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Indefinite quasilinear elliptic problems with subcritical and supercritical nonlinearities on unbounded domains

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Abstract

By using the fibering method, we study the existence of non-negative solutions for a class of indefinite quasilinear elliptic problems on unbounded domains with noncompact boundary, in the presence of competing subcritical and supercritical lower order nonlinearities.

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1. Introduction

Let Ω be a connected open set in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary and consider the quasilinear elliptic problem

$$-\operatorname{div}\big(\rho(x)|\nabla u|^{p-2}\nabla u\big) = f(x,u), \quad x \in \Omega, \tag{1}$$

$$k(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + h(x)|u|^{p-2}u = 0, \quad x \in \partial\Omega, \tag{2}$$

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where $1 < p < N$, $0 < \rho_0 \leqslant \rho(x) \in L^\infty(\Omega)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $k(x), h(x) \in L^\infty(\partial\Omega)$ and ν is the outward unit normal on $\partial\Omega$.

Equations of the type (1) arise in many and diverse contexts like differential geometry (e.g., in the scalar curvature problem and the Yamabe problem) [16], nonlinear elasticity [9], non-Newtonian fluid mechanics [10], glaciology [18], mathematical biology [4], and elsewhere. As a result, questions concerning the solvability of problem (1), (2) have received great attention, particularly after the seminal work of Brézis and Nirenberg [7]. Among the vast number of results recorded in the literature so far, the case which has been studied extensively concerns the class of *positive* or *non-negative* solutions under a variety of assumptions which usually imply that the nonlinear term $f(x,u)$ does not change sign in Ω. However, an exhaustive review of the existing bibliography is beyond our present scope and the interested reader should consult the survey in [3], as well as the references cited therein.

By contrast, considerably less seems to be known in the case where (1) has *indefinite* character, i.e., when $f(x,u)$ may change sign in Ω. Most notably, the semilinear case $p = 2$ with Neumann conditions was investigated in [6] for the model problem where Ω is bounded, $\rho(x) \equiv 1$ and

$$f(x,u) = -m(x)u + a(x)|u|^{q-2}u, \quad 1 < q < 2^* := \frac{2N}{N-2}.$$

By extending variational techniques to this framework, the authors succeeded there in delivering a rather complete picture of the situation. As it turns out, for instance, in the simplest noncoercive case where $m(x) \equiv 0$ the following two conditions

$$a(x) \text{ changes sign in } \Omega, \qquad \int_\Omega a(x)\,dx < 0,$$

are necessary and sufficient for the existence of a positive solution. Furthermore, their results were extended via degree theory in [5] to the class of equations whose left-hand side involves a general uniformly elliptic linear operator while the right-hand side is of the form $f(x,u) = a(x)g(u)$ and $g(u)$ has precise power-like growth at infinity.

A thorough treatment of the Dirichlet problem when Ω is bounded, $p = 2$, $\rho(x) \equiv 1$ and

$$f(x,u) = \lambda u + a(x)|u|^{q-2}u - b(x)|u|^{s-2}u, \quad 2 < q < s,$$

with $a(\cdot)$, $b(\cdot)$ being non-negative bounded functions, is provided in [2], also from a variational viewpoint. In particular, the analysis in [2] examines the influence of the competing terms $a|u|^{q-2}u$ and $b|u|^{s-2}u$ on the structure of the solution set when λ varies in a neighborhood of the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. When $\lambda = 0$, some interesting nonexistence phenomena are discussed in [17] depending on the smoothness properties of a nonconstant diffusion coefficient $\rho(x)$.

Several of the aforementioned results admit proper extensions to the quasilinear case $p \neq 2$. As a matter of fact, the Dirichlet problem in a bounded domain Ω (as well as the problem in the whole of $\mathbb{R}^N$) for the p-Laplace equation

$$-\Delta_p u := -\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) = \lambda g(x)|u|^{p-2}u + a(x)|u|^{q-2}u,$$

where $1 < p < q < p^* := \frac{Np}{N-p}$ and $g(\cdot)$, $a(\cdot)$ are sign-changing functions, was studied via the fibering method in [14]. More specifically, the analysis carried out there examines the issues of existence, nonexistence and multiplicity of positive solutions in connection to the first eigenvalue of $-\Delta_p$ and the obtained results generalize those concerning the semilinear case $p = 2$, with $g(x) \equiv 1$ [1]. When $\Omega = \mathbb{R}^N$, the same equation was also studied in [11,12] by applying a bifurcation-type approach in conjunction with critical point theory.

At the same time, when Ω is bounded, some results under Neumann or more general boundary conditions are also available. More precisely, the existence and nonexistence of non-negative solutions for the Neumann problem with

$$f(x,u) = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u, \quad q, s \in \left(1, p^*\right),$$

(p, q, s unequal), was studied in [21]; again by the fibering method; assuming that $a(\cdot)$ changes sign while $b(\cdot)$ is non-negative in Ω. In addition, existence and multiplicity results in connection to related eigenvalue problems were established in [8] and [15,22], under Neumann and mixed boundary conditions, respectively.

Our goal in this paper is to investigate the solvability, within the class of non-negative solutions, of the problem

$$-\operatorname{div}\left(\rho(x)|\nabla u|^{p-2}\nabla u\right) = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u - c(x)|u|^{t-2}u, \quad x \in \Omega, \tag{3}$$

$$\rho(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + h(x)|u|^{p-2}u = 0, \quad x \in \partial\Omega, \tag{4}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ with noncompact smooth boundary, $1 < p < N$, $q, s \in (1, p^*)$, $t > p^*$ (p, q, s unequal), $0 < \rho_0 \leqslant \rho(x) \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ and $0 < h(x) \in L^\infty(\partial\Omega)$. We assume further that the function $a(\cdot)$ changes sign while $b(\cdot)$ and $c(\cdot)$ remain non-negative in Ω (more precise conditions will be given in the next section).

The principal feature studied here is the interplay of subcritical and supercritical nonlinearities on the right-hand side of (3) whose competitive dominance affects essentially the indefinite character of the problem. Our approach is based on Pohozaev's fibering strategy [20] whose main advantage lies on the fact that it can separate the algebraic from the topological factors of the problem.

2. Preliminaries

Throughout the paper we assume that the following structure conditions are satisfied:

(Σ_0) $\begin{cases} 1 < p < N,\ 1 < q < p^*,\ 1 < s < p^* \ (p, q, s \text{ unequal}),\ t > p^*, \\ \rho \in L^\infty(\Omega) \cap L^\infty(\partial\Omega) \text{ and } 0 < \rho_0 \leqslant \rho(x) \text{ a.e. in } \Omega. \end{cases}$

(Σ_1) $\operatorname{meas}(\Omega_a^+) > 0$, where $\Omega_a^+ := \{x \in \Omega\colon a^+(x) > 0\}$ and there exist positive constants Λ_1, α_1, such that

$$\left|a(x)\right| \leqslant \frac{\Lambda_1}{(1 + |x|)^{\alpha_1}} \quad \text{a.e. in } \Omega.$$

(Σ_2) There exist positive constants Λ_2 and α_2, such that

$$0 \leqslant b(x) \leqslant \frac{\Lambda_2}{(1 + |x|)^{\alpha_2}} \quad \text{a.e. in } \Omega.$$

(Σ_3) $c \in L^{\infty}(\Omega)$ and $c(x) \geqslant 0$ a.e. in Ω.
(Σ_4) $h \in L^{\infty}(\partial\Omega)$ and

$$\frac{1}{K(1+|x|)^{p-1}} \leqslant h(x) \leqslant \frac{K}{(1+|x|)^{p-1}} \quad \text{a.e. on } \partial\Omega,$$

for some $K > 1$.

Let $w_\theta(x) := \frac{1}{(1+|x|)^\theta}$ where $\theta > 0$. For any $1 < \sigma < +\infty$ we define the weighted Lebesgue space $L^\sigma(w_\theta, \Omega) := \{u\colon \int_\Omega w_\theta |u|^\sigma\, dx < +\infty\}$ equipped with the norm

$$\|u\|_{w_\theta,\sigma} := \left(\int_\Omega w_\theta |u|^\sigma\, dx\right)^{1/\sigma}.$$

For the non-negative measurable function $c\colon \Omega \to \mathbb{R}$, the space $L^t(c, \Omega)$ is similarly defined. We associate with it the semi-norm $|u|_{c,t} := (\int_\Omega c|u|^t\, dx)^{1/t}$.

Furthermore, let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted on Ω. Then the weighted Sobolev space $E_p(\Omega)$ is defined as the completion of $C_\delta^\infty(\Omega)$ under the norm

$$|||u|||_{1,p} := \left(\int_\Omega |\nabla u|^p\, dx + \int_\Omega w_p |u|^p\, dx\right)^{1/p}. \tag{5}$$

The proposition below records all the embedding properties which are relevant to our purposes:

Theorem 1.

(i) *If*

$$p \leqslant \sigma \leqslant p^* := \frac{Np}{N-p} \quad \text{and} \quad N > \theta \geqslant N\left(1 - \frac{\sigma}{p^*}\right),$$

then the embedding $E_p(\Omega) \subseteq L^\sigma(w_\theta, \Omega)$ is continuous. In particular, if the upper bound for σ in the first inequality and the lower bound for θ in the second are strict, then the embedding is compact.

(ii) *If*

$$p \leqslant \tau \leqslant p^*\left(1 - \frac{1}{N}\right) \quad \text{and} \quad N > \eta \geqslant N\left(1 - \frac{\tau}{p^*}\right) - 1,$$

then the trace operator $E_p(\Omega) \to L^\tau(w_\eta, \partial\Omega)$ is continuous. Moreover, if the upper bound for τ in the first inequality and the lower bound for η in the second are strict, then the trace is compact.

(iii) *If*

$$1 < \sigma < p \quad \text{and} \quad \alpha > N\left(1 - \frac{\sigma}{p^*}\right),$$

then the embedding $L^p(w_p, \Omega) \subseteq L^\sigma(w_\alpha, \Omega)$ is continuous.

