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Geometry of quadratic differential systems in the neighborhood of infinity

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Abstract

In this article we give a complete global classification of the class $\mathbf{QS_{ess}}$ of planar, essentially quadratic differential systems (i.e. defined by relatively prime polynomials and whose points at infinity are not all singular), according to their topological behavior in the vicinity of infinity. This class depends on 12 parameters but due to the action of the affine group and re-scaling of time, the family actually depends on five parameters. Our classification theorem (Theorem 7.1) gives us a complete dictionary connecting very simple integer-valued invariants which encode the geometry of the systems in the vicinity of infinity, with algebraic invariants and comitants which are a powerful tool for computer algebra computations helpful in the route to obtain the full topological classification of the class **QS** of all quadratic differential systems.

Keywords: Poincaré compactification; Singular point; Phase portrait; Topological index; Intersection multiplicity; Linear group; Affine invariant

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1. Introduction

We consider real planar polynomial differential system, i.e. systems of the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y), \tag{S}$$

where p and q are polynomials in x and y with real coefficients $(p, q \in \mathbb{R}[x, y])$. In this article, a system of the above form with max(deg(p), deg(q)) = 2 will be called quadratic.

These are the simplest nonlinear differential systems. However, global problems regarding this class are difficult to solve. In 1900 Hilbert gave his list of 23 problems and one of them still unsolved, the second part of Hilbert's 16th problem, is on planar polynomial differential systems. This problem which asks for the maximum $H(n)$ of the numbers of limit cycles occurring in differential systems with $\max(\deg(p), \deg(q)) = n$ (for a discussion of this problem cf. [25]), is still unsolved even for quadratic differential systems. The interest is in the global behavior of all solutions in the whole plane and even at infinity (cf. [10]) and this for a whole family of systems, which is why this problem is so hard. The set **QS** of quadratic differential systems depends on 12 parameters, the coefficients of the two polynomials p and q. On **QS** acts the group of affine transformations and of changes of scale on the time axis. The orbit space of **QS** under the group action is five dimensional. But even five is a large number and it is expected that this class will yield over 2000 topologically distinct phase portraits. For this reason people began by studying particular subclasses of **QS** and in some cases a complete classification of phase portraits with respect to topological equivalence was obtained (quadratic systems with a center [18,23,31], quadratic Hamiltonian systems [1,7], quadratic chordal systems [9], quadratic systems with a weak focus of third order [2,14], etc.).

The goal in most of these articles was to obtain all topologically distinct phase portraits for that specific subclass of **QS** and whenever possible its bifurcation diagram. Two systems (S) and (S') are topologically equivalent if there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f carries orbits to orbits preserving (or reversing) their orientation. In most articles, the classifications were done by using specific charts and normal forms for the systems in these charts with respect to parameters satisfying certain inequalities or equations. The results are not readily applicable for systems given in normal forms with respect to other charts. Ever since Felix Klein gave his famous Erlangen program, we are used to calling a property geometric, if it is invariant under the action of some group. In this sense, most of the results obtained are not geometrical since they are chart-dependent.

Chart-independent classifications results were obtained by Sibirsky and his school (cf. [7,22,30]) using the algebraic invariant theory of differential equations developed by Sibirsky and his disciples (cf. [21,28,29,32]). Most of the articles of Sibirsky's school were published in Russian, only some appeared in translations which partly explains why this theory is rather unknown in the west. In these articles, invariants and comitants are introduced in their multi-index tensorial form, certain rather artificial polynomial combinations of these are chosen and classifications are given in terms of these combinations. The geometry of the systems remains hidden behind this technical language.

When studying truly global problems involving limit cycles such as for example Hilbert's 16th problem, second part, the perturbations of the systems possessing a center play an essential role. It is thus crucial to choose normal forms such that the algebraic varieties of systems possessing a center, computed with respect to these normal forms, be as simple as possible, some of them even linear varieties. This helps in the display of the bifurcation diagram of the systems with center on these algebraic varieties. We also need the global scheme of singularities finite and infinite for the class **QS**. However, a normal form good for the global study of singularities may yield algebraic varieties of systems with center which are complicated and on which the display of their bifurcation diagram turns out to be an impossible task. Vice versa, a normal form which yields simple looking algebraic varieties of systems with center may turn out to be very inconvenient for the study of singularities and for their blow-out. It is thus important to obtain the geometric global scheme of singularities finite or infinite in invariant form, i.e. independent of any specific normal form.

The goal of this article is to obtain the global geometric scheme of singularities at infinity, in invariant form, for the whole class **QS**. An analogous work for the finite singularities is presently in progress. We point out that for quadratic systems the points at infinity are solutions of a cubic form. We need their *simultaneous* study, in invariant form, and this not just for an individual system but for the whole class **QS**.

Furthermore, to easily grasp the geometry of the systems, simple invariants, simpler than the configuration space of Markus (cf.[15]) are needed. Such simple integer-valued invariants reflecting the geometry of the systems were used in [14,26].

In spite of their awesome character, polynomial invariants and comitants are a useful and very powerful computational tool, applicable to any canonical form, and they can be programmed on a computer. There is thus a need to merge the purely geometric invariants above mentioned with the algebraic invariant approach and we do this in the present work.

We briefly review now the history of the study of singularities at infinity of the class **QS**. Kooij and Reyn [13] obtained all possible local phase portraits around a single singular point at infinity of an arbitrary quadratic vector field. They did not consider the possible ways of combining such singularities so as to obtain a topological classification of quadratic systems in a neighborhood of the line at infinity. Nicolaev and Vulpe [16] obtained such a classification in terms of algebraic invariants and comitants and in [3] the affine invariant classification of quadratic system with respect to the possible distributions of the multiplicities of singularities at infinity was obtained by Baltag and Vulpe [3].

These classifications use the technical language of algebraic invariant theory developed by the school of Sibirsky ([6,28,32] etc.) and as previously indicated, the geometry of the systems remains hidden behind this technical language.

In this work we bring out the global geometry at infinity of the systems by using some global algebro-geometric concepts such as for example the notion of divisor and of zero-cycle on the line at infinity. We also combine the geometric approach in [14,26] with the algebraic invariant approach in [3,16] for the topological classification of quadratic systems in the neighborhood of infinity. A first version of this article, appeared in [27]. Our article proved to be very useful for the ongoing study of the family of quadratic systems with a second-order weak focus. We point out that in the attempt to merge the simple integer-valued invariants with the algebraic ones, the geometry of the systems led us to much simpler algebraic invariants than those in [16] and to simpler conditions in our classification Theorem 7.1.

The article is organized as follows: In Section 2 we consider the two compactifications of real planar polynomial systems and the foliations with singularities, on the real and complex projective planes, associated to these systems.

In Section 3 we describe the purely geometric objects, i.e. the divisors attached to the line at infinity, introduced in [26], which encode the multiplicities at infinity of the systems, and attach to these some integer-valued global affine invariants.

In Section 4 we consider group actions on quadratic differential systems and define algebraic invariants and comitants with respect to these group actions. We also give using a comitant, canonical forms for these differential systems according to their behavior at infinity.

In Section 5 we state and prove the classification theorem (Theorem 5.1) of the quadratic differential systems according to their multiplicity divisors at infinity and for each class we give the necessary and sufficient conditions in terms of algebraic invariants and comitants with respect to the group action. These conditions allow us to compute for any system and in any chart the types of the multiplicity divisors associated to the system.

In Section 6 we introduce new classifying tools, among them the index divisor encoding globally the topological indices of the singularities at infinity of any polynomial differential system without a line of singularities at infinity. We also introduce a divisor encoding globally the number of local separatrices bounding a hyperbolic sector of a singular point at infinity.

In Section 7 we state and prove the topological classification theorem (Theorem 7.1). This classification is expressed in both geometrical, affine integer-valued invariants, and in terms of algebraic invariants and comitants. A complete dictionary connecting the integer-valued geometric invariants with the algebraic invariants and comitants is given. One side of the dictionary displays the geometry of the systems and the other enables us to use the powerful tool of computer algebra to perform calculations useful in the route to obtain the full topological classification of the whole class of quadratic differential systems.

In the Appendix we list the invariants and comitants used in [16] and which are needed for the proofs of the main results. These are also listed for the purpose of comparison with the simpler algebraic invariants and comitants used in this article. Highlighting the geometry of the systems via the integer-valued invariants, helped us to choose better algebraic invariants and comitants than those in [16], closer to the geometry of the systems.

2. The two compactifications of real planar polynomial vector fields

A real planar polynomial system (S) can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathbb{R}^3$ with coordinates X, Y, Z. The central projection of the vector field $p\partial/\partial x + q\partial/\partial y$ on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. Poincaré indicated briefly in [20] that one can construct an analytic vector field $\mathcal{V}$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one induced by the phase curves of (S) via the central projection. A complete proof was given much later in [10]. The analytic vector field $\mathcal{V}$ on the whole sphere obtained in this way is called the Poincaré field associated to the system (S). The phase curves of $\mathcal{V}$ coincide in each chart with phase curves induced by planar polynomial vector fields, in particular in the chart corresponding to $Z = 1$, denoting the two coordinate axes x, y corresponding to the OX and OY directions, they coincide with the phase curves induced by (S). The two planar polynomial vector fields U, V associated to the charts for $X = 1$ (with local coordinates (u, z)) and for $Y = 1$ (with local coordinates (v, w)) and changes of coordinates $u = y/x,\ z = 1/x$, or $v = x/y,\ w = 1/y$ are as follows:

$$U \begin{cases} \dfrac{du}{dt} = C(1, u, z) \\ \dfrac{dz}{dt} = zP(1, u, z) \end{cases} \quad \text{and} \quad V \begin{cases} \dfrac{dv}{dt} = C(v, 1, w), \\ \dfrac{dw}{dt} = -wQ(v, 1, w), \end{cases}$$

where P, Q and C are defined further below.

By the compactification of the planar polynomial vector field associated to (S) we understand the restriction $\mathcal{V}|_{\mathcal{H}'}$ (where by $\mathcal{H}'$ we understand the upper hemisphere $\mathcal{H}$ completed with the equator) of the analytic vector field $\mathcal{V}$ on the sphere. We are interested in the behavior of the phase curves of (S) on $\mathbb{R}^2$ (or $\mathcal{V}|_{\mathcal{H}}$) completed with its points "at infinity", i.e. on the equator S^1 of S^2 for which we use U and V above. Since the vertical projection is a diffeomorphism of $\mathcal{H}'$ on the disk $\{(x, y) | x^2 + y^2 \leqslant 1\}$ we can view the phase portraits of our systems (S) on this disk, called the Poincaré disk.

We shall also use the compactifications (real or complex) associated to the foliations with singularities (real or complex) attached to a real polynomial system (S) (cf. [8] or [25]). These foliations can be described as follows: For a real polynomial system (S) with $n = \max(\deg(p), \deg(q))$ we associate to the two polynomials $p, q \in \mathbb{R}[x, y]$ defining (S), the homogeneous polynomials P, Q in X, Y, Z, of degree n with real coefficients, defined as follows:

$$P(X, Y, Z) = Z^n p(X/Z, Y/Z), \quad Q(X, Y, Z) = Z^n q(X/Z, Y/Z).$$

The real (respectively complex) foliations with singularities associated to (S) on the real (respectively complex) projective plane $\mathbb{P}^2(\mathbb{R})$ (respectively, $\mathbb{P}^2(\mathbb{C})$) are then described in homogeneous coordinates by the equation

$$A(X, Y, Z)\, dX + B(X, Y, Z)\, dY + C(X, Y, Z)\, dZ = 0, \tag{2.1}$$

where $A = ZQ$, $B = -ZP$, $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$ verify the following equality:

$$A(X, Y, Z)X + B(X, Y, Z)Y + C(X, Y, Z)Z = 0 \tag{2.2}$$

in $\mathbb{R}[X, Y, Z]$. (For more details see [8] or [25]).

Our goal in this work is to give a topological classification, in terms of both geometric and algebraic invariants, of the quadratic systems (S) and their compactification on H' in the neighborhood of the equator in the closed upper hemisphere H' of the Poincaré sphere. Correspondingly this yields a topological classification of the real foliations, in the neighborhood of the line at infinity associated to the imbedding of the affine plane:

$$j : \boldsymbol{a}^2(\mathbb{R}) = \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}),$$

where $j(x, y) = [x : y : 1]$. The line at infinity in this case is therefore $Z = 0$.

3. Divisors on the line at infinity encoding globally the multiplicities of singularities

In this section we consider real polynomial systems (S) with $n = \max(\deg(p), \deg(q))$ and their associated foliations with singularities, real or complex, defined in the previous section by Eq. (2.1) with coefficients A, B, C verifying (2.2).

Definition 3.1. For a system (S) we call divisor on the line at infinity, a formal expression of the form $D = \sum n(w)w$ where w is a point of the complex line $Z = 0$ of the complex protective plane, $n(w)$ is an integer and only a finite number of the numbers $n(w)$ are not zero. We call degree of the divisor D the integer $\deg(D) = \sum n(w)$. We call support of the divisor D the set $\mathrm{Supp}(D)$ of points w such that $n(w) \neq 0$.

For systems (S) two divisors on the line at infinity were introduced in [26]. These were applied in [14] for classifying topologically the quadratic systems with a weak focus of third order.

Definition 3.2. Assume that a system (S) is such that $p(x, y)$ and $q(x, y)$ are relatively prime over $\mathbb{C}$ and that $yp_n - xq_n$ is not identically zero (i.e. $Z \nmid C$). Here p_n (respectively q_n) is the sum of terms of degree n of p (respectively of q) in case at least one of them has a non-zero coefficient and zero otherwise.

The following divisor on the line at infinity is then well defined:

$$D_S(P, Q; Z) = \sum I_w(P, Q)w,$$

where the sum is taken for all points $w = [X : Y : 0]$ on the line $Z = 0$ and $I_w(P, Q)$ is the intersection number (or multiplicity of intersection) at w (cf. [11]) of the complex projective curves $P(X, Y, Z) = 0$ and $Q(X, Y, Z) = 0$.

We thus have $\mathrm{Supp}\,(D_S(P, Q; Z)) = \{w \in \{Z = 0\} | P(w) = 0 = Q(w)\}$.

The above divisor is a purely geometric object which encodes the contribution to the multiplicities of the singularities at infinity of system (S), arising from singularities in the finite plane, i.e. how many singular points in the finite plane could appear from those singularities at infinity in polynomial perturbations of degree n of (S).

Let us list a number of integer-valued invariants which are attached to this divisor.

Notation 3.1.

$$N_{\infty,f}(S) = \#\mathrm{Supp}\,(D_S(P,Q;Z));$$

$$\nu(S) = \max\{I_w(P,Q) | w \in \mathrm{Supp}\,(D_S(P,Q;Z))\};$$

$$\text{for every } m \leqslant \nu(S),\ s(m) = \#\{w \in \{Z=0\} | I_w(P,Q) = m\}.$$

Note that $N_{\infty,f}$ is the number of distinct infinite singularities of (S) which could produce finite singular points in a polynomial perturbation of degree n of (S).

We also need another divisor on the line at infinity which was used in [14,26] and which is defined as follows:

Definition 3.3. Suppose $Z \nmid C$ and consider

$$D_S(C,Z) = \sum I_w(C,Z)w,$$

where the sum is taken for all points $w = [X:Y:0]$ on the line $Z = 0$ of the complex projective plane.

Clearly for quadratic differential systems $\deg(D_S(C,Z)) = 3$.

Definition 3.4. A point w of the projective plane $\mathbb{P}^2(\mathbb{C})$ is said to be of multiplicity (r,s) for a system (S) if

$$(r,s) = (I_w(P,Q), I_w(C,Z)).$$

Following [26] we fuse the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

Definition 3.5. $$D_S = \sum \begin{pmatrix} I_w(P,Q) \\ I_w(C,Z) \end{pmatrix} w,$$

where w belongs to the line $Z = 0$ of the complex projective plane.

The above-defined divisor describes the number of singularities which could arise in a perturbation of (S) from singularities at infinity of (S) in both the finite plane and at infinity.

Definition 3.6. We call type of the divisor $D_S(P, Q; Z)$ the set

$$\{(s(m), m) | m \leqslant v(S)\}.$$

Remark 3.1. We observe that the types of $D_S(P, Q; Z)$ and of $D_S(C, Z)$ are affine invariants since both $I_w(P, Q)$ and $I_w(C, Z)$ remain invariant under the action of the affine group on systems (S) [19,24].

