^l)_+ r^{n-1-m(p+1)-\lambda_2} dr < \infty,$$

or

$$(\mathscr{A}4) \quad \int_1^\infty (K(r) - cr^l)_+ r^{n-1-mp-\lambda_2} dr < \infty$$

for some $c > 0$, where $k_\pm = \max(\pm k, 0)$. Then, there exists a positive constant $\alpha^* = \alpha^*(p, K)$ such that for each $\alpha \in (0, \alpha^*]$, Eq. (2.1) possesses a positive radial solution $_\alpha$ with $u_\alpha(0) = \alpha$ satisfying

$$\lim_{r \to \infty} r^m u_\alpha(r) = L(n, p, l, c),$$

and no two of them can intersect.

Proof. For simplicity, we assume $c = 1$. It follows from Proposition 2.3 that for each $\alpha > 0$, there exists a supersolution $\bar{u}_\alpha^+ > \bar{u}_\alpha$ of the equation $\Delta u + |x|^l u^p = 0$ satisfying $F_\alpha(r) := \bar{u}_\alpha^+(r) - \bar{u}_\alpha(r) = O(r^{-m-\lambda_2})$ at ∞

and

$$\Delta F_\alpha \leqslant -|x|^l((\bar{u}_\alpha^+)^p - \bar{u}_\alpha^p) \leqslant -p|x|^l \bar{u}_\alpha^{p-1} F_\alpha.$$

For all $\gamma > 0$, there exists a unique positive solution u_γ of (2.1) locally. First, we claim that for given $\beta > 0$ there exists $0 < \bar{\gamma} = \bar{\gamma}(\beta) < \beta$ such that for every $0 < \gamma \leqslant \bar{\gamma}$, $u_\gamma < \bar{u}_\beta$ in $\overline{B(R_\gamma)}$ whenever $u_\gamma > 0$ in $B(R_\gamma)$ for some $R_\gamma > 0$.

Suppose that for any $0 < \gamma < \beta$ there exists $0 < \tilde{\gamma} < \gamma$ such that $u_{\tilde{\gamma}} > 0$ in $B(R_{\tilde{\gamma}})$, $w_{\tilde{\gamma}}(r) := \bar{u}_\beta(r) - u_{\tilde{\gamma}}(r) > 0$ on $[0, R_{\tilde{\gamma}})$, but $w_{\tilde{\gamma}}(R_{\tilde{\gamma}}) = 0$ for some $R_{\tilde{\gamma}} > 0$. Then, $w_{\tilde{\gamma}}$ satisfies

$$\Delta w_{\tilde{\gamma}} = -|x|^l \bar{u}_\beta^p + K u_{\tilde{\gamma}}^p$$

in $\overline{B(R_{\tilde{\gamma}})}$. Fix $\alpha > \beta$. Applying Green's identity, we have

$$\begin{aligned} 0 &\leqslant \int_{B(R_{\tilde{\gamma}})} (w_{\tilde{\gamma}} \Delta F_\alpha - F_\alpha \Delta w_{\tilde{\gamma}}) \\ &\leqslant \int_{B(R_{\tilde{\gamma}})} \{-p|x|^l w_{\tilde{\gamma}} \bar{u}_\alpha^{p-1} F_\alpha + |x|^l \bar{u}_\beta^p F_\alpha - K u_{\tilde{\gamma}}^p F_\alpha\} \\ &\leqslant \int_{B(R_{\tilde{\gamma}})} \{-p|x|^l w_{\tilde{\gamma}} \bar{u}_\alpha^{p-1} F_\alpha + p|x|^l w_{\tilde{\gamma}} \bar{u}_\beta^{p-1} F_\alpha + (|x|^l - K) u_{\tilde{\gamma}}^p F_\alpha\} \end{aligned}$$

and

$$p \int_{B(R_{\tilde{\gamma}})} |x|^l w_{\tilde{\gamma}} [\bar{u}_\alpha^{p-1} - \bar{u}_\beta^{p-1}] F_\alpha \leqslant \int_{B(R_{\tilde{\gamma}})} (|x|^l - K) u_{\tilde{\gamma}}^p F_\alpha.$$

Since $\bar{u}_\beta > 0$ in $\mathbf{R}^n$ and $u_{\tilde{\gamma}} \leqslant \tilde{\gamma}$ on $[0, R_{\tilde{\gamma}}]$, we may assume that for small $\tilde{\gamma} > 0$, $R_{\tilde{\gamma}} > 1$ and $w_{\tilde{\gamma}} \geqslant \frac{1}{2} \bar{u}_\beta(1)$ in B_1. Hence, for small $\gamma > 0$, and thus for small $0 < \tilde{\gamma} \leqslant \gamma$, we have

$$\begin{aligned} \frac{p}{2} \bar{u}_\beta(1) \int_{B(1)} |x|^l [\bar{u}_\alpha^{p-1} - \bar{u}_\beta^{p-1}]\ F_\alpha &\leqslant \int_{B(R_{\tilde{\gamma}})} (|x|^l - K) u_{\tilde{\gamma}}^p F_\alpha \qquad (3.1) \\ &\leqslant \int_{B(R_{\tilde{\gamma}})} (K - |x|^l)_- \bar{u}_\beta^p F_\alpha. \end{aligned}$$

However, this is impossible because from (1.7), ($\mathscr{A}2$), and the Dominated Convergence Theorem the right-hand side of (3.1) goes to 0 as $\tilde{\gamma} \to 0$, while the left-hand side is a fixed positive constant, which verifies the claim. Therefore, there exists $0 < \bar{\gamma} < \beta$ such that for all $0 < \gamma \leqslant \bar{\gamma}$, $0 < u_\gamma < \bar{u}_\beta$ in $B(R_\gamma)$.