Proof. The first and second part of the theorem are just a restatement of [19, Theorem 1], while the third is a consequence of the inequality

$$\int_{\Omega} \frac{1}{(1+|x|)^{\alpha}}|u|^{\sigma}\,dx \leqslant \left(\int_{\Omega} \frac{1}{(1+|x|)^{d}}\,dx\right)^{\frac{p-\sigma}{p}} \left(\int_{\Omega} \frac{1}{(1+|x|)^{p}}|u|^{p}\,dx\right)^{\frac{\sigma}{p}},$$

where $d = \frac{(\alpha-\sigma)p}{p-\sigma} > N$. $\square$

Moreover, the following fact holds.

Lemma 2. [19, Lemma 2] *If $\rho(\cdot)$ and $h(\cdot)$ conform with (Σ_0) and (Σ_4), respectively, then*

$$\|u\|_{E_p} := \left(\int_{\Omega} \rho|\nabla u|^p\,dx + \int_{\partial\Omega} h|u|^p\,d\omega\right)^{1/p}$$

defines a norm in $E_p(\Omega)$ which is equivalent to $|||\cdot|||_{1,p}$.

We set $E(\Omega) := E_p(\Omega) \cap L^t(c,\Omega)$. Then, E endowed with the natural norm $\|\cdot\|_E := \|\cdot\|_{E_p} + |\cdot|_{c,t}$ becomes a Banach space.

Assume now that the exponents α_1, α_2 in conditions (Σ_1), (Σ_2), respectively, are such that the embeddings $E_p(\Omega) \subseteq L^q(w_{\alpha_1},\Omega)$ and $E_p(\Omega) \subseteq L^s(w_{\alpha_2},\Omega)$ hold (cf. Theorem 1) and consider the Euler–Lagrange functional $\Phi : E \to \mathbb{R}$ associated with problem (3), (4), defined as

$$\Phi(u) := \frac{1}{p}\left(\int_{\Omega} \rho|\nabla u|^p\,dx + \int_{\partial\Omega} h|u|^p\,d\omega\right) - \frac{1}{q}\int_{\Omega} a|u|^q\,dx + \frac{1}{s}\int_{\Omega} b|u|^s\,dx + \frac{1}{t}\int_{\Omega} c|u|^t\,dx. \tag{6}$$

Clearly, $\Phi(\cdot)$ is well defined in E. Furthermore, by applying standard arguments it can be easily verified that $\Phi \in C^1(E)$ and for any $\phi \in E$

$$\begin{aligned}\langle \Phi'(u), \phi\rangle = {} & \int_{\Omega} \rho|\nabla u|^{p-2}\nabla u \nabla\phi\,dx + \int_{\partial\Omega} h|u|^{p-2}u\phi\,d\omega \\ & - \int_{\Omega} \left(a|u|^{q-2}u - b|u|^{s-2}u - c|u|^{t-2}u\right)\phi\,dx.\end{aligned}$$

As usual, by a weak solution of problem (3), (4) we mean a critical point of $\Phi(\cdot)$.

According to the next proposition, if $a(\cdot)$ decays rapidly enough at infinity then any nontrivial non-negative weak solution of (3), (4) is essentially bounded on compact subsets of Ω and, therefore, strictly positive in Ω.

Lemma 3. *Suppose that conditions (Σ_0)–(Σ_4) hold and let $u \in E$ be a nontrivial non-negative weak solution of (3), (4). If $\alpha_1 > p$ then $w_p u \in L^{\infty}(\Omega)$ and $u > 0$ in Ω.*

Proof. Fix $m > 0$ and set $u_m(x) := \min\{u(x), m\}$. Then, multiply (3) with u_m^{kp+1}, where $k > 0$, integrate over Ω and use (4), (Σ_1)–(Σ_4), to get

$$\begin{aligned}&\int\limits_{\Omega} \rho(x)|\nabla u_m|^{p-2}\nabla u_m \nabla u_m^{kp+1}\,dx + \int\limits_{\partial\Omega} h(x)u^{p-1}u_m^{kp+1}\,d\omega\\ &\quad= \int\limits_{\Omega} \big(a(x)u^{q-1} - b(x)u^{s-1} - c(x)u^{t-1}\big)u_m^{kp+1}\,dx \leqslant \int\limits_{\Omega} a^+(x)u^{kp+q}\,dx. \qquad (7)\end{aligned}$$

On the other hand, by virtue of Theorem 1 and Lemma 2, we obtain

$$\begin{aligned}&\int\limits_{\Omega} \rho(x)|\nabla u_m|^{p-2}\nabla u_m \nabla u_m^{kp+1}\,dx + \int\limits_{\partial\Omega} h(x)u^{p-1}u_m^{kp+1}\,d\omega\\ &\quad= \frac{kp+1}{(k+1)^p}\int\limits_{\Omega} \rho(x)\big|\nabla\big(u_m^{k+1}\big)\big|^p\,dx + \int\limits_{\partial\Omega} h(x)u^{p-1}u_m^{kp+1}\,d\omega\\ &\quad\geqslant \frac{kp+1}{(k+1)^p}\bigg(\int\limits_{\Omega} \rho(x)\big|\nabla\big(u_m^{k+1}\big)\big|^p\,dx + \int\limits_{\partial\Omega} h(x)u_m^{(k+1)p}\,d\omega\bigg)\\ &\quad\geqslant c_1\frac{kp+1}{(k+1)^p}\bigg(\int\limits_{\Omega} \frac{1}{(1+|x|)^p}u_m^{(k+1)p^*}\,dx\bigg)^{\frac{p}{p^*}}, \qquad (8)\end{aligned}$$

where c_1 is a positive constant independent of k and m. Thus, by (7) and (8)

$$c_1\frac{kp+1}{(k+1)^p}\bigg(\int\limits_{\Omega} \frac{1}{(1+|x|)^p}u_m^{(k+1)p^*}\,dx\bigg)^{\frac{p}{p^*}} \leqslant \int\limits_{\Omega} a^+(x)u^{kp+q}\,dx. \qquad (9)$$

Assume first $q > p$. Then, for $q_1 = \frac{p^*}{p^*-q}$ and $\beta, \delta > 0$ to be determined later, we have formally

$$\begin{aligned}\int\limits_{\Omega} a^+(x)u^{kp+q}\,dx &\leqslant \Lambda_1\int\limits_{\Omega} \frac{1}{(1+|x|)^{\alpha_1}}u^{kp+q}\,dx\\ &\leqslant \Lambda_1\bigg(\int\limits_{\Omega}\bigg[\frac{1}{(1+|x|)^{\beta}}\bigg]^{q_1+\delta}dx\bigg)^{\frac{1}{q_1+\delta}}\bigg(\int\limits_{\Omega} u^{p^*}\,dx\bigg)^{\frac{\zeta-p(q_1+\delta)'}{\zeta(q_1+\delta)'}}\\ &\quad\times\bigg(\int\limits_{\Omega}\bigg[\frac{1}{(1+|x|)^{\alpha_1-\beta}}u\bigg]^{(k+1)\zeta}dx\bigg)^{\frac{p}{\zeta}}, \qquad (10)\end{aligned}$$

where $\zeta = \frac{pp^*(q_1+\delta)'}{p^*-(q-p)(q_1+\delta)'}$ and $(q_1+\delta)'$ is the Hölder-conjugate of $q_1+\delta$. Note that, on account of Theorem 1(i), $u \in L^{p^*}(\Omega)$. Therefore, since $\zeta \in (p(q_1+\delta)', p^*)$, by taking $\beta = \frac{\alpha_1(p^*-1)}{p^*}$ and selecting δ large enough so that $\beta(q_1+\delta) > N$, (10) yields

$$\int_\Omega a^+(x)u^{kp+q}\,dx \leqslant c_2\biggl(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1(k+1)\zeta/p^*}}u^{(k+1)\zeta}\,dx\biggr)^{\frac{p}{\zeta}}, \tag{11}$$

for some constant $c_2 > 0$ independent of k and m. Hence, upon combining (9) with (11) and letting $m \to +\infty$, we derive the estimate

$$\biggl(\int_\Omega \frac{1}{(1+|x|)^p}u^{(k+1)p^*}\,dx\biggr)^{\frac{1}{p^*}} \leqslant c_3\frac{k+1}{(kp+1)^{1/p}}\biggl(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1(k+1)\zeta/p^*}}u^{(k+1)\zeta}\,dx\biggr)^{\frac{1}{\zeta}}, \tag{12}$$

where $c_3 = (c_2/c_1)^{1/p}$. The above inequality can now serve as the backbone for a bootstrap procedure. We let $k := k_1 > 0$ so that $(k_1+1)\zeta = p^*$ and apply (12) recursively with $(k_n+1)\zeta = (k_{n-1}+1)p^*$, $n \geqslant 2$. Since $k_n = (\frac{p^*}{\zeta})^n - 1 \to +\infty$ while $\alpha_1 > p$, by continuing as in [13, Lemma 3.2] we eventually deduce that $w_p u \in L^\infty(\Omega)$.