Notation 3.2. Let us introduce for planar systems (S) the following notations:

$$\Delta_S = \deg D_S(P, Q; Z), \quad M_C = \max\{I_w(C, Z) | w \in \operatorname{Supp}(D_S(C, Z))\}.$$

Consider a real quadratic differential system (S):

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) \equiv p(x, y), \\ \frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) \equiv q(x, y). \end{aligned} \tag{3.1}$$

Suppose $\gcd(p, q) =$ constant, where p_i (respectively q_i) is the sum of terms in x and y of degree i of p (respectively of q) in case at least one such term has non-zero coefficient and zero otherwise. Recall that **QS** denotes the class of all real quadratic systems.

We want to list all possible divisors D_S for quadratic systems (S) and characterize in terms of invariants and comitants the types of these divisors. This would make possible for any given system and in any chart the computation of the type of its divisor D_S. To do this we need to construct invariants and comitants with respect to group actions, which we do in the next section.

4. Group actions on quadratic systems (3.1) and invariants and comitants with respect to these actions

4.1. Group actions on quadratic systems (3.1)

More explicitly systems (3.1) can be written in the form:

$$\begin{aligned} \frac{dx}{dt} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ \frac{dy}{dt} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + 2b_{11}xy + b_{02}y^2, \end{aligned}$$

and let $a = (a_{00}, \dots, b_{02})$. Consider the ring $\mathbb{R}[a_{00}, a_{10}, \dots, a_{02}, b_{00}, b_{10}, \dots, b_{02}, x, y]$ which we shall denote $\mathbb{R}[a, x, y]$.

On the set **QS** of all quadratic differential systems (3.1) acts the group $\mathrm{Aff}(2,\mathbb{R})$ of affine transformations on the plane. Indeed for every $g \in \mathrm{Aff}(2,\mathbb{R})$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$g : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B, \quad g^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - M^{-1}B,$$

where $M = \|M_{ij}\|$ is a 2×2 non-singular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in$ **QS** we can form its transformed system $\tilde{S} = gS$:

$$\frac{d\tilde{x}}{dt} = \tilde{p}(\tilde{x}, \tilde{y}), \quad \frac{d\tilde{y}}{dt} = \tilde{q}(\tilde{x}, \tilde{y}), \tag{$\tilde{S}$}$$

where

$$\begin{pmatrix} \tilde{p}(\tilde{x}, \tilde{y}) \\ \tilde{q}(\tilde{x}, \tilde{y}) \end{pmatrix} = M \begin{pmatrix} (p \circ g^{-1})(\tilde{x}, \tilde{y}) \\ (q \circ g^{-1})(\tilde{x}, \tilde{y}) \end{pmatrix}.$$

The map

$$\begin{aligned} \mathrm{Aff}(2,\mathbb{R}) \times \mathbf{QS} &\to \mathbf{QS} \\ (g, S) &\to \tilde{S} = gS \end{aligned}$$

verifies the axioms for a left group action. For every subgroup $G \subseteq \mathrm{Aff}(2,\mathbb{R})$ we have an induced action of G on **QS** . We can identify the set **QS** of systems (3.1) with a subset of $\mathbb{R}^{12}$ via the embedding $\mathbf{QS} \hookrightarrow \mathbb{R}^{12}$ which associates to each system (3.1) the 12-tuple $(a_{00}, \ldots, b_{02})$ of its coefficients.

On systems (S) such that $\max(\deg(p), \deg(q)) \leqslant 2$ we consider the action of the group $\mathrm{Aff}(2,\mathbb{R})$ which yields an action of this group on $\mathbb{R}^{12}$. For every $g \in \mathrm{Aff}(2,\mathbb{R})$ let $r_g : \mathbb{R}^{12} \to \mathbb{R}^{12}$ be the map which corresponds to g via this action. We know (cf. [30]) that r_g is linear and that the map $r : \mathrm{Aff}(2,\mathbb{R}) \to GL(12,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup G of $\mathrm{Aff}(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(12,\mathbb{R})$.

4.2. Invariants and comitants associated to the group actions

Definition 4.1. A polynomial $U(a, x, y) \in \mathbb{R}[a, x, y]$ is called a comitant of systems (3.1) with respect to a subgroup G of $\mathrm{Aff}(2,\mathbb{R})$, if there exists $\chi \in \mathbb{Z}$ such that for every $(g, \boldsymbol{a}) \in G \times \mathbb{R}^{12}$ and for every $(x, y) \in \mathbb{R}^2$ the following relation holds:

$$U(r_g(\boldsymbol{a}), g(x, y)) \equiv (\det g)^{-\chi} U(\boldsymbol{a}, x, y),$$

where $\det g = \det M$. If the polynomial U does not explicitly depend on x and y then it is called invariant. The number $\chi \in \mathbb{Z}$ is called the weight of the comitant $U(a, x, y)$. If $G = GL(2,\mathbb{R})$ (or $G = \mathrm{Aff}(2,\mathbb{R})$) then the comitant $U(a, x, y)$ of systems (3.1) is called GL-comitant (respectively, affine comitant).

Definition 4.2. A subset $X \subset \mathbb{R}^{12}$ will be called G-invariant, if for every $g \in G$ we have $r_g(X) \subseteq X$.

As it can easily be verified, the following polynomials are GL-comitants of system (3.1):

$$
\begin{aligned}
C_i(a,x,y) &= yp_i(x,y) - xq_i(x,y), \quad i = 0,1,2;\\
M(a,x,y) &= 2\,\mathrm{Hess}\,(C_2(a,x,y));\\
\eta(a) &= \mathrm{Discrim}\,(C_2(a,x,y));\\
K(a,x,y) &= \mathrm{Jacob}\,(p_2(x,y), q_2(x,y));\\
\mu_0(a) &= \mathrm{Res}_x(p_2,q_2)/y^4 = \mathrm{Discrim}\,(K(a,x,y))/16;\\
H(a,x,y) &= -\mathrm{Discrim}\,(\alpha p_2(x,y) + \beta q_2(x,y))|_{\{\alpha=y,\beta=-x\}};\\
L(a,x,y) &= 2K - 4H - M;\\
K_1(a,x,y) &= p_1(x,y)q_2(x,y) - p_2(x,y)q_1(x,y).
\end{aligned}
\tag{4.1}
$$

Let $T(2,\mathbb{R})$ be the subgroup of $\mathrm{Aff}(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(12,\mathbb{R})$, i.e. for every $\tau \in T(2,\mathbb{R})$, $\tau : x = \tilde{x} + \alpha,\ y = \tilde{y} + \beta$ we consider as above $r_\tau : \mathbb{R}^{12} \to \mathbb{R}^{12}$.

Definition 4.3. A GL-comitant $U(a,x,y)$ of systems (3.1) is called a T-comitant if for every $(\tau, \boldsymbol{a}) \in T(2,\mathbb{R}) \times \mathbb{R}^{12}$ and for every $(\tilde{x},\tilde{y}) \in \mathbb{R}^2$ the relation $U(r_\tau \cdot \boldsymbol{a}, \tilde{x}, \tilde{y}) = U(\boldsymbol{a}, \tilde{x}, \tilde{y})$ holds.

Let

$$U_i(a,x,y) = \sum_{j=0}^{d_i} U_{ij}(a) x^{d_i - j} y^j, \quad i = 1,\ldots,s$$

be a set of GL-comitants of systems (3.1) where d_i denotes the degree of the binary form $U_i(a,x,y)$ in x and y with coefficients in $\mathbb{R}[a]$ where $\mathbb{R}[a] = \mathbb{R}[a_{00},\ldots,b_{02}]$. We denote by

$$\mathcal{U} = \{U_{ij}(a) \in \mathbb{R}[a] | i = 1,\ldots,s, \ \ j = 0,1,\ldots,d_i\},$$

the set of the coefficients in $\mathbb{R}[a]$ of the GL-comitants $U_i(a,x,y), i = 1,\ldots,s,$ and by $V(\mathcal{U})$ its associated algebraic set:

$$V(\mathcal{U}) = \{\boldsymbol{a} \in \mathbb{R}^{12} | U_{ij}(\boldsymbol{a}) = 0 \ \forall U_{ij}(a) \in \mathcal{U}\}.$$

Definition 4.4. A GL-comitant $U(a,x,y)$ of systems (3.1) is called a conditional T-comitant (or CT-comitant) modulo $\langle U_1, U_2, \ldots, U_s \rangle$ if the following two conditions are

satisfied:
(i) the algebraic subset $V(\mathcal{U}) \subset \mathbb{R}^{12}$ is affinely invariant (see Definition 4.2);
(ii) for every $(\tau, \boldsymbol{a}) \in T(2,\mathbb{R}) \times V(\mathcal{U})$ we have $U(r_\tau \cdot \boldsymbol{a}, \tilde{x}, \tilde{y}) = U(\boldsymbol{a}, \tilde{x}, \tilde{y})$ in $\mathbb{R}[\tilde{x}, \tilde{y}]$.

In other words, a CT-comitant $U(a, x, y)$ modulo $\langle U_1, U_2, \ldots, U_s\rangle$ is a T-comitant on the algebraic subset $V(\mathcal{U}) \subset \mathbb{R}^{12}$.

The following proposition is straightforward.

Proposition 4.1. *Let $S \in$ **QS** and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P = 0$ and $Q = 0$ on the line $Z = 0$ are given by the common linear factors over $\mathbb{C}$ of p_2 and q_2. This yields the geometrical meaning of the comitants μ_0, K and H:*

$$\gcd\,(p_2(x,y), q_2(x,y)) = \begin{cases} constant & \text{iff } \mu_0(\boldsymbol{a}) \neq 0, \\ bx+cy & \text{iff } \mu_0 = 0,\, K(\boldsymbol{a},x,y) \neq 0, \\ (bx+cy)(dx+ey) & \text{iff } \begin{cases} \mu_0(\boldsymbol{a}) = 0,\, K(\boldsymbol{a},x,y) = 0, \\ and\;\; H(\boldsymbol{a},x,y) \neq 0; \end{cases} \\ (bx+cy)^2 & \text{iff } \begin{cases} \mu_0 = 0,\, K(\boldsymbol{a},x,y) = 0, \\ and\;\; H(\boldsymbol{a},x,y) = 0; \end{cases} \end{cases}$$

where $bx+cy$, $dx+ey \in \mathbb{C}[x,y]$ are some linear forms and $be - cd \neq 0$.

Definition 4.5. The polynomial $U(a, x, y) \in \mathbb{R}[a, x, y]$ has well determined sign on $V \subset \mathbb{R}^{12}$ with respect to x, y if for every fixed $\boldsymbol{a} \in V$, the sign of the polynomial function $U(\boldsymbol{a}, x, y)$ on $\mathbb{R}^2$ is constant where this function is not zero.

Observation 4.1. We draw the attention to the fact, that if a CT-comitant $U(a, x, y)$ of even weight is a binary form in x, y, of even degree in the coefficients of (3.1) and has well-determined sign on some affine invariant algebraic subset $V(\mathcal{U})$ then this property is conserved by any affine transformation and the sign is conserved.

4.3. Canonical forms of planar quadratic systems in the neighborhood of infinity

Lemma 4.1. *For a system (3.1) with $C_2(\boldsymbol{a}, x, y) \not\equiv 0$ the divisor $D_S(C, Z)$ is well defined and its type is determined by the corresponding conditions indicated in Table 1, where we write $q_1^c + q_2^c + q_3$ if two of the points, i.e. q_1^c, q_2^c, are complex but not real. Moreover, for each type of the divisor $D_S(C, Z)$ given by Table 1 the quadratic systems (3.1) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_{\mathrm{I}})$–$(\mathbf{S}_{\mathrm{IV}})$ corresponding to their behavior at infinity.*

Proof. The Table 1 follows easily from the definitions of $\eta(a)$ and $M(a, x, y)$ in (4.1). Let us consider the GL-comitant $C_2(a, x, y) \not\equiv 0$ simply as a cubic binary form in x and y. For every $\boldsymbol{a} \in \mathbb{R}^{12}$ the binary form $C_2(\boldsymbol{a}, x, y)$ can be reduced to one of the canonical forms given below, by a linear transformation, i.e. there exist $g \in GL(2, \mathbb{R})$: $g(x, y) = (u, v)$ such that the transformed binary form $gC_2(\boldsymbol{a}, x, y) = C_2(\boldsymbol{a}, g^{-1}(u, v))$

Table 1

M_C	Type of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants	Notation for the conditions
1	$q_1+q_2+q_3$	$\eta>0$	$(\mathcal{I}_1)$
	$q_1^c+q_2^c+q_3$	$\eta<0$	$(\mathcal{I}_2)$
2	$2q_1+q_2$	$\eta=0,\ M\neq 0$	$(\mathcal{I}_3)$
3	$3q$	$M=0$	$(\mathcal{I}_4)$

$$\begin{cases} \dfrac{dx}{dt}=k+cx+dy+gx^2+(h-1)xy, \\ \dfrac{dy}{dt}=l+ex+fy+(g-1)xy+hy^2; \end{cases} \tag{$\mathbf{S}_{\mathrm{I}}$}$$

$$\begin{cases} \dfrac{dx}{dt}=k+cx+dy+gx^2+(h+1)xy, \\ \dfrac{dy}{dt}=l+ex+fy-x^2+gxy+hy^2; \end{cases} \tag{$\mathbf{S}_{\mathrm{II}}$}$$

$$\begin{cases} \dfrac{dx}{dt}=k+cx+dy+gx^2+hxy, \\ \dfrac{dy}{dt}=l+ex+fy+(g-1)xy+hy^2; \end{cases} \tag{$\mathbf{S}_{\mathrm{III}}$}$$

$$\begin{cases} \dfrac{dx}{dt}=k+cx+dy+gx^2+hxy, \\ \dfrac{dy}{dt}=l+ex+fy-x^2+gxy+hy^2, \end{cases} \tag{$\mathbf{S}_{\mathrm{IV}}$}$$

is one of the following:

$$\text{(I) } xy(x-y); \quad \text{(II) } x(x^2+y^2); \quad \text{(III) } x^2y; \quad \text{(IV) } x^3, \tag{4.2}$$

which correspond to the types of the divisor $D_S(C,Z)$ indicated in Table 1. On the other hand, according to the Definition 4.1 of the GL-comitant, for $C_2(a,x,y)$ whose weight is $\chi=-1$, we have for $g\in \mathrm{GL}(2,\mathbb{R})$

$$C_2(r_g(\boldsymbol{a}), g(x,y)) = \det(g)C_2(\boldsymbol{a},x,y).$$

Using $g(x,y)=(u,v)$ we obtain

$$C_2(r_g(\boldsymbol{a}),u,v)=\lambda C_2(\boldsymbol{a}, g^{-1}(u,v)), \quad \lambda\in\mathbb{R},$$

where we may consider $\lambda=1$ by rescaling: $u=u_1/\lambda,\ v=v_1/\lambda$.

Thus, recalling that

$$p_2(x, y) = a_{20}x^2 + 2a_{11}x, y + a_{02}y^2, \quad q_2(x, y) = b_{20}x^2 + 2b_{11}x, y + b_{02}y^2,$$

for the first canonical form in (4.2) we have

$$C_2(\boldsymbol{a}, x, y) = -b_{20}x^3 + (a_{20} - 2b_{11})x^2y + (2a_{11} - b_{02})xy^2 + a_{02}y^3 = xy(x - y).$$

Identifying the coefficients of the above identity we get the canonical form ($\mathbf{S}_{\rm I}$): Analogously for the cases II, III and IV we obtain the canonical form ($\mathbf{S}_{\rm II}$), ($\mathbf{S}_{\rm III}$) and ($\mathbf{S}_{\rm IV}$) associated to the respective polynomials in (4.2). □

5. Classification of the quadratic systems according to the types of the multiplicity divisor $\mathcal{D}_S$

A specific type of a divisor D_S yields a class of quadratic systems (3.1). We want to list all possible types of the divisors D_S and for each specific type to determine the subset of **QS** where D_S has this type. We want to give this subset in terms of algebraic invariants and comitants so as to be able to check these conditions for every system (3.1) in any chart.

In order to construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [4], where

$$\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}
\end{aligned}$$

as well as the classical differential operator $(f, \varphi)^{(2)}$ acting on $\mathbb{R}[a, x, y]$ which is called *transvectant* of the second index (see, for example, [12,17]):

$$(f, \varphi)^{(2)} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2}.$$

Here $f(x, y)$ and $\varphi(x, y)$ are polynomials in x and y.