Regarding R_γ as the supremum of the set $\{R > 0 : u_\gamma > 0 \text{ in } B_R\}$, we observe that $R_\gamma \to \infty$ as $\gamma \to 0$. Indeed, for $0 \leqslant r < R_\gamma$,

$$\begin{aligned} u_\gamma(r) &= \gamma + \int_0^r u'_\gamma \, ds \\ &= \gamma - \int_0^r \int_0^s \left(\frac{t}{s}\right)^{n-1} K(t) u_\gamma^p(t) \, dt \, ds \\ &\geqslant \gamma - \gamma^p \int_0^r t^{n-1} K(t) \left[\int_t^r s^{1-n} \, ds\right] dt \\ &\geqslant \gamma \left[1 - \frac{\gamma^{p-1}}{n-2} \int_0^r tK(t) \, dt\right]. \end{aligned} \tag{3.2}$$

Thus, it follows from ($\mathscr{A}1$) that $R_\gamma \to \infty$ as $\gamma \to 0$ and moreover, for given $R > 0$ and $0 < d < 1$, there exists $0 < \tilde{\gamma} < \bar{\gamma}$ such that for $0 < \gamma < \tilde{\gamma}$, $u_\gamma > d\gamma$ in B_R.

Case 1: Consider the case that $r^{-l} K \leqslant p$ near ∞. Choose $R \geqslant 1$ so large that $r^{-l} K(r) \leqslant p$ for $r \in [R, \infty)$. Then, there exists $0 < \tilde{\gamma}_1 \leqslant \tilde{\gamma}$ such that for all $0 < \gamma < \tilde{\gamma}_1$, $R_\gamma \geqslant R$ and $u_\gamma(r) \geqslant \frac{\gamma}{2}$ on $[0, 1]$. Let J_β be the set of $0 < \gamma < \tilde{\gamma}_1$ satisfying

$$\frac{p}{2} \int_{B(1)} |x|^l \bar{u}_\beta^{p-1} F_\beta > \int_{B(R)} K u_\gamma^{p-1} F_\beta. \tag{3.3}$$

Then, J_β contains an interval, say $(0, \tilde{\gamma}_2]$. Suppose that $R_\gamma < \infty$ for some $0 < \gamma < \tilde{\gamma}_2$. From Green's identity, it follows that

$$\begin{aligned} 0 &\leqslant \int_{B(R_\gamma)} (u_\gamma \Delta F_\beta - F_\beta \Delta u_\gamma) \\ &\leqslant \int_{B(R_\gamma)} [-p|x|^l u_\gamma \bar{u}_\beta^{p-1} + K u_\gamma^p] F_\beta \\ &\leqslant \int_{B(R)} [-p|x|^l u_\gamma \bar{u}_\beta^{p-1} + K u_\gamma^p] F_\beta \\ &\quad + \int_{B(R_\gamma) \setminus B(R)} (K - p|x|^l) u_\gamma \bar{u}_\beta^{p-1} F_\beta. \end{aligned}$$

Then,

$$p \int_{B(1)} |x|^l u_\gamma \bar{u}_\beta^{p-1} F_\beta \leqslant p \int_{B(R)} |x|^l u_\gamma \bar{u}_\beta^{p-1} F_\beta \leqslant \int_{B(R)} K u_\gamma^p F_\beta.$$

Thus,

$$\frac{\gamma p}{2} \int_{B(1)} |x|^l \bar{u}_\beta^{p-1} F_\beta \leqslant \int_{B(R)} K u_\gamma^p F_\beta$$

and

$$\frac{p}{2}\int_{B(1)}|x|^l\bar{u}_\beta^{p-1}F_\beta \leqslant \int_{B(R)}Ku_\gamma^{p-1}F_\beta,$$

a contradiction. Therefore, $R_\gamma = \infty$ for all $0<\gamma<\tilde{\gamma}_2$, which implies that for every $0<\gamma<\tilde{\gamma}_2$, u_γ is an entire solution and thus, $0<u_\gamma<\bar{u}_\beta$ in $\mathbf{R}^n$.

Fix $0<\gamma<\tilde{\gamma}_2$. Next, we claim that there exists $0<\delta<\gamma$ such that $\bar{u}_\delta<u_\gamma$ in $\mathbf{R}^n$ and thus, for every $0<\varepsilon<\delta$, $0<\bar{u}_\varepsilon<u_\gamma$ in $\mathbf{R}^n$.