Next assume $q < p$. For $\gamma \in (0, a_1)$ and $\lambda > p$ we have formally

$$\begin{aligned}\int_\Omega a^+(x)u^{kp+q}\,dx &\leqslant \Lambda_1\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}}u^{kp+q}\,dx\\ &\leqslant \Lambda_1\biggl(\int_\Omega \frac{1}{(1+|x|)^{(\alpha_1-\gamma)\lambda/(\lambda-p)}}\,dx\biggr)^{\frac{\lambda-p}{\lambda}}\biggl(\int_\Omega \frac{1}{(1+|x|)^{\gamma\lambda/p}}u^{(kp+q)\frac{\lambda}{p}}\,dx\biggr)^{\frac{p}{\lambda}}.\end{aligned} \tag{13}$$

By restricting now $\lambda \in (p, \min\{\frac{p(N-p)}{N-\alpha_1}, p^*\})$ if $\alpha_1 \in (p, N)$ or $\lambda \in (p, p^*)$ if $\alpha_1 \geqslant N$ and taking $\gamma = \frac{p^2}{\lambda}$, (13) gives

$$\int_\Omega a^+(x)u^{kp+q}\,dx \leqslant \Lambda_1\biggl(\int_\Omega \frac{1}{(1+|x|)^{(\lambda\alpha_1-p^2)/(\lambda-p)}}\,dx\biggr)^{\frac{\lambda-p}{\lambda}}\biggl(\int_\Omega \frac{1}{(1+|x|)^p}u^{(k+\frac{q}{p})\lambda}\,dx\biggr)^{\frac{p}{\lambda}}. \tag{14}$$

Note that $\frac{\lambda\alpha_1-p^2}{\lambda-p} > N$. Thus, on combining (9) with (14) and letting $m \to +\infty$, we obtain the estimate

$$\biggl(\int_\Omega \frac{1}{(1+|x|)^p}u^{(k+1)p^*}\,dx\biggr)^{\frac{1}{p^*}} \leqslant c_4\frac{k+1}{(kp+1)^{1/p}}\biggl(\int_\Omega \frac{1}{(1+|x|)^p}u^{(k+\frac{q}{p})\lambda}\,dx\biggr)^{\frac{1}{\lambda}}, \tag{15}$$

for some $c_4 > 0$ independent of k. As before, (15) provides the key for a bootstrap argument. We let $k := k_1 > 0$ so that $(k_1 + \frac{q}{p})\lambda = p^*$ and apply (15) recursively with $(k_n + \frac{q}{p})\lambda = (k_{n-1}+1)p^*$,

$n \geqslant 2$. Since $k_n > (\frac{p^*}{\lambda})^n - \frac{q}{p} \to +\infty$, by continuing as in [13, Lemma 3.2] we finally conclude that $w_p u \in L^\infty(\Omega)$.

Thus $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and by virtue of the Harnack inequality [24], $u > 0$ in Ω. □

The previous lemma in conjunction with the regularity results obtained in [23] renders immediately the following:

Corollary 4. *Suppose that conditions* (Σ_0)–(Σ_4) *hold and let* $u \in E$ *be a non-negative solution of* (3), (4). *If* $\rho \in C^1(\Omega)$ *and* $\alpha_1 > p$ *then* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha \in (0,1)$.

3. Main results

Our search for critical points of the functional $\Phi(\cdot)$ will be based on the so-called one-parameter fibration of the underlying space E. To this end, we define the *extended functional* $\mathcal{F}: \mathbb{R} \times E \to \mathbb{R}$ by setting for any $r \in \mathbb{R}$ and $v \in E$

$$\mathcal{F}(r,v) := \Phi(rv) = \frac{|r|^p}{p}\|v\|^p_{E_p} - \frac{|r|^q}{q}\mathcal{A}(v) + \frac{|r|^s}{s}\mathcal{B}(v) + \frac{|r|^t}{t}\mathcal{C}(v), \tag{16}$$

where

$$\mathcal{A}(v) := \int_\Omega a(x)|v|^q\,dx, \qquad \mathcal{B}(v) := \int_\Omega b(x)|v|^s\,dx, \qquad \mathcal{C}(v) := \int_\Omega c(x)|v|^t\,dx.$$

If $u = rv$ is a critical point of $\Phi(\cdot)$, then necessarily

$$\mathcal{F}_r(r,v) = 0, \tag{17}$$

which is referred to as the *bifurcation equation* of the fibering scheme. If $r \neq 0$ then (17) is equivalent to

$$\Theta(r,v) = \|v\|^p_{E_p}, \tag{18}$$

where

$$\Theta(r,v) := \mathcal{A}(v)|r|^{q-p} - \mathcal{B}(v)|r|^{s-p} - \mathcal{C}(v)|r|^{t-p}. \tag{19}$$

Assume $r = r(v) \neq 0$ solves (18) for any v in some open subset $G \subseteq E \setminus \{0\}$ and $r \in C^1(G)$. Then the *reduced functional*

$$\begin{aligned}\widehat{\Phi}(v) &:= \Phi\big(r(v)v\big)\\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\mathcal{A}(v)\big|r(v)\big|^q + \left(\frac{1}{s} - \frac{1}{p}\right)\mathcal{B}(v)\big|r(v)\big|^s + \left(\frac{1}{t} - \frac{1}{p}\right)\mathcal{C}(v)\big|r(v)\big|^t \end{aligned} \tag{20}$$

is well defined and continuously differentiable in G. To compensate the introduction of the fibering parameter $r \in \mathbb{R}$, we impose also the constraint

$$H(v) = 1,$$

where $H : E \to \mathbb{R}$ is some appropriately chosen functional. The essence of the fibering method relies on the following key fact.

Lemma 5. [14, Lemma 3.4] *Let $H : E \to \mathbb{R}$ be a functional which is continuously Fréchet-differentiable in $E \setminus \{0\}$ and satisfies the conditions*:

$$\langle H'(v), v\rangle \neq 0 \quad \text{if } H(v) = 1,$$

and $H(0) = 0$. If $v \neq 0$ is a conditional critical point of $\widehat{\Phi}(\cdot)$ under the constraint $H(v) = 1$, then $u := r(v)v$ is a nonzero critical point of $\Phi(\cdot)$.

Throughout the paper, the fibering functional of our choice will be

$$H(v) := \|v\|_E = \|v\|_{E_p} + \mathcal{C}(v)^{1/t}. \tag{21}$$

Clearly, $\langle H'(v), v\rangle = 1$ for every $v \in S^1$, where

$$S^1 := \{v \in E\colon\ H(v) = 1\}. \tag{22}$$

Hence, by virtue of Lemma 5, the problem of finding solutions for (3), (4) will be reduced in the sequel to that of locating critical points of $\widehat{\Phi}(\cdot)$ on S^1. Note that since $\Theta(r, v)$ is even with respect to r, it suffices to look only for positive solutions $r(v)$ of (18). Thus, in what follows, $|r|$ will be tacitly replaced by r. Moreover, if $v \neq 0$ is a critical point of $\widehat{\Phi}(\cdot)$ then, by (18) and (20), so is $|v|$ and, by Lemma 5, $u = r(|v|)|v|$ is a nontrivial non-negative solution of (3), (4).

We proceed by defining the set

$$G_1 := \{v \in E\colon\ \mathcal{A}(v) > 0\}. \tag{23}$$

Due to assumption (Σ_1), $G_1 \neq \emptyset$.

Depending on the relative ordering of the exponents p, q, s, we partition our analysis into six separate cases.

Case 1: $q < p < s$.

Theorem 6. *Let assumptions (Σ_0)–(Σ_4) be satisfied where $q < p < s$,*

$$\alpha_1 \in \left(N\left(1 - \frac{q}{p^*}\right), +\infty\right) \quad \text{and} \quad \alpha_2 \in \left(N\left(1 - \frac{s}{p^*}\right), N\right).$$

Then problem (3), (4) admits a nontrivial non-negative solution $u \in E$. In particular, $u > 0$ in Ω. Furthermore, if $\rho \in C^1(\Omega)$ then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

Proof. Let us first rewrite the bifurcation equation (18) in the form

$$\|v\|^p_{E_p} r^{p-q} + \mathcal{B}(v) r^{s-q} + \mathcal{C}(v) r^{t-q} = \mathcal{A}(v). \tag{24}$$

It is clear that for every $v \in G_1$, (24) admits a unique solution $r(v) > 0$. Moreover, $r \in C^1(G_1)$ by the implicit function theorem [25, Theorem 4.B, p. 150]. At the same time, on account of

Theorem 1(iii), $\mathcal{A}(\cdot)$ is bounded on S^1. On the other hand, if $v \in S^1$ then, by (22), we should either have $\|v\|_{E_p} \geqslant \frac{1}{2}$ or $\mathcal{C}(v)^{1/t} \geqslant \frac{1}{2}$. Hence, by (24),

$$\min\left\{\frac{1}{2}r(v)^{p-q}, \frac{1}{2^t}r(v)^{t-q}\right\} \leqslant \mathcal{A}(v), \tag{25}$$

which implies that $r(\cdot)$ is bounded on $G_1 \cap S^1$. As a consequence, $\widehat{\Phi}(\cdot)$ is also bounded on $G_1 \cap S^1$. Furthermore, it is easily checked that for every $\mu > 0$ and every $v \in G_1$, $\mu r(\mu v) = r(v)$ while

$$\widehat{\Phi}(\mu v) = \widehat{\Phi}(v). \tag{26}$$

We now set

$$M := \inf_{v \in G_1 \cap S^1} \widehat{\Phi}(v), \tag{27}$$

and note that, by (20), $M < 0$. If $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_1 \cap S^1$, then, for a subsequence (still denoted by $\{v_n\}_{n\in\mathbb{N}}$), $v_n \to \tilde{v}$ weakly in $E_p(\Omega)$ and $c^{1/t}v_n \to c^{1/t}\tilde{v}$ weakly in $L^t(\Omega)$. By invoking Theorem 1(i), (iii) and our current hypotheses, we can assume further that $\{\mathcal{A}(v_n)\}_{n\in\mathbb{N}}$ and $\{\mathcal{B}(v_n)\}_{n\in\mathbb{N}}$ converge to $\mathcal{A}(\tilde{v}) \geqslant 0$ and $\mathcal{B}(\tilde{v}) \geqslant 0$, respectively, while, by (21) and (22),

$$0 \leqslant \mathcal{C}(\tilde{v}) \leqslant \liminf_{n\to+\infty} \mathcal{C}(v_n) \leqslant 1. \tag{28}$$

Moreover, up to a new subsequence, $r(v_n) \to \tilde{r}$. Therefore, by employing (16),

$$\Phi(\tilde{r}\tilde{v}) \leqslant \liminf_{n\to+\infty} \Phi\big(r(v_n)v_n\big) = M < 0, \tag{29}$$

and so $\tilde{r}\tilde{v} \neq 0$. Also, by applying (24) for the sequence $\{v_n\}_{n\in\mathbb{N}}$ and passing to the limit, we obtain

$$\|\tilde{v}\|_{E_p}^p \tilde{r}^{p-q} + \mathcal{B}(\tilde{v})\tilde{r}^{s-q} + \mathcal{C}(\tilde{v})\tilde{r}^{t-q} \leqslant \mathcal{A}(\tilde{v}), \tag{30}$$

which shows that $\mathcal{A}(\tilde{v}) > 0$ since, otherwise, $\tilde{r}\tilde{v} = 0$. Hence, $\tilde{v} \in G_1$ while the unique solution $r(\tilde{v})$ of (18) satisfies

$$\|\tilde{v}\|_{E_p}^p r(\tilde{v})^{p-q} + \mathcal{B}(\tilde{v})r(\tilde{v})^{s-q} + \mathcal{C}(\tilde{v})r(\tilde{v})^{t-q} = \mathcal{A}(\tilde{v}). \tag{31}$$