In [5] it is shown that if a polynomial $U \in \mathbb{R}[a, x, y]$ is a comitant of system (3.1) with respect to the group $\mathrm{GL}(2, \mathbb{R})$ then $\mathcal{L}(U)$ is also a GL-comitant. The same is true for the operator transvectant of two comitants f and φ.

So by using these operators and the GL-comitants $\mu_0(a)$, $M(a,x,y)$ and $K(a,x,y)$ we shall construct the following polynomials:

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i = 1,\dots,4,$$
$$\kappa(a) = (M,K)^{(2)}, \quad \kappa_1(a) = (M,C_1)^{(2)}, \tag{5.1}$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

These polynomials are in fact comitants of system (3.1) with respect to the group $GL(2,\mathbb{R})$.

To reveal the geometrical meaning of the comitants $\mu_i(a,x,y)$, $i = 0,1,\dots,4$ we use the following resultants whose calculation yield:

$$\mathrm{Res}_X(P,Q) = \mu_0 Y^4 + \mu_{10}Y^3Z + \mu_{20}Y^2Z^2 + \mu_{30}YZ^3 + \mu_{40}Z^4, \tag{5.2}$$

$$\mathrm{Res}_Y(P,Q) = \mu_0 X^4 + \mu_{01}X^3Z + \mu_{02}X^2Z^2 + \mu_{03}XZ^3 + \mu_{04}Z^4, \tag{5.3}$$

where $\mu_{ij} = \mu_{ij}(a) \in \mathbb{R}[a_{00},\dots,b_{02}]$.

On the other hand for μ_i, $i = 0,1,\dots,4$ from (5.1) we have

$$\begin{aligned}
\mu_0(a) &= \mu_0,\\
\mu_1(a,x,y) &= \mu_{10}x + \mu_{01}y,\\
\mu_2(a,x,y) &= \mu_{20}x^2 + \mu_{11}xy + \mu_{02}y^2,\\
\mu_3(a,x,y) &= \mu_{30}x^3 + \mu_{21}x^2y + \mu_{12}xy^2 + \mu_{03}y^3,\\
\mu_4(a,x,y) &= \mu_{40}x^4 + \mu_{31}x^3y + \mu_{22}x^2y^2 + \mu_{13}xy^3 + \mu_{04}y^4.
\end{aligned}$$

We observe that the leading coefficients of the comitants μ_i, $i = 0,1,\dots,4$ with respect to x (respectively y) are the corresponding coefficients in (5.2) (respectively (5.3)).

We draw the attention to the fact, that if the comitant $\mu_i(a,x,y)(i = 0,1,\dots,4)$ is not equal to zero then we may assume that its leading coefficients are both non-zero, as this can be obtained by applying a rotation of the phase plane of system (3.1). From here and (5.2), (5.3) and the above values of $\mu_i, i = 0,1,\dots,4$ we have:

Lemma 5.1. *The system $P(X,Y,Z) = Q(X,Y,Z) = 0$ possesses $m(=\Delta_S)(1\leqslant m\leqslant 4)$ solutions $[X_i : Y_i : Z_i]$ with $Z_i = 0$ $(i = 1,\dots,m)$ (considered with multiplicities) if and only if for every $i \in \{0,1,\dots,m-1\}$ we have $\mu_i(a,x,y) = 0$ in $\mathbb{R}[a,x,y]$ and $\mu_m(a,x,y) \neq 0$.*

Remark 5.1. It can easily be checked that the following identity holds:

$$\mu_4(a,X,Y) = \mathrm{Res}_Z(P(X,Y,Z),Q(X,Y,Z)).$$

Hence, clearly for any solution $[X_0 : Y_0 : Z_0]$ (including those with $Z_0 = 0$) of the system of equations $P(X, Y, Z) = Q(X, Y, Z) = 0$, the following relation is satisfied: $\mu_4(a, X_0, Y_0) = 0$.

We give below our theorem of classification of the types of all divisors D_S occurring in quadratic systems and we associate to each type the necessary and sufficient conditions in terms of algebraic invariants and comitants. The computation of these invariants and comitants can be programmed using symbolic manipulations and implemented on computers. Thus for any specific system (3.1) we can calculate explicitly its divisor type in whatever chart (3.1) is given.

Theorem 5.1. *We consider here the family* $\mathbf{QS_{ess}}$ *of all systems (S) in* $\mathbf{QS}$ *which are essentially quadratic, i.e.* $\gcd(P, Q) = 1$ *and* $Z\nmid C$. *All possible values which could be taken by* Δ_S *for such systems* (3.1) *are as listed in the first column of Table* 3. *For each value of* Δ_S, *all possibilities we have for* M_C, *are listed in the second column. For each combination* (Δ_S, M_C) *all the possibilities we have for the form of* D_S *are those indicated in the third column. For a specified* (Δ_S, M_C), *the necessary and sufficient conditions to have the form of* D_S *as indicated in the third column are those indicated in the corresponding fourth column.* (*We recall that* $\mathcal{I}_j$ *are the conditions indicated in Table* 1. *In the last column of Table* 3 *we denote by* Σ_i *the class of all quadratic systems which possess* (Δ_S, M_C, D_S) *as indicated in the first three columns.*)

Proof. We need to examine the four distinct cases corresponding to the canonical forms $(\mathbf{S}_{\mathrm{I}})$–$(\mathbf{S}_{\mathrm{IV}})$, respectively.

5.1. Systems of type $\mathbf{S}_{\mathrm{I}}$

For systems $(\mathbf{S}_{\mathrm{I}})$ we have $\mu_0 = gh(g + h - 1)$ and for $\mu_0 \neq 0$ according to Lemma 5.1 we have $\Delta_S = 0$ and, hence, we obtain a system of the class Σ_1 (see Table 2).

Let us consider now $\mu_0 = 0$. In this case we have $gh(g + h - 1) = 0$ and without loss of generality we may assume $g = 0$. Indeed, if $h = 0$ (respectively, $g + h - 1 = 0$) we can apply the linear transformation which will replace the straight line $y = 0$ with $x = 0$ (respectively, $y = 0$ with $y = x$). Let $g = 0$. By using the translation $x = x_1 + (f + eh)/2$, $y = y_1 + e/2$ we may assume $e = f = 0$. In this way system $(\mathbf{S}_{\mathrm{I}})$ will be brought to the following canonical form:

$$\dot{x} = k + cx + dy + (h - 1)xy, \quad \dot{y} = l - xy + hy^2, \tag{5.4}$$

for which we have

$$\mu_1 = ch(1 - h)y, \quad \kappa = 64h(1 - h), \quad K = 2h(h - 1)y^2.$$

For $\mu_1 \neq 0$, from Lemma 5.1 we obtain $\Delta_S = 1$ which leads us to the case Σ_5.

Considering $\mu_1 = 0$ we shall examine two cases: $\kappa \neq 0$ and $\kappa = 0$.

Table 2

Δ_S	M_C	Value of D_S	Necessary and sufficient conditions on the comitants	Σ_i
0	1	$\binom{0}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_0\neq 0,\ (\mathcal{I}_1)$	Σ_1
		$\binom{0}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_0\neq 0,\ (\mathcal{I}_2)$	Σ_2
	2	$\binom{0}{1}p+\binom{0}{2}q$	$\mu_0\neq 0,\ (\mathcal{I}_3)$	Σ_3
	3	$\binom{0}{3}p$	$\mu_0\neq 0,\ (\mathcal{I}_4)$	Σ_4
1	1	$\binom{1}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_1)$	Σ_5
		$\binom{1}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_2)$	Σ_6
	2	$\binom{1}{1}p+\binom{0}{2}q$	$\mu_0=0,\ \mu_1\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_7
		$\binom{0}{1}p+\binom{1}{2}q$	$\mu_0=0,\ \mu_1\neq 0,\ \kappa=0,\ (\mathcal{I}_3)$	Σ_8
	3	$\binom{1}{3}p$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_4)$	Σ_9
2	1	$\binom{2}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_1)$	Σ_{10}
		$\binom{1}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ (\mathcal{I}_1)$	Σ_{11}
		$\binom{2}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_2)$	Σ_{12}
		$\binom{0}{1}p+\binom{1}{1}q^c+\binom{1}{1}r^c$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ (\mathcal{I}_2)$	Σ_{13}
	2	$\binom{2}{1}p+\binom{0}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_{14}
		$\binom{1}{1}p+\binom{1}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ L=0,\ (\mathcal{I}_3)$	Σ_{15}
		$\binom{0}{1}p+\binom{2}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathcal{I}_3)$	Σ_{16}
	3	$\binom{2}{3}p$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ (\mathcal{I}_4)$	Σ_{17}
3	1	$\binom{3}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_1)$	Σ_{18}
		$\binom{2}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa=0,\ (\mathcal{I}_1)$	Σ_{19}
		$\binom{3}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ (\mathcal{I}_2)$	Σ_{20}
	2	$\binom{3}{1}p+\binom{0}{2}q$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_{21}
		$\binom{2}{1}p+\binom{1}{2}q$	$\mu_{0,1,2}=0, \mu_3\neq 0, \kappa=L=0, \kappa_1\neq 0, (\mathcal{I}_3)$	Σ_{22}
		$\binom{1}{1}p+\binom{2}{2}q$	$\mu_{0,1,2}=0, \mu_3\neq 0, \kappa=L=0, \kappa_1=0, (\mathcal{I}_3)$	Σ_{23}
		$\binom{0}{1}p+\binom{3}{2}q$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathcal{I}_3)$	Σ_{24}
	3	$\binom{3}{3}p$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ (\mathcal{I}_4)$	Σ_{25}

Table 2
(*Continued*)

Δ_S	M_C	Value of D_S	Necessary and sufficient conditions on the comitants	Σ_i
4	1	$\binom{4}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathscr{I}_1)$	Σ_{26}
		$\binom{3}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ K_1\neq 0,\ (\mathscr{I}_1)$	Σ_{27}
		$\binom{2}{1}p+\binom{2}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ K_1=0,\ (\mathscr{I}_1)$	Σ_{28}
		$\binom{4}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathscr{I}_2)$	Σ_{29}
		$\binom{0}{1}p+\binom{2}{1}q^c+\binom{2}{1}r^c$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ (\mathscr{I}_2)$	Σ_{30}
	2	$\binom{4}{1}p+\binom{0}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathscr{I}_3)$	Σ_{31}
		$\binom{3}{1}p+\binom{1}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=L=0,\ \kappa_1\neq 0,\ (\mathscr{I}_3)$	Σ_{32}
		$\binom{2}{1}p+\binom{2}{2}q$	$\mu_{0,1,2,3}=0, \mu_4\neq 0, \kappa=L=\kappa_1=0, K_1=0, (\mathscr{I}_3)$	Σ_{33}
		$\binom{1}{1}p+\binom{3}{2}q$	$\mu_{0,1,2,3}=0, \mu_4\neq 0, \kappa=L=\kappa_1=0, K_1\neq 0, (\mathscr{I}_3)$	Σ_{34}
		$\binom{0}{1}p+\binom{4}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathscr{I}_3)$	Σ_{35}
	3	$\binom{4}{3}p$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ (\mathscr{I}_4)$	Σ_{36}

5.1.1. Case $\kappa \neq 0$

As the condition $\kappa \neq 0$ is equivalent to condition $K \neq 0$, according to Proposition 4.1 we conclude that $\operatorname{Supp} D_S(P, Q; Z)$ contains exactly one point $p = [1 : 0 : 0]$ since $\gcd(p_2, q_2) = y$. By Lemma 5.1 its multiplicity $I_p(P, Q)$ depends of the number of vanishing comitants $\mu_i(a, x, y)$. In this way we obtain that a quadratic system belongs to the set Σ_{10} (respectively Σ_{18}; Σ_{26}) for $\mu_{0,1} = 0$, $\mu_2 \neq 0$ (respectively for $\mu_{0,1,2} = 0$, $\mu_3 \neq 0$; $\mu_{0,1,2,3} = 0$, $\mu_4 \neq 0$). We use the compact notation $\mu_{0,1,2} = 0$ for $\mu_0 = \mu_1 = \mu_2 = 0$.

5.1.2. Case $\kappa = 0$

In this case $h(h-1) = 0$ and analogously to the previous case, without loss of the generality we may assume $h = 0$. Thus, for system (5.4) we obtain

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy, \quad \mu_3 = (k-l)(dy-cx)xy,$$

$$\mu_4 = -xy[lc^2x^2 - (k-l)^2xy + 2lcdxy + ld^2y^2], \quad K_1 = -xy(cx+dy).$$

So, if $\mu_2 \neq 0$ taking into consideration Remark 5.1 and the value of the comitant μ_4, we obtain the case Σ_{11} in Table 3.

If $\mu_2 = 0$ and $\mu_3 \neq 0$ then $cd = 0$, $c^2 + d^2 \neq 0$ and clearly we arrive at the case Σ_{19}.

Let us now suppose that the conditions $\mu_2 = \mu_3 = 0$ hold.

5.1.2.1. $K_1 \neq 0$: Then $c^2 + d^2 \neq 0$ and from $\mu_3 = 0$ we obtain $k = l$ which yields either $\mu_4 = -ld^2xy^3$ (for $c = 0$) or $\mu_4 = -lc^2x^3y$ (for $d = 0$). Both these cases lead us to the case Σ_{27} in Table 3.

5.1.2.2. $K_1 = 0$: In this case it follows at once that $c = d = 0$ and, hence, $\mu_4 = 4(k-l)^2x^2y^2$. Thus taking into consideration Remark 5.1 we obtain the case Σ_{28}.

5.2. Systems of type ($\mathbf{S}_{\mathrm{II}}$)

For a canonical system ($\mathbf{S}_{\mathrm{II}}$) we obtain

$$\begin{aligned} \mu_0 &= -h[g^2 + (h+1)^2], \quad \kappa = -64[g^2 + (h+1)(1-3h)], \\ K &= 2(g^2 + h + 1)x^2 + 4ghxy + 2h(h+1)y^2 \end{aligned}$$

and for $\mu_0 \neq 0$ according to Lemma 5.1 we have $\Delta_S = 0$. Thus we obtain the case Σ_2 in Table 3.

Let us consider now $\mu_0 = 0$, i.e. $h[g^2 + (h+1)^2] = 0$.

5.2.1. Case $\kappa \neq 0$

In this case we have $h = 0$ and since the condition $\kappa \neq 0$ is equivalent to the condition $K \neq 0$, according to Proposition 4.1, $\operatorname{Supp} D_S(P, Q; Z)$ contains only one point, namely the real one. By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants μ_i. Therefore the quadratic system belongs to the set Σ_6 (respectively Σ_{12}; Σ_{20}; Σ_{29}) for $\mu_1 \neq 0$ (respectively for $\mu_1 = 0$, $\mu_2 \neq 0$; $\mu_{1,2} = 0$, $\mu_3 \neq 0$; $\mu_{1,2,3} = 0$, $\mu_4 \neq 0$).

5.2.2. Case $\kappa = 0$

The conditions $\mu_0 = \kappa = 0$ yield $g = 0$, $h = -1$ and translating the origin of coordinates at the point $(e/4, f/4)$ the system ($\mathbf{S}_{\mathrm{II}}$) will be brought to the form

$$\dot{x} = k + cx + dy, \quad \dot{y} = l - x^2 - y^2, \tag{5.5}$$

for which

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = (c^2 + d^2)(x^2 + y^2),$$

$$\mu_4 = (x^2 + y^2)[(k^2 - c^2l)x^2 - 2cdlxy + (k^2 - d^2l)y^2].$$

Thus, according to the Remark 5.1, for $\mu_2 \neq 0$ we obtain the case Σ_{13}.

Let us admit that condition $\mu_2 = 0$ is satisfied. Then $c = d = 0$ and for systems (5.5) we have $\mu_3 = 0$, $\mu_4 = k^2(x^2 + y^2)^2$. This leads us to the case Σ_{30}.

5.3. *Systems of type* ($\mathbf{S}_{\rm III}$)

For canonical systems ($\mathbf{S}_{\rm III}$) one can calculate

$$\mu_0 = gh^2, \quad \kappa = -64h^2, \quad K = 2[g(g-1)x^2 + 2ghxy + h^2y^2].$$

It is quite clear that for $\mu_0 \neq 0$ we have $\Delta_S = 0$ and this leads us to the case Σ_3.

Suppose $\mu_0 = 0$. We examine the two cases: $\kappa \neq 0$ and $\kappa = 0$.

5.3.1. *Case* $\kappa \neq 0$

Then $h \neq 0$ which yields $g = 0$ and thus for the systems ($\mathbf{S}_{\rm III}$) we have $\gcd(p_2, q_2) = y$. So, taking into consideration the Remark 5.1 and the fact that for the systems ($\mathbf{S}_{\rm III}$) the polynomial $C_2(x, y) = x^2y$ we obtain the case Σ_7 if $\mu_1 \neq 0$.