Suppose that there exist $\varepsilon_j>0$ going to 0 and $r_{\varepsilon_j}>0$ going to ∞ as $j\to\infty$ such that for each $j\geqslant 1$, $0<\varepsilon_j<\gamma$, $\tilde{w}_{\varepsilon_j} = u_\gamma - \bar{u}_{\varepsilon_j}>0$ in $B(r_{\varepsilon_j})$, and $\tilde{w}_{\varepsilon_j}(r_{\varepsilon_j}) = 0$. By Green's identity,

$$\begin{aligned}
0 &\leqslant \int_{B(r_{\varepsilon_j})}(\tilde{w}_{\varepsilon_j}\Delta F_\beta - F_\beta\Delta\tilde{w}_{\varepsilon_j})\\
&\leqslant \int_{B(r_{\varepsilon_j})}\{-p|x|^l\tilde{w}_{\varepsilon_j}\bar{u}_\beta^{p-1}F_\beta + Ku_\gamma^pF_\beta - |x|^l\bar{u}_{\varepsilon_j}^pF_\beta\}
\end{aligned}$$

and

$$\begin{aligned}
0 &\leqslant \int_{B(r_{\varepsilon_j})}\{p|x|^l\tilde{w}_{\varepsilon_j}\bar{u}_\beta^{p-1}F_\beta - |x|^l(u_\gamma^p - \bar{u}_{\varepsilon_j}^p)F_\beta\}\\
&\leqslant \int_{B(r_{\varepsilon_j})}(K-|x|^l)u_\gamma^pF_\beta\\
&\leqslant \int_{B(r_{\varepsilon_j})}(K-|x|^l)\bar{u}_\beta^pF_\beta. \qquad (3.4)
\end{aligned}$$

Since the integrand of (3.4) is positive, it follows by Fatou's Lemma and the Dominated Convergence Theorem with ($\mathscr{A}2$) and ($\mathscr{A}3$) that

$$0\leqslant \int_{\mathbf{R}^n}[p|x|^lu_\gamma\bar{u}_\beta^{p-1}F_\beta - |x|^lu_\gamma^pF_\beta]\leqslant \int_{\mathbf{R}^n}(K-|x|^l)u_\gamma^pF_\beta<\infty.$$

Hence,

$$\int_{\mathbf{R}^n}(p|x|^l\bar{u}_\beta^{p-1} - Ku_\gamma^{p-1})u_\gamma F_\beta\leqslant 0$$

and thus

$$\int_{B(R)}(p|x|^l\bar{u}_\beta^{p-1} - Ku_\gamma^{p-1})u_\gamma F_\beta\leqslant 0.$$

Therefore,

$$\frac{p}{2}\int_{B(1)}|x|^l\bar{u}_\beta^{p-1}F_\beta \leqslant \int_{B(R)} Ku_\gamma^{p-1}F_\beta,$$

which contradicts (3.3).

Case 2: For $\beta > 0$, let I_β be the set of $0<\gamma<\bar{\gamma}(\beta)$ satisfying

$$\frac{p}{2}\int_{B(1)}|x|^l[\bar{u}_\beta^{p-1} - u_\gamma^{p-1}]F_\beta > \int_{B(R_\gamma)}(K-|x|^l)_+u_\gamma^{p-1}F_\beta.$$

Then, $I_\beta \supset (0,\gamma_\beta)$ for some $\gamma_\beta > 0$ since, from (1.7) and ($\mathscr{A}$4), the right-hand side goes to 0 as $\gamma \to 0$ by the Dominated Convergence Theorem while the left-hand side is bounded below a positive constant which is irrelevant to γ when $\gamma > 0$ is small.

It follows from (3.2) that there exists $0<\hat{\gamma}\leqslant\gamma_\beta$ such that for all $0<\gamma<\hat{\gamma}$, $R_\gamma > 1$, and $u_\gamma(r)\geqslant\frac{3}{4}\gamma$ on $[0,1]$.

We now claim that for small $0<\gamma<\hat{\gamma}$ so that $u_\gamma(r)\geqslant\frac{3}{4}\gamma$ for $0\leqslant r\leqslant 1$, there exists $0<\eta<\gamma$ such that $u_\gamma > \bar{u}_\eta$ in $\mathbf{R}^n$. Suppose that there exists $0<\hat{\gamma}_1<\hat{\gamma}$ such that for each $0<\eta<\hat{\gamma}_1$ there exists $r_\eta > 0$ satisfying $\hat{w}_\eta(r) = u_{\hat{\gamma}_1}(r) - \bar{u}_\eta(r) > 0$ in $[0,r_\eta)$ and $\hat{w}_\eta(r_\eta) = 0$. From Green's identity,

$$\begin{aligned}0 &\leqslant \int_{B(r_\eta)}(\hat{w}_\eta\Delta F_\beta - F_\beta\Delta\hat{w}_\eta)\\ &\leqslant \int_{B(r_\eta)}\{-p|x|^l\hat{w}_\eta\bar{u}_\beta^{p-1}F_\beta + Ku_{\hat{\gamma}_1}^pF_\beta - |x|^l\bar{u}_\eta^pF_\beta\}\end{aligned}$$

and

$$\begin{aligned}\int_{B(r_\eta)}p|x|^l\hat{w}_\eta[\bar{u}_\beta^{p-1} - u_{\hat{\gamma}_1}^{p-1}]F_\beta &\leqslant \int_{B(r_\eta)}[p|x|^l\hat{w}_\eta\bar{u}_\beta^{p-1} - |x|^l(u_{\hat{\gamma}_1}^p - \bar{u}_\eta^p)]F_\beta\\ &\leqslant \int_{B(r_\eta)}(K-|x|^l)_+u_{\hat{\gamma}_1}^pF_\beta.\end{aligned}$$