On comparing (30) with (31), we deduce $\tilde{r} \leqslant r(\tilde{v})$. We claim that $\tilde{r} = r(\tilde{v})$. Indeed, if $\tilde{r} < r(\tilde{v})$, then, since the function $z \mapsto \Phi(z\tilde{v})$, $z \in (0, r(\tilde{v})]$, is strictly decreasing, in view of (29), we should have

$$M \geqslant \Phi(\tilde{r}\tilde{v}) > \Phi\big(r(\tilde{v})\tilde{v}\big) = \widehat{\Phi}(\tilde{v}). \tag{32}$$

But then, by virtue of (26) and (32),

$$\widehat{\Phi}\left(\frac{\tilde{v}}{\|\tilde{v}\|_E}\right) = \widehat{\Phi}(\tilde{v}) < M,$$

which contradicts (27). As a result, by passing to the limit in (24) (when applied to $\{v_n\}_{n\in\mathbb{N}}$) and using (31), we deduce

$$\lim_{n\to+\infty}\left\{\|v_n\|_{E_p}^p+\mathcal{C}(v_n)r(v_n)^{t-p}\right\}=\|\tilde{v}\|_{E_p}^p+\mathcal{C}(\tilde{v})r(\tilde{v})^{t-p}, \tag{33}$$

from which it readily follows that $\|v_n\|_{E_p}\to\|\tilde{v}\|_{E_p}$ and $\mathcal{C}(v_n)\to\mathcal{C}(\tilde{v})$. Consequently, $\tilde{v}\in S^1$ and $\widehat{\Phi}(\tilde{v})=M$. Because now $|\tilde{v}|$ is also a minimizer of $\widehat{\Phi}(\cdot)$, we may assume $\tilde{v}\geqslant 0$. Then, Lemma 5 guarantees that $u=r(\tilde{v})\tilde{v}$ is a nontrivial non-negative solution of (3), (4). In particular, by verifying that $\alpha_1>p$ and recalling Lemma 3, $u>0$ in Ω. Furthermore, if $\rho\in C^1(\Omega)$ then $u\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, by Corollary 4. □

Case 2: $q<s<p$.

Arguing exactly as in the proof of the previous theorem one can establish:

Theorem 7. *Let assumptions* (Σ_0)–(Σ_4) *be satisfied where* $q<s<p$,

$$\alpha_1\in\left(N\left(1-\frac{q}{p^*}\right),+\infty\right)\quad\text{and}\quad\alpha_2\in\left(N\left(1-\frac{s}{p^*}\right),+\infty\right).$$

Then problem (3), (4) *admits a nontrivial non-negative solution* $u\in E$. *In particular,* $u>0$ *in* Ω. *Furthermore, if* $\rho\in C^1(\Omega)$ *then* $u\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha\in(0,1)$.

Case 3: $s<q<p$.

Theorem 8. *Let assumptions* (Σ_0)–(Σ_4) *be satisfied where* $s<q<p$,

$$\alpha_1\in\left(N\left(1-\frac{q}{p^*}\right),+\infty\right)\quad\text{and}\quad\alpha_2\in\left(N\left(1-\frac{s}{p^*}\right),+\infty\right).$$

Moreover, suppose that the following condition holds:

(Σ_5) $V:=(\operatorname{supp}a^+\setminus\operatorname{supp}b)^o\neq\emptyset.$

Then problem (3), (4) *admits a nontrivial non-negative solution* $u\in E$. *In particular,* $u>0$ *in* Ω. *Furthermore, if* $\rho\in C^1(\Omega)$ *then* $u\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha\in(0,1)$.

Proof. Let $v\in G_1$. If $\mathcal{B}(v)=0$ then the bifurcation equation (18) has a unique solution $r(v)>0$ which satisfies

$$\|v\|_{E_p}^p r^{p-q}+\mathcal{C}(v)r^{t-q}=\mathcal{A}(v).$$

On the other hand, if $\mathcal{B}(v)>0$ then the function $\Theta(\cdot,v)$ (see (19)) has a unique critical point $r_*:=r_*(v)$ which corresponds to a global maximum and satisfies

$$(p-s)\mathcal{B}(v)=(p-q)\mathcal{A}(v)r_*^{q-s}+(t-p)\mathcal{C}(v)r_*^{t-s}. \tag{34}$$

Clearly, if $\|v\|_{E_p}^p < \Theta(r_*(v), v)$ then (18) has exactly two positive solutions $r_1(v)$, $r_2(v)$ with $r_1(v) < r_*(v) < r_2(v)$. We set $r := r(v)$ to be the unique solution of (18) in case $\mathcal{B}(v) = 0$ or the greater solution r_2 in case $\mathcal{B}(v) > 0$. Note that, if $\mathcal{B}(v) > 0$ then

$$r^{p-s+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)r^{q-s} - (s-p)\mathcal{B}(v) - (t-p)\mathcal{C}(v)r^{t-s},$$

which, on account of (34), yields

$$r^{p-s+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)\big(r^{q-s} - r_*^{q-s}\big) - (t-p)\mathcal{C}(v)\big(r^{t-s} - r_*^{t-s}\big) < 0,$$

while if $\mathcal{B}(v) = 0$,

$$r^{p+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)r^q - (t-p)\mathcal{C}(v)r^t < 0.$$

Thus, $r(\cdot)$ is continuously differentiable by the implicit function theorem. We now define

$$G_2 := \big\{v \in G_1\colon \mathcal{B}(v) = 0\big\} \cup \big\{v \in G_1\colon \mathcal{B}(v) > 0 \text{ and } \|v\|_{E_p}^p < \Theta\big(r_*(v), v\big)\big\}, \tag{35}$$

where $r_*(v)$ is determined by (34). Under assumptions (Σ_1) and (Σ_5), $G_2 \neq \emptyset$ since for any $v \in E$ with $\operatorname{supp} v \subseteq V$ there holds $\mathcal{A}(v) > 0$ and $\mathcal{B}(v) = 0$. We claim that G_2 is also open. Indeed, let $\hat{v} \in G_2$ and assume that there is a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq E \setminus G_2$ with $v_n \to \hat{v}$ strongly in E. By a straightforward continuity argument we may assume, without loss of generality, that $\mathcal{B}(\hat{v}) = 0$ while $\mathcal{B}(v_n) > 0$ for every $n \in \mathbb{N}$. Therefore,

$$\|v_n\|_{E_p}^p \geqslant \Theta\big(r_*(v_n), v_n\big) \quad \text{for every } n \in \mathbb{N}. \tag{36}$$

Moreover, since $\mathcal{A}(\hat{v}) > 0$, on account of (34), $r_*(v_n) \to 0$. At the same time, on combining (19) with (34), we get

$$\Theta\big(r_*(v_n), v_n\big) = \frac{q-s}{p-s}\mathcal{A}(v_n)r_*(v_n)^{q-p} - \frac{t-s}{p-s}\mathcal{C}(v_n)r_*(v_n)^{t-p},$$

and so $\lim_{n\to+\infty}\Theta(r_*(v_n), v_n) = +\infty$, in contradiction to (36).

Next, we set

$$M := \inf_{v\in G_2\cap S^1} \widehat{\Phi}(v).$$

By the same reasoning employed in the proof of Theorem 6, it follows that $r(\cdot)$, as well as $\widehat{\Phi}(\cdot)$ are bounded on $G_2 \cap S^1$ while, on account of (20) and (Σ_5), $M < 0$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a minimizing sequence in $G_2 \cap S^1$. By invoking Theorem 1(i) and (iii) and our current hypotheses, we can assume that, for a subsequence (still denoted by $\{v_n\}_{n\in\mathbb{N}}$), there is $\tilde{v} \in E$ so that $\mathcal{A}(v_n) \to \mathcal{A}(\tilde{v}) \geqslant 0$, $\mathcal{B}(v_n) \to \mathcal{B}(\tilde{v}) \geqslant 0$ while (28) holds. Furthermore, up to a new subsequence, $r(v_n) \to \tilde{r}$. In particular, $\tilde{r} > 0$ for if $\tilde{r} = 0$ then, by (20), $M = \lim_{n\to+\infty}\widehat{\Phi}(v_n) = 0$; a contradiction. In return, this implies $\mathcal{A}(\tilde{v}) > 0$. Indeed, from the bifurcation equation (18) we have

$$\|v_n\|_{E_p}^p r(v_n)^{p-q} \leqslant \mathcal{A}(v_n),$$

and by passing to the limit,

$$\|\tilde{v}\|_{E_p}^p \tilde{r}^{p-q} \leqslant \liminf_{n\to+\infty}\big(\|v_n\|_{E_p}^p r(v_n)^{p-q}\big) \leqslant \lim_{n\to+\infty} \mathcal{A}(v_n) = \mathcal{A}(\tilde{v}). \tag{37}$$

Thus, if $\mathcal{A}(\tilde{v}) = 0$ then $\tilde{v} = 0$ since $\tilde{r} > 0$. However, this induces contradiction because, upon using (16), we should have $0 = \Phi(0) \leqslant \liminf_{n\to+\infty} \Phi(r(v_n)v_n) = M$.