On the other hand the condition $h \neq 0$ implies $K \neq 0$. Hence, by Proposition 4.1 and Lemma 5.1, Supp $D_S(P, Q; Z)$ contains exactly one point [1:0:0] of the multiplicity $(\Delta_S, 1)$. Consequently we conclude that the quadratic system belongs to the set Σ_{14} (respectively, Σ_{21}; Σ_{31}) for $\mu_1 = 0$, $\mu_2 \neq 0$ (respectively, $\mu_{1,2} = 0$, $\mu_3 \neq 0$; $\mu_{1,2,3} = 0$, $\mu_4 \neq 0$).

5.3.2. *Case* $\kappa = 0$

In this case $h = 0$ and for systems ($\mathbf{S}_{\rm III}$) with $p_2 = gx^2$, $q_2 = (g-1)xy$ we have

$$\mu_0 = 0, \quad \mu_1 = dg(g-1)^2x, \quad L = 8gx^2,$$

and $\gcd(p_2, q_2) = x$. By Lemma 5.1 for $\mu_1 \neq 0$ the quadratic systems belong to the set Σ_8.

Supposing $\mu_1 = 0$ we shall consider two subcases: $L \neq 0$ and $L = 0$.

5.3.2.1. Subcase $L \neq 0$: Then $g \neq 0$ and hence $\gcd(p_2, q_2) = x$ for $g \neq 1$ and $\gcd(p_2, q_2) = x^2$ for $g = 1$. Hence in both cases by Proposition 4.1 and Lemma 5.1, Supp $D_S(P, Q; Z)$ contains exactly one point [0:1:0] whose multiplicity depends of the number of vanishing comitants $\mu_i(a, x, y)$. Therefore we conclude that the quadratic systems belong to the set Σ_{16} (respectively Σ_{24}; Σ_{35}) for $\mu_2 \neq 0$ (respectively $\mu_2 = 0$, $\mu_3 \neq 0$; $\mu_{2,3} = 0$, $\mu_4 \neq 0$).

5.3.2.2. Subcase $L = 0$: For the systems ($\mathbf{S}_{\rm III}$) we have $g = 0$ and applying the translation of the phase plane (to obtain $e = f = 0$) these systems can be brought to the form

$$\dot{x} = k + cx + dy, \quad \dot{y} = l - xy. \tag{5.6}$$

For systems (5.6) we have $\mu_0 = \mu_1 = 0$ and

$$\mu_2 = -cdxy, \quad \mu_3 = -kxy(cx - dy), \quad \kappa_1 = -32d,$$
$$\mu_4 = -xy[c^2lx^2 + (2cdl - k^2)xy + d^2ly^2].$$

So, if $\mu_2 \neq 0$ by the Remark 5.1 and Lemma 5.1 systems (5.6) belong to the class Σ_{15}.

Let us suppose that the condition $\mu_2 = 0$ holds.

5.3.2.2.1. If $\kappa_1 \neq 0$ then $d \neq 0$ which implies $c = 0$. Then $\mu_3 = dkxy^2$ and taking into consideration the factorization of the comitant μ_4, we obtain the case Σ_{22} for $\mu_3 \neq 0$ and the case Σ_{32} for $\mu_3 = 0$, $\mu_4 \neq 0$.

5.3.2.2.2. Let us suppose $\kappa_1 = 0$. Then $d = 0$ and for the system (5.6) we obtain

$$\mu_3 = -ckx^2y, \quad \mu_4 = -x^2y(c^2lx - k^2y), \quad K_1 = -cx^2y.$$

Therefore, if $\mu_3 \neq 0$ by Remark 5.1 and Lemma 5.1 systems (5.6) belong to the class Σ_{23}. If $\mu_3 = 0$ we obtain $ck = 0$ and we need to distinguish two cases: $K_1 \neq 0$ and $K_1 = 0$.

The condition $K \neq 0$ yields $c \neq 0$ and, hence, $k = 0$. This leads us to the case Σ_{34}. If $K_1 = 0$ then $c = 0$ and we obtain the case Σ_{33}.

5.4. Systems of type ($\mathbf{S}_{\mathrm{IV}}$)

Note that for systems of the type ($\mathbf{S}_{\mathrm{IV}}$) we have $D_S(C, Z) = 3q$. So, Supp $D_S(P, Q; Z)$ could contain only the point [0:1:0]. By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants μ_i. Therefore we obtain that the quadratic system belongs to the set Σ_4 (respectively Σ_9; Σ_{17}; Σ_{25}; Σ_{36}) for $\mu_0 \neq 0$ (respectively for $\mu_0 = 0$, $\mu_1 \neq 0$; $\mu_{0,1} = 0$, $\mu_2 \neq 0$; $\mu_{0,1,2} = 0$, $\mu_3 \neq 0$; $\mu_{0,1,2,3} = 0$, $\mu_4 \neq 0$).

As all cases are examined, Theorem 5.1 is proved. □

6. Divisors encoding the topology of singularities at infinity

We now need to consider the topological types of the singularities at infinity of quadratic systems. For this we shall introduce a third divisor at infinity:

Definition 6.1. We call index divisor on the real line at infinity of $\mathbb{R}^2$, associated to a real system (S) such that $Z \nmid C$, the expression $\sum i(w)w$ where w is a singular point on the line at infinity $Z = 0$ of system (S) and $i(w)$ is the topological index (cf. [14]) of w, i.e. $i(w)$ is the topological index of one of the two opposite singular points w, w' of $\mathcal{V}$ on S^2.

Remark 6.1. This is a well-defined divisor which could be extended trivially to a divisor $\sum j(w)w,\ w \in \{Z=0\}$ on the line at infinity $Z=0$ of $\mathbb{C}^2$ by letting

$$j(w) = \begin{cases} i(w) & \text{if} \quad w \in \mathbb{P}^2(\mathbb{R}), \\ 0 & \text{if} \quad w \in \mathbb{P}^2(\mathbb{C}) \backslash \mathbb{P}^2(\mathbb{R}), \end{cases}$$

where we identify $\mathbb{P}^2(\mathbb{R})$ with its image via the inclusion $\mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{P}^2(\mathbb{C})$ induced by $\mathbb{R} \hookrightarrow \mathbb{C}$.

Notation 6.1. We denote by $I(S)$ the above divisor on $Z=0$ in $\mathbb{P}^2(\mathbb{C})$, i.e. $I(S) = \sum j(w)w$.

Notation 6.2. We denote by $N_{\mathbb{C}}(S)$ (respectively, by $N_{\mathbb{R}}(S)$) the total number of distinct singular points, be they real or complex (respectively, real), on the line at infinity $Z=0$ of the complex (respectively, real) foliation with singularities associated to (S).

We need to see how the divisor $I(S) = \sum j(w)w$ and the divisors $D_S(P,Q;Z) = \sum I_w(P,Q)w$ and $D_S(C,Z) = \sum I_w(C,Z)p$ constructed in Section 3 are combined. For this we shall fuse these three divisors on the complex line at infinity into just one but with the values in the abelian group $\mathbb{Z}^3$:

Notation 6.3. Let us consider the following divisor with the value in $\mathbb{Z}^3$ on $Z=0$:

$$\mathcal{D}_S = \sum_w (I_w(C,Z), I_w(P,Q), j(w))w,$$

where w belongs to the line $Z=0$ of the complex projective plane.

We cannot detect the multiplicities of the singularities at infinity of a system $S(\lambda)$ for the parameter value λ from just the phase portrait of $S(\lambda)$. On the other hand $\mathcal{D}_{S(\lambda)}$ has dynamic qualities since it gives us some information about what could happen to the phase portraits in the neighborhood of λ. For example if $w \in \{Z=0\}$ and if $I_w(P,Q) = 2$ for $S(\lambda_0)$, then we know that in the neighborhood of λ_0 the phase portraits of $S(\lambda)$ will have 2 finite points arising from w in the neighborhood of w.

We denote by $\mathcal{H}'$ and $\mathcal{H}$ the following sets:

$$\mathcal{H}' = \{X^2+Y^2+Z^2=1 | Z \geqslant 0\}, \quad \mathcal{H} = \{X^2+Y^2+Z^2=1 | Z>0\}.$$

For (S) in **QS** satisfying the hypothesis of Theorem 5.1 let $\sigma(S)$ be the set of all $n_\infty = 2N_R(S)$ real singular points at infinity considered on the equator S^1 of the Poincaré sphere.

We consider the function $n_{\text{sect}} : \sigma(S) \to \mathbb{N}$ where $n_{\text{sect}}(w)$ is the number of distinct local sectors of the point $w \in S^1$ on $\mathcal{H}$.

Let $w \in \sigma(S)$ and let $\rho(S) = (w_1, w_2, \ldots, w_{n_\infty})$ be the ordered sequence of singularities of S on S^1, enumerated when S^1 is described in the positive sense and such that $w_1 = w$.

Let $O_S(w) = (n_{\text{sect}}(w_1),\ n_{\text{sect}}(w_2), \ldots, n_{\text{sect}}(w_{n_\infty}))$. Then we have:

$$O_S(w_i) = (n_{\text{sect}}(w_i), n_{\text{sect}}(w_{i+1}), \ldots, n_{\text{sect}}(w_{n_\infty}), n_{\text{sect}}(w_1), \ldots, n_{\text{sect}}(w_{i-1})).$$

Notation 6.4. We denote by $O(S)$ anyone of the sequences $O_S(w_i)$.

Notation 6.5. We denote by $\max(n_{\text{sect}})$ the maximum value of the function n_{sect}, by $N_{\max}(n_{\text{sect}}) = \#\{w \in S^1 | n_{\text{sect}}(w) = \max(n_{\text{sect}})\}$ and by $N_{\text{hsect}}(S)$ the total number of hyperbolic sectors in $\mathcal{H}'$ of singularities at infinity of a system $(S) \in \mathbf{QS_{ess}}$.

Definition 6.2. Let $h_1(w_1)$ and $h_2(w_2)$ be two distinct hyperbolic sectors of singularities at infinity w_1, w_2 of a system $(S) \in \mathbf{QS_{ess}}$. (i) We say that $h_1(w_1)$ and $h_2(w_2)$ are finitely adjacent if $w_1 = w_2 = w$ and the two sectors $h_1(w_1)$ and $h_2(w_2)$ have a common border which is a separatrix of w in the finite plane.

(ii) We say that $h_1(w_1)$ and $h_2(w_2)$ are adjacent at infinity if w_1 and w_2 are opposite points of S^1 and w_1 (also w_2) as a point of S^2 has two hyperbolic sectors with a common border, part of the equator.

Notation 6.6. We shall use the following notation

$$N_{\text{hsect}}^{f\infty a} = (N_{\text{hsect}}^{f-a}, N_{\text{hsect}}^{\infty-a}),$$

where N_{hsect}^{f-a} (respectively $N_{\text{hsect}}^{\infty-a}$) is the total number of finitely adjacent couples of hyperbolic sectors (respectively adjacent at infinity).

7. Classification of quadratic differential systems according to their behavior in the neighborhood of infinity

The study of the geometry of the systems yields a simpler set of algebraic invariants than those used in [16]. We refine here the invariants which appeared in [16] so as to reveal the geometry of the systems.

We now need to relate the geometrical invariants defined in the previous section to their algebraic counterparts, i.e. the comitants and algebraic invariants.

To do this we construct below the GL-comitants which we need, by using the following basic ones:

$$C_i = yp_i(x, y) - xq_i(x, y), \quad i = 0, 1, 2,$$

$$D_i = \frac{\partial}{\partial x} p_i(x, y) + \frac{\partial}{\partial y} q_i(x, y), \quad i = 1, 2, \quad J_1 = \text{Jacob}(C_0, D_2),$$

$$J_2 = \text{Jacob}(C_0, C_2), \quad J_3 = \text{Discrim}(C_1), \quad J_4 = \text{Jacob}(C_1, D_2).$$

Using comitants (4.1) and (5.1) we constructed in Sections 4 and 5 we define the following new polynomials:

$$\begin{aligned} N &= K + H, \quad R = L + 8K, \quad \kappa_2 = -J_1, \quad \xi = M - 2K, \\ K_2 &= 4\mathrm{Jacob}(J_2, \xi) + 3\mathrm{Jacob}(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2), \\ K_3 &= 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(C_1D_2 + K_1). \end{aligned} \tag{7.1}$$

All these polynomials are GL-comitants, being obtained from simpler GL-comitants.

In the statement of the next Theorem *Fig*. j for $j = 1, \ldots, 40$ will denote a phase portrait in the vicinity of infinity of a quadratic system in $\mathbf{QS_{ess}}$. The notation for the figures in [16] was Fig j, $j = 1, \ldots, 40$. The correspondence between the two notations is indicated in columns 6 and 7 in Table 3.

In our next theorem we relate the geometry at infinity of quadratic systems with algebraic and geometric invariants.

Theorem 7.1 (*The classification theorem*). *We consider here the family* $\mathbf{QS_{ess}}$ *of all systems* (S) *in* $\mathbf{QS}$ *which are essentially quadratic, i.e.* $\gcd(P, Q) = 1$ *and* $Z \nmid C$.

(A) *The phase portraits in the vicinity of infinity of the class* $\mathbf{QS_{ess}}$ *are classified topologically by the integer-valued affine invariant* $\mathcal{J} = (O, N_{\text{hsect}}, N_{\text{hsect}}^{f\infty a})$ *which expresses geometrical properties of the systems, e.g. number of real singularities, number of their sectors and the way in which these numbers are concatenated, etc. The classification appears in Table 3 with the corresponding phase portraits in Table* 5, *where they are listed for each value of* $N_{\mathbb{R}}(S)$ *in order of increasing topological complexity.*

(B) *The geometrical properties in the neighborhood of infinity of quadratic systems* (S) *in* $\mathbf{QS_{ess}}$ *are expressed in terms of algebraic invariants and comitants as indicated in Table* 4, *which contains the full information regarding multiplicities and indices of the singularities at infinity for all quadratic differential systems in* $\mathbf{QS_{ess}}$. *The conditions appearing in the last column of Table 4 are affinely invariant.*

The proof is based on the Theorem 5.1 as well as on the invariant classification of quadratic systems at infinity given in [16], subject to some corrections as we shall indicate below.

We point out that the affinely invariant conditions occurring in part **B** of the theorem, greatly simplify the analogous conditions in [16].