Since $\bar{u}_\eta$ is monotonically decreasing to 0 as η decreases to 0 so that $\bar{u}_\eta \to 0$ uniformly on $[0,R]$ for any fixed $R>0$, we may assume that $r_\eta > 1$ and $\hat{w}_\eta(r)\geqslant\frac{3}{4}\hat{\gamma}_1 - \bar{u}_\eta(r)\geqslant\frac{1}{2}\hat{\gamma}_1$ in B_1 if $\eta > 0$ is small enough. Then, we have

$$\frac{p}{2}\int_{B(1)}|x|^l[\bar{u}_\beta^{p-1} - u_{\hat{\gamma}_1}^{p-1}]F_\beta \leqslant \int_{B(R_\gamma)}(K-|x|^l)_+u_{\hat{\gamma}_1}^{p-1}F_\beta,$$

which is impossible because $\hat{\gamma}_1 \in I_\beta$.

Repeating the above arguments, we find a decreasing sequence $\{u_{\gamma_i}\}$ of positive solutions of (2.1) such that there exists a positive decreasing sequence $\{\alpha_i\}$ going to 0 as $i \to 0$ satisfying $u_{\gamma_i} > \bar{u}_{\alpha_i} > u_{\gamma_{i+1}}$ in $\mathbf{R}^n$ for each $i \geqslant 1$. By virtue of Lemma 2.5, the proof is complete. ∎

We apply Proposition 3.1 to the nonradial case under another assumption:

(K3) For $N \geqslant 3$, the infimum $K_1(r)$ and the supremum $K_2(r)$ of $K(x)$ on $\{x = (x_1, x_2) \in \mathbf{R}^{n-N} \times \mathbf{R}^N : |x_2| = r\}$ are continuous on $(0, \infty)$, and $\int_0 rK_2(r)dr < \infty$.

THEOREM 3.2. *Let $p \geqslant p_c(N, l)$ with $N \geqslant 3$ and $l > -2$. Assume that $K \geqslant 0$ satisfies* (K1), (K3) *and, for some $c > 0$, K_1 holds* ($\mathscr{A}2$) *while K_2 holds either* ($\mathscr{A}4$) *or* ($\mathscr{A}3$) *and $r^{-l}K_2(r) \leqslant cp$ near ∞. Then, Eq.* (1.2) *possesses infinitely many positive entire solutions satisfying*

$$\lim_{|x_2| \to \infty} |x_2|^m u(x_1, x_2) = L(N, p, l, c) \tag{3.5}$$

uniformly in $x_1 \in \mathbf{R}^{n-N}$ and no two of them can intersect.

Proof. Applying Proposition 3.1 to K_1 and K_2, we have a positive radial solution w_1, w_2 of $\Delta w + K_1 w^p = 0$ in $\mathbf{R}^N$ and a positive radial solution v_1, v_2 of $\Delta v + K_2 v^p = 0$ in $\mathbf{R}^N$ satisfying

$$v_1 > \bar{u}_{\alpha_1} > w_1 > \bar{u}_{\eta_1} > v_2 > \bar{u}_{\alpha_2} > w_2 \qquad \text{in } \mathbf{R}^N,$$

where $\bar{u}_{\alpha_1}, \ \bar{u}_{\eta_1}, \ \bar{u}_{\alpha_2}$ are solutions of (1.6) in $\mathbf{R}^N$. Since $\tilde{v}_i(x_1, x_2) := v_i(|x_2|)$ and $\tilde{w}_i(x_1, x_2) := w_i(|x_2|)$ are supersolutions and subsolutions of (1.2) in $\mathbf{R}^n \backslash \{0\}$ respectively, by the standard super- and subsolution method there exist solutions u_i of (1.2) in $\mathbf{R}^n \backslash \{0\}$ such that

$$\tilde{v}_i \geqslant u_i \geqslant \tilde{w}_i, \qquad i = 1, 2.$$

It is easy to see that u_i are weak solutions of (1.2) in $\mathbf{R}^n$ and entire solutions. Repeating the above procedure, we construct infinitely many ordered positive entire solutions satisfying the asymptotic behavior (3.5). ∎