We proceed now to show that $\tilde{v} \in G_2$. Of course, if $\mathcal{B}(\tilde{v}) = 0$ then $\tilde{v} \in G_2$ automatically. Let us therefore assume $\mathcal{B}(\tilde{v}) > 0$. By our earlier discussion, for every v_n with n large enough, the function $\Theta(\cdot, v_n)$ has a unique critical point $r_*(v_n) > 0$ which corresponds to a global maximum and satisfies

$$(p-s)\mathcal{B}(v_n) = (p-q)\mathcal{A}(v_n)r_*(v_n)^{q-s} + (t-p)\mathcal{C}(v_n)r_*(v_n)^{t-s}. \tag{38}$$

Since $r_*(v_n) < r(v_n)$, the sequence $\{r_*(v_n)\}_{n\in\mathbb{N}}$ is also bounded. Thus, up to a further subsequence, $r_*(v_n) \to \tilde{r}_*$. In particular, by our assumption $\mathcal{B}(\tilde{v}) > 0$ and (38), $\tilde{r}_* > 0$. Hence,

$$\begin{aligned}
\|\tilde{v}\|_{E_p}^p &\leqslant \limsup_{n\to+\infty} \|v_n\|_{E_p}^p \leqslant \limsup_{n\to+\infty} \Theta\big(r_*(v_n), v_n\big) \\
&\leqslant \limsup_{n\to+\infty}\big\{\mathcal{A}(v_n)r_*(v_n)^{q-p} - \mathcal{B}(v_n)r_*(v_n)^{s-p}\big\} - \liminf_{n\to+\infty} \mathcal{C}(v_n)r_*(v_n)^{t-p} \\
&\leqslant \mathcal{A}(\tilde{v})\tilde{r}_*^{q-p} - \mathcal{B}(\tilde{v})\tilde{r}_*^{s-p} - \mathcal{C}(\tilde{v})\tilde{r}_*^{t-p} = \Theta(\tilde{r}_*, \tilde{v}),
\end{aligned} \tag{39}$$

and so, a fortiori,

$$\|\tilde{v}\|_{E_p}^p \leqslant \Theta\big(r_*(\tilde{v}), \tilde{v}\big). \tag{40}$$

We claim that strict inequality holds in (40). Indeed, let us suppose

$$\|\tilde{v}\|_{E_p}^p = \Theta\big(r_*(\tilde{v}), \tilde{v}\big). \tag{41}$$

Since $\tilde{r} > 0$, by applying (18) for $v = v_n$ and passing to the limit, we also obtain

$$\begin{aligned}
\|\tilde{v}\|_{E_p}^p &\leqslant \limsup_{n\to+\infty} \|v_n\|_{E_p}^p \leqslant \limsup_{n\to+\infty} \Theta\big(r(v_n), v_n\big) \\
&\leqslant \limsup_{n\to+\infty}\big\{\mathcal{A}(v_n)r(v_n)^{q-p} - \mathcal{B}(v_n)r(v_n)^{s-p}\big\} - \liminf_{n\to+\infty} \mathcal{C}(v_n)r(v_n)^{t-p} \\
&\leqslant \mathcal{A}(\tilde{v})\tilde{r}^{q-p} - \mathcal{B}(\tilde{v})\tilde{r}^{s-p} - \mathcal{C}(\tilde{v})\tilde{r}^{t-p} = \Theta(\tilde{r}, \tilde{v}).
\end{aligned} \tag{42}$$

Consequently, on comparing (39), (41) and (42), we should have $\tilde{r} = \tilde{r}_* = r_*(\tilde{v})$. On the other hand, by passing to the limit in (38) and recalling that $r_*(\tilde{v})$ satisfies

$$(p-s)\mathcal{B}(\tilde{v}) = (p-q)\mathcal{A}(\tilde{v})r_*(\tilde{v})^{q-s} + (t-p)\mathcal{C}(\tilde{v})r_*(\tilde{v})^{t-s},$$

we immediately infer that, along a subsequence, $\mathcal{C}(v_n) \to \mathcal{C}(\tilde{v})$ and

$$\mathcal{B}(\tilde{v}) = \frac{p-q}{p-s}\mathcal{A}(\tilde{v})\tilde{r}^{q-s} + \frac{t-p}{p-s}\mathcal{C}(\tilde{v})\tilde{r}^{t-s}. \tag{43}$$

But then, by combining (20) with (43), we deduce

$$M = \lim_{n\to+\infty} \widehat{\Phi}(v_n) = \frac{(q-s)(p-q)}{qsp}\mathcal{A}(\tilde{v})\tilde{r}^q + \frac{(t-s)(t-p)}{tsp}\mathcal{C}(\tilde{v})\tilde{r}^t > 0,$$

which is a contradiction. Therefore, $\tilde{v} \in G_2$ as claimed.

We shall show in the sequel that $\tilde{r} = r(\tilde{v})$. Since $\tilde{v} \neq 0$, we set $\mu := \|\tilde{v}\|_E^{-1}$ and assume first $\mathcal{B}(\tilde{v}) > 0$. Clearly, $\mu\tilde{v} \in G_1 \cap S^1$ while, by using (34), it is easy to verify

$$\mu r_*(\mu\tilde{v}) = r_*(\tilde{v}). \tag{44}$$

Moreover, on account of (19), (35) and (44), we have

$$\|\tilde{v}\|_{E_p}^p < \Theta\big(r_*(\tilde{v}), \tilde{v}\big) = \Theta\big(\mu r_*(\mu\tilde{v}), \tilde{v}\big) = \mu^{-p}\Theta\big(r_*(\mu\tilde{v}), \mu\tilde{v}\big),$$

and so

$$\|\mu\tilde{v}\|_{E_p}^p < \Theta\big(r_*(\mu\tilde{v}), \mu\tilde{v}\big),$$

which implies $\mu\tilde{v} \in G_2 \cap S^1$. Furthermore, by (18) and (19), $r(\mu\tilde{v})$ satisfies

$$\Theta\big(\mu r(\mu\tilde{v}), \tilde{v}\big) = \|\tilde{v}\|_{E_p}^p = \Theta\big(r(\tilde{v}), \tilde{v}\big). \tag{45}$$

Thus, since $\mu r(\mu\tilde{v}) > \mu r_*(\mu\tilde{v}) = r_*(\tilde{v})$ and $r(\tilde{v}) > r_*(\tilde{v})$, (45) yields

$$\mu r(\mu\tilde{v}) = r(\tilde{v}). \tag{46}$$

On the other hand, by virtue of (42),

$$\Theta\big(r(\tilde{v}), \tilde{v}\big) = \|\tilde{v}\|_{E_p}^p \leqslant \Theta(\tilde{r}, \tilde{v}), \tag{47}$$

and so $\tilde{r} \leqslant r(\tilde{v})$. Suppose $\tilde{r} < r(\tilde{v})$. Then, upon using (16), (46) and noticing that the function

$$\psi(z) := \frac{\partial}{\partial z}\Phi(z\tilde{v}) = z^{p-1}\big\{\|\tilde{v}\|_{E_p}^p - \Theta(z, \tilde{v})\big\}, \quad z > \tilde{r}, \tag{48}$$

is strictly negative for $z \in (\tilde{r}, r(\tilde{v}))$, we deduce

$$M = \liminf_{n\to+\infty} \Phi\big(r(v_n)v_n\big) \geqslant \Phi(\tilde{r}\tilde{v}) > \Phi\big(r(\tilde{v})\tilde{v}\big) = \Phi\big(r(\mu\tilde{v})\mu\tilde{v}\big) = \widehat{\Phi}(\mu\tilde{v}),$$

which is impossible. To show that $\tilde{r} = r(\tilde{v})$ when $\mathcal{B}(\tilde{v}) = 0$ is easier. Indeed, since the function $z \mapsto \Theta(z, \tilde{v})$, $z > 0$, is now strictly decreasing, the bifurcation equation (18) implies directly $\mu r(\mu\tilde{v}) = r(\tilde{v})$. The desired conclusion then follows by applying the same reasoning as before and observing that the function $\psi(z)$, as defined by (48), is again strictly negative for $z \in (\tilde{r}, r(\tilde{v}))$. Hence, by passing to the limit in (18) (when applied to $\{v_n\}_{n\in\mathbb{N}}$), we rederive (33) from which we infer that $\tilde{v} \in S^1$ and $\widehat{\Phi}(\tilde{v}) = M$. Arguing similarly to Case 1, one can now establish all the assertions of the theorem. $\square$

In the analysis of the remaining three cases we find it necessary to strengthen our hypothesis (Σ_3) by assuming that the function $c(\cdot)$ is essentially bounded away from zero on compact subsets of $\Omega_a^+ := \{x \in \Omega\colon\ a^+(x) > 0\}$ while it may decay to zero on Ω_a^+, as $|x| \to +\infty$, at a controlled rate if, of course, Ω_a^+ is unbounded. For this purpose, (Σ_3) will be replaced henceforth by the following condition:

(Σ_6) $c \in L^\infty(\Omega)$, $c(\cdot) \geqslant 0$ a.e. in $\Omega \setminus \Omega_a^+$ while

$$\frac{\Gamma}{(1+|x|)^\gamma} \leqslant c(x) \quad \text{a.e. in } \Omega_a^+, \tag{49}$$

where $\gamma \in (0, \frac{t}{q}(\alpha_1 - N(1-\frac{q}{t})))$ and $\Gamma > 0$.

Apart from this change, the rest of our hypotheses (Σ_0), (Σ_1), (Σ_2) and (Σ_4) will continue to remain in force. Note, however, that if (Σ_6) holds then an indirect restriction on the possible values of the exponent α_1 in (Σ_1) is imposed, namely,

$$\alpha_1 > N\left(1 - \frac{q}{t}\right).$$

Nevertheless, as we shall see in the sequel, this is merely a reflection of the fact that when $q > p$, the derivation of a priori bounds for solutions of the bifurcation equation (18) under the fibering constraint (22), hinges on a rather delicate balance between the competing effects induced by the terms $a(x)|u|^{q-2}u$ and $-c(x)|u|^{t-2}u$ on the right-hand side of (3). Another simple, yet important, implication of (Σ_1) and (Σ_6) is that $\mathcal{C}(v) > 0$ whenever $\mathcal{A}(v) > 0$.

Case 4: $s < p < q$.