Remark 7.1 (*Corrections to Nikolaev and Vulpe [16]*). In the statement of Theorem 2 (a), b)) in [5, p. 92] $\Delta_m > 0$ must be replaced by $\Delta_m < 0$ and conversely. Since this theorem was used in [16] we have to note that several expressions in the sequences of the invariant conditions given in [16] must be taken with opposite sign, more precisely:

- Fig. 4. the inequality $FS_1 > 0$ must be replaced by $FS_1 < 0$;
- Fig. 5. the inequality $FS_1 < 0$ must be replaced by $FS_1 > 0$;

Table 3

$N_{\mathbb{R}}(S)$	$\max(n_{sect})$	$N_{\max}(n_{sect})$	$O(S)$	N_{hsect}	# of Figures New	# of Figures Old	$N_{hsect}^{f\infty a}$
3	1	6	(1,1,1,1,1,1)	0	1	2	
	2	1	(2,1,1,1,1,1)	2	2	4	
		2	(2,2,1,1,1,1)	4	3	7	
			(2,1,2,1,1,1)	4	4	6	
			(2,1,1,2,1,1)	4	5	1	
		3	(2,2,1,1,2,1)	6	6	5	
		4	(2,2,1,2,2,1)	8	7	3	
2	1	4	(1,1,1,1)	2	8	22	
				1	9	12	
				0	10	18	
	2	1	(2,1,1,1)	3	11	15	
				2	12	26	(2,0)
					13	16	(0,2)
				1	14	23	
		2	(2,2,1,1)	3	15	29	
			(2,1,2,1)	5	16	13	
				4	17	20	
				2	18	8	(0,1)
					19	21	(0,0)
		3	(2,1,2,2)	4	20	10	(2,2)
					21	25	(2,0)
		4	(2,2,2,2)	6	22	9	
	3	1	(3,1,1,1)	4	23	11	
				3	24	28	
			(3,1,2,1)	4	25	24	
			(3,2,1,2)	5	26	14	
		2	(3,1,3,1)	6	27	19	
				2	28	27	
			(3,2,3,2)	6	29	17	

Table 3
(*Continued*)

$N_{\mathbb{R}}(S)$	$\max(n_{sect})$	$N_{\max}(n_{sect})$	$O(S)$	N_{hsect}	# of Figures New	# of Figures Old	$N_{hsect}^{f\infty a}$
1	1	2	(1,1)	0	30	30	
	2	1	(2,1)	2	31	32	
				1	32	34	
				0	33	38	
		2	(2,2)	4	34	31	
				2	35	40	(2,0)
					36	39	(0,2)
	3	1	(3,1)	2	37	33	(2,0)
					38	37	(0,0)
			(3,2)	3	39	36	
		2	(3,3)	4	40	35	

Table 4

Figures	Value of $\mathscr{D}_S$	Necessary and sufficient conditions
Fig. 1	$(1,0,1)p+(1,0,1)q+(1,0,1)r$	$\eta>0,\ \mu_0<0,\ \kappa>0$
	$(1,2,1)p+(1,0,1)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1}=0,\ \mu_2<0,\ \kappa>0$
Fig. 2	$(1,1,0)p+(1,0,1)q+(1,0,1)r$	$\eta>0,\ \mu_0=0,\ \mu_1\neq 0,\ \kappa>0$
	$(1,3,1)p+(1,0,1)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa>0$
	$(1,2,1)p+(1,1,1)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1,2}=\kappa=0,\ \mu_3K_1<0$
Fig. 3	$(1,1,0)p+(1,1,0)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1}=\kappa=0,\ \mu_2L<0$
	$(1,3,0)p+(1,1,0)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1,2,3}=\kappa=0,\ \mu_4L<0,\ K_1\neq 0$
Fig. 4	$(1,1,0)p+(1,1,0)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1}=\kappa=0,\ \mu_2L>0$
	$(1,3,0)p+(1,1,0)q+(1,0,1)r$	$\eta>0,\ \mu_{0,1,2,3}=\kappa=0,\ \mu_4L>0,\ K_1\neq 0$
Fig. 5	$(1,0,1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_0>0$
	$(1,2,1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_{0,1}=0,\ \mu_2>0,\ \kappa<0$
	$(1,4,1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa<0$
	$(1,0,1)p+(1,0,1)q+(1,2,-1)r$	$\eta>0,\ \mu_{0,1}=0,\ \mu_2>0,\ \kappa>0$
	$(1,0,1)p+(1,0,1)q+(1,4,-1)r$	$\eta>0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa>0$
	$(1,2,1)p+(1,0,1)q+(1,2,-1)r$	$\eta>0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=K_1=0$

Table 4
(*Continued*)

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 6	$(1,1,0)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_0=0,\ \mu_1\neq 0,\ \kappa<0$
	$(1,3,1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa<0$
	$(1,2,1)p+(1,1,1)q+(1,0,-1)r$	$\eta>0,\ \mu_{0,1,2}=\kappa=0,\ \mu_3K_1>0$
Fig. 7	$(1,0,-1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_0<0,\ \kappa<0$
	$(1,2,-1)p+(1,0,1)q+(1,0,-1)r$	$\eta>0,\ \mu_{0,1}=0,\ \mu_2<0,\ \kappa<0$
Fig. 8	$(2,2,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,$ $\mu_2>0,\ L>0,\ K_2<0$
	$(2,4,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,$ $\mu_4>0,\ L>0,\ K=0,\ K_2<0$
	$(2,2,0)\,p+(1,2,1)\,q$	$\eta=0, M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0$ $\mu_4\neq 0,\ L=K_1=0,\ \kappa_2<0$
Fig. 9	$(2,1,1)\,p+(1,0,1)\,q$	$\eta=0, M\mu_1\neq 0,\ \mu_0=\kappa=0, L>0, K<0$
	$(2,3,1)\,p+(1,0,1)\,q$	$\eta=0, M\kappa_1L\neq 0, \mu_{0,1,2}=\kappa=0, \mu_3K_1<0$
	$(2,1,1)\,p+(1,2,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=L=0,$ $\kappa_1\neq 0,\ \mu_3K_1<0$
Fig. 10	$(2,2,2)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,$ $\mu_2<0,\ L>0,\ K<0$
Fig. 11	$(2,1,1)\,p+(1,1,0)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=L=0,\ \mu_2\neq 0$
	$(2,1,1)\,p+(1,3,0)\,q$	$\eta=0, M\neq 0, \mu_{0,1,2,3}=\kappa=L=0, \mu_4\kappa_1\neq 0$
Fig. 12	$(2,2,2)\,p+(1,1,0)\,q$	$\eta=0, M\neq 0, \mu_{0,1,2}=\kappa=\kappa_1=0,$ $L=0,\ \mu_3K_1<0$
Fig. 13	$(2,2,1)\,p+(1,0,1)\,q$	$\eta=0, M\neq 0, \mu_{0,1}=\kappa=0, \mu_2\neq 0, \kappa_1L\neq 0$
	$(2,4,1)\,p+(1,3,0)\,q$	$\eta=0, M\mu_4\neq 0, \mu_{0,1,2,3}=\kappa=0, \kappa_1L\neq 0$
Fig. 14	$(2,3,1)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0, \mu_{0,1,2}=\kappa=\kappa_1=0,$ $\mu_3\neq 0,\ L>0,\ K<0$
Fig. 15	$(2,3,1)\,p+(1,1,0)\,q$	$\eta=0,\ M\mu_4K_1\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=L=0$
Fig. 16	$(2,1,1)\,p+(1,0,-1)\,q$	$\eta=0, M\mu_1\neq 0, \mu_0=\kappa=0, L<0, N\leq 0$
	$(2,1,1)\,p+(1,2,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=L=0,$ $\kappa_1\neq 0,\ \mu_3K_1>0$

Table 4
(*Continued*)

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 17	$(2,2,2)\,p+(1,0,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,\ \mu_2>0,\ L<0$
	$(2,4,2)\,p+(1,0,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,\ \mu_4>0,\ L<0$
	$(2,2,2)\,p+(1,2,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=L=K_1=0,\ \mu_4\neq 0,\ \kappa_2>0$
Fig. 18	$(2,0,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_0>0$
	$(2,0,0)\,p+(1,2,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\ \mu_2>0,\ \kappa\neq 0$
	$(2,0,0)\,p+(1,4,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0$
	$(2,4,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,$ $\mu_4>0,\ L>0,\ K\neq 0,\ R\geq 0$
		$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,$ $\mu_4>0,\ L>0,\ K=0,\ K_2\geq 0$
Fig. 19	$(2,2,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,\ \mu_2>0,\ L>0,\ K_2\geq 0$
Fig. 20	$(2,0,0)\,p+(1,1,0)\,q$	$\eta=0,\ M\neq 0,\ \mu_0=0,\ \mu_1\neq 0,\ \kappa\neq 0$
	$(2,0,0)\,p+(1,3,0)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa\neq 0$
Fig. 21	$(2,2,0)\,p+(1,1,0)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=\kappa_1=L=0,\ \mu_3K_1>0$
Fig. 22	$(2,0,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_0<0$
	$(2,0,0)\,p+(1,2,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\ \mu_2<0,\ \kappa\neq 0$
Fig. 23	$(2,1,-1)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\ \mu_1\neq 0,\ L>0,\ K>0$
	$(2,3,-1)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=0,\ \kappa_1L\neq 0,\ \mu_3K_1>0$
Fig. 24	$(2,4,0)\,p+(1,0,1)\,q$	$\eta=0,\ ML\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,\ \mu_4<0$
Fig. 25	$(2,3,-1)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=\kappa_1=0,$ $\mu_3\neq 0,\ L>0,\ K>0$
Fig. 26	$(2,1,1)\,p+(1,0,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\ \mu_1\neq 0,\ L<0,\ N>0$
	$(2,3,1)\,p+(1,0,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=\kappa_1=0,\ \mu_3\neq 0,\ L<0$
Fig. 27	$(2,2,-2)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,$ $\mu_2<0,\ L>0,\ K>0$
Fig. 28	$(2,4,0)\,p+(1,0,1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=\kappa_1=0,$ $\mu_4>0,\ L>0,\ K\neq 0,\ R<0$
Fig. 29	$(2,2,0)\,p+(1,0,-1)\,q$	$\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=\kappa_1=0,\ \mu_2<0,\ L<0$

Table 4
(*Continued*)

Figures	Value of $\mathscr{D}_S$	Necessary and sufficient conditions
Fig. 30	$(1,0,1)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_0>0$
	$(1,2,1)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_{0,1}=0,\ \mu_2>0,\ \kappa\neq 0$
	$(1,4,1)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0$
	$(1,0,1)p+(1,1,0)q^c+(1,1,0)r^c$	$\eta<0,\ \mu_{0,1}=\kappa=0,\ \mu_2\neq 0$
	$(1,0,1)p+(1,2,0)q^c+(1,2,0)r^c$	$\eta<0,\ \mu_{0,1,2,3}=\kappa=0,\ \mu_4\neq 0$
	$(3,0,1)\,p$	$M=0,\ \mu_0>0$
	$(3,2,1)\,p$	$M=0,\ \mu_{0,1}=0,\ \mu_2>0,\ K\neq 0,\ K_2<0$
		$M=0,\ \mu_{0,1}=0,\ \mu_2>0,\ K=0$
	$(3,4,1)\,p$	$M=0,\ \mu_{0,1,2,3}=0,\ \mu_4>0,\ K_3\geq 0$
Fig. 31	$(1,1,0)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_0=0,\ \mu_1\neq 0$
	$(1,3,0)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_{0,1,2}=0,\ \mu_3\neq 0$
	$(3,3,0)\,p$	$M=0,\ \mu_{0,1,2}=K=0,\ \mu_3K_1>0,\ K_3\geq 0$
Fig. 32	$(3,2,1)\,p$	$M=0,\ \mu_{0,1}=0,\ \mu_2>0,\ K\neq 0,\ K_2\geq 0$
	$(3,4,1)\,p$	$M=0,\ \mu_{0,1,2,3}=K=0,\ \mu_4>0,\ K_3<0$
Fig. 33	$(3,3,2)\,p$	$M=0,\ \mu_{0,1,2}=K=0,\ \mu_3K_1<0$
Fig. 34	$(1,0,-1)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_0<0$
	$(1,2,-1)p+(1,0,0)q^c+(1,0,0)r^c$	$\eta<0,\ \mu_{0,1}=0,\ \mu_2<0,\ \kappa\neq 0$
	$(3,0,-1)\,p$	$M=0,\ \mu_0<0$
Fig. 35	$(3,4,1)\,p$	$M=0,\ \mu_{0,1,2,3}=0,\ \mu_4<0$
Fig. 36	$(3,4,1)\,p$	$M=0,\ \mu_{0,1,2,3}=0,\ \mu_4>0,\ K\neq 0,\ K_3<0$
Fig. 37	$(3,1,0)\,p$	$M=0,\ \mu_0=0,\ \mu_1\neq 0$
	$(3,3,0)\,p$	$M=0,\ \mu_{0,1,2}=0,\ \mu_3K\neq 0,\ K_3>0$
Fig. 38	$(3,3,0)\,p$	$M=0,\ \mu_{0,1,2}=K=0,\ \mu_3K_1>0,\ K_3<0$
Fig. 39	$(3,3,0)\,p$	$M=0,\ \mu_{0,1,2}=0,\ \mu_3K\neq 0,\ K_3<0$
Fig. 40	$(3,2,-1)\,p$	$M=0,\ \mu_{0,1}=0,\ \mu_2<0$

- Fig. 6. the inequality $GA < 0$ must be replaced by $GA > 0$;
- Fig. 7. the inequality $GA > 0$ must be replaced by $GA < 0$;
- Fig. 37. the inequality $S_3 < 0$ must be replaced by $S_3 < 0$, $FS_1 < 0$;
- Fig. 38. the inequalities $S_3 > 0$, $FS_1 < 0$ must be replaced by $S_3 > 0$, $FS_1 < 0$.

Furthermore the saddle-node given in Fig. 29 of [16] is not correctly placed. The correct phase portrait is given here in Fig. 15.

Proof of the Theorem 5.1. (A) The phase portraits in the vicinity of infinity of $\mathbf{QS_{ess}}$ where obtained in [16]. All calculations were done again for this article and as we indicated in Remark 7.1, all phase portraits obtained in [16] with exception of Fig. 29 turned out to be correct. Fig. 29 in [16] needed to be modified at one of its singularities and we give the respective corrected figure in Table 5 (Fig. 15).

In [16, p. 481–484], the phase portraits appeared as they were obtained from calculations and not listed according to their geometry. To draw attention to the geometry we list them here for each possible value of $N_{\mathbb{R}}(S)$ according to their topological complexity. In Table 3 we first place the number $N_{\mathbb{R}}(S)$ of real singularities of the real foliation on $\mathbb{P}^2(\mathbb{R})$, followed by the maximum number $\max(n_{\text{sect}})$ of sectors of singularities. Although these numbers could be read on the value of $O(S)$, we place them in separate columns as they are important invariants for the geometry at infinity of the systems. We complete the table going through all phase portraits and listing $O(S)$ which by itself determines uniquely 27 of the 40 phase portraits. To distinguish the remaining 13 phase portraits we use the invariant $N_{\text{hsect}}^{f\infty a} = (N_{\text{hsect}}^{f-a}, N_{\text{hsect}}^{\infty-a})$ whose values we place in the last column, thus completing the classification.

(B) As in the proof of part (A) we use the results in [16] subject to the modifications in Remark 7.1. Since some letters appear both here and in [16] but not always with the same meaning, we shall use the convention to apply "tilde" to letters which are used to denote comitants in [16].

The proof of part (B) proceeds in 3 steps:

(I) In this step we replace the conditions in [16] subject to the modifications in Remark 7.1 with conditions involving newly defined comitants and invariants as we shall indicate below.

(II) In this part we simplify the conditions obtained in step (I) in order to obtain the corresponding conditions in the last column of Table 4.

(III) We prove that these last conditions are affinely invariant.

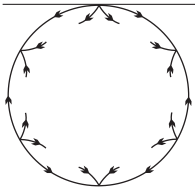
Proof of step (I). First of all we shall prove that the comitants used in [16] (see Appendix) can be replaced respectively by the comitants used here as follows:

$$\tilde{\mu} \Rightarrow \mu_0, \quad \tilde{H} \Rightarrow \mu_1, \quad \tilde{G} \Rightarrow \mu_2, \quad \tilde{F} \Rightarrow \mu_3, \quad \tilde{V} \Rightarrow \mu_4, \quad \tilde{L} \Rightarrow C_2, \quad \tilde{M} \Rightarrow M,$$

$$\tilde{\eta} \Rightarrow \eta, \quad \tilde{\theta} \Rightarrow \kappa, \quad \tilde{N} \Rightarrow K, \quad \tilde{S}_1 \Rightarrow K_1, \quad \tilde{A} \Rightarrow L, \tilde{A} + 4\tilde{N} \Rightarrow R,$$

$$\tilde{A} + \tilde{N} \Rightarrow N, \quad \tilde{\sigma} \Rightarrow \kappa_1, \quad \tilde{S}_2 \Rightarrow K_2, \tilde{S}_3 \Rightarrow K_3, \quad \tilde{S}_4 \Rightarrow \kappa_2. \tag{7.2}$$

Table 5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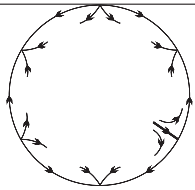
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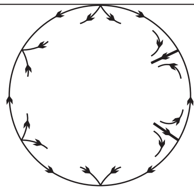
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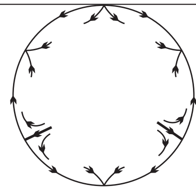


Figure 4

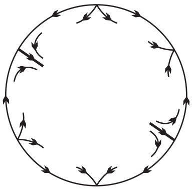
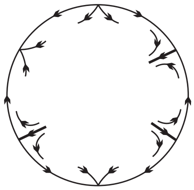


Figure 5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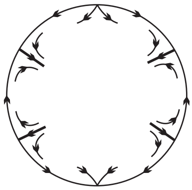
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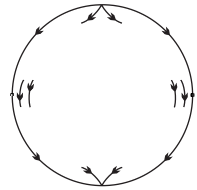
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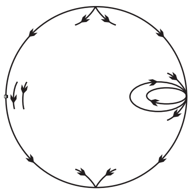
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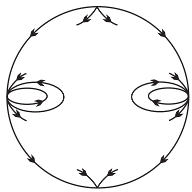
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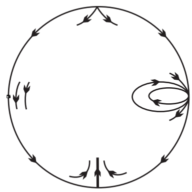
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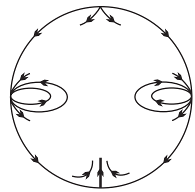
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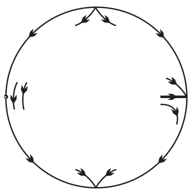
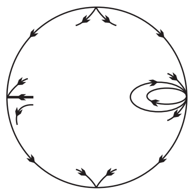