An immediate consequence of Theorem 3.2 is the following

COROLLARY 3.3. *Let $p \geqslant p_c(N, l)$ with $N \geqslant 3$ and $l > -2$. Suppose that $K \geqslant 0$ satisfies* (K1), (K3) *and that there exists $c > 0$ such that*

$$K_i(r) = cr^l + O(r^{-d}) \text{ at } \infty, \qquad i = 1, 2,$$

for some constant $d > N - \frac{p+1}{p-1}(2 + l) - \lambda_2(N, p, l)$, where K_1, K_2 are defined in (K3). *Then, the same result as in Theorem 3.2 holds.*

We now translate Theorem 3.2 into the context of Riemannian geometry. Let (M, g) be an n-dimensional Riemannian manifold and K be a given function. The scalar curvature problem is to find a metric g_1 on M conformal to g such that the corresponding scalar curvature to g_1 is K. The introduction of $u > 0$ by $g_1 = u^{4/(n-2)}g$, $n \geqslant 3$, leads to the equation

$$\frac{4(n-1)}{n-2}\Delta_g - ku + Ku^{\frac{n+2}{n-2}} = 0, \tag{3.6}$$

where Δ_g denotes the Laplace–Beltrami operator on M in the g metric and k is the scalar curvature of (M, g). If $M = \mathbf{R}^n$ and $g = \Sigma_{i=1}^n dx_i^2$ is the standard metric, then Eq. (3.6) reduces to

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \qquad \text{in } \mathbf{R}^n.$$

We write $x = (x_1, x_2) \in \mathbf{R}^{n-N} \times \mathbf{R}^N = \mathbf{R}^n$, $N \geqslant 3$.

THEOREM 3.4. *Let $\frac{n+2}{n-2} \geqslant p_c(N, l)$ with $N \geqslant 3$ and $l > -2$. Assume that K satisfies* (K1), (K3), *and*

$$\int_1^\infty (K_1(r) - cr^l)_- r^{N-1-\frac{n(2+l)}{2}-\lambda_2}\, dr < \infty,$$

and that $r^{-l}K_2(r) \leqslant \frac{N+2}{N-2}c$ near ∞,

$$\int_1^\infty (K_2(r) - cr^l)_+ r^{N-1-\frac{n(2+l)}{2}-\lambda_2}\, dr < \infty,$$

or

$$\int_1^\infty (K_2(r) - cr^l)_+ r^{N-1-\frac{(n+2)(2+l)}{4}-\lambda_2}\, dr < \infty,$$

for a constant $c > 0$, where $\lambda_2 = \lambda_2(N, \frac{n+2}{n-2}, l)$ and K_1, K_2 are defined in (K3). *Then, there exist infinitely many Riemannian metrics g_1 on $\mathbf{R}^n$ with the following properties*:

(i) *K is the scalar curvature of g_1;*

(ii) *g_1 is conformal to the standard metric g on $\mathbf{R}^n$;*

(iii) *g_1 is complete.*

4. INHOMOGENEOUS EQUATION

In this section, we are concerned with infinite multiplicity for the inhomogeneous Eq. (1.1). Under the assumptions on K as in Proposition 3.1, Eq. (2.1) with $p \geqslant p_c(n, l)$ and $l > -2$ has a family $\{u_\alpha\}$ of positive radial solutions indexed by $\alpha \in (0, \alpha^*]$ for some $\alpha^* > 0$ such that $u_\alpha(0) = \alpha$ and u_α is monotonically increasing with respect to α. For $\alpha \in (0, \alpha^*]$ with $\alpha^* > 0$ small, set $W(\alpha, t) := r^m u_\alpha(r) - L,\ t = \log r$, and

$$D(\alpha, t) := e^{\lambda_1 t} W(\alpha, t) \qquad \text{for } p > p_c,$$

$$D(\alpha, t) := t^{-1} e^{\lambda_1 t} W(\alpha, t) \qquad \text{for } p = p_c.$$

From the proof of Proposition 3.1, we observe that for each $\alpha \in (0, \alpha^*]$, there exist $\gamma < \alpha$ and $\beta > \alpha$ such that $\bar{u}_\gamma \leqslant u_\alpha \leqslant \bar{u}_\beta$ in $\mathbf{R}^n$ and, thus, $r^m u_\alpha(r) \to L$ as $r \to \infty$. Moreover, it follows from (2.2), (2.3), and (2.4) that for fixed $0 < a < \alpha^*$, $D(\alpha, t)$ are uniformly bounded above and below near $+\infty$ on $[a, \alpha^*]$; that is, there exists $M = M(a, p)$ such that for all $\alpha \in [a, \alpha^*]$,

$$|W(\alpha, t)| \leqslant M e^{-\lambda_1 t} \qquad \text{for } p > p_c \tag{4.1}$$

and

$$|W(\alpha, t)| \leqslant M t e^{-\lambda_1 t} \qquad \text{for } p = p_c. \tag{4.2}$$

For fixed $-\infty < t < +\infty$, $D(\alpha, t)$ is continuous with respect to α. The next observation is that $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$, which seems of independent interest. To verify this, we need only the condition

$$\int_1^\infty |K(r) - cr^l| r^{n-1-m(p+1)-\lambda_2} dr < \infty. \tag{4.3}$$

Lemma 4.1. *For given $0 < a < \alpha^*$, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$.*