Let $v \in G_1$. Then the function $\Theta(\cdot, v)$ has a unique critical point $r_* := r_*(v) > 0$ which corresponds to a global maximum and satisfies

$$(q-p)\mathcal{A}(v)r_*^{q-t} + (p-s)\mathcal{B}(v)r_*^{s-t} = (t-p)\mathcal{C}(v). \tag{50}$$

In particular, by combining (19) with (50), we obtain

$$\Theta\big(r_*(v), v\big) = \frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p} - \frac{t-s}{t-p}\mathcal{B}(v)r_*(v)^{s-p}. \tag{51}$$

Therefore, for every $v \in G_3$, where

$$G_3 := \big\{v \in G_1\colon\ \|v\|_{E_p}^p < \Theta\big(r_*(v), v\big)\big\}, \tag{52}$$

the bifurcation equation (18) has exactly two positive solutions $r_1(v)$, $r_2(v)$ where $r_1(v) < r_*(v) < r_2(v)$. We set $r := r(v)$ to be the greater solution r_2. Since

$$r^{p-t+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)r^{q-t} + (p-s)\mathcal{B}(v)r^{s-t} - (t-p)\mathcal{C}(v),$$

by virtue of (50), we find

$$r^{p-t+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)\big(r^{q-t} - r_*^{q-t}\big) + (p-s)\mathcal{B}(v)\big(r^{s-t} - r_*^{s-t}\big) < 0.$$

Hence, $r \in C^1(G_3)$ by the implicit function theorem.

We now define the set

$$G_4 := \left\{ v \in G_1\colon\ \|v\|_{E_p}^p < \frac{p}{q}\frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p} - \frac{p}{s}\frac{t-s}{t-p}\mathcal{B}(v)r_*(v)^{s-p} \right\}, \tag{53}$$

and assume $G_4 \neq \emptyset$. Since $\frac{p}{q} < 1$ and $\frac{p}{s} > 1$, by using (51), we immediately see that $G_4 \subseteq G_3$ and so G_3 is also nonempty. At the same time, by using (50), the following scaling property holds

$$\mu r_*(\mu v) = r_*(v), \quad \text{for any } \mu > 0 \text{ and } v \in G_1. \tag{54}$$

Consequently, if $v \in G_4$ then, from (53), $\mu v \in G_4$, as well. In particular, by choosing $\mu = \|v\|_E^{-1}$, $G_4 \cap S^1 \neq \emptyset$. On the other hand, $\widehat{\Phi}(v) < 0$ for any $v \in G_4$. Indeed, recalling that $r(v) > r_*(v)$, (53) yields directly

$$\|v\|_{E_p}^p < \frac{p}{q}\frac{t-q}{t-p}\mathcal{A}(v)r(v)^{q-p} - \frac{p}{s}\frac{t-s}{t-p}\mathcal{B}(v)r(v)^{s-p},$$

which, after a straightforward rearrangement using (18) and (20), proves the assertion.

We proceed to show that under the validity of (Σ_6),

$$\inf_{v\in G_3\cap S^1} C(v) > 0. \tag{55}$$

As it turns out, (55) provides the key ingredient needed in the investigation of the constrained minimization problem for the reduced functional $\widehat{\Phi}(\cdot)$ in the present, as well as in the remaining two cases. In proving (55), note first that if $v \in G_3$ then, (19) in conjunction with (50) and (52) imply

$$0 < \Theta\big(r_*(v), v\big) = \frac{t-s}{p-s}\left(\frac{q-s}{t-s}\mathcal{A}(v) - \mathcal{C}(v)r_*(v)^{t-q}\right)r_*(v)^{q-p},$$

and so

$$r_*(v) < \left(\frac{q-s}{t-s}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}. \tag{56}$$

Hence, by using (51), (52), (56) and noticing that $q > p$, we get

$$\begin{aligned}\|v\|_{E_p}^p &< \frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p} - \frac{t-s}{t-p}\mathcal{B}(v)r_*(v)^{s-p} \\ &\leqslant \frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p} < \frac{t-q}{t-p}\mathcal{A}(v)\left(\frac{q-s}{t-s}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{q-p}{t-q}}.\end{aligned} \tag{57}$$

In particular, if $v \in G_3 \cap S^1$ then, on account of (21), (22) and (57), we deduce

$$\mathcal{A}(v)^{t-p} > \left(\frac{t-p}{t-q}\right)^{t-q}\left(\frac{t-s}{q-s}\right)^{q-p}\mathcal{C}(v)^{q-p}\big(1-\mathcal{C}(v)^{\frac{1}{t}}\big)^{p(t-q)}. \tag{58}$$

On the other hand, by employing assumptions (Σ_1) and (Σ_6), we obtain via Hölder's inequality

$$\begin{aligned}
\mathcal{A}(v) &\leqslant \int\limits_{\Omega_a^+} a^+(x)|v|^q\,dx \leqslant \Lambda_1 \int\limits_{\Omega_a^+} \frac{1}{(1+|x|)^{\alpha_1}}|v|^q\,dx \\
&\leqslant \Lambda_1 \Bigg(\int\limits_{\Omega_a^+} \frac{1}{(1+|x|)^{(\alpha_1-\frac{\gamma q}{t})\frac{t}{t-q}}}\,dx\Bigg)^{\frac{t-q}{t}} \Bigg(\int\limits_{\Omega_a^+} \frac{1}{(1+|x|)^{\gamma}}|v|^t\,dx\Bigg)^{\frac{q}{t}} \\
&\leqslant \frac{\Lambda_1}{\Gamma^{\frac{q}{t}}} \Bigg(\int\limits_{\Omega_a^+} \frac{1}{(1+|x|)^{(\alpha_1-\frac{\gamma q}{t})\frac{t}{t-q}}}\,dx\Bigg)^{\frac{t-q}{t}} \mathcal{C}(v)^{\frac{q}{t}}.
\end{aligned} \tag{59}$$

Observe now that, since $(\alpha_1 - \frac{\gamma q}{t})\frac{t}{t-q} > N$, the last integral in (59) converges. Thus, on combining (58) with (59), we find out that for some constant $\xi > 0$, depending only on p, q, s, t, Λ_1 and Γ,

$$\mathcal{C}(v) > \Bigg\{1 + \xi\Bigg(\int\limits_{\Omega_a^+} \frac{1}{(1+|x|)^{(\alpha_1-\frac{\gamma q}{t})\frac{t}{t-q}}}\,dx\Bigg)^{\frac{t-p}{tp}}\Bigg\}^{-t} \quad \text{for any } v \in G_3 \cap S^1, \tag{60}$$

whence (55) follows.

Next, suppose $\alpha_1 \in (N(1-\frac{q}{t}), N)$ and $\alpha_2 \in (N(1-\frac{s}{p^*}), +\infty)$. Then, due to Theorem 1(i), (iii) and since $t > p^*$, we infer that $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ are bounded on S^1. At the same time, from the bifurcation equation (cf. (24)) we have

$$r(v) < \left(\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}, \tag{61}$$

and so, by virtue of (55), $r(\cdot)$ must be bounded on $G_3 \cap S^1$. As a result, $\widehat{\Phi}(\cdot)$ is also bounded on $G_3 \cap S^1$.

We are now ready to consider the variational problem

$$M := \inf_{v \in G_3 \cap S^1} \widehat{\Phi}(v) < 0.$$

If $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_3 \cap S^1$ then, by invoking again Theorem 1(i) and (iii), there exists $\tilde{v} \in E$ such that, for a subsequence (not relabelled), $\mathcal{A}(v_n) \to \mathcal{A}(\tilde{v}) \geqslant 0$, $\mathcal{B}(v_n) \to \mathcal{B}(\tilde{v}) \geqslant 0$ while, by (21), (22) and (60),

$$0 < \mathcal{C}(\tilde{v}) \leqslant \liminf_{n\to+\infty} \mathcal{C}(v_n) \leqslant 1. \tag{62}$$

Since $r_*(v_n) < r(v_n)$, on account of (61), up to a further subsequence, $r(v_n) \to \tilde{r}$ and $r_*(v_n) \to \tilde{r}_*$. Clearly, $\tilde{r} > 0$ since $M = \lim_{n\to+\infty} \widehat{\Phi}(v_n) < 0$. In return, $\mathcal{A}(\tilde{v}) > 0$ because, otherwise, (61) and (62), would imply $\tilde{r}=0$. Furthermore, $\tilde{r}_*>0$ since, by (50),

$$r_*(v_n) \geqslant \left(\frac{q-p}{t-p}\frac{\mathcal{A}(v_n)}{\mathcal{C}(v_n)}\right)^{\frac{1}{t-q}}.$$

We claim that $\tilde{v}\in G_3$. Indeed, if not, then, by applying the same arguments as in the proof of Theorem 8, we would have $\tilde{r}=\tilde{r}_*=r_*(\tilde{v})$, while, along a subsequence, $\mathcal{C}(v_n)\to\mathcal{C}(\tilde{v})$ where, by (50),

$$\mathcal{C}(\tilde{v}) = \frac{q-p}{t-p}\mathcal{A}(\tilde{v})\tilde{r}^{q-t} + \frac{p-s}{t-p}\mathcal{B}(\tilde{v})\tilde{r}^{s-t}. \tag{63}$$

This, however, leads to contradiction since, (20) in conjunction with (63) yields

$$M = \lim_{n\to+\infty}\widehat{\Phi}(v_n) = \frac{(t-q)(q-p)}{tqp}\mathcal{A}(\tilde{v})\tilde{r}^q + \frac{(t-s)(p-s)}{tsp}\mathcal{B}(\tilde{v})\tilde{r}^s > 0.$$

A similar reasoning as in Case 3 shows that $r(\tilde{v})=\tilde{r}$. Finally, by passing to the limit in (18) we rederive (33) which, in return, implies $\tilde{v}\in S^1$ and $\widehat{\Phi}(\tilde{v})=M$. Hence, by recalling Lemmas 5, 3 and Corollary 4, we have the following:

Theorem 9. *Let $s<p<q$ and assume that conditions (Σ_0), (Σ_1), (Σ_2), (Σ_4) and (Σ_6) hold, with*

$$\alpha_1 \in \left(N\left(1-\frac{q}{t}\right), N\right) \quad \text{and} \quad \alpha_2 \in \left(N\left(1-\frac{s}{p^*}\right), +\infty\right).$$

Assume further that the set G_4 (as defined by (53)) is nonempty. Then problem (3), (4) admits a nontrivial non-negative solution $u\in E$. In particular, if $\alpha_1>p$ then $u>0$ in Ω. If also $\rho\in C^1(\Omega)$ then $u\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha\in(0,1)$.