Figure 13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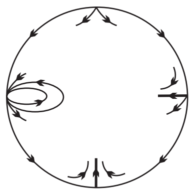
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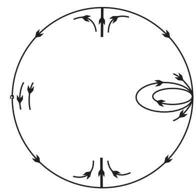


Figure 16

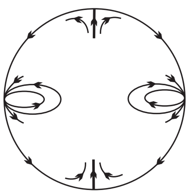
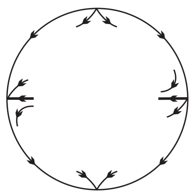


Figure 17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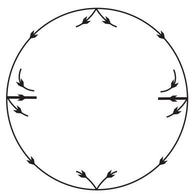


Figure 19

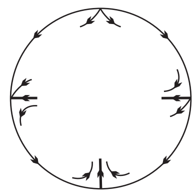
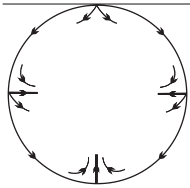


Figure 20

Indeed, firstly the following relations among the comitants (7.2) hold:

$$\mu_0 = \tilde{\mu}, \quad \mu_1 = 2\tilde{H}, \quad \mu_2 = \tilde{G}, \quad \mu_3 = \tilde{F},$$
$$\mu_4 = \tilde{V}, \quad C_2 = \tilde{L}, \quad M = 8\tilde{M},$$

Table 5
(*Continued*)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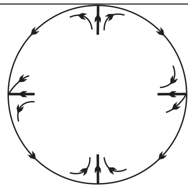
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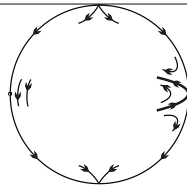
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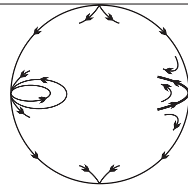
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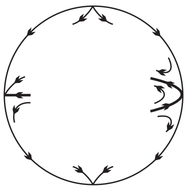
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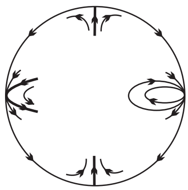
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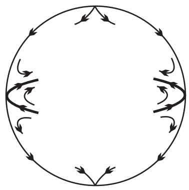
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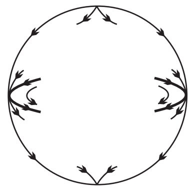
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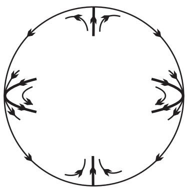
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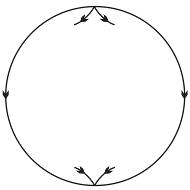
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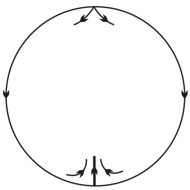
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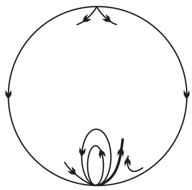
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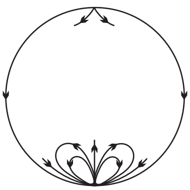
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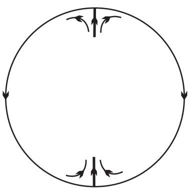
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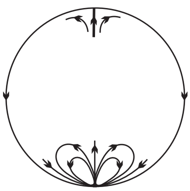
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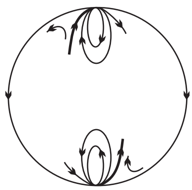
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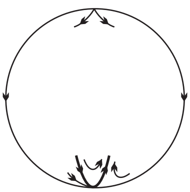
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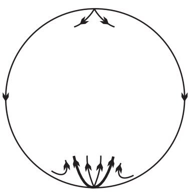
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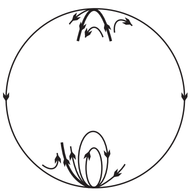
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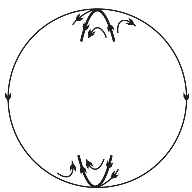
Figure 40

$$\eta = \tilde{\eta}, \quad \kappa = 64\tilde{\theta}, \quad K = 4\tilde{N}, \quad K_1 = \tilde{S}_1,$$
$$L = 8\tilde{A}, \quad R = 8(\tilde{A} + 4\tilde{N}). \tag{7.3}$$

Therefore we only have to compare the conditions involving the comitants

$$N, \quad \kappa_1, \quad \kappa_2, \quad K_2, \quad K_3 \tag{7.4}$$

and show the corresponding equivalence with the conditions involving the comitants

$$\tilde{A}+\tilde{N},\quad \tilde{\sigma},\quad \tilde{S}_4,\quad \tilde{S}_2,\quad \tilde{S}_3 \tag{7.5}$$

in [16], respectively.

We point out that all comitants (7.5) are only used for the systems ($\mathbf{S}_{\mathrm{III}}$) and ($\mathbf{S}_{\mathrm{IV}}$). So, in what follows we shall examine each one of this cases.

We first consider the systems of form ($\mathbf{S}_{\mathrm{III}}$).

In this case we have four singularities on the equator (i.e. $\eta = 0$, $M \neq 0$). The phase portraits in the vicinity of infinity of these systems are given by one of the Figs. 8–29 both here and in [16]. One can observe, that all comitants (7.4) (respectively, (7.5)) are used for systems ($\mathbf{S}_{\mathrm{III}}$) only in the case when $\kappa = 0$ (respectively, $\tilde{\theta} = 0$). In this case for systems ($\mathbf{S}_{\mathrm{III}}$) the condition $\kappa = -64h^2 = 0$ yields $h = 0$ and we obtain the systems

$$\dot{x} = k + cx + dy + gx^2,\quad \dot{y} = l + ex + fy + (g-1)xy, \tag{7.6}$$

for which $L = 8gx^2$ and

$$\kappa_1 = 32d,\, \tilde{\sigma} = -\frac{d}{4}(5g^2 - 2g + 1);\, N = (g-1)(g+1)x^2,\, \tilde{A}+\tilde{N} = \frac{1}{2}g(g+1)x^2.$$

Clearly, the condition $\kappa_1 = 0$ is equivalent to $\tilde{\sigma} = 0$. We now compare the signs of N and $\tilde{A}+\tilde{N}$. As in Table 4 the comitant N appears only in two cases (i.e. Figs. 16 and 26) and in these cases the condition $L < 0$ (i.e. $g < 0$) is used, from the expressions of N and $\tilde{A}+\tilde{N}$ above we obtain $\mathrm{sign}(N) = \mathrm{sign}(\tilde{A}+\tilde{N})$.

We observe from Table 4 that the comitant K_2 is applied for systems ($\mathbf{S}_{\mathrm{III}}$) only when $\kappa = \kappa_1 = 0$, $L \neq 0$. Since $\kappa_1 = 0$ implies $d = 0$ systems (7.6) become

$$\dot{x} = k + cx + dy + gx^2,\quad \dot{y} = l + ex + fy + (g-1)xy, \tag{7.7}$$

and we calculate $K_2 = 48(g^2 - g + 2)(c^2 - 4gk)x^2$, $\tilde{S}_2 = 2g^2(c^2 - 4gk)x^2$. Hence, K_2 has a well-determined sign and since for every g we have $g^2 - g + 2 > 0$, from $L \neq 0$ we obtain $\mathrm{sign}(K_2) = \mathrm{sign}(\tilde{S}_2)$.

We note that the invariant $\kappa_2(a)$ is here used only to distinguish Figs. 8 and 17 in the case when systems ($\mathbf{S}_{\mathrm{III}}$) belong to the class Σ_{33} in Table 3. Since for this class the conditions $\kappa = L = K_1 = 0$ hold for systems ($\mathbf{S}_{\mathrm{III}}$), we obtain respectively $h = g = c^2 + d^2 = 0$. So, the systems ($\mathbf{S}_{\mathrm{III}}$) become

$$\dot{x} = k,\quad \dot{y} = l + ex + fy - xy, \tag{7.8}$$

for which $\kappa_2 = -k$, $\tilde{S}_4 = -2k$ and, hence, $\mathrm{sign}(\kappa_2) = \mathrm{sign}(\tilde{S}_4)$.

It remains to consider systems of the form $(\mathbf{S}_{\mathrm{IV}})$ For the Figs. 30–40 which can occur for this class of systems, only the comitants $\tilde{S}_2$ and $\tilde{S}_3$ of (7.5) were used in [16]. Hence we only have to examine the conditions given in terms of comitants K_2 and K_3 from (7.4).

We observe that the comitant K_2 is used to distinguish Figs. 30 and 32 when we also have $K \neq 0$. In this case the systems $(\mathbf{S}_{\mathrm{IV}})$ belong to the class Σ_{17} in Table 3 with conditions $\mu_0 = \mu_1 = 0$. For systems $(\mathbf{S}_{\mathrm{IV}})$ we have $\mu_0 = -8h^3$. Hence $h = 0$ and the systems $(\mathbf{S}_{\mathrm{IV}})$ become

$$\dot{x} = k + cx + dy + 2gx^2, \quad \dot{y} = l + ex + fy - x^2 + 2gxy, \tag{7.9}$$

for which $K = 2g^2x^2$, $\mu_1 = 8dg^3x$. As $K \neq 0$, the condition $\mu_1 = 0$ implies $d = 0$ and we obtain the systems

$$\dot{x} = k + cx + dy + 2gx^2, \quad \dot{y} = l + ex + fy - x^2 + 2gxy, \tag{7.10}$$

for which we have: $K_2 = 24g^2(c^2 - 8gk)x^2$, $\tilde{S}_2 = 4g^2(c^2 - 8gk)x^2$. Thus, in the case under consideration the comitant K_2 has a well-determined sign and $\mathrm{sign}(K_2) = \mathrm{sign}(\tilde{S}_2)$.

We examine now the comitant K_3 which is applied for systems $(\mathbf{S}_{\mathrm{IV}})$ only in the cases when $\Delta_S \geqslant 3$, i.e. $\mu_{0,1,2} = 0$. So, we shall consider systems (7.9) for which $\mu_0 = 0$ and we examine two subcases: $K \neq 0$ and $K = 0$.

If $K \neq 0$ then $g \neq 0$ and for systems (7.9) the condition $\mu_1 = 0$ gives $d = 0$. Moreover we may assume $e = f = 0$ via a translation. So, we obtain the systems

$$\dot{x} = k + cx + 2gx^2, \quad \dot{y} = l - x^2 + 2gxy, \tag{7.11}$$

for which $\mu_2 = 8g^3kx^2$ and as $g \neq 0$ the condition $\mu_2 = 0$ yields $k = 0$. Then for systems (7.11) we obtain $K_3 = -12g^2lx^6$, $\tilde{S}_3 = -12g^2lx^6$. Hence K_3 has a well-determined sign and $\mathrm{sign}(K_3) = \mathrm{sign}(\tilde{S}_3)$.

Assume now $K = 0$, i.e. $g = 0$ and for systems (7.9) we obtain $\mu_1 = 0$, $\mu_2 = d^2x^2$. Thus, the condition $\mu_2 = 0$ yields $d = 0$ and we obtain the following systems:

$$\dot{x} = k + cx, \quad \dot{y} = l + ex + fy - x^2, \tag{7.12}$$

for which $K_3 = 3f(2c - f)x^6 = \tilde{S}_3$.

Proof of step (II). We show below how some of the conditions in [16] can be substituted by simpler ones in Table 4. To do this we shall prove the following five lemmas.

Lemma 7.1. *Let* $\tilde{\mathfrak{C}}$ *be the conjunction of the all the conditions*: $\tilde{\eta} = \tilde{\mu} = \tilde{H} = \tilde{\theta} = \tilde{\sigma} = 0$ and $\tilde{M}\tilde{G}\tilde{A} \neq 0$. *Let* $\mathfrak{C}$ *be the conjunction of the following conditions*: $\eta = \mu_0 =$

$\mu_1 = \kappa = \kappa_1 = 0$ and $M\mu_2 L \neq 0$. *We have the following equivalences*:

$$\begin{array}{lll} Fig.\ 8: & \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A}>0, \tilde{S}_2 < 0 & \Leftrightarrow\ \mathfrak{C}, \mu_2 > 0, L>0, K_2<0; \\ Fig.\ 10: & \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} > 0, \tilde{S}_2 > 0, \tilde{N} < 0 & \Leftrightarrow\ \mathfrak{C}, \mu_2 > 0, L > 0, K < 0; \\ Fig.\ 17: & \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A} < 0, (\tilde{S}_2 \leqslant 0) \vee (\tilde{G} > 0, \tilde{S}_2 > 0) & \Leftrightarrow\ \mathfrak{C}, \mu_2 > 0, L < 0; \\ Fig.\ 19: & \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A} > 0, (\tilde{S}_2 = 0) \vee (\tilde{G} > 0, \tilde{S}_2 > 0) & \Leftrightarrow\ \mathfrak{C}, \mu_2 > 0, L > 0, K_2 \geqslant 0; \\ Fig.\ 27: & \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} > 0, \tilde{S}_2 > 0, \tilde{N} > 0 & \Leftrightarrow\ \mathfrak{C}, \mu_2 < 0, L > 0, K > 0; \\ Fig.\ 29: & \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} < 0, \tilde{S}_2 > 0 & \Leftrightarrow\ \mathfrak{C}, \mu_2 < 0, L < 0. \end{array}$$

Proof. According to (7.3) the conditions $\tilde{\mathfrak{C}}$ and $\mathfrak{C}$ are equivalent. We are in the class of systems $(\mathbf{S}_{\rm III})$ for which we must apply the conditions on the right, i.e. $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, and $\kappa = \kappa_1 = 0$, $L \neq 0$. For systems $(\mathbf{S}_{\rm III})$ we have $\kappa = -64h^2$, $\kappa_1 = -32d$ and hence conditions $\kappa = \kappa_1 = 0$ yield $h = d = 0$. Then

$$\mu_0 = \mu_1 = 0, \ \ \mu_2 = g[f^2 g + cf(g-1) + k(g-1)^2]x^2 \neq 0$$

and since $g \neq 0$ we may assume $c = 0$ via a translation. Hence we get the systems

$$\dot{x} = k + gx^2, \quad \dot{y} = l + ex + fy + (g-1)xy, \tag{7.13}$$

for which

$$\begin{aligned} \mu_{0,1} &= 0, \ \ \mu_2 = g[f^2 g + k(g-1)^2]x^2 \tilde{G} \neq 0, \ \ L = gx^2 = 8\tilde{A} \neq 0, \\ K &= 2g(g-1)x^2 = 4\tilde{N}, K_2 = -192gk(g^2 - g + 2)x^2, \tilde{S}_2 = -8g^3 k. \end{aligned} \tag{7.14}$$

We observe, that $\text{sign}(K_2) = \text{sign}(\tilde{S}_2)$ because the discriminant of the quadratic polynomial $g^2 - g + 2$ is negative. We shall consider two cases: $L < 0$ and $L > 0$.

Case $L < 0$: If $\mu_2 < 0$ (then $\tilde{G} < 0$) from (7.14) it follows that $\tilde{S}_2 > 0$ and hence we obtain the conditions indicates on the left in the lemma, which correspond to Fig. 29. Thus the conditions $L < 0$ and $\mu_2 < 0$ lead to Fig. 29.

Assume $\mu_2 > 0$ (then $\tilde{G} > 0$). If either $K_2 > 0$ (then $\tilde{S}_2 > 0$) or $K_2 \leqslant 0$ (then $\tilde{S}_2 \leqslant 0$) we obtain the conditions on the left for Fig. 17. Taking into account that for $\mu_2 \neq 0$ from (7.14) it follows that the condition $\tilde{S}_2 \leqslant 0$ implies $\mu_2 > 0$ (then $\tilde{G} > 0$) we conclude, that the conditions $L < 0$ and $\mu_2 > 0$ lead to Fig. 17.

Case $L > 0$: Suppose firstly $\mu_2 < 0$. Then $\tilde{G} < 0$ and from (7.14) we have $\tilde{S}_2 > 0$ and $N \neq 0$ (i.e. $K \neq 0$). Hence we obtain the conditions for Fig. 10 (on the left in the lemma) if $K < 0$ and for Fig. 27 if $K > 0$.