Proof. Setting $W(\alpha, t) := r^m u_\alpha(r) - L,\ t = \log r$, we see that W satisfies

$$W_{tt} + (n - 2 - 2m) W_t + c(p-1) L^{p-1} W + c g(W) + h(e^t) e^{-lt} (W + L)^p = 0,$$

where $h(r) := K(r) - cr^l$ and $g(s) := (s + L)^p - L^p - pL^{p-1}s$ such that for s near 0,

$$g(s) = \frac{p(p-1)}{2} L^{p-2} s^2 + O(s^3). \tag{4.4}$$

Case 1: Let $p > p_c$. Then, $D(\alpha, t) = e^{\lambda_1 t} W(\alpha, t)$ holds that

$$D_{tt} + (\lambda_2 - \lambda_1) D_t + e^{\lambda_1 t} [c g(W) + h(e^t) e^{-lt} (W + L)^p] = 0 \tag{4.5}$$

and

$$(D_t e^{(\lambda_2-\lambda_1)t})_t = -e^{\lambda_2 t}[cg(W) + h(e^t)e^{-lt}(W+L)^p]. \tag{4.6}$$

Integrating (4.6) over $[T, t]$, we have

$$\begin{aligned} D_t(\alpha, t) &= e^{-(\lambda_2-\lambda_1)t}\{e^{(\lambda_2-\lambda_1)T} D_t(\alpha, T) \\ &\quad - \int_T^t e^{\lambda_2 s}[cg(W) + h(e^s)e^{-ls}(W+L)^p]\, ds\}. \end{aligned} \tag{4.7}$$

It follows from (4.1) and (4.4) that for any $0 < \varepsilon < \min\{\lambda_1, \lambda_2 - \lambda_1\}$ and for some $M_1 > 0$,

$$\begin{aligned} e^{(\lambda_1-\lambda_2)t} \int_T^t ce^{\lambda_2 s}|g(W(s))|\, ds &\leqslant e^{(\lambda_1-\lambda_2)t} \int_T^t cM_1 e^{(\lambda_2-2\lambda_1)s} ds \\ &\leqslant cM_1 e^{-\varepsilon t} \int_T^t e^{-(\lambda_1-\varepsilon)s} ds, \end{aligned}$$

which goes to 0 as $t \to +\infty$.

On the other hand, from (4.3), we have

$$\begin{aligned} e^{(\lambda_1-\lambda_2)t} \int_T^t e^{(\lambda_2-l)s}|h(e^s)|\, ds &= e^{(\lambda_1-\lambda_2)t} \int_T^t e^{(\lambda_2-\lambda_1)s} e^{(\lambda_1-l)s}|h(e^s)|\, ds \\ &\leqslant \int_T^\infty e^{(\lambda_1-l)s}|h(e^s)|\, ds < \infty. \end{aligned}$$

Hence, the function

$$F(t) := e^{(\lambda_1-\lambda_2)t} \int_T^t e^{(\lambda_2-l)s}|h(e^s)|\, ds$$

is bounded and holds that

$$F'(t) = (\lambda_1 - \lambda_2)F(t) + e^{(\lambda_1-l)t}|h(e^t)|.$$

Then,

$$\begin{aligned} (\lambda_2 - \lambda_1) \int_T^t F(s)\, ds &= F(T) - F(t) + \int_T^t e^{(\lambda_1-l)s}|h(e^s)|\, ds \\ &\leqslant F(T) + \int_T^\infty e^{(\lambda_1-l)s}|h(e^s)|\, ds < \infty, \end{aligned} \tag{4.8}$$

and thus

$$\int_T^{+\infty} F(s)\,ds < \infty. \tag{4.9}$$

Therefore, from (4.8), $F(t)$ converges as $t \to +\infty$ which in turn implies that from (4.9),

$$\lim_{t\to+\infty} F(t) = 0. \tag{4.10}$$

Hence, by (4.7) and (4.10), $D_t(\alpha, t)$ converges uniformly to 0 on $[a, \alpha^*]$ as $t \to +\infty$. Integrating (4.5) over $[T, t]$, we see that

$$(\lambda_2 - \lambda_1)(D(\alpha, t) - D(\alpha, T)) = D_t(\alpha, T) - D_t(\alpha, t) \\ - \int_T^t e^{\lambda_1 s}[cg(W) + h(e^s)e^{-ls}(W+L)^p]\,ds.$$

Then, it follows immediately that $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$.