Remark 10. It is easy to verify that $G_4\neq\emptyset$ if, for example, condition (Σ_5) holds (cf. Theorem 8) and $a^+(\cdot)$ is large with respect to $c(\cdot)$ in the sense that the inequality

$$\mathcal{A}(v)^{t-p} > \zeta\,\mathcal{C}(v)^{q-p}\|v\|_{E_p}^{p(t-q)},$$

is satisfied for some $v\in G_1$ with $\operatorname{supp} v\subseteq V$, where $\zeta=(\frac{q}{p})^{t-q}\frac{(t-p)^{t-p}}{(t-q)^{t-q}(q-p)^{q-p}}$.

Case 5: $p<q<s$.

Since this case shares some of the characteristics presented already in Case 4, we shall only give a brief outline of the analysis by paying more attention to those details which are different. Indeed, if $v\in G_1$ then also here the function $\Theta(\cdot,v)$ has a unique critical point $r_*:=r_*(v)>0$ which corresponds to a global maximum and satisfies

$$(q-p)\mathcal{A}(v) = (s-p)\mathcal{B}(v)r_*^{s-q} + (t-p)\mathcal{C}(v)r_*^{t-q}. \tag{64}$$

Moreover, if $\mathcal{B}(v) > 0$ then from (64) we get

$$r_*(v) < \left(\frac{q-p}{s-p}\frac{\mathcal{A}(v)}{\mathcal{B}(v)}\right)^{\frac{1}{s-q}},$$

and so

$$\begin{aligned} r_*(v)^{p-q}\Theta\big(r_*(v), v\big) &= \frac{t-q}{t-p}\mathcal{A}(v) - \frac{t-s}{t-p}\mathcal{B}(v)r_*(v)^{s-q} \\ &> \frac{t-q}{t-p}\mathcal{A}(v) - \frac{t-s}{t-p}\frac{q-p}{s-p}\mathcal{A}(v) = \frac{s-q}{s-p}\mathcal{A}(v) > 0, \end{aligned} \tag{65}$$

while if $\mathcal{B}(v) = 0$,

$$r_*(v) = \left(\frac{q-p}{t-p}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}},$$

and

$$r_*(v)^{p-q}\Theta\big(r_*(v), v\big) = \frac{t-q}{t-p}\mathcal{A}(v) > 0. \tag{66}$$

Thus, if $v \in G_3$ (cf. (52)), the bifurcation equation (18) has exactly two positive solutions $r_1(v)$, $r_2(v)$ with $r_1(v) < r_*(v) < r_2(v)$. As before, we set $r := r(v)$ to be the greater solution r_2. Furthermore, by using (64), we find

$$r^{p-q+1}\Theta_r(r, v) = (s-p)\mathcal{B}(v)\big(r_*^{s-q} - r^{s-q}\big) + (t-p)\mathcal{C}(v)\big(r_*^{t-q} - r^{t-q}\big) < 0,$$

and so $r \in C^1(G_3)$. Next, we define the set

$$G_5 := \left\{v \in G_1\colon\ \|v\|_{E_p}^p < \frac{p}{q}\frac{s-q}{s-p}\mathcal{A}(v)r_*(v)^{q-p}\right\}, \tag{67}$$

and assume $G_5 \neq \emptyset$. Since $\frac{p}{q} < 1$ and $\frac{s-q}{s-p} < \frac{t-q}{t-p}$, by virtue of (65) and (66), we immediately see that $G_5 \subseteq G_3$ and so $G_3 \neq \emptyset$, as well. Moreover, by (67), $G_5 \cap S^1 \neq \emptyset$ since $r_*(\cdot)$ enjoys again the scaling property (54). Furthermore, $\widehat{\Phi}(v) < 0$ for any $v \in G_5$. Indeed, since $r(v) > r_*(v)$, (67) yields

$$\|v\|_{E_p}^p < \frac{p}{q}\frac{s-q}{s-p}\mathcal{A}(v)r(v)^{q-p},$$

which, after a straightforward rearrangement using (18), becomes

$$\left(\frac{1}{p} - \frac{1}{q}\right)\mathcal{A}(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right)\mathcal{B}(v)r(v)^s + \left(\frac{1}{s} - \frac{1}{p}\right)\mathcal{C}(v)r(v)^t < 0.$$

The assertion now follows on observing that $\frac{1}{t} < \frac{1}{s}$ and recalling (20). On the other hand, given (Σ_6), the procedure employed in proving (55) carries over to the present case with only one modification: the needed upper estimate for $r_*(v)$ (cf. (56)) is now implied directly by (64);

namely, we have

$$r_*(v) \leqslant \left(\frac{q-p}{t-p}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}.$$

As a consequence, one eventually rederives estimate (60), albeit with a different constant $\xi > 0$. At the same time, if $\alpha_1 \in (N(1-\frac{q}{t}), N)$ and $\alpha_2 \in (N(1-\frac{s}{p^*}), N)$ then, by Theorem 1(i), $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ are bounded on S^1. Moreover, by virtue of (55) and (61), $r(\cdot)$, as well as $\widehat{\Phi}(\cdot)$, are bounded on $G_3 \cap S^1$.

The investigation of the variational problem

$$M := \inf_{v \in G_3 \cap S^1} \widehat{\Phi}(v) < 0,$$

can now be pursued along lines similar to those already presented in Case 4. As a matter of fact, the necessary modifications in the analysis are rather straightforward. Nevertheless, for the convenience of the reader who wishes to keep up with all the details, we find it worth mentioning that in proving $\tilde{v} \in G_3$:

(i) there is no need to show also $\tilde{r}_* > 0$, because here p is the smallest of all the exponents involved and so no problem arises when passing to the limit in (19), if $v = v_n$, $n \in \mathbb{N}$;
(ii) having shown that $\tilde{r} = \tilde{r}_* = r_*(\tilde{v})$ and, along a subsequence, $\mathcal{C}(v_n) \to \mathcal{C}(\tilde{v})$, one must eliminate $\mathcal{A}(\tilde{v})$ from the expression concerning $\lim_{n\to+\infty} \widehat{\Phi}(v_n)$ to reach the impossibility

$$M = \lim_{n\to+\infty} \widehat{\Phi}(v_n) = \frac{(s-q)(s-p)}{sqp}\mathcal{B}(\tilde{v})\tilde{r}^s + \frac{(t-q)(t-p)}{tqp}\mathcal{C}(\tilde{v})\tilde{r}^t > 0.$$

In summary, the following holds:

Theorem 11. *Let $p < q < s$ and assume that conditions (Σ_0), (Σ_1), (Σ_2), (Σ_4) and (Σ_6) hold, with*

$$\alpha_1 \in \left(N\left(1-\frac{q}{t}\right), N\right) \quad \text{and} \quad \alpha_2 \in \left(N\left(1-\frac{s}{p^*}\right), N\right).$$

Assume further that the set G_5 (as defined by (67)) is nonempty. Then problem (3), (4) admits a nontrivial non-negative solution $u \in E$. In particular, if $\alpha_1 > p$ then $u > 0$ in Ω. If also $\rho \in C^1(\Omega)$ then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

Remark 12. It is easily checked that $G_5 \neq \emptyset$ if, for example, the same conditions stated in Remark 10 hold but with $\zeta = (\frac{q}{p})^{t-q}(\frac{s-p}{s-q})^{t-q}(\frac{t-p}{q-p})^{q-p}$.

Case 6: $p < s < q$.

This is the most complicated case of all for it combines, on the one hand, the delicate competition induced by the subcritical and supercritical nonlinearities on the right-hand side of (3) with a richer qualitative behavior of the function $\Theta(\cdot, v)$, on the other. As a matter of fact, it becomes here necessary to examine in parallel two essentially different subcases.

Let $v \in G_1$ and assume first $\mathcal{B}(v) > 0$. We set

$$T(r, v) := r^{p-s}\Theta(r, v) = \mathcal{A}(v)r^{q-s} - \mathcal{B}(v) - \mathcal{C}(v)r^{t-s}, \quad r \geqslant 0. \tag{68}$$

Clearly, $T(0, v) = -\mathcal{B}(v) < 0$ while $\lim_{r\to+\infty} T(r, v) = -\infty$. Moreover, $T(\cdot, v)$ has a unique critical point $\bar{r}(v) > 0$ which corresponds to a global maximum and

$$T\big(\bar{r}(v), v\big) = \frac{t-q}{t-s}\mathcal{A}(v)\bar{r}(v)^{q-s} - \mathcal{B}(v), \tag{69}$$

where

$$\bar{r}(v) := \left(\frac{q-s}{t-s}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}. \tag{70}$$

Therefore, on account of (68), $\Theta(r, v) > 0$ for some $r > 0$ if and only if $T(\bar{r}(v), v) > 0$, that is

$$\bar{r}(v) > \hat{r}(v) := \left(\frac{t-s}{t-q}\frac{\mathcal{B}(v)}{\mathcal{A}(v)}\right)^{\frac{1}{q-s}}. \tag{71}$$

Let condition (71) holds. Then, an elementary investigation of the function

$$\varphi(r) := r^{p-s+1}\Theta_r(r, v) = (q-p)\mathcal{A}(v)r^{q-s} - (s-p)\mathcal{B}(v) - (t-p)\mathcal{C}(v)r^{t-s}$$

shows that $\varphi(\cdot)$ has exactly two positive roots $r_{1^*}(v)$ and $r_{2^*}(v)$ with $r_{1^*}(v) < r_{2^*}(v)$. As a matter of fact, $r_{1^*}(v)$ is a point of local minimum of $\Theta(\cdot, v)$ with $\Theta(r_{1^*}(v), v) < 0$ and $r_{2^*}(v)$ is a point of global maximum of $\Theta(\cdot, v)$ with $\Theta(r_{2^*}(v), v) > 0$. We set $r_*(v) := r_{2^*}(v)$ and claim that

$$\bar{r}(v) < r_*(v). \tag{72}$$

Indeed, by (68) we have

$$r^{s-p}T_r(r, v) = \Theta_r(r, v) + (p-s)\frac{\Theta(r, v)}{r}, \quad r > 0,$$

and since $T_r(\bar{r}(v), v) = 0$ while $\Theta(\bar{r}(v), v) = \bar{r}(v)^{s-p}T(\bar{r}(v), v) > 0$, we deduce

$$\Theta_r\big(\bar{r}(v), v\big) = (s-p)\frac{\Theta(\bar{r}(v), v)}{\bar{r}(v)} > 0,$$

whence (72) immediately follows.