Assume now $\mu_2 > 0$ (then $\tilde{G} > 0$). From (7.14) we obtain $\tilde{S}_2 \leqslant 0$ (then $K_2 \leqslant 0$) which yields $\mu_2 > 0$. Hence we conclude, that the conditions $L > 0$, $\mu_2 > 0$, $K_2 \geqslant 0$ lead to Fig. 19, whereas the conditions $L > 0$, $\mu_2 > 0$, $K_2 < 0$ lead to the Fig. 8. $\square$

Lemma 7.2. *Let $\tilde{\mathfrak{C}}_1$ be the conjunction of the following conditions*: $\tilde{\eta} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{F} = \tilde{\theta} = \tilde{\sigma} = 0$ *and* $\tilde{M}\tilde{V}\tilde{A} \neq 0$. *Let* $\mathfrak{C}_1$ *be the conjunction of the following conditions*: $\eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$ *and* $M\mu_4 L \neq 0$. *We have the following equivalences*:

$$\begin{array}{lll} Fig.\ 8: & \tilde{\mathfrak{C}}_1, \tilde{V} \neq 0, \tilde{A} \neq 0, \tilde{N} = 0, \tilde{S}_2 < 0 & \Leftrightarrow\ \mathfrak{C}_1, \mu_4 > 0, L > 0, K_2 < 0; \\ Fig.\ 17: & \tilde{\mathfrak{C}}_1, \tilde{V} \neq 0, \tilde{N} \neq 0, \tilde{A} < 0 & \Leftrightarrow\ \mathfrak{C}_1, \mu_4 > 0, L < 0; \end{array}$$

$$Fig.\ 18: \left[\begin{array}{l} \tilde{\mathfrak{C}}_1, \tilde{A}\tilde{V} \neq 0, (\tilde{N} = 0, \tilde{S}_2 = 0) \\ \vee (\tilde{N} \neq 0, \tilde{A} > 0, \tilde{A} + 4\tilde{N} \geqslant 0) \\ \vee (\tilde{N} \neq 0, \tilde{V} > 0, \tilde{S}_2 > 0) \end{array}\right] \Leftrightarrow \left[\begin{array}{l} \mathfrak{C}_1, \mu_4 > 0, L > 0, \\ (R \geqslant 0, K \neq 0) \vee \\ (K_2 \geqslant 0, K = 0) \end{array}\right];$$

$$\begin{array}{lll} Fig.\ 24: & \tilde{\mathfrak{C}}_1, \tilde{N} = 0, \tilde{A} \neq 0, \tilde{V} < 0 & \Leftrightarrow\ \mathfrak{C}_1, \mu_4 < 0, L \neq 0; \\ Fig.\ 28: & \tilde{\mathfrak{C}}_1, \tilde{V}\tilde{N} \neq 0, \tilde{A} > 0, \tilde{A} + 4\tilde{N} < 0 & \Leftrightarrow\ \mathfrak{C}_1, \mu_4 > 0, L > 0, R < 0. \end{array}$$

Proof. We are in the class of systems ($\mathbf{S}_{\mathrm{III}}$) for which we must set the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \mu_4 \neq 0$ and $\kappa = \kappa_1 = 0$, $L = 8\tilde{A} \neq 0$. It was shown before (see p. 35) that for systems ($\mathbf{S}_{\mathrm{III}}$) the conditions $\kappa = \kappa_1 = 0$ yield $h = d = 0$. Then $L = gx^2 \neq 0$ and $K = 2g(g-1)x^2$ and we shall construct two canonical forms corresponding to the cases $K \neq 0$ and $K = 0$.

Assume firstly $K \neq 0$. Then $g - 1 \neq 0$ and we may assume $e = f = 0$ due to a translation. Therefore considering the conditions $h = d = e = f = 0$, for systems ($\mathbf{S}_{\mathrm{III}}$) calculations yield: $\mu_0 = \mu_1 = 0\mu_2 = gk(g-1)^2$ and by $g(g-1) \neq 0$ the condition $\mu_2 = 0$ yields $k = 0$. This implies $\mu_3 = -clg(g-1)x^3$, $\mu_4 = lx^3[lg^2x + c^2(g-1)y]$. Hence, the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ yield $c = 0$ and we get the systems

$$\dot{x} = gx^2, \quad \dot{y} = l + (g-1)xy, \tag{7.15}$$

for which

$$\mu_{0,1,2,3} = 0, \mu_4 = g^2l^2x^4 = \tilde{V}, \quad L = 8gx^2 = 8\tilde{A} \neq 0, \quad K_2 = 0 = \tilde{S}_2,$$

$$K = 2g(g-1)x^2 = 4\tilde{N} \neq 0, R = 8g(2g-1)x^2 = 8(\tilde{A} + 4\tilde{N}). \tag{7.16}$$

Suppose now that the condition $K = 2g(g-1)x^2 = 0$ holds. Since $L = gx^2 \neq 0$ this yields $g = 1$ and we may assume $c = 0$ via a translation. Then we obtain $\mu_2 = f^2x^2 = 0$ which implies $f = 0$ and we get the systems

$$\dot{x} = k + x^2, \quad \dot{y} = l + ex, \tag{7.17}$$

for which

$$\mu_{0,1,2,3}=0,\quad \mu_4=(l^2+ke^2)x^4=\tilde V\neq 0,\quad L=8x^2=8\tilde A,$$

$$K=0=\tilde N,\, R=8x^2=8(\tilde A+4\tilde N),\quad K_2=-384kx^2=48\tilde S_2. \tag{7.18}$$

We shall consider two cases: $\mu_4<0$ and $\mu_4>0$.

Case $\mu_4<0$: Then $\tilde V<0$ and from (7.16) and (7.18) we have the conditions $\tilde N=0$ and $\tilde S_2>0$. Hence the conditions $\mu_4<0$ and $L\neq 0$ lead to the conditions in the lemma corresponding to Fig. 24.

Case $\mu_4>0$: In this case $\tilde V>0$ and we shall examine two subcases: $L<0$ and $L>0$.

Subcase $L<0$: Then $\tilde A<0$. From (7.16) and (7.18) we conclude that $\tilde N\neq 0$ and we obtain the conditions corresponding to Fig. 17. Hence we conclude that for $\mu_4>0$ and $L<0$ we get Fig. 17.

Subcase $L>0$: Hence $\tilde A>0$.

(a) If $R<0$ (then $\tilde A+4\tilde N<0$) from (7.16) and (7.18) we obtain $\tilde N\neq 0$ and hence we get the conditions for Fig. 28.

(b) Assume now $R\geqslant 0$. If $K\neq 0$ (then $\tilde N\neq 0$) we obtain one sequence of conditions for Fig. 18, and namely: $\tilde N\neq 0$, $\tilde A>0$ and $\tilde A+4\tilde N\geqslant 0$.

Suppose $K=0$ (i.e. $\tilde N=0$). If in addition $K_2<0$ (then $\tilde S_2<0$) then we obtain the conditions for Fig. 8. From (7.16) and (7.18) we obtain that the condition $K_2<0$ implies $\tilde N=0$. Then we conclude, that for $\mu>0$, $L>0$ and $K_2<0$ we obtain the conditions for Fig. 8.

Assuming $K_2\geqslant 0$ (then $\tilde S_2\geqslant 0$) and taking into account that we are in the case $\mu_4>0$, we get two of the series of conditions for Fig. 18, which can be combined into the following series: $\mu_4>0$, $K=0$, $L>0$, $K_2\leqslant 0$. $\square$

Lemma 7.3. *Let* $\tilde{\mathfrak{C}}_2$ *be the conjunction of all the conditions*: $\tilde M=\tilde\mu=\tilde H=0$ *and* $\tilde L\tilde G\neq 0$. *Let* $\mathfrak{C}_2$ *be the conjunction of the following conditions*: $M=\mu_0=\mu_1=0$ *and* $C_2\mu_2\neq 0$. *We have the following equivalences*:

$$\begin{array}{lll}
Fig.\ 30: \tilde{\mathfrak{C}}_2, \tilde G\neq 0, (\tilde N\neq 0, \tilde S_2<0)\vee(\tilde N=0) & \Leftrightarrow & \mathfrak{C}_2, \mu_2>0, (K\neq 0, K_2<0) \\
 & & \vee (K=0); \\
Fig.\ 32: \tilde{\mathfrak{C}}_2, \tilde G\tilde N\neq 0, (\tilde G>0, \tilde S_2>0)\vee(\tilde S_2=0) & \Leftrightarrow & \mathfrak{C}_2, \mu_2>0, K\neq 0, K_2\geqslant 0; \\
Fig.\ 40: \tilde{\mathfrak{C}}_2, \tilde G<0, \tilde N\neq 0, \tilde S_2>0 & \Leftrightarrow & \mathfrak{C}_2, \mu_2<0.
\end{array}$$

Proof. We are in the class of systems $(\mathbf{S}_{\mathrm{IV}})$ for which we must set the conditions $\mu_0=\mu_1=0$, $\mu_2\neq 0$. We have $\mu_0=-h^3=0$ which implies $h=0$ and then $\mu_1=dg^3x$ and $K=2gx^2$. We shall consider two subcases: $K\neq 0$ and $K=0$.

Assume firstly $K\neq 0$. Then $g\neq 0$ and the condition $\mu_1=0$ yields $d=0$. We can assume $g=1$ and $e=f=0$ due to the rescaling $x\to x/g, y\to y/g^2$ and a translation. Then we get the systems

$$\dot x=k+cx+x^2,\quad \dot y=l-x^2+xy, \tag{7.19}$$

for which

$$\mu_{0,1} = 0, \mu_2 = kx^2 = \tilde{G} \neq 0, K = 2x^2 = 4\tilde{N}, \quad K_2 = 48(c^2 - 4k)x^2 = 24\tilde{S}_2. \quad (7.20)$$

Admit now $K = 0$. Hence $g = 0$ and we can assume $e = 0$ due to a translation. Then we obtain the systems

$$\dot{x} = k + cx + dy, \quad \dot{y} = l + fy - x^2, \quad (7.21)$$

for which

$$\mu_{0,1,} = 0, \quad \mu_2 = d^2x^2 = \tilde{G} \neq 0, K = 0 = \tilde{N}, L_2 = 0 = \tilde{S}_2. \quad (7.22)$$

Case $\mu_2 < 0$: From (7.20) and (7.22) it follows that the condition $\mu_2 < 0$ implies $\tilde{N} \neq 0$ and $\tilde{S}_2 > 0$. Hence we obtain the conditions for Fig. 40 and we conclude that the condition $\mu_2 < 0$ immediately leads to the conditions for Fig. 40.

Case $\mu_2 > 0$ (i.e. $\tilde{G} > 0$): Assume that the condition $K \neq 0$ holds (then $\tilde{N} \neq 0$). If $K_2 < 0$ we have $\tilde{S}_2 < 0$ and then we obtain the conditions for Fig. 30. If either $K_2 > 0$ or $K_2 = 0$ via $\tilde{G} > 0$ in both cases we get Fig. 32

Suppose $K = 0$ (i.e. $\tilde{N} = 0$. In this case we obtain the conditions $\tilde{G} \neq 0$, $\tilde{N} = 0$ which lead to Fig. 30. Note that from (7.20) and (7.22) it follows that the condition $K = 0$ implies $\mu_2 > 0$. $\square$

Lemma 7.4. *Let* $\tilde{\mathfrak{C}}_3$ *be the conjunction of the following conditions*: $\tilde{M} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{N} = 0$ *and* $\tilde{L}\tilde{F} \neq 0$. *Let* $\mathfrak{C}_3$ *be the conjunction of the following conditions*: $M = \mu_0 = \mu_1 = \mu_2 = K = 0$ *and* $C_2\mu_3 \neq 0$. *We have the following equivalences*:

$$Fig.\ 31: \begin{bmatrix} \tilde{\mathfrak{C}}_3, \tilde{F} \neq 0, (\tilde{S}_3 = 0) \\ \vee (\tilde{F}\tilde{S}_1 > 0, \tilde{S}_3 > 0) \end{bmatrix} \Leftrightarrow \mathfrak{C}_3, \mu_3 K_1 > 0, K_3 \geqslant 0;$$

$$Fig.\ 33: \tilde{\mathfrak{C}}_3, \tilde{F}\tilde{S}_1 < 0, \tilde{S}_3 < 0 \Leftrightarrow \mathfrak{C}_3, \mu_3 K_1 < 0;$$

$$Fig.\ 38: \tilde{\mathfrak{C}}_3, \tilde{F}\tilde{S}_1 > 0, \tilde{S}_3 < 0 \Leftrightarrow \mathfrak{C}_3, \mu_3 K_1 > 0, K_3 < 0.$$

Proof. We are in the class of systems ($\mathbf{S}_{\mathrm{IV}}$) for which we must set the conditions $\mu_0 = \mu_1 = \mu_2 = 0 = K, \mu_3 \neq 0$. We have $\mu_0 = -h^3 = 0$ hence $h = 0$ and then $K = 2gx^2$. The condition $K = 0$ yields $g = 0$ and this leads to systems (7.21) for which the condition $\mu_2 = d^2x^2 = 0$ yields $d = 0$. Hence we obtain the systems

$$\dot{x} = k + cx, \quad \dot{y} = l + fy - x^2, \quad (7.23)$$

for which

$$\mu_{0,1,2} = 0, \quad \mu_3 = -c^2 f d^2 x^3 = \tilde{F} \neq 0, \quad K = 0 = \tilde{N},$$
$$K_1 = -cx^3 = \tilde{S}_1, \quad K_3 = 6f(2c - f)x^6 = \tilde{S}_3. \quad (7.24)$$

We note that $\mu_3 K_1 = c^3 f x^6 \neq 0$ and hence sign $(\mu_3 K_1) = \mathrm{sign}(cf) = \mathrm{sign}(\tilde{F}\tilde{S}_1)$.

Case $\mu_3 K_1 < 0$: From (7.24) we obtain $\tilde{S}_3 < 0$ and hence we conclude that the condition $\mu_3 K_1 < 0$ leads to the conditions for Fig. 33.

Case $\mu_3 K_1 > 0$: For $K_3 < 0$ (then $\tilde{S}_3 < 0$) we obtain the conditions for Fig. 38. If either $K_3 > 0$ or $K_3 = 0$ we observe that in both cases we get the conditions for Fig. 31. From (7.24) it follows that the condition $K_3 = 0$ implies $\mu_3 K_1 > 0$. Therefore we conclude that the conditions $\mu_3 K_1 > 0$ and $K_3 \geqslant 0$ lead to the conditions for Fig. 31. $\square$

Lemma 7.5. *Let* $\tilde{\mathfrak{C}}_4$ *be the conjunction of the following conditions*: $\tilde{M} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{F} = 0$ *and* $\tilde{L}\tilde{V} \neq 0$. *Let* $\mathfrak{C}_4$ *be the conjunction of the following conditions*: $M = \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ *and* $C_2\mu_4 \neq 0$. *We have the following equivalences*:

$$Fig.\ 30: \left[\begin{array}{l} \tilde{\mathfrak{C}}_4,\, \tilde{V} \neq 0,\, (\tilde{N} \neq 0,\, \tilde{S}_3 > 0) \\ \vee\, (\tilde{N} = \tilde{S}_1 = \tilde{S}_3 = 0) \vee \\ (\tilde{N} = 0,\, \tilde{S}_1 \neq 0,\, \tilde{V} > 0) \end{array}\right] \Leftrightarrow \mathfrak{C}_4,\, \mu_4 > 0,\, K_3 \geqslant 0;$$

$$Fig.\ 32: \tilde{\mathfrak{C}}_4,\, \tilde{V} \neq 0,\, \tilde{N} = \tilde{S}_1 = 0,\, \tilde{S}_3 \neq 0 \Leftrightarrow \mathfrak{C}_4,\, \mu_4 > 0,\, K_3 < 0,\, K = 0;$$

$$Fig.\ 35: \tilde{\mathfrak{C}}_4,\, \tilde{V} < 0,\, \tilde{N} = 0,\, \tilde{S}_1 \neq 0 \Leftrightarrow \mathfrak{C}_4,\, \mu_4 < 0;$$

$$Fig.\ 36: \tilde{\mathfrak{C}}_4,\, \tilde{V} \neq 0,\, \tilde{N} \neq 0,\, \tilde{S}_3 < 0 \Leftrightarrow \mathfrak{C}_4,\, \mu_4 > 0,\, K_3 < 0,\, K \neq 0.$$

Proof. We are in the class of systems $(\mathbf{S}_{\mathrm{IV}})$ for which we must set the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \mu_4 \neq 0$. We have $\mu_0 = -h^3 = 0$ which implies $h = 0$ and then $\mu_1 = dg^3 x$ and $K = 2gx^2$. We shall consider two subcases: $K \neq 0$ and $K = 0$.