Case 2: Let $p = p_c$. Then, $D(\alpha, t) = t^{-1}e^{\lambda_1 t}W(\alpha, t)$ satisfies

$$D_{tt} + \frac{2}{t}D_t + \frac{e^{\lambda_1 t}}{t}[cg(W) + h(e^t)e^{-lt}(W+L)^p] = 0 \tag{4.11}$$

and

$$(t^2 D_t)_t = -te^{\lambda_1 t}[cg(W) + h(e^t)e^{-lt}(W+L)^p]. \tag{4.12}$$

Integrating (4.12) over $[T, t]$, we have

$$tD_t(\alpha, t) = t^{-1}\left\{T^2 D_t(\alpha, T) - \int_T^t se^{\lambda_1 s}[cg(W) + h(e^s)e^{-ls}(W+L)^p]\,ds\right\}. \tag{4.13}$$

First, note that from (4.2) and (4.4),

$$t^{-1}\int_T^t cse^{\lambda_1 s}|g(W)|\,ds \leqslant t^{-1}\int_T^t cM_2 se^{-\lambda_1 s}ds$$

for some $M_2 > 0$. Second, letting

$$G(t) := t^{-1}\int_T^t se^{(\lambda_1 - l)s}|h(e^s)|\,ds,$$

we have

$$G'(t) = -t^{-1}G(t) + e^{(\lambda_1 - l)t}|h(e^t)|.$$

Then,

$$G(t) - G(T) = -\int_T^t s^{-1}G(s)\,ds + \int_T^t e^{(\lambda_1 - l)s}|h(e^s)|\,ds. \tag{4.14}$$

Hence, we have

$$\int_T^{+\infty} \frac{G(s)}{s}\,ds \leqslant G(T) + \int_T^{+\infty} e^{(\lambda_1 - l)s}|h(e^s)|\,ds < \infty, \tag{4.15}$$

which implies that by (4.14), $G(t)$ converges as $t \to +\infty$ and, thus, to 0 by (4.15) again. Thus, from (4.13), $tD_t(\alpha, t)$ converges uniformly to 0 on $[a, \alpha^*]$ as $t \to +\infty$. Multiplying (4.11) by t and integrating over $[T, t]$, we have

$$\begin{aligned} D(\alpha, t) = {} & D(\alpha, T) + TD_t(\alpha, T) - tD_t(\alpha, t) \\ & - \int_T^t e^{\lambda_1 s}[cg(W) + h(e^s)e^{-ls}(W + L)^p]\,ds. \end{aligned}$$

Therefore, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$. ∎

An immediate consequence of Lemma 4.1 is that the limit of $D(\alpha, t)$ as $t \to +\infty$ is continuous.

PROPOSITION 4.2. *Let $p \geqslant p_c(n, l)$ with $l > -2$. Suppose the assumptions of Proposition* 3.1. *Then, $D(\alpha) := \lim_{t\to+\infty} D(\alpha, t)$ is continuous for $\alpha > 0$ small. Moreover, $D(\alpha) \to -\infty$ as $\alpha \to 0$.*

The continuity of $D(\alpha)$ is crucial in establishing the following main result.

THEOREM 4.3. *Let $p \geqslant p_c(n, l)$ with $l > -2$. Assume that $K \geqslant 0$ and f satisfy* (K1) *and* (f1) *respectively. Suppose there exist radial functions $H^{\pm}$ such that*

(i) $H^{\pm}(r) \geqslant 0$, $H^{\pm}(r) \in \mathbf{C}((0, \infty))$, *and* $\int_0 rH^{\pm}(r)dr < \infty$;

(ii) $\max(\pm f(x), 0) \leqslant (1 + |x|^{mp})^{-1}H^{\pm}(|x|)$;

(iii) $H^- \leqslant K^-$ *and*

$$\int_{B^c} (K^- - H^- - c|x|^l)_- |x|^{-m(p+1)-\lambda_2}\,dx < \infty;$$

(iv) $H^+(r) = O(r^l)$, $K^+(r) < c\,pr^l$ *near* ∞ (*or* $\leqslant c\,pr^l$ *in the case* $H^+ \equiv 0$),

$$\int_{B^c} (K^+ + H^+ - c|x|^l)_+ |x|^{-m(p+1)-\lambda_2} dx < \infty,$$

or

$$\int_{B^c} (K^+ + H^+ - c|x|^l)_+ |x|^{-mp-\lambda_2} dx < \infty,$$

for some $c > 0$, *where* $K^-(r) := \inf_{|x|=r} K(x)$, $K^+(r) := \sup_{|x|=r} K(x)$, $\lambda_2 = \lambda_2(n, p, l)$, *and* B^c *is the complement of a ball* B *centered at* 0. *Then, there exists* $\mu_* > 0$ *such that, for every* $\mu \in [0, \mu_*)$, *Eq.* (1.1) *has infinitely many positive entire solutions with the asymptotic behavior*

$$\lim_{|x| \to \infty} |x|^m u(x) = L(n, p, l, c).$$

Proof. To construct super- and subsolutions of Eq. (1.1), we consider the homogeneous problem

$$v'' + \tfrac{n-1}{r} v' + (K^{\pm} \pm H^{\pm}) v^p = 0 \quad \text{in } (0, \infty); \quad v(0) = \alpha > 0. \tag{4.16}$$

We may assume that $K^+ + H^+ \leqslant c\,p|x|^l$ near ∞ in the first case of (iv) by taking $\mu > 0$ small, and we consider only the case that $K^- - H^- \not\equiv cr^l \not\equiv K^+ + H^+$ and $f \not\equiv 0$ because the other cases can be handled similarly. By $v_\alpha^{\pm}$, we denote the solutions respectively. From Proposition 3.1, there exists $\alpha^* > 0$ such that for each $\alpha \in (0, \alpha^*]$ there exist positive entire solutions $v_\alpha^{\pm}$ of (4.16) respectively which increase as α increases and which are below $\bar{u}_\theta$ for some $\theta > \alpha^*$. Moreover, for given $\alpha \in (0, \alpha^*]$, there exist $0 < \eta < \gamma < \xi < \alpha$ such that

$$\bar{u}_\eta < v_\gamma^- < \bar{u}_\xi < v_\alpha^+ \qquad \text{in } \mathbf{R}^n.$$