Next, let $v \in G_1$ and assume $\mathcal{B}(v) = 0$. Then $\Theta(\cdot, v)$ has a unique critical point $r_*(v) > 0$ which corresponds to a global maximum and

$$\Theta\big(r_*(v), v\big) = \frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p}, \tag{73}$$

where

$$r_*(v) := \left(\frac{q-p}{t-p}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}. \tag{74}$$

In particular, by direct verification, (72) holds in this case, as well. Because now $r_*(v)$ must satisfy in both cases the equation $\Theta_r(r,v)=0$ or, equivalently,

$$(q-p)\mathcal{A}(v) = (s-p)\mathcal{B}(v)r_*(v)^{s-q} + (t-p)\mathcal{C}(v)r_*(v)^{t-q}, \tag{75}$$

the following inequality is always true

$$r_*(v) \leqslant \left(\frac{q-p}{t-p}\frac{\mathcal{A}(v)}{\mathcal{C}(v)}\right)^{\frac{1}{t-q}}. \tag{76}$$

From the above discussion we conclude that if G_3 is defined as in (52) and if condition (71) holds, then for any $v \in G_3$ Eq. (18) has two positive solutions $r_1(v), r_2(v)$ with $r_1(v) < r_*(v) < r_2(v)$. As before, we set $r(v) := r_2(v)$. Since $\Theta_r(r,v) < 0$ for all $r > r_*(v)$, by the implicit function theorem, $r \in C^1(G_3)$.

We now define the set

$$G_6 := \left\{v \in G_1\colon\ \|v\|_{E_p}^p < \frac{p}{s}\frac{t-s}{t-p}\left(\frac{s}{q}\frac{t-q}{t-s}\mathcal{A}(v)\bar{r}(v)^{q-s} - \mathcal{B}(v)\right)\bar{r}(v)^{s-p}\right\}, \tag{77}$$

and assume $G_6 \neq \emptyset$. Observe that the nonemptiness of G_6 implies a stronger condition than (71), namely,

$$\bar{r}(v) > \left(\frac{q}{s}\frac{t-s}{t-q}\frac{\mathcal{B}(v)}{\mathcal{A}(v)}\right)^{\frac{1}{q-s}}.$$

We proceed to show that $G_6 \subseteq G_3$. Indeed, let $v \in G_6$ and assume first $\mathcal{B}(v) > 0$. Then, since $\frac{p}{s} < 1$, $\frac{t-s}{t-p} < 1$ and $\frac{s}{q} < 1$, from (68), (69), (72) and (77) we obtain

$$\begin{aligned}
\|v\|_{E_p}^p &< \left(\frac{s}{q}\frac{t-q}{t-s}\mathcal{A}(v)\bar{r}(v)^{q-s} - \mathcal{B}(v)\right)\bar{r}(v)^{s-p} < \left(\frac{t-q}{t-s}\mathcal{A}(v)\bar{r}(v)^{q-s} - \mathcal{B}(v)\right)\bar{r}(v)^{s-p} \\
&= T\big(\bar{r}(v), v\big)\bar{r}(v)^{s-p} = \Theta\big(\bar{r}(v), v\big) < \Theta\big(r_*(v), v\big),
\end{aligned}$$

and so $v \in G_3$. Next, let $v \in G_6$ and assume $\mathcal{B}(v) = 0$. Then, from (72), (73) and (77),

$$\|v\|_{E_p}^p < \frac{p}{q}\frac{t-q}{t-p}\mathcal{A}(v)\bar{r}(v)^{q-p} < \frac{t-q}{t-p}\mathcal{A}(v)r_*(v)^{q-p} = \Theta\big(r_*(v), v\big),$$

which implies again $v \in G_3$. Notice also that $G_6 \cap S^1 \neq \emptyset$ since $\bar{r}(v)$ enjoys the scaling property: $\mu\bar{r}(\mu v) = \bar{r}(v)$ for any $\mu > 0$ and $v \in G_1$. Moreover, since $\bar{r}(v) < r_*(v) < r(v)$, for any $v \in G_6$ we clearly have

$$\|v\|_{E_p}^p < \frac{p}{s}\frac{t-s}{t-p}\left(\frac{s}{q}\frac{t-q}{t-s}\mathcal{A}(v)r(v)^{q-s} - \mathcal{B}(v)\right)r(v)^{s-p},$$

which, after using (18) and (20), is equivalent to

$$\widehat{\Phi}(v) < 0, \quad v \in G_6.$$

On the other hand, by employing the upper estimate (76) for $r_*(v)$, we easily confirm (retracing the corresponding steps already presented in Case 4) that the key inequality (55) holds here, as well. If now $\alpha_1 \in (N(1-\frac{q}{t}), N)$ and $\alpha_2 \in (N(1-\frac{s}{p^*}),\ N)$ then, by Theorem 1(i), $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ are bounded on S^1. Furthermore, by virtue of (55) and (61), $r(\cdot)$, as well as $\widehat{\Phi}(\cdot)$, are bounded on $G_3 \cap S^1$.

The ground is thus well-prepared to consider the variational problem

$$M := \inf_{v \in G_3 \cap S^1} \widehat{\Phi}(v) < 0. \tag{78}$$

If $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_3 \cap S^1$ then, by invoking again Theorem 1(i), there exists $\tilde{v} \in E$ such that, for a subsequence (not relabelled), $\mathcal{A}(v_n) \to \mathcal{A}(\tilde{v}) \geqslant 0$, $\mathcal{B}(v_n) \to \mathcal{B}(\tilde{v}) \geqslant 0$ while (62) holds. Moreover, up to a further subsequence, $r(v_n) \to \tilde{r}$ and $r_*(v_n) \to \tilde{r}_*$. Due to (20), (61), (62) and (78), $\tilde{r} > 0$ and $\mathcal{A}(\tilde{v}) > 0$. In particular, by virtue of (71) and (72), we must have

$$\tilde{r} \geqslant \tilde{r}_* \geqslant \hat{r}(\tilde{v}) = \left(\frac{t-s}{t-q}\frac{\mathcal{B}(\tilde{v})}{\mathcal{A}(\tilde{v})}\right)^{\frac{1}{q-s}}. \tag{79}$$

We claim $\tilde{v} \in G_3$. Indeed, if not, then, as in the proof of Theorem 8, $\tilde{r} = \tilde{r}_* = r_*(\tilde{v})$ where $r_*(\tilde{v})$ is the (unique) point of global maximum of $\Theta(\cdot, \tilde{v})$ and satisfies

$$(q-p)\mathcal{A}(\tilde{v}) = (s-p)\mathcal{B}(\tilde{v})r_*(\tilde{v})^{s-q} + (t-p)\mathcal{C}(\tilde{v})r_*(\tilde{v})^{t-q}.$$

Consequently, by passing to the limit in (75) when $v = v_n$, $n \in \mathbb{N}$, and noticing that the function $z \mapsto (q-p)\mathcal{A}(\tilde{v})z^{q-t} - (s-p)\mathcal{B}(\tilde{v})z^{s-t}$, $z > 0$, is strictly decreasing for $z \geqslant \hat{r}(\tilde{v})$, we infer that, for a subsequence, $\mathcal{C}(v_n) \to \mathcal{C}(\tilde{v})$ where,

$$\mathcal{C}(\tilde{v}) = \frac{q-p}{t-p}\mathcal{A}(\tilde{v})\tilde{r}^{q-t} - \frac{s-p}{t-p}\mathcal{B}(\tilde{v})\tilde{r}^{s-t}. \tag{80}$$

But then, on combining (20) with (80), we deduce

$$0 > M = \lim_{n\to+\infty}\widehat{\Phi}(v_n) = \frac{(t-q)(q-p)}{tqp}\left(\tilde{r}^{q-s} - \frac{q}{s}\frac{s-p}{q-p}\frac{t-s}{t-q}\frac{\mathcal{B}(\tilde{v})}{\mathcal{A}(\tilde{v})}\right)\mathcal{A}(\tilde{v})\tilde{r}^s,$$

which is impossible upon verifying that $\frac{q}{s}\frac{s-p}{q-p} < 1$ and taking into account the lower estimate (79). A similar reasoning as in Case 3 shows $r(\tilde{v}) = \tilde{r}$, $\tilde{v} \in S^1$ and $\widehat{\Phi}(\tilde{v}) = M$. Hence, by recalling Lemmas 5, 3 and Corollary 4, we have established the following

Theorem 13. *Let $p < s < q$ and assume that conditions* (Σ_0), (Σ_1), (Σ_2), (Σ_4) *and* (Σ_6) *hold, with*

$$\alpha_1 \in \left(N\left(1 - \frac{q}{t}\right), N\right) \quad \text{and} \quad \alpha_2 \in \left(N\left(1 - \frac{s}{p^*}\right), N\right).$$

Assume further that the set G_6 (as defined by (77)*) is nonempty. Then problem* (3), (4) *admits a nontrivial non-negative solution $u \in E$. In particular, if $\alpha_1 > p$ then $u > 0$ in Ω. If also $\rho \in C^1(\Omega)$ then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.*

Remark 14. It is readily confirmed that $G_6 \neq \emptyset$ if, for example, the same conditions stated in Remark 10 hold but with $\zeta = (\frac{q}{p})^{t-q}(\frac{t-p}{t-q})^{t-q}(\frac{t-s}{q-s})^{q-p}$.

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