If $K \neq 0$ then the condition $\mu_1 = 0$ leads to systems (7.19) for which $\mu_2 = kx^2$. Hence the condition $\mu_2 = 0$ yields $k = 0$ and we calculate: $\mu_3 = -clx^3$ and $\mu_4 = -l(c^2x - lx - c^2y)x^3$. Hence the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ yield $c = 0, l \neq 0$ and we obtain the systems

$$\dot{x} = x^2, \quad \dot{y} = l - x^2 + xy, \tag{7.25}$$

for which

$$\begin{aligned} &\mu_{0,1,2,3} = 0, \quad \mu_4 = l^2x^4 = \tilde{V} \neq 0, \quad K = \frac{1}{2}x^2 = 4\tilde{N}, \\ &K_3 = -6lx^6 = \tilde{S}_3 \neq 0. \end{aligned} \tag{7.26}$$

Admit now that $K = 0$. This leads to systems (7.23) for which the condition $\mu_3 = -c^2 f d^2 x^3 = 0$ yields $cf = 0$. Then we get the systems

$$\dot{x} = k + cx, \quad \dot{y} = l + fy - x^2, \tag{7.27}$$

with $cf = 0$ and

$$\begin{aligned} &\mu_{0,1,2,3} = 0, \quad \mu_4 = (k^2 - c^2 l)x^4 = \tilde{V} \neq 0, \quad K = 0 = \tilde{N}, \\ &K_1 = -cx^3 = \tilde{S}_1, \quad K_3 = -6f^2x^6 = \tilde{S}_3, \quad K_1K_3 = 0. \end{aligned} \tag{7.28}$$

Table 6

Case	GL-comitants	Degree in a	Degree in x and y	Weight	Algebraic subset $V(*)$
1	$\eta(a), \mu_0(a), \kappa(a)$	4	0	2	$V(0)$
2	$C_2(a,x,y)$	1	3	−1	$V(0)$
3	$K(a,x,y)$	2	2	0	$V(0)$
4	$L(a,x,y)$	2	2	0	$V(0)$
5	$M(a,x,y)$	2	2	0	$V(0)$
6	$N(a,x,y)$	2	2	0	$V(0)$
7	$R(a,x,y)$	2	2	0	$V(0)$
8	$\kappa_1(a)$	3	0	1	$V(\eta,\kappa)$
9	$\kappa_2(a)$	2	0	0	$V(\eta,\kappa,L,K_1)$
10	$K_2(a,x,y)$	4	2	0	$V(\eta,\mu_0,\mu_1,\kappa,\kappa_1)$
11	$K_3(a,x,y)$	4	6	−2	$V(M,\mu_0,\mu_1,\mu_2)$
12	$K_1(a,x,y)$	2	3	−1	$V(K)$
13	$\mu_1(a,x,y)$	4	1	1	$V(\mu_0)$
14	$\mu_2(a,x,y)$	4	2	0	$V(\mu_0,\mu_1)$
15	$\mu_3(a,x,y)$	4	3	−1	$V(\mu_0,\mu_1,\mu_2)$
16	$\mu_4(a,x,y)$	4	4	−2	$V(\mu_0,\mu_1,\mu_2,\mu_3)$

Case $\mu_4 < 0$ (i.e. $\tilde{V} < 0$): From (7.26) and (7.28) we obtain that the condition $\mu_4 < 0$ implies $\tilde{N} = 0$ and $\tilde{S}_1 \neq 0$. Hence for $\mu_4 < 0$ we obtain the conditions for Fig. 35.

Case $\mu_4 > 0$: Then $\tilde{N} > 0$ and we shall consider 3 subcases: $K_3 < 0$, $K_3 > 0$ and $K_3 = 0$.

Subcase $K_3 < 0$: If $K \neq 0$ then $\tilde{N} \neq 0$ and we have the conditions for Fig. 36. Suppose $K = 0$, i.e. $\tilde{N} = 0$. Then by $K_3 \neq 0$ from (7.28) we have $\tilde{S}_1 = 0$. Therefore we conclude that conditions $K_3 < 0$ and $K = 0$ lead to the Fig. 32.

Subcase $K_3 > 0$: Then $\tilde{S}_3 > 0$ and from (7.26) and (7.28) we conclude that $K \neq 0$, i.e. $\tilde{N} \neq 0$. Hence we obtain one series of the conditions for Fig. 30.

Subcase $K_3 = 0$: Then $\tilde{S}_3 = 0$ and according to (7.26) and (7.28) we have $K = 0$. This leads to systems (7.27) for which the condition $K_3 = 0$ yields $f = 0$. Then we have either $K_1 \neq 0$ (i.e. $\tilde{S}_1 \neq 0$) or $K_1 = 0$ (i.e. $\tilde{S}_1 = 0$). Since the conditions $\tilde{V} > 0$ and $\tilde{S}_3 = 0$ hold, both cases lead to the conditions for Fig. 30.

Lemma 7.1 is proved and this completes the proof of the step (II).

Proof of step (III). We draw the attention to the fact that all the constructed polynomials which were used in Theorems 5.1 and 7.1 are GL-comitants. But in fact we are interested in the action of the affine group Aff(2,ℝ) on these systems. We shall prove the following lemma.

Lemma 7.6. *The polynomials which are used in Theorems* 5.1 *or* 7.1 *have the properties indicated in the Table* 6. *In the last column are indicated the algebraic sets on which the* GL-*comitants on the left are* CT-*comitants*. *The Table* 6 *shows us that all conditions included in the statements of Theorems* 5.1 *or* 7.1 *are affinely invariant.*

Proof. (I) *Cases* 1–7: The polynomials $\eta(a), \kappa(a), \mu_0(a), K(a,x,y), L(a,x,y), M(a,x,y), N(a,x,y)$ and $R(a,x,y)$ are T-comitants, because these GL-comitants were constructed only by using the coefficients of the polynomials $p_2(x,y)$ and $q_2(x,y)$.

(II) *Cases* 8–11: (a) We consider the GL-invariant $\kappa_1(a)$ which according to Table 4 was used only in the class of systems ($\mathbf{S}_{\mathrm{III}}$). It was shown before (see p. 33) that for $\kappa = 0$ systems ($\mathbf{S}_{\mathrm{III}}$) can be brought by an affine transformation to systems (7.6) for which $\kappa_1 = -32d$. On the other hand for any system in the orbit under the translation group action of a system (7.6) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ we obtain $\kappa_1(\boldsymbol{a}) = -32d$. Hence the value of κ_1 does not depend of the vector defining the translations. Therefore we conclude that the polynomial κ_1 is a CT-comitant modulo $\langle \eta, \kappa \rangle$.

(b) We consider now the GL-invariant $\kappa_2(a)$. From Table 4 we observe that $\kappa_2(a)$ is only applied to distinguish the Figs. 8 and 17 when for systems ($\mathbf{S}_{\mathrm{III}}$) the conditions $\kappa = L = K_1 = 0$ hold. As it was shown before (see p. 33) for $\kappa = L = K_1 = 0$ the systems ($\mathbf{S}_{\mathrm{III}}$) can be brought by an affine transformation to systems (7.8) for which $\kappa_2 = -\kappa$. On the other hand for any system in the orbit under the translation group action of a system (7.8) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ we obtain $\kappa_2(a) = -k$. Hence we conclude that the polynomial κ_2 is a CT-comitant modulo $\langle \eta, \kappa, L, K_1 \rangle$.

(c) We examine now the GL-invariant $K_2(a)$ which was used in cases ($\mathbf{S}_{\mathrm{III}}$) and ($\mathbf{S}_{\mathrm{IV}}$). Assume firstly $\eta = 0$ and $M \neq 0$ i.e. we are in the class of systems ($\mathbf{S}_{\mathrm{III}}$). We have shown before (see p. 33) that for $\kappa = \kappa_1 = 0$ the systems ($\mathbf{S}_{\mathrm{III}}$) can be brought by an affine transformation to systems (7.7) for which $K_2 = 48(g^2 - g + 2)(c^2 - 4gk)x^2$. Suppose now that the conditions $M = 0$ and $C_2 \neq 0$ hold, i.e. we are in the class of systems ($\mathbf{S}_{\mathrm{IV}}$). It was shown before (see p. 34) that for $\mu_0 = \mu_1 = 0$ systems ($\mathbf{S}_{\mathrm{IV}}$) can be brought by an affine transformation to systems (7.10) for which $K_2 = 24g^2(c^2 - 8gk)x^2$.

On the other hand for any system in the orbit under the translation group action of a system (7.7) (respectively, of a system (7.10)) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ (respectively, $\boldsymbol{a}_1 \in \mathbb{R}^{12}$) we obtain $K_2(\boldsymbol{a}, x, y) = 48(g^2 - g + 2)(c^2 - 4gk)x^2$ (respectively, $K_2(\boldsymbol{a}_1, x, y) = 24g^2(c^2 - 8gk)x^2$). Calculations yield that for system (7.7) (respectively, for system (7.10)) we have $\mu_0 = \mu_1 = 0$ (respectively $\kappa = \kappa_1 = 0$). Hence we conclude that the GL-comitant $K_2(a,x,y)$ is a CT-comitant modulo $\langle \eta, \mu_0, \mu_1, \kappa, \kappa_1 \rangle$.

(d) We examine now the comitant K_3 which is applied for systems ($\mathbf{S}_{\mathrm{IV}}$) only in the cases when $\Delta_S \geqslant 3$, i.e. $\mu_0 = \mu_1 = \mu_2 = 0$. It was shown before (see p. 34) that for $\mu_0 = \mu_1 = \mu_2 = 0$ systems ($\mathbf{S}_{\mathrm{IV}}$) can be brought by an affine transformation either to systems (7.11) for $K \neq 0$ or to systems (7.12) for $K = 0$. Calculations yield, that for any system in the orbit under the translation group action of a system (7.11) (respectively, of a system (7.12)) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ (respectively, $\boldsymbol{a}_1 \in \mathbb{R}^{12}$) we obtain $K_3(\boldsymbol{a}, x, y) = -12g^2lx^6$ (respectively, $K_3(\boldsymbol{a}_1, x, y) = 3f(2c - f)x^6$). Hence in both cases the values of K_3 do not depend of the vector defining the translations. Therefore the GL-comitant $K_3(a,x,y)$ is a CT-comitant modulo $\langle M\mu_0, \mu_1, \mu_2 \rangle$.

(III) Cases 12–16: Let $\tau \in T(2, \mathbb{R})$ be the translation: $x = \tilde{x} + \alpha, y = \tilde{y} + \beta$ and consider a quadratic system (3.1) which corresponds to a point $\boldsymbol{a} \in \mathbb{R}^{12}$. It is sufficient to verify that the following relations occur, where $\xi = \tilde{x}\beta - \tilde{y}\alpha$:

$$K_1(r_\tau \cdot a, \tilde{x}, \tilde{y}) = K_1(a, \tilde{x}, \tilde{y}) - \xi K(a, \tilde{x}, \tilde{y}),$$

$$\mu_s(r_\tau \cdot \boldsymbol{a}, \tilde{x}, \tilde{y}) = \mu_s(\boldsymbol{a}, \tilde{x}, \tilde{y}) + \sum_{k=0}^{s-1} \binom{4-k}{s-k} \xi^{s-k} \mu_k(\boldsymbol{a}, \tilde{x}, \tilde{y}), \quad s = 1, 2, 3, 4.$$

So, Lemma 7.6 is proved and this completes the proof of the Theorem 7.1. $\square$

Appendix

Let us consider the tensorial form of quadratic system:

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2).$$

The following invariants and comitants, defined by polynomials of J_i, R_i which are tensorially defined GL-comitants, were used in [16] for the classification in the neighborhood of infinity of quadratic differential systems:

$$2\tilde{\mu} = J_4, \quad \tilde{\sigma} = J_7, \quad 2\tilde{\theta} = J_5, \quad \tilde{L} = R_{12},$$
$$2\tilde{M} = 9R_3 + 6R_6 - 8R_{11}^2, \quad \tilde{S}_1 = R_5,$$

$$\tilde{S}_2 = 2J_1^2 R_6 + 2J_1 R_1^2 - 2J_2 R_6 + J_2 R_{11}^2 + 8J_3 R_3 - 8J_3 R_6 - 4R_7 - R_8, \quad \tilde{H} = R_{13},$$

$$\tilde{S}_3 = R_{12}^2(7J_2 - 6J_1^2 - 8J_3) - R_{12}(10J_1 R_5 + 4R_1 R_{10} - 6R_3 R_9) + 4R_3 R_{10}^2 - 4R_5^2,$$

$$\tilde{S}_4 = 4J_3 - J_2, \quad \tilde{V} = R_4^2 - R_2 R_5, \quad 2\tilde{A} = 2R_6 - 3R_3,$$
$$2\tilde{\eta} = J_4 + 20J_5 - 8J_6$$

$$2\tilde{N} = R_3, \quad 2\tilde{G} = 2R_1^2 - 2J_2 R_3 + 4R_7 + R_8,$$
$$2\tilde{F} = J_2 R_5 + 4R_2 R_3 + 4R_1 R_4,$$

where

$$J_1 = a^\alpha_\alpha, \quad J_2 = a^\alpha_p a^\beta_q \varepsilon_{\alpha\beta}\varepsilon^{pq}, \quad J_3 = a^\alpha a^\beta_{\alpha\beta},$$
$$J_4 = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{sn} a^\delta_{lm} \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}\varepsilon^{mn},$$

$$J_5 = a^{\alpha}_{\gamma} a^{\beta}_{\delta r} a^{\gamma}_{qk} a^{\delta}_{sl} \varepsilon_{\alpha\beta} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl}, \quad J_6 = a^{\alpha}_{pr} a^{\beta}_{\alpha q} a^{\gamma}_{\delta s} a^{\delta}_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs},$$
$$J_7 = a^{\alpha}_{p} a^{\beta}_{\gamma q} a^{\gamma}_{\alpha\beta} \varepsilon^{pq},$$

$$R_1 = x^{\alpha} a^{\beta}_{q} a^{\gamma}_{p\alpha} \varepsilon_{\beta\gamma} \varepsilon^{pq}, \quad R_2 = x^{\alpha} a^{\beta} a^{\gamma}_{\alpha} \varepsilon_{\beta\gamma}, \quad R_3 = x^{\alpha} x^{\beta} a^{\gamma}_{p\alpha} a^{\delta}_{q\beta} \varepsilon_{\gamma\delta} \varepsilon^{pq},$$
$$R_4 = x^{\alpha} x^{\beta} a^{\gamma} a^{\delta}_{\alpha\beta} \varepsilon_{\gamma\delta},$$

$$R_5 = x^{\alpha} x^{\beta} x^{\gamma} a^{\delta}_{\alpha} a^{\mu}_{\beta\gamma} \varepsilon_{\delta\mu}, \quad R_6 = x^{\alpha} x^{\beta} a^{\gamma}_{\alpha\beta} a^{\delta}_{\gamma\delta},$$
$$R_7 = x^{\alpha} x^{\beta} a^{\gamma} a^{\delta}_{\alpha p} a^{\mu}_{\beta s} a^{\nu}_{qr} \varepsilon_{\gamma\delta} \varepsilon_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs},$$

$$R_8 = x^{\alpha} x^{\beta} a^{\gamma}_{\alpha} a^{\delta}_{\beta} a^{\mu}_{pr} a^{\nu}_{qs} \varepsilon_{\gamma\mu} \varepsilon_{\delta\nu} \varepsilon^{pq} \varepsilon^{rs}, \quad R_9 = x^{\alpha} a^{\beta} \varepsilon_{\beta\alpha},$$
$$R_{10} = x^{\alpha} x^{\beta} a^{\gamma}_{\alpha} \varepsilon_{\gamma\beta},$$

$$R_{11} = x^{\alpha} a^{\beta}_{\alpha\beta}, \quad R_{12} = x^{\alpha} x^{\beta} x^{\gamma} a^{\delta}_{\alpha\beta} \varepsilon_{\delta\gamma},$$
$$R_{13} = x^{\alpha} a^{\beta}_{p} a^{\gamma}_{ar} a^{\delta}_{qk} a^{\mu}_{sl} \varepsilon_{\beta\gamma} \varepsilon_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl},$$

and

$$\varepsilon^{11} = \varepsilon^{22} = \varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21} = -\varepsilon_{21} = 1.$$

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