Define

$$\gamma_\alpha = \sup\{\beta \in (\eta, \alpha) : v_\beta^- < v_\alpha^+ \quad \text{in } \mathbf{R}^n\}.$$

Obviously, $v_{\gamma_\alpha}^- \leqslant v_\alpha^+$. Then, the strong maximum principle implies that $v_{\gamma_\alpha}^- < v_\alpha^+$ in $\mathbf{R}^n$. By Lemma 4.1, we may set

$$D^-(\gamma_\alpha) := \lim_{r \to \infty} r^{\lambda_1}(v_{\gamma_\alpha}^-(r) - L) \qquad \text{and} \qquad D^+(\alpha) := \lim_{r \to \infty} r^{\lambda_1}(v_\alpha^+(r) - L)$$

if $p > p_c$, and

$$D^-(\gamma_\alpha) := \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(v^-_{\gamma_\alpha}(r) - L) \qquad \text{and} \qquad D^+(\alpha) := \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(v^+_\alpha(r) - L)$$

if $p = p_c$. Then, it follows from Proposition 4.2 that $D^-(\gamma_\alpha) = D^+(\alpha)$. Indeed, if $D^-(\gamma_\alpha) < D^+(\alpha)$, then $v^-_{\gamma_\alpha} < v^+_\alpha$ near ∞. Hence, the continuity of D^- implies that there exist $R > 0$ and $\delta > 0$ such that if $0 < \beta - \gamma_\alpha < \delta$ and $\beta < \alpha$, then $v^-_\beta(r) < v^+_\alpha(r)$ for $r \in [R, \infty)$. Since v^-_β is monotonically decreasing to $v^-_{\gamma_\alpha}$ as β decreases to γ_α and $v^-_\beta \to v^-_{\gamma_\alpha}$ uniformly on $[0, R]$, there exists $\gamma_\alpha < \gamma_1 < \beta$ such that $v^-_{\gamma_1} < v^+_\alpha$ in $\mathbf{R}^n$, which contradicts the definition of γ_α.

Fix $\alpha_1 \in (0, \alpha^*]$. From the proof of Proposition 3.1, there exist $0 < \eta_1 < \gamma_{\alpha_1}$ and $0 < \eta_2 < \alpha_2 < \frac{\eta_1}{2}$ such that

$$\bar{u}_{\eta_2} < v^-_{\gamma_{\alpha_2}} < v^+_{\alpha_2} < \bar{u}_{\eta_1/2} < \bar{u}_{\eta_1} < v^-_{\gamma_{\alpha_1}} \qquad \text{in } \mathbf{R}^n.$$

Since a_α in (2.4) is strictly increasing as α increases, we have $D^-(\gamma_{\alpha_2}) = D^+(\alpha_2) < D^-(\gamma_{\alpha_1}) = D^+(\alpha_1)$. By the continuity of D^+, $D^+([\alpha_2, \alpha_1]) = [D^+(\alpha_2), D^+(\alpha_1)]$. We apply (ii) and (1.7) to find $\mu^\pm$ satisfying

$$\mu^+ f_+ \leqslant H^+(v^+_{\alpha_1})^p, \qquad \mu^- f_- \leqslant H^-(v^-_{\gamma_{\alpha_1}})^p.$$

For each $0 \leqslant \mu \leqslant \min\{\mu^+, \mu^-\}$, we conclude by the super- and subsolution method (see [5,10]) that for every $\alpha \in [\alpha_2, \alpha_1]$, Eq. (1.1) possesses a positive entire solution u_α satisfying

$$v^-_{\gamma_\alpha} < u_\alpha < v^+_\alpha \qquad \text{in } \mathbf{R}^n,$$

and moreover,

$$\lim_{|x|\to\infty} |x|^m u_\alpha(x) = L(n, p, l, c).$$

Every u_α is characterized by the asymptotic behavior

$$\lim_{|x|\to\infty} |x|^{\lambda_1}(u_\alpha(x) - L) = D^+(\alpha)$$

if $p > p_c$ and

$$\lim_{|x|\to\infty} \frac{|x|^{\lambda_1}}{\log|x|}(u_\alpha(x) - L) = D^+(\alpha)$$

if $p = p_c$. ∎

Theorem 1.1 follows from Theorem 4.3 by taking

$$H^{\pm}(|x|) = (1 + |x|^{mp})F^{\pm}(|x|),$$

where $F^{\pm}(r) := \max_{|x|=r} f_{\pm}(x)$. The first case in (iv) is applied to deduce Theorem 1.1. Combining (K2) and (f3), we see that the integral conditions in (iii) and (iv) are satisfied.

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